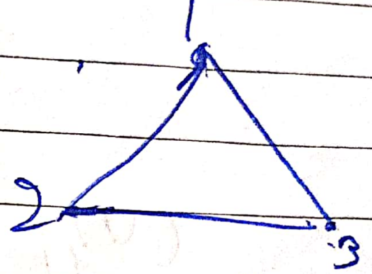
LECTURE 14

Let us combinatorially define $\partial_p \sigma = \sum_{i=0}^{p} \{v_0, \ldots, \hat{v_i}, \ldots, v_p\}$

That is, all the boundary faces of σ are in Δ. Example take,



$\sigma = \{1, 2, 3\}$

$\sigma \notin \Delta$

$\partial_2 \Delta = 12 + 13 + 23 \in C_1(\Delta)$

Then $\forall j \notin \{p-1, p\}$

$\boxed{\text{Preserve}}$ $\beta_j(\Delta \cup \sigma) = \beta_j(\Delta)$ → Adding a face, every Betti no. is preserved

Further, (1) $\partial_p \sigma \in im(\partial_p)$

$\Rightarrow \beta_p(\Delta \cup \sigma) = \beta_p(\Delta) + 1$ $\boxed{\text{Creation}}$

$\beta_{p-1}(\Delta \cup \sigma) = \beta_{p-1}(\Delta)$

Other hand,

(2) $\partial_p \sigma \notin \mathrm{im}(\partial_p)$

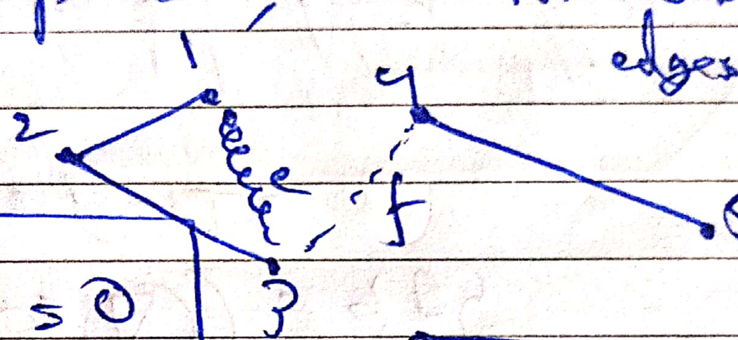
$\Rightarrow \beta_{p-1}(\Delta \cup \sigma) = \beta_{p-1}(\Delta) - 1$

$\beta_p(\Delta \cup \sigma) = \beta_p(\Delta)$

Example $p = 1$

We choose to add edges in same component or in different components

Δ



old $\beta_1 = 0$
new $\beta_1 = 1$

old $\beta_0 = 2$
new $\beta_0 = 2$

For ....

Old $\beta_1 = 0$
new $\beta_1 = 0$

old $\beta_0 = 2$
new $\beta_0 = 1$

For ....

So, suppose $\partial_p(e) = 1+3$

$\mathrm{im}\,\partial_1(\Delta) = \{0, 1+2, 2+3, 4+5, 1+3, 1+3+4+5\}$

So, $\partial_p(e) \in \mathrm{im}\,\partial_1(\Delta)$

So this should lead to creation.

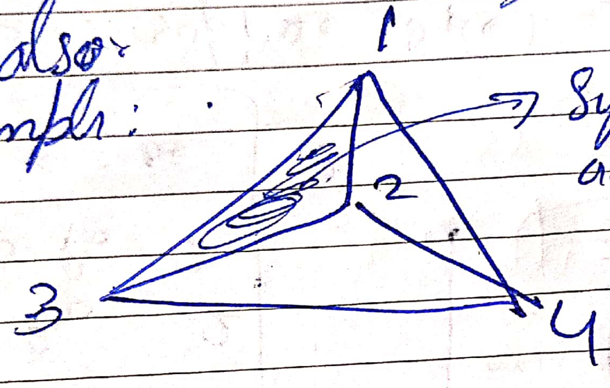
Now $f = \{3,4\}$

$\partial_1 f = 3+4 \notin im\cdot(\partial_1(\Delta))$

Hence leads to destruction of β_0

this holds true in higher dimension also.

Example:



Suppose one want to add $\sigma = 123$

$\partial_2 \sigma = 12+13+23$

$im\cdot(\partial_2(\Delta)) = \{0\}$

$\beta_1 = \beta - 1$

Incremental Algorithms to compute Betti numbers of a given simplicial complex

Input: Δ. Output: $\beta_p(\Delta)$ $\forall p \in \mathbb{Z}$

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Initializations: Order faces in Δ as $\sigma_1, \dots, \sigma_m$ s.t.

$$\Delta_i := \{\sigma_1, \dots, \sigma_i\}$$ is a simplicial complex for all i.

$\Delta_0 := \phi$ (empty simplicial complex)

Update rule

for $i = 1$ to m.

set $p = \dim \sigma_i$

if $p = 0$

$\beta_0 = \beta_0 + 1$

elseif $\boxed{\partial_p \sigma_i \in \text{im}(\partial_p(\Delta_{i-1}))}$

$\beta_p = \beta_p + 1$

else.

$\beta_{p-1} = \beta_{p-1} - 1$

end if

end for

this is quite hard

Example:

$\Delta = \{1, 2, 3, 12, 13, 2\dots$

$\Delta_1 = \{12\}$ X

Not SC

But () are as a)

$\Delta_0 = \{1, 3, 12, 3, 123\}$

$\Delta_1 = \{1\}$

$\Delta_2 = \{1, 2\}$

$\Delta_3 = \{1, 2, 12\}$

We know,

$\partial_p : C_p(\Delta_{i-1}) \longmapsto C_{p-1}(\Delta_{i-1})$

Matrix representation $[\partial_p(\Delta_{i-1})] = A$

We know, $\partial_p \sigma_i \in C_{p-1}(\Delta_{i-1})$

even if $\sigma_i \notin \Delta_{i-1}$

Find ~~vector representation~~ of $\partial_p \sigma_i$ in the basis of $C_{p-1}(\Delta_{i-1})$ → say its b

check if $b \in$ col space A

$\boxed{\text{need to find } Ax = B}$ → computation what we need.

has a solution in $\mathbb{Z}_2^{f_p(\Delta)}$

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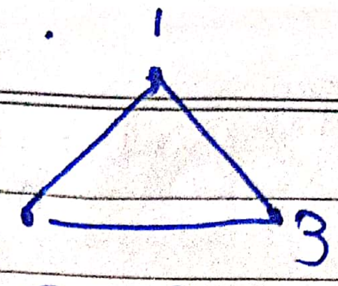
Example :



Let $\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3,$

$\sigma_4 = 12, \sigma_5 = 23, \sigma_6 = 13$

Step 1 :

$\Delta_0 = \phi$ & $\beta_p = 0 \ \forall p \in \mathbb{Z}$

Since $\dim(\sigma_1) = 0$, $\beta_0 = 0 + 1 = 1$

Step 2: $\Delta_1 = \{1\}$

$\sigma_2 = \{2\}$

$\dim(\sigma_2) = 0$

$\beta_0 = 1 + 1 = 2$

Step 3: $\Delta_2 = \{1, 2\}$

$\sigma_3 = \{3\}$ same as above,

$\beta_0 = 2 + 1 = 3$

Step 4: $\Delta_3 = \{1, 2, 3\}$ & $\sigma_4 = \{1, 2\}$

$\partial_2 \sigma_4 = 1 + 2 \notin \operatorname{im}(\partial_2(\Delta_3))$

$\qquad\qquad\qquad\qquad \| \quad \{0\}$

Hence $\beta_0 = 3 - 1 = 2$

Step 5: $\Delta_4 = \{1, 2, 3, 12\}$

$\sigma_5 = \{23\}$

$\partial_1 \sigma_5 = 2+3 \notin im(\partial_1(\Delta_4))$

$\{0, 1+2\}$

$\therefore \beta_0 = 2-1 = 1$

Step 6: $\Delta_5 = \{1, 2, 3, 12, 23\}$

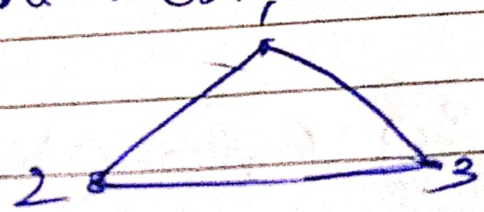
$\sigma_6 = \{13\}$

$\partial_1 \sigma_6 \in im(\partial_1(\Delta_5)) = \{0, 1+2, 2+3, 1+3\}$

Hence, $\beta_1 = 0+1 = 1$

The final Betti number are $\beta_0 = 1$, $\beta_1 = 1$, $\beta_j = 0 \ \forall j \in \{0, 2\}$

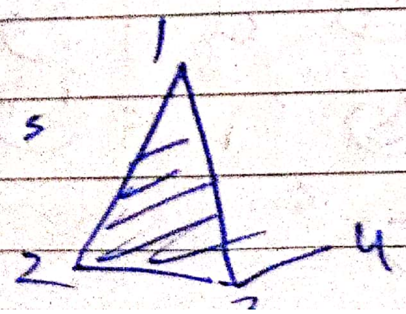
Note these are the Betti numbers corresponding to



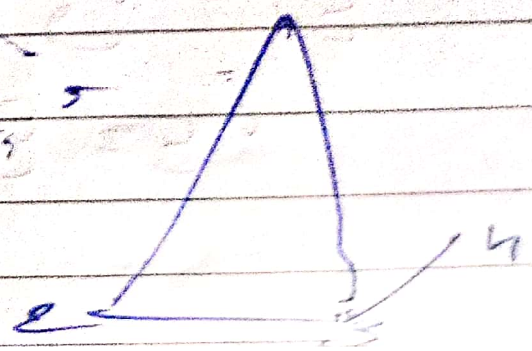
Proof of Proposition: From the corollary

to

M =



K_1 =



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$M = K_1 \cup \sigma$, where $\sigma = 123$

We will not write $K_2 = \{\sigma\}$, since K_2 will not be a simplicial complex

Instead, we will set

$$K_2 = \{1, 2, 3, 12, 13, 23, 123\}$$

So, K_2 includes σ

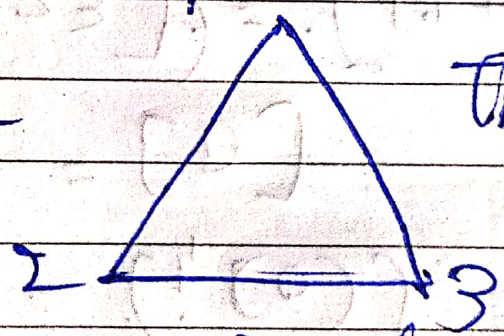
Let $n = 1$,

From MVT,

$$\beta_1(M) = \beta_1(K_1) + \beta(K_2) - \beta_1(L)$$
$$+ \dim(\ker \nu_1) + \dim(\ker \nu_0) - (t$$

$\beta_1(K_1) = 1$, $\beta_1(K_2) = 0$

$L = K_1 \cap K_2$ Thus $\beta_1(L) =$



Next $\nu_1 = (i_1^*, j_1^*)$, where

$$i_1^* : H_1(L) \longrightarrow H_1(K_1)$$
$$j_1^* : H_1(L) \longrightarrow H_1(K_2)$$

Here $i_1 : C_1(L) \longrightarrow C_1(k_1)$
$j_1 : C_1(L) \longrightarrow C_1(k_2)$

Clearly, $H_1(L) = \{[0], [12+13+23]\}$

so, $i_1^*([0]) = [i_1[0]]$
$= [0]$

$i_1^*([12+13+23]) = [12+13+23]$

$H_1(k_2) = \{[0]\}$
$H_1(k_1) = \{[0], [12+13+23]\}$

On other hand,

$H_1(k_1) = \{[0]\}$

$j_1^*([0]) = [j_1(0)]$
$= [0]$

$j_1^*([12+13+23]) = [j_1(12+13+23)]$
$= [12+13+23]$

From the desk of because $12+13+23 \in \mathrm{im}(\partial_2)$ DATE

$$\Rightarrow 12+3+23 \in B_1$$

$$\Rightarrow 12+3+23 \sim 0$$

Hence $\dim \ker(\nu_1) = 0$ as $\ker \nu_1 = \{[0]\}$

because $\nu_1([0]) = [i_1^*(0), j_1^*(0)]$

$$= [[0],[0]]$$

$$\nu_1^*([12+13+23]) = (j_1^*([12+13+23]), j_1^*([12+13+23]))$$

$$= ([12+13+23], [0])$$

Finally $\nu_0 = (i_0^*, j_0^*)$

$$i_0^* : H_0(L) \longrightarrow H_0(K_1)$$

$$j_0^* : H_0(L) \longrightarrow H_0(K_2)$$

$$H_0(L) = Z_0(L)/B_0(L) = \{[0]\}$$

From this we have $\dim(\ker \nu_0) = 0$

Hence $B_1(M) = 1 + 0 - 1 + 0 = 0$

$$= 0$$

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