LECTURE 5

~~Quotients~~ Relation:

A binary relation betⁿ sets X & Y is a subset R of cartesian product $X \times Y$.

Equivalence Relation: A relation R on X (i.e. between X & itself) is said to be an equivalent relation if it is:

(a) reflexive: $x \sim x \ \forall x \in X$

(b) symmetric: $x \sim y \Rightarrow y \sim x$

(c) transitive: $x \sim y$ & $y \sim z \Rightarrow x \sim z$, $\forall x, y$ & z

The equivalence class of an element is

$$[x] = \{y \in X : x \sim y\}$$

Example: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$x \sim y$ if $x \bmod 2 = y \bmod 2$

So, essentially its saying that, odd & even are placed in ~~the~~ two different classes.

~~Then~~ N

Let G be an abelian group ~~&~~ N be its subgroup. Then, N induces an equivalence relation R on G, where $x \sim y$ if $y - x \in N$

a) First, $0 \in N$, since N is a group itself, Then $\forall\, a \in G$, $a \sim a$, as $a - a = 0$, & $0 \in N$

b) Now, suppose $x \sim y$ i.e $y - x \in N$

But N is a subgroup, $y - x$ must have an additive inverse, so $-(y - x) \Rightarrow x - y \in N$

$\Rightarrow y \sim x \Rightarrow$ ~~Reflexive~~ symmetric

c) suppose, $a \sim b$, $b \sim c$, i.e
$b - a \in N$, & $c - b \in N$

Since, N is a subgroup, it is closed under addition so, $(b - a) + (c - b)$
$= b - \cancel{a} + c - b$
$= c - a \in N$

$\Rightarrow a \sim c \Rightarrow$ Transitive

Hence Equivalence Relation

Let $G/N := \{[a] : a \in G\}$

$$[a] + [b] := [a + b]$$

One can check that $(G/N, +)$ is an abelian group.

defined in $(\mathbb{Z}, +)$

$[0] + [0] := [0 + 0] = [0]$

$[0] + [1] := [0 + 1] = [1]$

$[1] + [0] := [1 + 0] = [1]$

$[1] + [1] := [1 + 1] = [2]$
$= [0]$

Now, let V be a vector space & S be a subspace, then

$$V/S := \{[x] : x \in V\}.$$

It has same field as of parent space

So, $[x] + [y] := [x+y]$ &

$$\alpha[x] = [\alpha x]$$

Example: Let $V = \mathbb{R}^2$ &

$$S = \{\alpha(1,1) : \alpha \in \mathbb{R}\}.$$

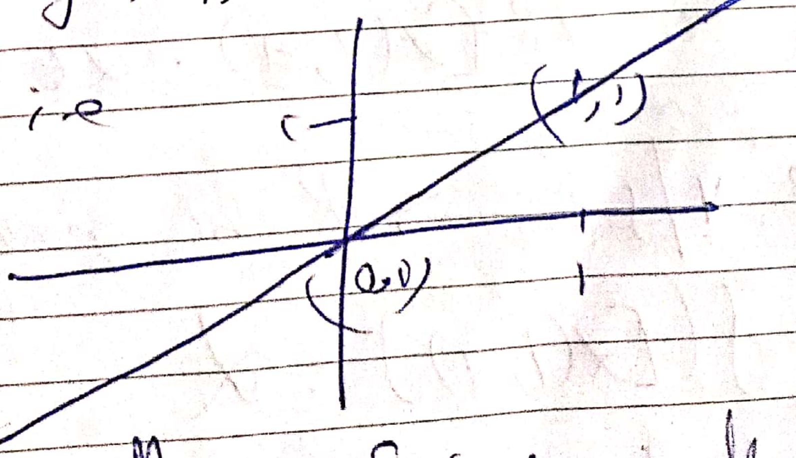
Clearly, $\alpha(1,1) + \beta(1,1) = (\alpha+\beta)(1,1)$

So, closed under linear combinations, hence S is closed.

Let R be a relation in V where xRy if $y - x \in S$

i.e. $y - x = \alpha\cdot(1,1)$ for some $\alpha \in \mathbb{R}$

So, every $y - x$ is on the line from $(0,0)$ to $(1,1)$ i.e.



& every element in S is equivalent

$(\alpha - \alpha)(1,1) \in S$

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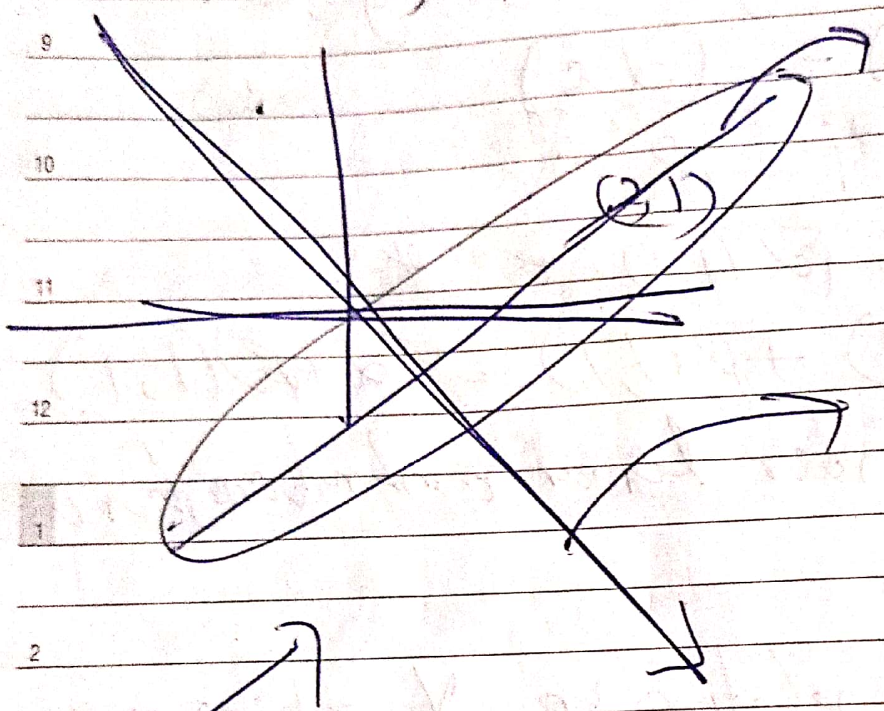
Let $x = (2,1)$

Then $x \sim y \Leftrightarrow y - x \in S \Leftrightarrow y - x = \alpha(1,1)$

$\Rightarrow y = x + \alpha(1,1)$

Thus $[x] = \{x + \alpha(1,1)\}$

So, Now for a line $(2,1)$, i.e.



All equivalence class (same!!!)

Wherever, this intersects, we have, choose the point of intersection as the representative of the equivalence.

Then $V/S = \{[x] : x \in R\}$ → this x is picked s.t. it satisfies $x = \alpha(1,-1)$

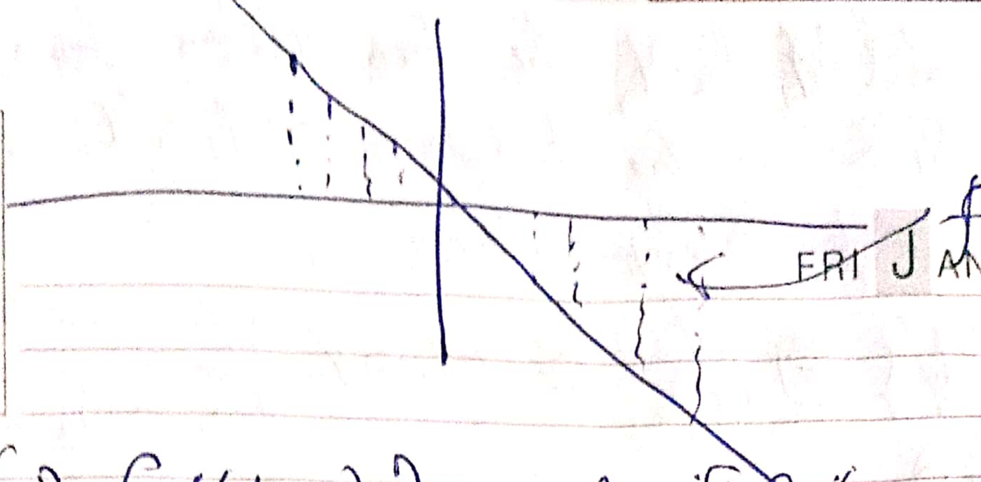
$= \{[x] : x = \alpha(1,-1)\}$

$= \{[\alpha(1,-1)] : \alpha \in R\}$

Let $f : V/S \longrightarrow \mathbb{R}$ be given by

$f([\alpha(1,-1)]) = \alpha$

So, any point x is creates an equivalence class $[x]$



$$= f\left(\lambda_1[\alpha(1,-1)] + \lambda_2[\beta(1,-1)]\right)$$

$$= f\left([\lambda_1\alpha(1,-1) + \lambda_2\beta(1,-1)]\right)$$

$$= f\left([(\lambda_1\alpha + \lambda_2\beta)(1,-1)]\right)$$

$$= \lambda_1\alpha + \lambda_2\beta$$

$$= \lambda_1 f([\alpha(1,-1)]) + \lambda_2 f([\beta(1,-1)])$$

Thus f is a homomorphism

Further $f([\alpha(1,-1)]) = \alpha = \beta = f([\beta(1,-1)])$

$\Rightarrow \beta = \alpha$

& For each $\alpha \in \mathbb{R}$, $\exists$ $[\alpha(1,-1)]$ s.t.

$$f([\alpha(1,-1)]) = \alpha$$

Hence, f is isomorphism. Hence, V/S & $\mathbb{R}$ are isomorphic

i.e. $V/S \cong \mathbb{R}$

Theorem: Let V_1 & V_2 be two vector spaces. Let S_1 & S_2 be subspaces of V_1 & V_2 respectively. Then

$$V_1/S_1 \oplus V_2/S_2 \cong (V_1 \oplus V_2)/(S_1 \oplus S_2)$$

Proof: Let

$$\varphi : V_1/S_1 \oplus V_2/S_2 \longmapsto (V_1 \oplus V_2)/(S_1 \oplus S_2)$$

be given by:

$$\varphi(([v_1], [v_2])) \longmapsto [(v_1, v_2)]$$

HW (i) Check φ is well defined.
(ii) φ is a bijection
(iii) φ is a homomorphism

Theorem: Let V be a finite dimensional vector space. Let S be a subspace. Then,

$$\dim(V/S) = \dim(V) - \dim(S)$$

Proof: Let V be n-dimensional space & S be k-dimensional i.e. $\dim(V) = n$, $\dim(S) = k$

$\Rightarrow S$ has a basis of form $\{v_1, \dots, v_k\}$

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From Linear Algebra, can be extended into a basis for V. Thus V has a basis of the form

01 SUN FEB $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$

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We need to show

$$\dim(V/S) = n-k$$

Suffices to show, that V/S has a basis with $(n-k)$

9 elements. Let $v \in V$, Then,

10 $$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Let $v' = \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n$

11 Claim: $v' \sim v$. (meaning $v - v' \in S$)

12

So, $v - v' = \alpha_1 v_1 + \dots + \alpha_k v_k \in S$

1

Thus $\underbrace{[v] = [v']}$

2

$$= [\alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n]$$

3

Applying defn of + & α. for equivalence

4 class

5

$$= \alpha_{k+1}[v_{k+1}] + \dots + \alpha_n [v_n]$$

6

Hence $V/S = \{[v]; v \in V\}$,

it follows that $\{[v_{k+1}], \dots, [v_n]\}$ spans V/k

2015 To show linear independence.

M	30	2	9	16	23
T	31	3	10	17	24
W		4	11	18	25
T		5	12	19	26
F		6	13	20	27
S		7	14	21	28
S	1	8	15	22	29

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Suppose $\exists\ \beta_{k+1}, \ldots, \beta_n \in F$ (not all zero)

$$\beta_{k+1}[v_{k+1}] + \cdots + \beta_n [v_n] = [0]$$

Hence,

$$[\beta_{k+1} v_{k+1} + \cdots + \beta_n v_n] = [0]$$

$$\Rightarrow 0 \sim \beta_k v_{k+1} + \cdots + \beta_n v_n$$

$$\Rightarrow \beta_{k+1} v_{k+1} + \cdots + \beta_n v_n \in S$$

Thus, $\beta_{k+1} v_{k+1} + \cdots + \beta_n v_n = c_1 v_1 + \cdots + c_k v_k$ for some $c_1, \ldots, c_k$,

This leads to contradiction since $\{v_1, \ldots, v_n\}$ is a basis for V. i.e

$$\beta_{k+1} v_{k+1} + \cdots + \beta_n v_n - (c_1 v_1 + \cdots + c_k v_k) = 0$$

not all c_i & β_i are zero,

Hence $\{[v_{k+1}], \ldots, [v_n]\}$ is linearly independent in V/S. Hence we conclude that a basis for V/S is $\{[v_{k+1}], \ldots, [v_n]\}$ i.e

$$\dim(V/S) = n-k = \dim(V) - \dim(S)$$