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	7	14	21	28	F	0
1	8	15	22	29	S	1
2	9	16	23	30	S	5

LECTURE 8:

Let Δ be an abstract simplicial complex & $p \leq \dim(\Delta)$ = dimension

Aim: Construct the vector space $C_p(\Delta)$ corresponding to the p-faces in Δ.

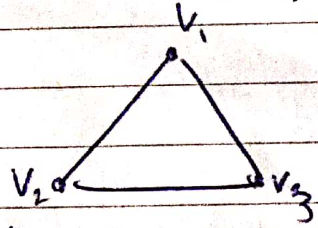
Recall, a p-face or a p-dimensional face is one with $p+1$ vertices.

Let $\mathcal{F}^p(\Delta)$ be set of p-dimensional faces in Δ.

Let $C_p(\Delta) = \{ \sum_{i=1}^{f_p} a_i \sigma_i : a_i \in \mathbb{Z}_2, \& \ \sigma_i \in \mathcal{F}^p(\Delta) \}$,

where $f_p = |\mathcal{F}^p(\Delta)| = \#$ no. of p-dimensional faces

Example: Let $\Delta = \{v_1, v_2, v_3, v_1v_2, v_1v_3, v_2v_3\}$



Consider $p=0$ case, then,

$\mathcal{F}^0(\Delta) = \{v_1, v_2, v_3\}$, $\mathcal{F}^1(\Delta) = \{v_1v_2, v_1v_3, v_2v_3\}$

$\mathcal{F}^2(\Delta) = \{\emptyset\}$

$C^0(\Delta) = \{v_1, v_2, v_3, v_1+v_2, v_2+v_3, v_3+v_1, v_1+v_2+v_3, 0\}$

$\mathcal{F}^1(\Delta) = \{v_1v_2, v_1v_3, v_2v_3\}$

$C^1(\Delta) = \{v_1v_2, v_1v_3, v_2v_3, v_1v_2 + v_2v_3, v_{12} + v_{13},$

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$v_{23} + v_{31},\ v_3 v_1 + v_1 v_2 + v_2 v_3 \}$

Define $+ : C_p(\Delta) \times C_p(\Delta) \longrightarrow C_p(\Delta)$

so that

$$\sum_{i=1}^{f_p} a_i \sigma_i + \sum_{i=1}^{f_p} a_i' \sigma_i = \sum_{i=1}^{f_p} (a_i \oplus a_i') \sigma_i$$

addition is already defined for the field $\mathbb{Z}_2$

Define $\cdot : \mathbb{Z}_2 \times C_p(\Delta) \longrightarrow C_p(\Delta)$

$$z \cdot \left(\sum_{i=1}^{f_p} a_i \sigma_i\right) = \sum_{i=1}^{f_p} \underbrace{(z \cdot a_i)}\sigma_i$$

$\forall z \in \mathbb{Z}_2$ with defined as $z, a_i \in \mathbb{Z}_2$

Claim: $(C_p(\Delta) \mathbb{Z}_2, +, \cdot)$ is a vector space

Proof: We first show that $(C_p(\Delta), +)$ is an abelian group

Associativity

$$\left(\sum_{i=1}^{f_p} a_i \sigma_i + \sum_{i=1}^{f_p} a_i' \sigma_i\right) + \left(\sum_{i=1}^{f_p} a_i'' \sigma_i\right).$$

$$= \sum (a_i + a_i') \sigma_i + \sum a_i'' \sigma_i$$

$$= \sum_{i=1}^{f_p} ((a_i + a_i') + a_i'') \sigma_i$$

Since field is associative, it does not matter

Thus + is associative.

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Identity element: The identity element is

$$0 = 0\sigma_1 + \dots + 0\sigma_{f_p}$$

This is true as:

$$\sum_{i=1}^{f_p} a_i\sigma_i + 0 = \sum_{i=1}^{\sigma_i} (a_i + 0)\cdot\sigma_i$$

$$= \sum_{i=1}^{f_p} a_i\sigma_i$$

Inverse: Each element is its own inverse

$$\sum_{i=1}^{f_p} a_i\sigma_i + \sum_{i=1}^{f_p} a_i'\sigma_i = \sum_{i=1}^{f_p} (a_i + a_i')\sigma_i$$

$$= \sum_{i=1}^{f_p} 0\,\sigma_i = 0$$

Because $a_i + a_i' \in \mathbb{Z}_2$

Now verify identity (multiplicative)

Identity: Observe that

$$1\cdot\left(\sum_{i=1}^{f_p} a_i\sigma_i\right) = \sum_{i=1}^{f_p} (1\cdot a_i)\sigma_i$$

$$= \sum_{i=1}^{f_p} a_i\sigma_i$$

Compatibility: $\forall x, y \in \mathbb{Z}_2$

$$x\cdot\left(y\cdot\sum_{i=1}^{f_p} a_i\sigma_i\right) = x\cdot\sum_{i=1}^{f_p} (y\cdot a_i)\sigma_i$$

$$= \sum (x\cdot y\cdot a_i)\sigma_i$$

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$= (x \cdot y) \sum_{i=1}^{\beta_p} \sigma_i$, as desired

Distributivity: $\forall x, y \in \mathbb{Z}_2$

$$(x+y) \cdot \sum_{i=1}^{\beta_p} a_i \sigma_i$$

$$= \sum_{i=1}^{\beta_p} ((x+y) a_i) \sigma_i$$

$$= \sum (x \cdot a_i + y \cdot a_i) \sigma_i$$

$$= \sum x \cdot a_i \sigma_i + \sum y \cdot a_i \sigma_i$$

$$= x \cdot \sum a_i \sigma_i + y \cdot \sum a_i \sigma_i,$$

as desired

Similarly, $x \cdot \left(\sum_{i=1}^{\beta_p} a_i \sigma_i + \sum a_i' \cdot \sigma_i \right)$

$$= x \cdot \left(\sum (a_i + a_i') \sigma_i \right)$$

$$= \sum x \cdot (a_i + a_i') \sigma_i)$$

$$= \sum (x \cdot a_i + x \cdot a_i') \sigma_i$$

$$= \sum (x \cdot a_i) \sigma + \sum (x \cdot a_i') \sigma_i$$

$$= x \cdot \sum a_i \sigma_i + x \sum a_i' \sigma_i$$

$\boxed{\text{Hence, } C_p(\Delta) \text{ is a } \mathbb{Z}_2 \text{ vector space}}$

So, what about dimension

Dimension of $C_p(\Delta)$. It is easy that $\{\sigma_1, \ldots, \sigma_{\beta_p}\}$ span $C_p(\Delta)$ since every element in $C_p(\Delta)$ is of the form

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$\sum_{i=1} a_i \sigma_i \cdot \forall a_i \in \mathbb{Z}_2$

Those $\{\sigma_1, \dots, \sigma_{f_p}\}$ are p-chains i.e.

$\sigma_i = 0\sigma_1 + \dots + 1\sigma_i + \dots + 0\sigma_{f_p}$ $\forall i, 0 \le i \le f_p$

So, take $\sum_{i=1}^{f_p} a_i\sigma_i = a_1 \cdot (\sigma_1) + \dots + a_{f_p} \cdot (\sigma_{f_p})$

$\sum_{i=1}^{f_p} a_i \sigma_i = 0$ if $0\sigma_1 + 0\sigma_2 + \dots + 0\sigma_{f_p}$

otherwise if $a_i = 1$ for some i,

then $\sum_{i=1}^{f_p} a_i \sigma_i = 0 \cdot \sigma_1 + \dots + 1\sigma_i + \dots + 0\sigma_{f_p}$

Hence, $\{\sigma_1, \dots, \sigma_{f_p}\}$ is linearly independent,

therefore $\dim(C_p(\Delta)) = f_p = |\mathcal{F}^p(\Delta)|$

Claim: $C_p(\Delta) \cong \mathbb{Z}_2^{f_p}$

Let $\varphi : C_p(\Delta) \longmapsto \mathbb{Z}_2^{f_p}$ be given by

$\varphi\left(\sum_{i=1}^{f_p} a_i \sigma_i\right) = (a_1, \dots, a_{f_p})$

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then;

$\varphi\left(x \sum a_i \sigma_i + y \sum a_i' \sigma_i\right)$

$= \varphi\left(\sum_{i=1}^{f_p} (x a_i + y a_i') \sigma_i\right)$

$= (x a_1 + y a_1', \dots, x a_{f_p} + y a_{f_p}')$

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$= x\cdot(a_1,\dots,a_{f_p}) + y\cdot(a_1',\dots,a_{f_p}')$

$= x\cdot\varphi\left(\sum_{i=1}^{f_p} a_i\sigma_i\right) + y\cdot\varphi\left(\sum_{i=1}^{f_p} a_i'\sigma_i\right)$

Thus φ is a homomorphism

$$\varphi\left(\sum_{i=1}^{f_p} a_i\sigma_i\right) = \varphi\left(\sum_{i=1}^{f_p} a_i'\sigma_i\right)$$

$$\iff (a_1,\dots,a_{f_p}) = (a_1',\dots,a_{f_p}')$$

$$\iff a_i = a_i' \quad \forall\, i=1,\dots,f_p$$

Hence injective

Similarly, for any $(a_1,\dots,a_{f_p}) \in \mathbb{Z}_2^{f_p}$

By definition of $C^p(\Delta)$, it contains ~~a~~

$\varphi\left(\sum_{i=1}^{f_p} a_i\sigma_i'\right) = (a_1,\dots,a_{f_p})$ Hence φ is surjective

$\Rightarrow \varphi$ is an isomorphism. $\Rightarrow C^p(\Delta) \cong \mathbb{Z}_2^{p}$

The above discussion was for $0 \le p \le \dim(\Delta)$

For $p<0$ and $p>\dim(\Delta)$, we set,

$C_p(\Delta) = \{0\}$, the trivial $\mathbb{Z}_2$-vector space. This is referred to the usual homology set. Sometimes we set $C_{-1}(\Delta) = \mathbb{Z}_2$

In such cases, we say we work with reduced homology.

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Our goal now is somehow count the number of holes.

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31	3	10	17	24	M	A
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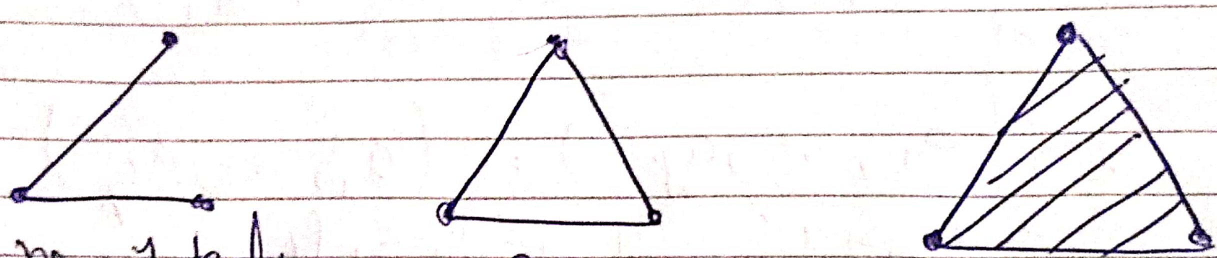
A p-hole is a $(p+1)$ dimensional hollow enclosed region.

9

10



no 1-hole since no 2-dim hollow enclosed region

one 1-hole

no 1-hole since enclosed region filled

11

12

1