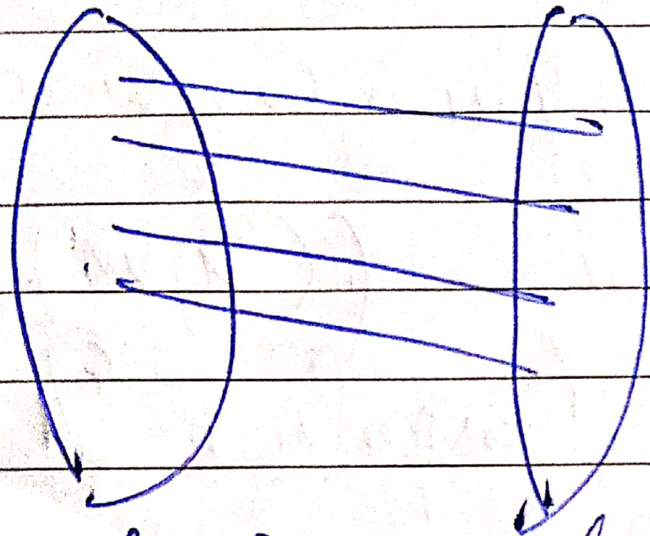
LECTURE 12

From the desk of DATE

Consider again, the short exact sequence

$$0 \xrightarrow{\eta_4} A_3 \xrightarrow{\eta_3} A_2 \xrightarrow{\eta_2} A_1 \xrightarrow{\eta_1} 0$$

So,



Maps to some subset of A_2

So $\mathrm{Im}(\eta_3)$ is a subspace of A_2 &
$\mathrm{Ker}(\eta_2)$ is a " " "

So, since $\mathrm{Ker}(\eta_2) = \mathrm{Im}(\eta_3)$

$$\Rightarrow A_2/\mathrm{Im}(\eta_3) = A_2/\mathrm{Ker}(\eta_2)$$
$$\cong \mathrm{Im}(\eta_2)$$

We know, $\dim(A_2/\mathrm{Ker}(\eta_3)) = \dim(A_2) - \dim(\mathrm{Ker}(\eta_2))$

But $\dim(A_2) - \dim(\mathrm{Ker}(\eta_2)) = \dim(\mathrm{Im}(\eta_2))$

Hence, $\dim(\operatorname{Im}(\eta_2)) = \dim(A_2/\ker(\eta_2))$

$\Rightarrow A_2/\ker(\eta_2) \cong \operatorname{Im}(\eta_2)$

& from the sequence, we know $A_1 = \operatorname{Im}(\eta_2)$

$$\Rightarrow A_2/\ker(\eta_2) \cong \operatorname{Im}(\eta_2) = A_1$$

Hence, $\dim(A_2)/\operatorname{Im}(\eta_3)$ — 1

$$= \dim(A_2) - \dim(\operatorname{Im}(\eta_3)) = \dim(A_1)$$

Since η_3 is injective,

$\dim(\operatorname{Im}(\eta_3)) = \dim(A_3)$ — 2

As everything is mapped to everything to A_1

Substituting 2 in 1

$$\dim(A_2) - \dim(A_3) = \dim(A_1)$$

i.e. $\dim(A_2) = \dim(A_1) + \dim(A_3)$

Similarly, one can show that if

$$A_4 \xrightarrow{\eta_4} A_3 \longmapsto A_2 \longrightarrow A_1 \xrightarrow{\eta_1} A_0$$

is an exact sequence.

$$0 \longmapsto \operatorname{cok} \eta_4 \longmapsto A_2 \longrightarrow \ker \eta_1 \longmapsto 0$$

is a short exact sequence where

$$\operatorname{cok} \eta_4 = A_3 / \operatorname{im}(\eta_4)$$

where cok means cokernel of η_4

Chain complex

Recall that C_p is the p-th chain space &

$$\partial_p : C_p \longmapsto C_{p-1}$$

is a vector space homomorphism

The chain complex is a sequence

$$\cdots \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots$$

$$\cdots C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \cdots$$

Chain maps:

Let X & Y be two chain

$$\cdots X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots$$

$$\cdots Y_{n+1} \xrightarrow{g_{n+1}} Y_n \xrightarrow{g_n} Y_{n-1} \xrightarrow{g_{n-1}} \cdots$$

So, we associate this "chains" to X & Y.

A chain map $\varphi : X \longrightarrow Y$ is a sequence of homomorphisms

$$\varphi : X_n \longrightarrow Y_n, \; n \in \mathbb{Z},$$

s.t the following diagram commutes

$$\begin{tikzcd}
\cdots X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\varphi_{n+1}"] & X_n \arrow[r, "f_n"] \arrow[d, "\varphi_n"] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\varphi_{n-1}"] & \cdots \\
\cdots Y_{n+1} \arrow[r, "g_{n+1}"] & Y_n \arrow[r, "g_n"] & Y_{n-1} \arrow[r, "g_{n-1}"] & \cdots
\end{tikzcd}$$

i.e

$$g \circ \varphi_n = \varphi_{n-1} \circ f_n \ \forall n$$

i.e ,,,,,, & oooo go ~~to~~ the same place

claim 1: If $x \in \ker(f_n)$, then $\varphi_n(x) \in \ker(g_n)$ — (1)

Proof: Since $x \in \ker(f_n)$

$$f_n(x) = 0$$

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d, "\varphi_n"] & X_{n-1} \arrow[d, "\varphi_{n-1}"] \\
Y_n \arrow[r, "g_n"] & Y_{n-1}
\end{tikzcd}$$

Since φ_{n-1} is a homomorphism

$$\varphi_{n-1} \circ f_n(x) = 0$$

& we know diagram commutes

$$g_n \circ \varphi_n = \varphi_{n-1} \circ f_n(x)$$
$$= 0$$

$$g_n(\varphi_n(x)) = 0 \Rightarrow \varphi_n(x) \in \ker(g_n)$$

Claim 2: If $x \in im(f_{n+1})$, then

$$\varphi_n(x) \in im(g_{n+1}) \quad -(2)$$

$$\begin{tikzcd} X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\varphi_{n+1}"'] & X_n \arrow[d, "\varphi_n"] \\ Y_{n+1} \arrow[r, "g_{n+1}"] & Y_n \end{tikzcd}$$

Proof: $x \in im(f_{n+1})$

$\Rightarrow \exists\, x' \in X_{n+1}$ s.t.

$x = f_{n+1}(x')$

Since diagram commutes

$\varphi_n(f_{n+1}(x')) = g_{n+1}(\varphi_{n+1}(x'))$

$\Rightarrow \varphi_n(x) = g_{n+1}(\varphi_{n+1}(x'))$

$\Rightarrow$ that is $\varphi_n(x)$ is in image of

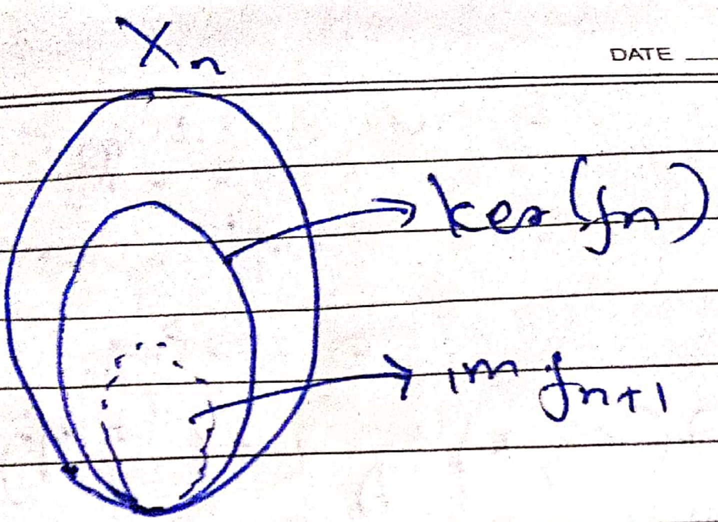
g_{n+1} $\blacksquare$

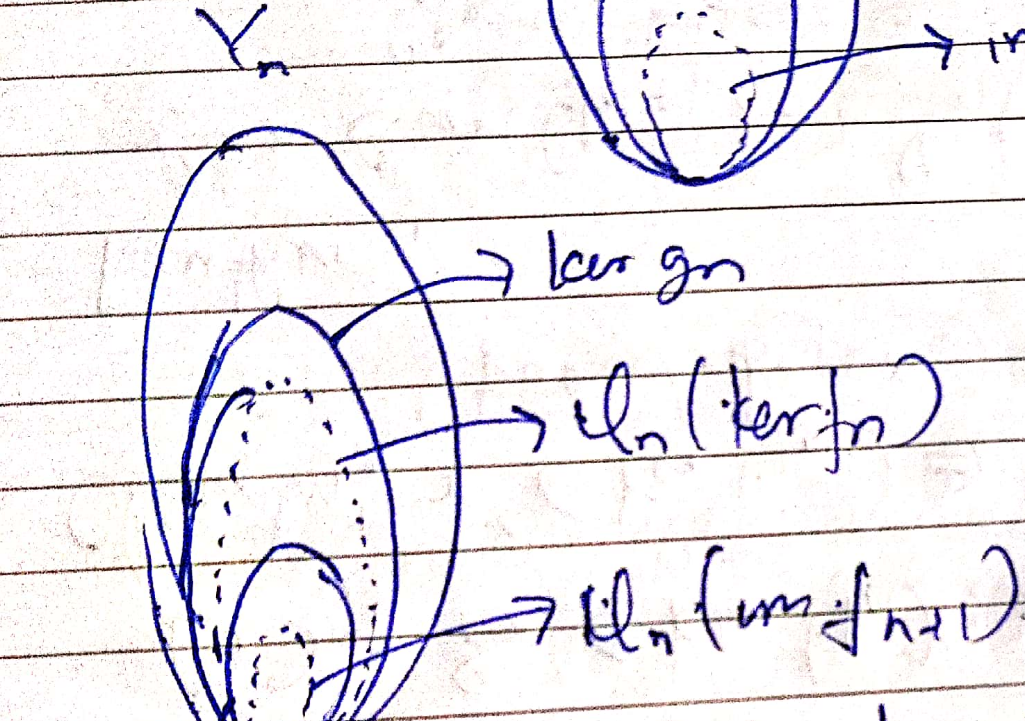
As usual; since $f^2 \equiv 0$ & $g^2 \equiv 0$,

we have $im\, f_{n+1} \subseteq \ker(f_n)$

$$im\, g_{n+1} \subseteq \ker(g_n)$$

Pictorially, we have,





Comment: It is possible that

~~$H_n(\operatorname{im} f_{n+1})$~~

$H_n \cdot g_{n+1}$

$g(\ker f_n) \subseteq \operatorname{im} g_{n+1}$

Based on the above discussion, we can define a induced homomorphisms ~~In similar spirit of~~ homology groups from before, we can define,

$H_n(X) := \ker(f_n)/\operatorname{im}(f_{n+1})$

$H_n(Y) := \ker(g_n)/\operatorname{im}(g_{n+1})$

We can define something like this:

$$\begin{tikzcd} H_n(X) \arrow[r] \arrow[d] & H_{n-1}(X) \arrow[d] \cdots \\ H_n(X') & H_{n-1}(X') \cdots \end{tikzcd}$$

We want to find if there is an homomorphism betn them

Consider $[x] \in H_n(X)$

$\Rightarrow x \in \ker(f_n)$ &

$[x] = \{x' \in \ker(f_n) : x'-x \in \text{im}(f_{n+1})$

Clearly, $x' \in \ker(f_n) \Rightarrow \varphi_n(x') \in \ker(g$

(proved before) ①

Further,

$x'-x \in \text{im}(f_{n+1}) \Rightarrow \varphi_n(x'-x) \in \text{im}(g_n$

(from before ②)

but φ is a homomorphism

$\Rightarrow \varphi_n(x') - \varphi(x) \in \text{im}(g_{n+1})$

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Thus if $x \sim x'$ in $\ker(g_n)$

then $\varphi_n(x') \sim \varphi_n(x)$

In other words, if

$[x] = \{x' \in \ker(g_n) : x' \sim x\}$

& $\varphi_n(x)$ & $\varphi_n(x')$ are elements of $\ker(g_n)$

so, & $\varphi_n(x') - \varphi(x) \in \operatorname{im}(g_{n+1})$

then $\varphi_n([x]) \subseteq [\varphi_n(x)]$.

$= \{y \in \ker(g_n)$ s.t

$y \sim \varphi_n(x)\}$