LECTURE 9

$$H_p := Z_p / B_p$$

i.e. $H_p = \{[c] : c \in Z_p\}$, where,

$$[c] = \{c' \in Z_p : c' - c \in B_p\}$$

Since H_p is obtained by taking the quotient of Z_p with B_p, it follows that Z_p itself is a $\mathbb{Z}_2$-vector space.

Finally, we said the p-th Betti number $\beta_p := \dim(H_p)$ counts # of $(p+1)$ dim holes

$\cdot v_1$ $\quad$ $\cdot v_2$

Last time, it was showed that H_p is a trivial vector space for all $p \leq -2$ & $p \geq 2$.

It remains to determine H_p for $p \in \{1, 0, -1\}$

Suppose $p = 1$. Then, $Z = \ker(\partial_1)$ & $B_1 = \operatorname{im}(\partial_2)$

$C_2 = C_1 = \{0\}$

& $C_0 = \{0, v_0, v_{-1}, v_0 + v_1\}$

Now, $\partial_2(0) = 0$. Hence
$B_1 = \operatorname{im}(\partial_2) = \{0\}$

Further $\partial_1(0) = 0$, Therefore,
$Z_1 = \ker(\partial_1) = \{0\}$

This shows that $H_1 = \{[0]\}$, where $[0] = \{0\}$.

$B_1 = 0$

Now, for $p = -1$, I would homology
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$C_{-1} = \{0\}$, $C_0 = \{0, v_0, v_1, v_0 + v_1\}$ &

$C_{-2} = \{0\}$

$\partial_{-1}(0) = 0$, Hence, $Z_{-1} = \ker(\partial_{-1}) = \{0\}$

Separately, $\partial_0(c) = 0 \ \forall \ c \in C_0$

$B_{-1} = Im(\partial_0) = \{0\}$

This shows that $H_{-1} = \{[0]\}$

Finally, consider $p = 0$, Here, $C_1 = 0$,

$C_{-1} = 0$ & $C_0 = \{0, v_0, v_1, v_0 + v_1\}$

Then, $\partial_1(0) = 0$, Hence $B_0 = Im(\partial_1)$
$= 0$

On the other hand,

$\partial_0(c) = 0 \ \forall \ c \in C_0$.

Hence $Z_0 = C_0$

$H_0 = Z_0 / B_0$
$= \{[0], [v_0], [v_1], [v_1 + v_0]\}$

$[v_0] = \{ c \in \mathbb{Z}_0$ s.t $c - v_0 \in B_0$

Check, $0 - v_0 = -v_0 \neq 0 \notin B_0$

$v_1 - v_0 = -v_1 - v_0 \neq 0 \notin B_0$

$v_0 + v_1 - v_0 = v_1 \neq 0 \notin B_0$

$v_0 - v_0 = 0 \in B_0$

So, the equivalence classes make sense, so they are listed separately

Thus basis is $\{[v_0], [v_1]\}$ forms a basis for H_0, it follows that

$B_0 = \dim(H_0) = 2$

How to compute equivalence classes in general

If $Q = V/S$, then

$Q = \{[v] : v \in V\}$

$[v] = v + S = \{v + s \mid s \in S\}$

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$[v] = \{v \in V : v' \sim v\}$

$= \{v' \in V : v' - v \in S\}$

~~Hence, it suffices to show, that $v \sim v'$ iff~~

$v' = v + s$

Let us redo with the reduced homology

Now $C_{-1} = \mathbb{Z}_2$. This will only affect H_p for $p \in \{0, -1, -2\}$, while the rest remain the same.

Let $p = -2$. Then, $C_{-1} = \mathbb{Z}_2$

Here $B_{-2} = \cancel{\mathbb{Z}_{-2}} = \{0\}$. Hence, H_{-2} is trivial so, kernel $(\partial_{-2}) = \{0\} = Z_{-2}$

$B_{-2} = \mathrm{im}(\partial_{-1}) = \{0\}$.

So, $B_{-2} = \{0\}$

Now, let $p = -1$, Then

$C_0 = \{0, v_0, v_1, v_0 + v_1\}$.

$C_{-1} = \mathbb{Z}_2$, & $C_{-2} = \{0\}$

~~Observe that $\partial_0(0) = 0$ & ∂.~~

~~Im (∂)~~

Observe that $\partial_0(0) = 0$ &

$\partial_0(v_0 + v_1) = 0$

while $\partial_0(v_0) = 1$ & $\partial_0(v_1) = 1$

Hence, $B_{-1} = \{0, 1\}$

$\partial_{-1}(0) = \partial_{-1}(1) = 0$

$\Rightarrow Z_{-1} = \{0, 1\}$

Hence, $H_{-1} = \{[0], [1]\}$

$[0] : \{e \in Z_{-1}$ s.t $e - 0 \in B_{-1}\}$

~~only possibly zero 0, 1~~

$0 - 0 = 0 \in B_{-1}$

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But $1 - 0 = 1 \notin B_{-1}$

~~Now for $[1] = \{c \in Z_{-1}$ s.t. $c - 1 \in B_{-1}\}$~~

~~Also, only possible for $[0]$ as $0 - 1 = 1 \notin B_{-1}$ but $1 - 1 = 0 \in B_{-1}$~~

~~So we can write~~ $[0] = \{0+0, 0+1\}$
$= \{0, 1\}$

$[1] = \{1+0, 1+1\}$
$= \{0, 1\}$

Therefore, $H_{-1} = \{[0]\}$ & is again the trivial space. (Think of ~~this as the additive~~ identity.

Finally, suppose, $p = 0$,

$C_1 = \{0\}$, $C_{-1} = \mathbb{Z}_2$ &

$C_0 = \{0, v_0, v_1, v_0 + v_1\}$

Here $\partial_1(0) = 0$, Thus $B_0 = \{0\}$

Separately $\partial_0(0) = 0$ & $\partial_0(v_0 + v_1) = 0$

Hence, $Z_0 = \{0, v_0 + v_1\}$

This implies, $H_0 = \{[0], [v_0 + v_1]\}$

where, $[0] = \{0\}$ & $[v_0 + v_1] = \{v_0 + v_1\}$

$B_0 = \{0\}$

$[0] = \{c \in Z_0$ s.t $c - 0 \in B_0\}$

this is true only for $c = 0$

as $v_0 + v_1 - 0 = v_0 + v_1 \notin B_0$

Similarly for $[v_0 + v_1]$ equivalence

Dim of H_0 i.e. Basis of H_0

is $[v_0 + v_1]$ as $[0] = 0 \cdot [v_0 + v_1]$

& $[v_0 + v_1] = 1 \cdot [v_0 + v_1]$

Hence $\dim(H_0) = 1$

$\beta_0 = 1$

Comparison between usual & reduced Homology

Let $\tilde{\beta}_p$ denote the p-th Betti number in reduced homology. Then

$\tilde{\beta}_p = \beta_p \ \forall \ p \neq 0$

$\tilde{\beta}_0 = \beta_0 - 1$

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Now: interpretation, Usual $B_p = 0 \; \forall p \neq 0$, $B_0 = 2$ | Reduced $B_p = 0 \; \forall p \neq$ $B_0 = 1$

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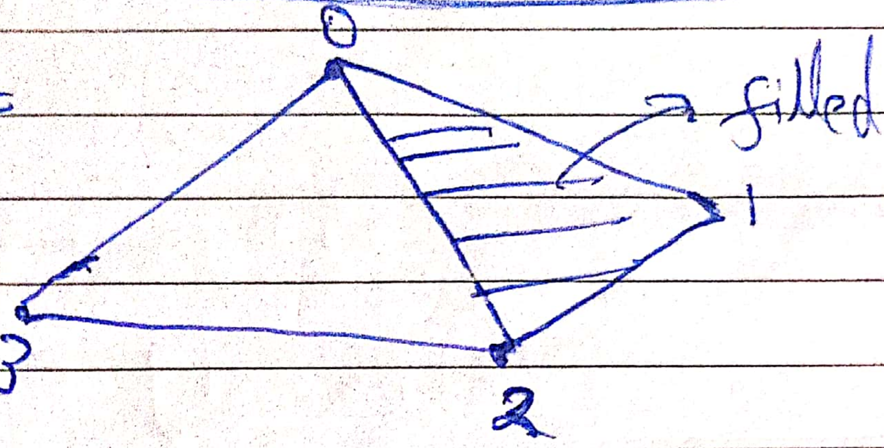
(2) Δ =



$\Delta = \{0, 1, 2, 3, 01, 02, 12, 03, 23, 012\}$

Goal $B_p \; \forall p \in \mathbb{Z}$

$\forall p \geq 4$, $B_p = 0$, since

$C_{p+1} \to C_p \to C_{p-1}$

$Z_p = B_p = \{0\}$, $H_p = \{[0]\}$

$\forall p \leq -2$, $B_p = 0$

Now, we need to compute for $p \in \{-1, 0, 1, 2, 3\}$

$C_3 \to$	$C_2 \to$	C_1
$\{0\}$	012	0
	0	01
		02
		12
		03
		23
		Σ of all combinations

$Im(\partial_3) = B_2$

$Ker(\partial_2) = Z_2$

$\partial_3(0) = 0$

$\partial_2(0) = 0$

$\partial_2(012) = 01 + 12 + 02$

$Ker(\partial_2) = 0 = Z_2$

$Im(\partial_3) = 0 = B_2$

$H_2 = \{[0]\}$ so, $dim(H_2) = 0 = \beta_2$