Let A = $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \dots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, x = $\begin{pmatrix} x_1 \\ \vdots \\ x_2 \end{pmatrix}$ and b = $\begin{pmatrix} b_1 \\ \vdots \\ b_2 \end{pmatrix}$ the matrix of coefficients of

equation Ax = b. For a specific column j∈ $\{1\colon n\}$ of A, give the solution $\tilde{x}$ of the system

$$\tilde{A}\tilde{x} = a_j$$

In terms of x where $\tilde{A} = A - a_j e_j^T + b e_j^T$(b has been interchanged with the j-th column of A)
As we can calculate the determinant of A, The idea that came to my mind for finding the answer for $\tilde{x}_j \in (\tilde{x}_1, \dots, \tilde{x}_n)$ is to calculate the determinant of $\tilde{A}$ and then calculate

$$\tilde{x}_j = (detA)^{-1}\det(A \overset{j}{\leftarrow} b)$$

Where $A \overset{j}{\leftarrow} b$ is the matrix $\tilde{A}$(b has been interchanged with the j-th column of A), equally :

$$\tilde{x}_j = (detA)^{-1}\det(\tilde{A})$$

For proving that the above equation satisfying the result, first of all we need to define two theorems.

Theorem 1.
A system of equations has a solution if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix.

Proof.
We know rank of the matrix is dimension of the span of columns of the matrix. Now if Ax=b has solution, then it means that some linear combination of columns of A gives us b, which implies that b lies in $span(A)$ and so $rank(A|b) = rank(A)$. You can argue similarly in the reverse direction.

Theorem 2.
(I) If the matrix A^* is obtained from a square matrix A by swapping two rows or two columns, then $detA^* = -detA$.
(II) If the matrix A^* is obtained by A multiplying the i-th row, or the j-th column by the scalar c, then $detA^* = c\, detA$.
(III) If the matrix A^* is obtained by A by replacing the k-th row A_k by A_k + cA_i, or the k-th column A^k by A^k + cA^i, with $i \neq k$, then $detA^* = detA$.

Proof.
We prove all statements by induction. The case $n = 2$ is easily check directly.
We assume $n \geq 3$ and (I)–(III) are true for all matrices of size $n - 1 \times n - 1$.

I) We prove the case when $j = i + 1$, i.e., we are interchanging two consecutive rows.
Let $l \in \{1, \dots, n\} \setminus \{i, j\}$. Then $A(l)$ is obtained from $B(l)$by interchanging two of its rows and by our assumption

$$cof(A)_{1,l} = -cof(B)_{1,l}.$$

Now consider $a_{1i} cof(A)_{1,l}$ We have that $a_{1,i} = b_{1,j}$ and also that $A(i) = B(j)$. Since $j = i + 1$, we have

$$(-1)^{1+j} = (-1)^{1+i++1} = -(-1)^{1+i}$$

and therefore $a_{1i}cof(A)_{1,i} = -b_{1j}cof(B)_{1,j}$ and $a_{1j}cof(A)_{1,j} = -b_{1i}cof(B)_{1,i}$.

we see that if in the formula for $detA$ we change the sign of each of the summands we obtain the formula for $detB$.

$$\det A = \sum_{l=1}^{n} a_{1i}cof(A)_{1,l} = -\sum_{l=1}^{n} b_{1l}B_{1l} = \det B$$

We have therefore proved the case of (1) when $j = i + 1$. In order to prove the general case, one needs the following fact. If $i < j$, then in order to interchange ith and jth row one can proceed by interchanging two adjacent rows $2(j - i) + 1$ times:
First swap ith and $i + 1$st, then $i + 1$st and $i + 2$nd, and so on. After one interchanges $j - 1$st and jth row, we have ith row in position of jth and lth row in position of $l - 1$st for $i +$. Then proceed backwards swapping adjacent rows until everything is in place.
Since $2(j - i) + 1$ is an odd number $(-1)^{2(j-i)+1} = -1$ and we have that $detA = -detB$.

II)
This is like (I) but much easier. Assume that (II) is true for all n−1×n−1 matrices. We have that $a_{ji} = kb_{ji}$ for $1 \leq j \leq n$. In particular $a_{1i} = kb_{1i}$, and for $l \neq i$ matrix $A(l)$ is obtained from $B(l)$ by multiplying one of its rows by k. Therefore $cof(A)_{1l} = kcof(B)_{1l}$ for $l \neq i$ and for all l we have $a_{1l}cof(A)_{1l} = kb_{1l}cof(B)_{1l}$. so we have $detA = kdetB$.

III)
First we use a lemma
Lemma.
If two rows of A are identical then $detA = 0$.
Proof lemma.
This is a consequence of (I). If two rows of A are identical, then A is equal to the matrix obtained by interchanging those two rows and therefore by (I) $detA = -detA$. This implies $detA = 0$

Now, Assume (III) is true for all $n - 1 \times n - 1$ matrices and fix A and B such that A is obtained by multiplying i-th row of B by k and adding it to j-th row of B $(i \neq j)$ then $detA = detB$.
If $k = 0$ then $A = B$ and there is nothing to prove, so we may assume $k \neq 0$.
Let C be the matrix obtained by replacing the j-th row of B by the i-th row of B multiplied by k.
We have

$$detA = detB + detC$$

and we ‘only’ need to show that $detC = 0$. But i-th and j-th rows of C are proportional. If D is obtained by multiplying the j-th row of C by $\frac{1}{K}$ then by (II) we have $detC = \frac{1}{k}detD$ (recall that $k \neq 0$). But i-th and j-th rows of D are identical, hence by (lemma) we have $detD = 0$ and therefore $detC = 0$.

Thinking the matrix of coefficients A portioned by columns, we write A = $(A^1 \dots A^n)$, recall that another way to write the system AX=b is b = $x_1 A^1 + \cdots + x_n A^n$.
Now, by **Theorem 2** part (II), we get

$$x_h \det A = det(A^1 \dots x_h A^h \dots A^n)$$

As a sequence of **Theorem 2** part (III), the addition of $x_j A^j$ to the h-th column of

$$(A^1 \dots x_h A^h \dots A^n) = \left(A \overset{h}{\leftarrow} x_h A^h\right)$$

does not affect its determinant whenever $j \neq h$. Hence

$$x_h \det A = det\left(A \overset{h}{\leftarrow} x_h A^h\right) = (A \underbrace{\overset{h}{\leftarrow} x_h A^h) + \sum_{j \neq h} x_j\, A^j}_{b}$$

It follows that $x_h = (\det A)^{-1} \det(A \overset{h}{\leftarrow} b)$.

By all the theorems that we discussed, we can see that for updating the solution based on $\tilde{x}_j \in (\tilde{x}_1, \dots, \tilde{x}_n)$ we can calculate

$$\tilde{x}_j = (detA)^{-1} \det(A \overset{j}{\leftarrow} b)$$

If we start from $j = 1$, we can reach out to the first element of x. By continuing the process and increasing j up to n, we will obtain the solution of the equation.
For computing calculations if we reach k elements of x $(1 \leq k < n)$. The $n \times 1$ matrix of x is represented by $n - k$ variables. Because the variables are strictly less than the $n \times 1$ matrix of b. As a result it can be solved by regular Gauss-Jordan method or by any methods that can be done by computers. This algorithm that we talked about it is known as Cramer's rule.

MATLAB code:

```
n = 5;
A = randn(n);
disp("This is matrix A :");
disp(A);
b= randn(n,1);
disp("This is vector solution b :");
disp(b);
w = det(A);
syms x [n 1];
c = A;
disp("The first initialization :")
disp(x)
m = zeros(n,1);
for i= 1:n
  c(:,i) = b;
  xs = det(c)/w;
  x = eval(subs(x, x(i), xs));
  m(i) = xs;
  c = A;
  if i~=n
      disp("Then the answer updates :")
      fprintf("%.4f\n", m(1:i)),disp(x(i+1:n));
  end
end
disp("This is final answer :");
disp(x);
```