LECTURE 13 Given above: the chain map induces the chain map

$$\varphi^* : H(X) \longmapsto H(Y)$$

So, we have

$$\begin{tikzcd}
\cdots H_{n+1}(X) \arrow[r, "f_{n+1}"] \arrow[d, "\varphi^*_{n+1}"] & H_n(X) \arrow[r, "f_n"] \arrow[d, "\varphi^*_n"] & H_{n-1}(X) \arrow[d, "\varphi^*_{n-1}"] \\
\cdots H_{n+1}(Y) \arrow[r, "g_{n+1}"] & H_n(Y) \arrow[r, "g_n"] & H_{n-1}(Y)
\end{tikzcd}$$

where: $f_n([x]) = [f_n(x)] \; \forall [x] \in H_n(X)$

$g_n([y]) = [g_n(y)] \; \forall [y] \in H_n(Y)$

& $\varphi^*([x]) = [\varphi_n(x_n)] \; \forall [x] \in H_n(X)$

We will show that each of the above is well-defined.

Remember, $[x] = \{x' \in \ker(f_n) : x' \sim x\}$

$= \{x' \in \ker(f_n) \; x' - x \in \mathrm{im}(f_{n+1})\}$

Same, $[y] = \{y' \in \ker(g_n) ; y' \sim y\}$

$= \{y' \in \ker(g_n) ; y' - y \in \mathrm{im}(g_{n+1})\}$

So, let's consider f_n.

We wish to show that if

$[x] = [x']$ for $x, x' \in Z_n(X)$

then $f_n([x]) = f_n([x'])$

$\Rightarrow$ Since $[x] = [x']$, then

From the def. of $x' \sim x$ i.e $x' - x \in im(f_{n+1})$

Thus $x' = x + b$ for some $b \in im(f_{n+1})$

Hence, $f_n(x') = f_n(x) + f_n(b)$ ($\because$ homomorphism)

$= f_n(x) + 0$ ($\because$ ~~...~~ $f^2 \equiv 0$)

$= f_n(x)$

Hence $f_n([x]) = [f_n(x)] = [f_n(x')]$

$= f_n([x'])$

Similarly, for g_n also.

Now, we now show φ_n^* is well-defined

That is, if $[x] = [x']$, then,

$\varphi_n^*([x]) = \varphi_n^*([x'])$

So, x & $x' \in Z_n(x)$

$x' - x \in im(f_{n+1})$. Hence $\exists b$ such that

s.t $x' = x + b$.

Unlike f_n, $\varphi_n(x)$ & $\varphi_n(x')$ need not be same

However, $x \in \ker(f_n) \Rightarrow \varphi_n(x) \in \ker(g_n)$

$b \in \mathrm{im}(f_{n+1}) \Rightarrow \varphi_n(b) \in \mathrm{im}(g_{n+1})$

$\Rightarrow \varphi_n(x) \sim \varphi_n(x')$

Since, $\varphi(x') - \varphi_n(x) = \varphi_n(b)$

& $\varphi_n(b) \in \mathrm{im}(g_{n+1})$

So, $\varphi_n(x)$ & $\varphi_n(x') \in \ker(g_n)$

$\cancel{[\varphi(x)] = \{\varphi(x'') \in \ker(g_n) : \varphi(x'')}$

$[\varphi(x')] = \{\varphi(x'') \in \ker(g_n) : \varphi(x'') \sim \varphi(x')\}$

$\Rightarrow [\varphi_n(x)] = [\varphi_n(x')]$

It remains to show that

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X) \arrow[r, "f_{n+1}"] \arrow[d, "\varphi^*_{n+1}"] & H_n(X) \arrow[r, "f_n"] \arrow[d, "\varphi^*_n"] & H_{n-1}(X) \arrow[r, "f_{n-1}"] \arrow[d, "\varphi^*_{n-1}"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Y) \arrow[r, "g_{n+1}"] & H_n(Y) \arrow[r, "g_n"] & H_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Commutes.

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Now, $\varphi_{n-1}^{*} \circ f_n([x]) = \varphi_{n-1}^{*}([f_n(x)])$

$= [\varphi_{n-1}(f_n(x))]$

$g_n \circ \varphi_n^{*}([x]) = g_n([\varphi_n(x)])$

$= [g_n(\varphi_n(x))]$ — ③

Since the original diagram commutes,

i.e $g \circ \varphi_n = \varphi_{n-1} \circ f_n \ \forall n$

& since our homomorphism were well defined i.e f_n, g_n & φ_n^{*}. We have

③ commutes.

Mayer-Vietoris Theorem:

Let M be a simplicial complexes & K_1, K_2 subcomplexes s.t $M = K_1 \cup K_2$

Let $L = K_1 \cap K_2$

Then, the following sequence is exact

$$H_n(L) \xrightarrow{v_n} H_n(K_1) \oplus H_n(K_2) \longrightarrow H_n(M) \longrightarrow H_{n-1}(L) \xrightarrow{v_{n-1}} H_{n-1}(K_1) \oplus H_{n-1}(K_2) \longrightarrow \cdots$$

where $v_n = (i_n^*, j_n^*)$ with i_n^* & j_n^* being the homomorphisms induced by

$i : L \hookrightarrow K_1$

& $j : L \hookrightarrow K_2$

Recall:

$A_4 \xrightarrow{h_4} A_3 \longrightarrow A_2 \longrightarrow A_1 \xrightarrow{h_1} A_0$

is exact then

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$0 \longrightarrow cok(\eta_4) \longrightarrow A_2 \longrightarrow ker(\eta_1) \longrightarrow 0$

is a short exact sequence

where $cok(\eta_4) = A_3 / im(\eta_4)$.

From properties of a short exact sequence it follows that

$$\dim(A_2) = \dim(cok(\eta_4)) + \dim(ker\,\eta_1) \quad (1)$$

Since $cok(\eta_4) = A_3/im(\eta_4)$,

$$\dim(cok(\eta_4)) = \dim(A_3) - \dim(im(\eta_4)) \quad (2)$$

From Rank Nullity theorem

$$\dim(A_4) = \dim(ker(\eta_4)) + \dim(im(\eta_4)) \quad (3)$$

Hence,

$$\dim(A_2) = \dim(A_3) - \dim(im(\eta_4)) + \dim(ker(\eta_1)) \quad (\because (2))$$

$$= \dim(A_3) + \dim(ker(\eta_4)) - \dim(A_4) + \dim(ker\,\eta_1) \quad (\because (3)) \quad (*)$$

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Corollary, For all n,

$$\beta(M) = \beta_n(K_1) + \beta_n(K_2) - \beta_n(L) + \dim(\ker \nu_n) + \dim(\ker \nu_{n-1})$$

Proof: From Mayer-Vietoris theorem,

$$\overset{(A_4)}{H_n(L)} \xrightarrow{\nu_n} \overset{(A_3)}{H_n(K_1) \oplus H_n(K_2)} \to \overset{(A_2)}{H_n(M)} \to \overset{(A_1)}{H_{n-1}(L)} \xrightarrow{\nu_{n-1}} \overset{(A_0)}{H_{n-1}(K_1) \oplus H_{n-1}(K_2)}$$

is an exact sequence.

so using $(*)$ & replacing the above sequence

So, $\dim(H_n(M)) = \beta_n(M)$

So, $\beta_n(M) = \beta_n(K_1) + \beta_n(K_2) + \dim \ker \nu_{n-1}$
$+ \dim(\ker \nu_n)$
$- \dim(H_n(L)) / \beta_n(L)$

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Claim: Let Δ be a simplicial complex on vertex set V & $\sigma \subseteq V$ be such $|\sigma| = p+1$ for some $p \geq 0$. Furthermore, suppose $\sigma \notin \Delta$ by $\partial_p \sigma \in C_{p-1}(\Delta)$, i.e, all boundary faces of σ are in Δ.

Then, $\forall j \in \{p-1, p\}$

$$\beta_j(\Delta \cup \sigma) = \beta_j(\Delta)$$

Furthermore,

(i) $\partial_p \sigma \in \mathrm{im}(\partial_p) \Rightarrow \beta_p(\Delta \cup \sigma) = \beta_p(\Delta) + 1$

$$\beta_{p-1}(\Delta \cup \sigma) = \beta_{p-1}(\Delta)$$

(2) $\partial_p \sigma \notin \mathrm{im}(\partial_p) \Rightarrow$

$$\beta_{p-1}(\Delta \cup \sigma) = \beta_{p-1}(\Delta) - 1$$

$$\beta_p(\Delta \cup \sigma) = \beta_p(\Delta)$$