3) (a) Let X, Y, Z be topological spaces: $f_1 : X \to Y$ be a homotopy equivalence i-e $\exists\, g_1 : Y \to X$ s.t $f_1 g_1 \simeq \mathbb{1}_Y$ & $g_1 f_1 \simeq \mathbb{1}_X$

Similarly $f_2 : Y \to Z$ is a homotopy equivalence s.t $g_2 : Z \to Y$ s.t $g_2 f_1 \simeq \mathbb{1}_Y$ & $f_2 g_2 \simeq \mathbb{1}_Z$

We need to show: $f_2 f_1 : X \to Z$ is a homotopy equivalence, take:

$$(f_2 f_1) \circ (g_1 g_2) = f_2 \circ (f_1 \circ g_1) g_2 \simeq f_2 \circ \mathbb{1}_Y \circ g_2$$
$$= f_2 \circ g_2$$
$$\simeq \mathbb{1}_Z$$

The above is continuous as it is a function composition

Similarly, $(g_1 g_2) \circ (f_2 f_1) = g_1 (g_2 \circ f_2) f_1$

$$\simeq g_1 \circ \mathbb{1}_Z \circ f_1 = g_1 \circ f_1 \simeq \mathbb{1}_X$$

b) $f \simeq f$. since $\varphi(x, t) : X \times [0,1] \to X$ is a homotopy between f & itself. Hence reflexive. Suppose $f \simeq g$, $\exists$ a homotopy $\varphi(x,t) : X \times [0,1] \to Y$ s.t $\varphi(x, 0) = f(x)$ & $\varphi(x, 1) = g(x)$. But $\varphi(x, 1-t)$ is a homotopy between g & f.

Hence $f \simeq g$ & $g \simeq f$. Suppose $f \simeq g$ and $g \simeq h$. $\exists$ $\varphi_1 : X \times [0,1] \to Y$ and $\varphi_2 : Y \times [0,1] \to Z$ s.t. $\varphi_1(x,0) = f(x)$, $\varphi_1(x,1) = g(x)$, $\varphi_2(x,0) = g(x)$ & $\varphi_2(x,1) = h(x)$. Define

$$\varphi_3 : X \times [0,1] \to Z$$

$$= \begin{cases} \varphi_1(x, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \varphi_2(x, 2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

$\varphi_3(x,0) = \varphi_1(x,0) = f(x)$ & $\varphi_3(x,1) = \varphi_2(x,1) = h(x)$. φ_3 is continuous as the piecewise functions are continuous.

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(30) $f : X \to Y$ is a homotopy equivalence with homotopy inverse $g : Y \to X$ & suppose $h : X \to Y$ is homotopic to f.

$$1_Y \simeq fg \simeq hg,$$

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Similarly $1_X \simeq gf \simeq gh$.

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$\Rightarrow \exists$ a map s.t. $hg \simeq \mathbb{1}$ & $gh \simeq \mathbb{1} \Rightarrow h$ is a homotopy equivalence