LECTURE 8:

From the desk of

DATE

Goal: Count number of holes of different dimensions

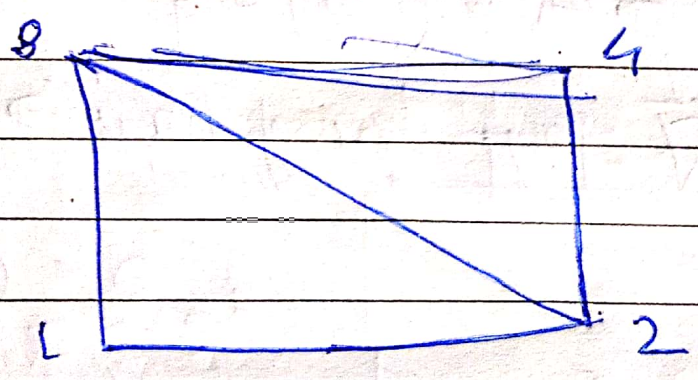
∂_0 = Usual homology, $\{0\}$

$\tilde{\partial}$: Reduced homology, $C_{-1} = \mathbb{Z}_2$

Significance of ∂_p : Ker(∂_p) contains useful information from a perspective of homology

Issues with ∂_p

i) What happens if there is more than one collection of p-face that enclose (p+1) dimensional holes?



$\Delta = \{1, 2, 3, 4, 12, 13, 23, 24, 34\}$

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Clearly, $\partial_1(12+13+23)=0$

Also, $\partial_1(23+34+24)=0 \to 0\in C_0$

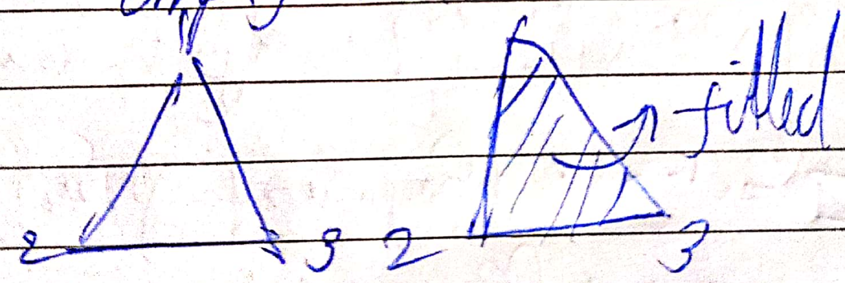
However, $\partial_1(12+24+34+13)=0$

In fact $\partial_1(N)=0$.

Intuitively one may think that Δ has only two holes, $\ker(\partial_1)$ definitely has more than two elements.

thus, simply counting the # of elements in Kernel is not good idea to count $(p+1)$ dimensional holes

2) ∂_p cannot distinguish bet' a filled hole & an 'empty' hole.



In both cases, $C_0=\{1,2,3,1+2,2+3,1+2+3\}$

$C_1 = \{0, 12, 13, 23, 12+13, 12+23, 13+23, 12+13+23\}$

Thus, elements in $\ker(\partial_p)$ can either represent 'hollow' p-cycles or 'filled' p-cycles

Resolution:

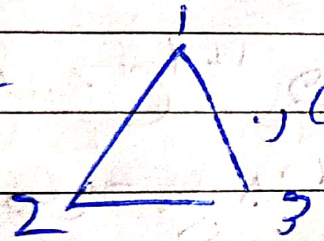
i) Genus observation. For complex on the right

$12+13+23 = \partial_2(123)$

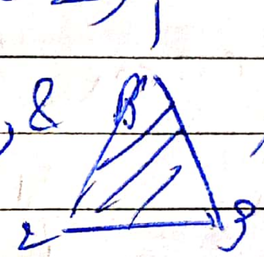
Thus, $12+13+23 \in \ker(\partial_1)$

but also $\in \mathrm{im}(\partial_2)$

So, for



, call it A, &



, call it B,

$12+13+23 \notin \mathrm{im}(\partial_2)$

$C_2 = \{0\}$

$\partial_2(0) = 0 \in C_1$

$\in C_2$

For B,

$12+13+23 \in \mathrm{im}(\partial_2)$

So, in fact, we can combine this, and generalize it

Fundamental Lemma of Homology

$$\boxed{\partial^2 = 0}$$

i.e $\forall c \in C_{p+1}$.

$\partial_p \partial_{p+1}(c') = 0$

Equivalently,

$im(\partial_{p+1}) \subseteq Ker(\partial_p)$

So, $\partial_p \circ \partial_{p+1}(c')$

$\partial_p(c'') = c'''$

$c' \in C_{p+1}$,

$c'' \in C_p$,

$c''' \in C_{p-1}$

$C_{p+1} = \{\sum a_i \sigma_i .$
$a_i \in \mathbb{Z}_2$ &
$\sigma_i \in \mathcal{F}^{p+1}(\Delta)\}$

Proof: To begin with, we show,

$\partial_p \partial_{p+1} \sigma = 0$

for any $(p+1)$-face σ:

Let $\sigma = \{v_0, \ldots, v_{p+1}\}$

Then, $\partial_{p+1} \sigma = \sum\limits_{j=0} \{v_0, \ldots, \hat{v}_j, \ldots, v_{p+1}\}$

Now, $\partial_p \partial_{p+1} \sigma$

$= \sum \partial_p \{v_0, \ldots, \hat{v}_j, \ldots, v_{p+1}\}$

Now, $\partial_p \partial_{p+1} \sigma = \sum\limits_{j=0}^{p+1} \partial_p \{v_0, \ldots, \hat{v}_j, \ldots, v_{p+1}\}$

~~Now, $\partial_p \partial_{p+1} \sigma = \sum\limits_{j=0}^{p+1} \{v_0, \ldots, \hat{v}_j, \ldots, v_{p+1}\}$~~

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From the desk of $= \sum_{j=0}^{p+1} \sum_{i \neq j} \partial \{v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{p+1}\}$ DATE

$= 0$ (which is this true?)

suppose (example), 3-face $\in C_3$,

$\partial_3 \{0,1,2,3\}$

$\Rightarrow \{0,1,2\}, + \{0,1,3\}, + \{0,2,3\},$

$\{1,2,3\},$ Apply ∂_2 to this

~~$\partial_2 \{0,1,2\} + \partial_2 \{0$~~

$= (01 + 12 + 02) + (01 + 03 + 03) +$
$(02 + 03 + 23) + (12 + 13 + 23)$

Proof is done, so, for any $c \in C_{p+1}$

so, Now, take $c = \sum_{i=0}^{p+1} a_i \sigma_i$

Hence we have

$$\partial_p \partial_{p+1} (c) = \partial_p \partial_{p+1} \left(\sum_{i=0}^{p+1} a_i \sigma_i\right)$$

$$= \sum_{i=0}^{p+1} a_i \partial_p \partial_{p+1} (c_i)$$

$$= \sum a_i \cdot 0$$

$$= 0$$

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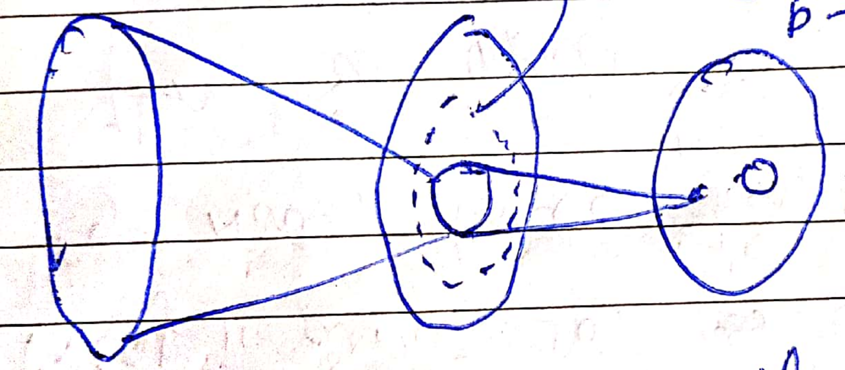
Specifically, $B_p := \operatorname{im}(\partial_{p+1})$ is a subspace of $Z_p := \ker(\partial_p)$.

This follows because ∂_{p+1} is a vector space homomorphism.



$\operatorname{Im}(\partial_{p+1})$ will be a subspace of C_p.

We will refer to the elements of B_p as the p-boundaries

Clearly, each p-boundary is also a p-cycle but they are the curious kinds

$$B_p \subseteq Z_p \subseteq C_p$$

Every p-chain in B_p is a p-cycle.

Consider, to focus on the interesting things, we let

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Since H_p is a vector space, we can talk about its dimension

The p-Betti number i.e β_p, denotes this number

$\beta_p = \dim(H_p)$

$\#p$-holes $= \#(p+1)$-dimensional holes

Claim: β_p counts the number of $(p+1)$-dimensional holes in a given simplex complex

$= \#$ of elements in basis of H_p

We will look at a few examples to verify this claim

every element in B_p is to be interpreted as zero

$\forall c \in B_p$, $0 \sim c$, show

$c - 0 \in B_p$
$\;\;\, \parallel$
$c \in B_p$.

to show $[0] = B_p$

Hence, $0 \sim c$

Example: v_1 v_2

We will compute the Betti numbers under the mod 2 homology setup.

Now, $C_0 = \{v_1, v_2, v_1+v_2, 0\}$

$C_p = \{0\} \ \forall p \neq 0$

Let $p \geq 2$, Then

$C_{p+1} = C_p = C_{p-1} = \{0\}$

Then, $\partial_{p+1}(0) = 0$ & $\partial_p(0) = 0$

Hence $\ker(\partial_p) = \{0\}$

$\operatorname{Im}(\partial_{p+1}) = \{0\}$

Other words, $C_p = Z_p = B_p = \{0\}$

Now, $H_p = Z_p / B_p$

That is $H_p = \{[x] : x \in Z_p\}$,

$[x] = \{y \in Z_p : x \sim y\}$

$= \{y \in Z_p : y - x \in B_p\}$

Thus $H_p = \{[0]\}$, where $[0] = \{0\}$ since

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$0 - 0 = 0 \in B_p$

Note $[0] + [0] = [0+0] = [0]$

Also $0 \cdot [0] = [0 \cdot 0] = [0]$

$1 \cdot [0] = [1 \cdot 0] = [0]$

H_p is a $\mathbb{Z}_2$-vector space (over field $\mathbb{Z}_2$)

Therefore H_j is the trivial vector space.

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