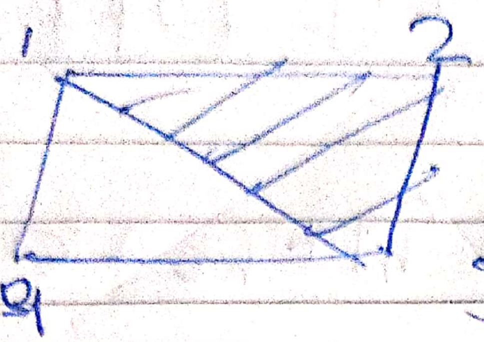
LECTURE 10:

Last time, we started computing Betti Numbers



Now, $\beta_p = 0 \ \forall\ p > 2$ & $\forall\ p \leq -1$ both in usual & reduced homology case.

This can shown as in first example.

Today we will focus on β_1.

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

C_2	C_1	C_0
$\{0,$	0	0
$123\}$	12	1
	23	2
	14	⋮
	12 + 13	1 + 2
	⋮	⋮
	12 + 13 + 14	1 + 2 + 3 + 4
	⋮	
	12 + 13 + 23 + 34 + 14 }	

$\mathrm{Im}(\partial_2) = B_1 = \{0, 12+23+13\}$

$Z_1 = \{0, 12+23+13, 14+34+13, 12+23+14+34\}$

Let $x \sim y$ if $y - x \in B_1$ then,

$0 \sim 12+23+13$

Hence $[0] = \{0, 12+23+13\}$

$= [12+23+13]$

Also, $13+34+14 \sim 12+23+34+14$.

since $13+34+14 = 12+23+34+14+12+23+13$

Thus $[13+34+14]$

$= \{13+34+14, 12+23+34+14\}$

$\Rightarrow H_1 = \{[0], [13+34+14]\}$

Clearly, $\{[13+34+14]\}$ is a basis

Hence $B_1 = 1$

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Next compute B_0 in usual homology

i.e. $C_{-1} = \{0\}$, $Z_0 = \ker(\partial_1) = C_0$

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1}$$

$C_1 =$ 0, 12, 13, 14, +, ⋮ all combination

$B_0 = \mathrm{Im}(\partial_1) = \{0, 1+2, 1+3, 1+4, 2+3+3+4\ (2+4), 1+2+3+4\}$

comes from $\partial_1(23+34)$

Clearly $\{1+2, 1+3, 1+4\}$ forms a basis for B_0 $\Rightarrow \dim(B_0) = 3$

Seperately, $\dim(Z_0) = 4$ $(1, 2, 3, 4)$

Hence, $B_0 = \dim(H_0)$
$= \dim(Z_0) - \dim(B_0)$
$= 1$

 Notice that we didn't explicitly write down

∂_0, yet we managed to compute β_0

Notice that β_0 won't change since C_1, C_0 & ∂_1 didn't change

But $Z_0 = \{0, 1+2, 1+4, 1+3, 2+3, 3+4, 2+4, 1+2+3+4\}$

$\dim(Z_0) = 3$,

$\Rightarrow \beta_0 = 3 - \cancel{3} = 0$

In general in both usual & reduced homology, B_0 & hence $\dim(B_0)$ is the same

What changed was Z_0 in reduced homology case is 1 less than that in usual homology

As one can see, computing Betti numbers gets harder as the complexity of simplicial.

This motivates the need for computational tools.

Since $\mathbb{Q}$, C_p, Z_p & B_p are all vector spaces & D_p is a linear map, it makes sense to look for tools from linear algebra.

Matrix Representation of linear operator

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow] & V' \arrow[d, leftrightarrow] \\ \mathbb{R}^n \arrow[r, "{[T]_B^{B'}}"] & \mathbb{R}^m \end{tikzcd}$$

The above diagram commutes i.e $\forall\, v \in V$

$$[T]_B^{B'} [v]_B = [T(v)]_{B'}$$

A similar diagram exists for vector space over any field $\mathbb{F}$

$$\begin{tikzcd}
C_p \arrow[r] \arrow[d] & C_{p-1} \arrow[d, leftrightarrow] \\
\mathbb{Z}_2^{f_p} \arrow[r, "{[\partial_p]_S^{S'}}"] & \mathbb{Z}_2
\end{tikzcd}$$

$f_p = \dim(C_p)$

The commutative property of this diagram implies

$$[\partial_p]_S^{S'} [c]_S = [\partial_p(c)]_{S'}$$

$\forall\, c \in C_p$

Here S is the standard basis for C_p i.e.,

$$S = \{\sigma_1, \dots, \sigma_{f_p}\}$$

Similarly,

$$S' = \{\tau_1, \dots, \tau_{f_{p-1}}\}$$

Note we assume some ordering among

& $(p-1)$ dimensional faces in the given

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To find the ith column of $[\partial_p]_S^{S'}$ we can look at value of

$$[\partial_p]_S^{S'} e_i, \text{ where } e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \in \mathbb{Z}_2$$

Now, $e_i = [\sigma_i]_S$.

Also, $[\partial_p]_S^{S'} [\sigma_i]_S = [\partial_p(\sigma_i)]_{S'}$

Essentially Applying ∂_p operator to the faces

Size of $[\partial_p]_S^{S'}$ is $f_{p-1} \times f_p$

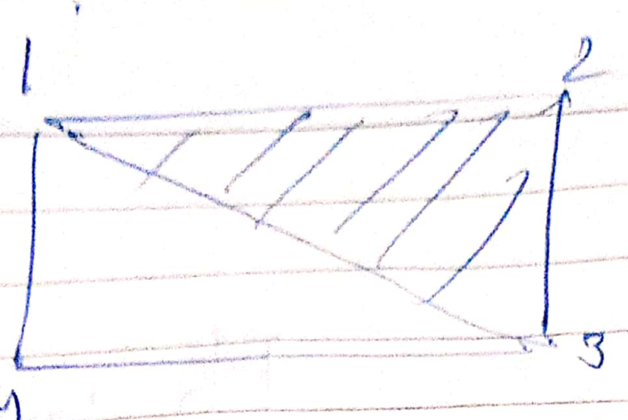
So, we have:

$$\begin{array}{c} \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{f_{p-1}} \end{array} \begin{array}{c} \sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_{f_p} \\ \begin{bmatrix} & & & \\ & & & \\ & & \square & \\ & & & \end{bmatrix} \end{array}$$

Further

$$a_{ij} = \begin{cases} 1 & \text{if } \tau_i \text{ is a face of } \sigma_i \end{cases}$$

Example :



$C_2 = \{123 ; 0\}$

$C_1 = \{12, 14, 13, 23, 34, \ldots 0, \ldots 12+14\}$

$$\partial_2 = \begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 34 \end{array} \begin{array}{c} 123 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

$$\partial_1 = \begin{array}{c|ccccc} & 12 & 13 & 14 & 23 & 34 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 0 & 1 \end{array}$$

Now $\beta_k = \dim(Z_k) - \dim(B_k)$

$= \dim(\ker(\partial_k)) - \dim(\operatorname{Im}(\partial_{k+1}))$

= rank of null space of ∂_k − rank of column space of ∂_{k+1}