LECTURE 11

S & S' is the standard basis for C_p & C_{p-1} respectively

$[\partial_p]_S^{S'}$ is a $f_{p-1} \times f_p$ matrix &

$$[\partial_p]_S^{S'} [c]_S = [\partial_p(c)]_{S'} \quad \forall c \in C_p$$

Using the fact,

$$[\partial_p]_S^{S'} e_j = [\partial_p]_S^{S'} \cdot [\sigma_j]_S$$

$$= [\partial_p(\sigma_j)]_{S'}$$

It follows that ij-th element of $[\partial_p]_S^{S'}$ is 1 if τ_i is a face of σ_j

Computing Betti numbers using matrix representation of boundary.

Recall that:

$$\beta_K = \dim(H_K) = \dim(Z_K) - \dim(B_K)$$

$$= \dim(\ker(\partial_K)) - \dim(\mathrm{Im}(\partial_{K+1}))$$

= rank of nullspace of ∂_K − rank of image space/column rank of ∂_{K+1}

In above example:

$$\partial_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

We will now use a series of row operations to get this matrix in RREF.

In linear algebra, we studied:

① exchange row

② multiply a row with a scalar

(3) take scalar multiple of a row & add to another row.

While work with matrices in $\mathbb{Z}_2$ since the the non-zero scalar on $\mathbb{Z}_2$ is only 1, the above operations reduce to the following:
(1) exchange rows & (2) Add one row to others

So (*) can be reduced to:

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ The column rank is 1

Hence rank of null space of $\partial_2 = 0$ from rank-nullity theorem.
Similarly ∂_1 can be reduced as follows:

as:

$$\partial_1 = \begin{array}{c|ccccc} & 12 & 13 & 14 & 23 & 34 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 0 & 1 \end{array}$$

$R_2 \to R_1 + R_2$

$$② = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$R_3 \to R_3 + R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$R_2 \to R_2 + R_3$

$R_4 \to R_4 + R_3$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this column rank is 3 of ∂_1 while null rank is 2

So, Dim. of H_1 is $\beta_1 = 3 - 2 = 1$

Similarly, $\beta_0 = \dim(Z_0) - \dim(B_0)$

$$= \dim(C_0) - \dim(\operatorname{Im}(\partial_0))$$

$$= 4 - \text{column rank of } \partial_1$$

$$= 4 - 3 = 1$$

~~Other~~ hand

$$\beta_2 = \dim(Z_2) - \dim(B_2)$$

$$= 0 - 0 = 0$$

Above calculations concerned the usual homology setup. Let us now at reduced homology case.

Here only ∂_0 & ∂_{-1} change. The matrix corresponding to ∂_{-1} is the empty matrix. However it is easy to see that:

$$\dim(Z_{-1}) = \dim(C_{-1}) = 1$$

With regards to ∂_0, observe:

$$C_0 \xrightarrow{\partial_0} C_{-1}$$

Now for $Im(\partial_0)$

With regards to ∂_0, observe,

$$\begin{tikzcd} C_0 \arrow[r, mapsto, "\partial_0"] \arrow[d, leftrightarrow] & C_{-1} = \{0,1\} \\ \mathbb{Z}_2^{f_0} \arrow[r, mapsto] & \mathbb{Z}_2 \arrow[u] \end{tikzcd}$$

Hence, the matrix corresponding to ∂_0 in our example. ∂_0 size is $1 \times f_0$

$$\partial_0 \text{ matrix} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ 1 & \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Now, matrix is in RREF

Hence, it is easy to see that column rank of ∂_0 is 1, while rank of its null space is 3

– Column rank of ∂_1

β_0 = rank of null space of ∂_0

$= 3 - 3 = 0$

Mayer Victoris Sequence:

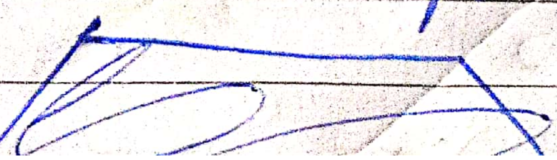
Named after two Austrian mathematicians
Mayer served as Einstein's assistant

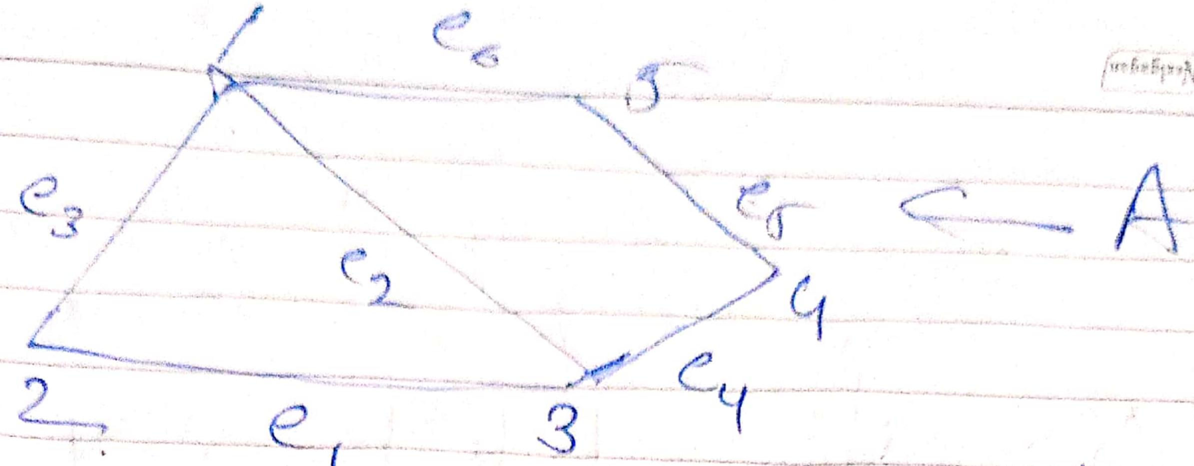
Mayer Victorix sequence is an algebraic tool that is useful to compute homology groups of complicated topological spaces (in our case simplicial complexes)

Let us say X is the given complicated simplicial complex.
Then the idea is to compute the homology of X is to decompose it into smaller & simpler simplicial subcomplexes, say A & B, whose homology is relatively easier to compute.

Eg: Let Δ be a simplicial complex & $\sigma \notin \Delta$ s.t $|\sigma| = p+1$ & $\partial_p \sigma \in C_{p-1}$

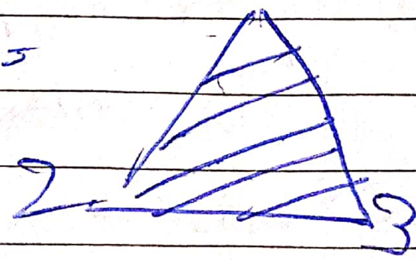
This implies all the boundary faces of σ are in Δ except σ itself.





Imagine the triangle $\triangle_{2\ 3}^{1}$ is filled after we have already calculated the homology of previous figure above. Calculating the homotopy groups again will be cumbersome.

Let B =



$X = A \cup B$

We need to know, the Mayer-Vietoris sequence.

From the desk of

Mayer Vietoris sequence

DATE

Let $A_0, A_1, \ldots, A_m$ be a sequence of vector space & $\eta_i : A_i \longmapsto A_{i-1}$ homomorphism between them.

The sequence

$$A_m \xmapsto{\eta_m} A_{m-1} \longmapsto \cdots A_1 \xrightarrow{\eta_1} A_0$$

is said to be exact is

$$\ker(\eta_{m-1}) = im(\eta_m)$$

If $m = 4$ & A_0 & A_4 are trivial vector space, then

$$0 \xmapsto{\eta_4} A_3 \xmapsto{\eta_3} A_2 \xmapsto{\eta_2} A_1 \xmapsto{\eta_1}$$

is a short exact sequence.

Now, $im(\eta_4) = 0$

Since this is a short exact sequence it follows that

$$\ker(\eta_3) = 0$$

$\Rightarrow \eta_3$ must be an injective.

Suppose not, Then, $\exists v \neq v'$ s.t.

$\eta_3(v) = \eta(v')$.

$\eta_3(v) - \eta_3(v') = 0$

$\eta_3(v - v') = 0$

$\Rightarrow v - v' \in \ker(\eta_3)$, but since $v \neq v' \Rightarrow v - v' \neq 0$, so, contradicting the fact that $\ker(\eta_3) = 0$

Hence η_3 must be injective.

$\ker(\eta_1) = A$

$\operatorname{im}(\eta_2) = \ker(\eta) = A$

η_2 must be a surjective map.

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