You may try question 1 as an exercise. However, in this assignment you are only required to do question 2. The deadline is Saturday, October 31, 11:59pm.

Assignment on Nonlinear Equations

1. One classical method for solving *cubics* is Cardano's solution. The *cubic* equation $x^3+ax^2+bx+c=0$ is transformed to a reduced form $y^3+py+q=0$ by the substitution $x=y-\frac{a}{3}$. The coefficients in the reduced form are $p=b-\frac{a^2}{3}$, $q=c-\frac{ab}{3}+2\left(\frac{a}{3}\right)^3$. A real root of the *reduced* form is given by $y_1=\left[-\frac{q}{2}+s\right]^{\frac{1}{3}}+\left[-\frac{q}{2}-s\right]^{\frac{1}{3}}$, where $s=\left[\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2\right]^{\frac{1}{2}}$. Then a real root of the original equation is given by $x_1=y_1-\frac{a}{3}$. The other two roots can be found by similar formulas or by *factoring* out x_1 (*deflating* x_1) and solving the resulting *quadratic* equation.

   Using Matlab do the following:

   (a) Apply Cardano's method to find the *real root* of $x^3+3x^2+\delta^2x+3\delta^2=0$, for various values of δ. Investigate the loss of accuracy from roundoff for large $\delta=10^{10:19}$, observe the results when δ is about the reciprocal of *machine unit [Hint: You can use Matlab's "eps" for this].*

   (b) Apply Newton's method to the same equation for the same values of δ. Investigate the effects of roundoff error and the choice of starting value. Explain the rapid convergence of Newton's method for very large δ.

2. Consider the ecliptic which is the plane of the Earth's orbit around the Sun with the Sun at the origin $(0,0)$ occupying one of the foci of the Earth's elliptical orbit. Also assume that Mercury's orbit is on the same plane, and let

You do only this question.

$$
\begin{aligned}
x_M(t) &= -11.9084+57.9117\cos(2\pi t/87.97)\\
y_M(t) &= 56.6741\sin(2\pi t/87.97)
\end{aligned}
$$

$$
\begin{aligned}
x_E(t) &= -2.4987+149.6041\cos(2\pi t/365.25)\\
y_E(t) &= 149.5832\sin(2\pi t/365.25)
\end{aligned}
$$

   be the coordinates of Mercury (M) and Earth (E) at time t (in Earth days). The planets M, E are in *Opposition* if they, along with Sun (S) appear on the same line, with the Sun in the middle, that is M-S-E. They are in *Conjunction* if the Sun appears on the edge, that is S-M-E. Assuming that M and E are in *conjunction* at $t=0$, which is what the equations imply, write MATLAB code that will use the Secant method to compute the time of 10 consecutive *Oppositions* and the spacing between them (in Earth days). Produce a table with these information. The code should include a MATLAB function that will accept as input a function, say $f(x)$, two initial approximations of a possible root and a tolerance and will produce the root.

   Note that you are basically asked to compute ten roots of a specific nonlinear equation that you should generate. Although initial approximations for every root should be found, the possibility of finding a pattern that may lead to the initial approximation of more than one roots (perhaps all) should be investigated.

Appendix In the first question you will need to compute the cubic root of a real number x. When x is negative, Matlab returns a complex number. Although this is not wrong, returning a real value may be a more useful choice, in my view. Next we discuss the underlying mathematical problem and explain how to get the real root.

Background Consider the complex number $z = x + iy$, with $(x, y) \in \mathbb{R} \times \mathbb{R}$, then $|z| = \sqrt{x^2 + y^2}$. If $|z| \neq 0$ we may express z in trigonometric form as

$$z = \sqrt{x^2 + y^2} \left(\frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}} \right).$$

Since now

$$\frac{x}{\sqrt{x^2 + y^2}} \in [-1, 1], \quad \frac{y}{\sqrt{x^2 + y^2}} \in [-1, 1]$$

and

$$\left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2 = 1$$

we may conclude that

$$\exists \theta \in (-\pi, \pi] : \cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}. \tag{1}$$

There is an infinite number of angles $\varphi \equiv 2k\pi + \theta$ with $k \in \mathbb{Z}$, that satisfy (1). We call the angle in $(-\pi, \pi]$ that satisfies (1) the **argument** of z, and we denote it by

$$\arg z = \theta \wedge \theta \in (-\pi, \pi].$$

Note that if $\theta \in \{0, \pi\}$, then $z \in \mathbb{R}$.

Example 1 *If $z = 1 + i\sqrt{3}$ then $\theta = \arg z$ should satisfy the system*

$$\left\{ \begin{array}{c} \cos\theta = \frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \\ \theta \in (-\pi, \pi] \end{array} \right\} \Longleftrightarrow \theta = \frac{\pi}{3} = 60^\circ$$

We may therefore conclude that

Remark 2 *Every complex number $z = x + iy \neq 0$ has a unique* ***trigonometric form***

$$z = \rho (\cos\theta + i \sin\theta)$$

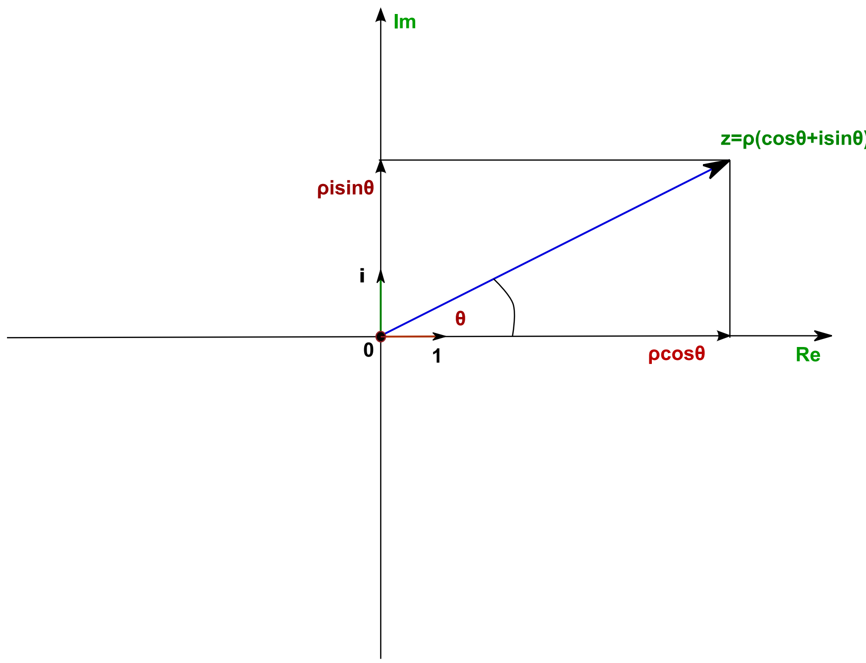
with $(\rho, \theta) = (|z|, \arg z) \in \mathbb{R}^+ \times (-\pi, \pi]$ and vise versa.

Remark 3 *If $z = 0$ we may write*

$$z = |z| (\cos\theta + i \sin\theta), \textit{ for } \forall\theta \in (-\pi, \pi]$$

Therefore if $z = 0$ its ***argument*** *cannot be defined uniquely.*

Remark 4 *Every complex number $z = \rho(\cos\theta + i\sin\theta)$ can also be visualized as a vector in the complex plane, which also makes clear the fact that the* ***argument*** *of $z = 0$ is undefined.*



Remark 5 *A complex number $z = x + iy \neq 0$ can be written in a more* ***general trigonometric form*** *as*

$$z = \rho\left(\cos\left(2k\pi + \theta\right) + i\sin\left(2k\pi + \theta\right)\right),\ \rho = |z| \wedge \theta = \arg z \wedge k \in \mathbb{Z}$$

Remark 6 *Assume $z_1 = \rho_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = \rho_2(\cos\theta_2 + i\sin\theta_2)$ then the following properties are satisfied:*

1. $\rho_1(\cos\theta_1 + i\sin\theta_1) = \rho_2(\cos\theta_2 + i\sin\theta_2) \Leftrightarrow \{\rho_1 = \rho_2 \wedge \theta_1 = \theta_2 + 2k\pi\}$
2. $z_1 z_2 = \rho_1\rho_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
3. *If $z_1 \neq 0$ then $z_1^{-1} = \rho_1^{-1}(\cos(-\theta_1) + i\sin(-\theta_1))$*
4. *If $z_2 \neq 0$ then $\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$*
5. *$z_1^n = \rho_1^n(\cos(n\theta_1) + i\sin(n\theta_1))$, with $n \in \mathbb{Q}$ (set of rational numbers). For $n \in \mathbb{N}_0$ we have De Moivre's theorem.*

Theorem 7 *If $a = \rho(\cos\theta + i\sin\theta) \in \mathbb{C}$, with $a \neq 0$ the equation*

$$z^n = a$$

has exactly n distinct roots given by

$$z = \sqrt[n]{\rho}\left(\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right),\ \textit{for } k = 0 : n - 1$$

Definition 8 *The* n*th root of a complex number* $a = \rho(\cos\theta + i\sin\theta)$ *is a* correspondence $\sqrt[n]{}$ *that maps a number from* $\mathbb{C}$ *to* n *distinct numbers in* $\mathbb{C}$*, and it is defined by*

$$\sqrt[n]{\rho}\left(\cos\left(\frac{2k\pi+\theta}{n}\right)+i\sin\left(\frac{2k\pi+\theta}{n}\right)\right),\ \textit{for } k = 0 : n-1$$

Note that $\sqrt[n]{}$ *is* ***not*** *a* function*, since it maps one number to* n *numbers.*

Example 9 *Compute* $\sqrt[3]{8i}$*.*

Solution 10 *Since* $8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ *we have*

$$\begin{aligned}
\sqrt[3]{8i} &= \sqrt[3]{8}\left(\cos\left(\frac{2k\pi+\frac{\pi}{2}}{3}\right)+i\sin\left(\frac{2k\pi+\frac{\pi}{2}}{3}\right)\right),\ \textit{for } k\in\{0,1,2\}\\
&= 2\left(\cos\left(\frac{2k\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{2k\pi}{3}+\frac{\pi}{6}\right)\right),\ \textit{for } k\in\{0,1,2\}\\
&\in \left\{\sqrt{3}+i, -\sqrt{3}+i, -2i\right\}
\end{aligned}$$

Remark 11 *In the special case where* $a \in \mathbb{R}-\{0\}$*,* $\sqrt[n]{a}$ *may also be given by*

$$\sqrt[n]{a} = \left\{\begin{array}{c} \sqrt[n]{|a|}\left(\cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right)\right),\ \textit{if } a>0,\ \textit{since } \theta=0 \\ \sqrt[n]{|a|}\left(\cos\left(\frac{2k\pi+\pi}{n}\right)+i\sin\left(\frac{2k\pi+\pi}{n}\right)\right),\ \textit{if } a<0,\ \textit{since } \theta=\pi \end{array}\right.,\ \textit{for } k=0:n-1$$

Example 12 *Compute* $\sqrt[3]{-8}$*.*

Solution 13 *Since* $-8 = 8(\cos\pi + i\sin\pi)$ *we have*

$$\begin{aligned}
\sqrt[3]{-8} &= \sqrt[3]{8}\left(\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)\right),\ \textit{for } k=0:2\\
&= \left\{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right),\ 2(\cos\pi+i\sin\pi),\ 2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)\right\}\\
&= \left\{1+i\sqrt{3},\ -2,\ 1-i\sqrt{3}\right\}
\end{aligned}$$

When $\sqrt[3]{-8}$ *was given to* Matlab*, it returned* $1+i\sqrt{3}$ *which is not wrong but rather surprising. That is, one may wonder why not returning the real value* -2 *instead. I think* Matlab *(wrongly, in my view) attempts to return a "positive" value (a value in the first quarter of the complex plane), following probably a different definition where* $\sqrt[2]{a}$ *is the positive number* $b : b^2 = a$*. This however, makes* Matlab's *function ambiguous. For example* $\sqrt[9]{-8}$ *has two such "positive" values, namely* $1.1839+0.43092i$ *and* $0.62996+1.0911i$*, yet* Matlab *returns* $1.1839+0.43092i$*. Note however that if one computes* $-\sqrt[9]{8}$ Matlab *will return* -1.2599*, which suggests a way of fixing the "problem" in the assignment.*