LECTURE 7

From the desk of

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For identifying holes, we define a vector space homomorphism b/w different C_p's.

Specifically, for $1 \le p \le \dim(\Delta)$, let

$\partial_p : C_p \longmapsto C_{p-1}$ be the boundary operator, i.e. if $\sigma = \{v_0, \dots, v_p\}$ then

$$\partial_p \sigma = \sum_{j=0}^{p} \{v_0, \dots, \hat{v}_j, \dots v_p\},$$

Here the $\wedge$ symbol denotes the corresponding vertex is dropped

Example $\partial_1(\{v_1, v_2\}) = \{v_2\} + \{v_1\}$

for $\partial_1 : C_1 \longmapsto C_0$, it removes all the ~~other~~ sets depending on the dimension

Further, for all other elements in C_p, the definition is given by linear expansion.

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Say $\partial_2 : C_2 \longmapsto C_1$

$$\partial_2(\{v_1, v_2, v_3\}) = \{v_2, v_3\} + \{v_1, v_3\} + \{v_1, v_2\}$$

Now, we need to consider the p-chains such that the $a_i \in \mathbb{Z}_2$ where $a_i \neq 0$ for i = atleast 2 indeses

So, we want:

$$\partial_p\left(\sum_i a_i \sigma_i\right) = \sum a_i \cdot \partial_p \sigma_i$$

We now test is ∂_p is linear.

Let $c = \sum a_i \sigma_i$ & $c' = \sum a_i' \sigma_i$

Then, for, $x, x' \in \mathbb{Z}_2$,

$x \cdot c + x' \cdot c'$,

Suppose:

$$\partial_1(\{v_1, v_2\} + \{v_1, v_3\} + \{v_2, v_3\})$$

$$= v_1 + v_2 + v_1 + v_3 + v_2 + v_3$$

$$= 0.$$

$= 0$

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Again, it takes ours pattern of our interest its 0. Hence, the kernel is somewhat established or gives an idea.

$$\Rightarrow \left(\sum x \cdot a_i\right)\sigma_i + \left(\sum x' a_i\right)\sigma_i$$

$$= \sum_i (x a_i + x \cdot a_i')\sigma_i$$

Hence $\partial_p (x \cdot c + x' \cdot c')$

$$= \sum_i ((x a_i + x' a_i')) \partial_p \sigma_i$$

$$= \sum_i \left[(x a_i) \partial_p \sigma_i + (x' a_i) \cdot \partial_p \sigma_i\right]$$

$$= \sum (x a_i) \cdot \partial_p \sigma_i + \sum_i (x' a_i') \cdot \partial_p \sigma_i$$

Addition is associative & commutative for p-chains $\forall p$ $0 \le p \le \dim(D)$

Now, $= \sum x \cdot (a_i \partial_p \sigma_i) +$

$$\sum x' \cdot (a_i' \cdot \partial_p \sigma_i)$$

Follows from compatibility of

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$= x \cdot \sum a_i \partial_p \sigma_i + x' \cdot \sum a_i \cdot \partial_p \sigma_i$

Follows from distributivity of $\cdot$ & $+$.

$= x \cdot \partial_p c + x' \cdot \partial_p c'$

Hence, ∂_p is linear i.e a vector space homomorphism

It still remains to define ∂_p for $p \leq 0$ & $p \geq \dim(\Delta) + 1$.

Suppose $p \leq -2$ or $p \geq \dim(\Delta) + 2$.

Then, $C_p = C_{p-1} = \{0\}$.

Hence one can trivially define,

$$\partial_p(0) = 0$$

additive identity on C_p

additive identity in C_{p-1}

Note, that a linear map has to be always map the identity of the domain vector space to the identity of codomain space.

Let $T: V \to V'$ be a linear map &

~~Suppose:~~

~~$T(0) = y \neq 0$~~

$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$

$T(0_V) = T(0_V) - T(0_V)$

$= 0_{V'}$

$\Rightarrow$ Mapping $0_V \to 0_{V'}$.

Let $p = \dim(\Delta) + 1$, then $C_p = \{0\}$,

However, $C_{p-1} = \{ \sum_i a_i \sigma_i : a_i \in \mathbb{Z}_2$ &

$\sigma_i \in \mathcal{F}^k(\Delta) \}$,

Since we want homomorphism

Set $\partial_p(0) = 0$. It is trivial to check for homomorphism so, take.

$a_1 0 + a_2 0 = 0 + 0 = 0$

$\Rightarrow \partial_p(0) = 0 = a_1 . 0 + a_2 . 0$

$= a . \partial_p(0) + a . \partial_p(0)$

Now consider the usual homology,

Then, $C_{-1} = \{0\}$.

Then, $\partial_{-1} : C_{-1} \to C_{-2}$ is defined

$\partial_{-1}(0) = 0$.

On the other hand

$\partial_0 : C_0 \to C_{-1}$

$\partial_0(\sigma) = 0 \quad \forall \sigma \in C_0$

One can check that both ∂_0 & ∂_{-1} are homomorphism.

Let's consider the reduced homology case.

Then, $C_{-1} = \mathbb{Z}_2$

In this case $\partial_{-1} : C_{-1} \longmapsto C_{-2}$ is defined using the relation

$$\partial_{-1}(0) = \partial_{-1}(1) = 0$$

On the other hand,

$$\partial_0 : C_0 \longmapsto C_{-1}$$

is defined with

$$\partial_0(\sum a_i \sigma_i) = \sum a_i \text{ s.t } \sigma_i \in \mathcal{F}^0(\Delta)$$

One can again check that ∂_0 & ∂_{-1} are homomorphisms.

$$\partial_0(0) = 0$$

$\partial_0(v_i)$ for some 0-dim vertex, and it will be mapped to 1

All even no. of sums i.e get mapped to 0, & odd to 1.

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to 0, & odd to 1.

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Now, ∂_p takes ~~every~~ p-chains of interest to 0.

Now, $\partial_2(\{v_0, v_1, v_2\} + \{v_0, v_1, v_3\} + \{v_0, v_2, v_3\} + \{v_1, v_2, v_3\})$

$= \partial_2(\{v_0, v_1, v_2\}) + \partial_2(\{v_0, v_1, v_3\}) + \partial_2(\{v_0, v_2, v_3\}) + \partial_2(\{v_1, v_2, v_3\})$

$= (\{v_0, v_1\} + \{v_0, v_2\} + \{v_1, v_2\})$
$+ (\{v_0, v_1\} + \{v_0, v_3\} + \{v_1, v_3\})$
$+ (\{v_0, v_2\} + \{v_0, v_3\} + \{v_2, v_3\})$
$+ (\{v_1, v_2\} + \{v_1, v_3\} + \{v_2, v_3\})$ — (1)

Here we are just dropping vertices

So, we have $\{v_0, v_1\} + \{v_0, v_1\} = (1+1)\{v_1, v_2\}$
$= 0 \cdot \{v_1, v_2\}$
$= 0$

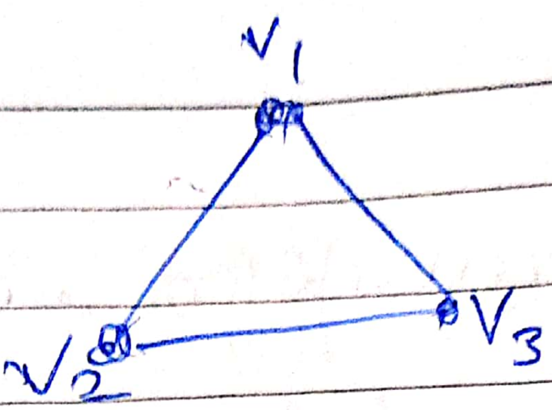
Same with others

So, (1) = 0

Furthermore, a collection of p faces that does:

For example,

Let Δ =



& $c = \{v_1, v_2\} + \{v_2, v_3\}$

Then $\partial_1 c = v_1 + v_2 + v_2 + v_3$

$$= v_1 + v_3 \neq 0$$

In other words,

$\text{Ker}(\partial_p) := \{c \in C_p \text{ s.t } \partial_p(c) = 0\}$

~~contains useful information about $(p+1)$-dimensional holes.~~

$\text{Ker}(\partial_p)$ is a subspace of C_p

Henceforth we will refer to elements of $Z_p := \text{Kernel}(\partial_p)$ as p-cycles