Full-Information Online Learning with Adversarial Reward

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The Expert Problem

Alternative protocol:

Environment **decides** the reward vector r_t (not revealing)

Learner chooses an expert a_t

Environment reveals r_t

Given: set of experts $\mathcal{A} = \{1, \dots, A\}$

For time $t = 1, 2, \dots, T$:

Learner chooses a distribution over experts $p_t \in \Delta_{\mathcal{A}}$

Environment **decides** and reveals the reward vector $r_t = (r_t(1), \dots, r_t(A))$

Adversarial environment: $r_1(a), \dots, r_T(a)$ do not have the same mean

$$\text{Regret} = \max_{a \in \mathcal{A}} \sum_{t=1}^{T} r_t(a) - \sum_{t=1}^{T} \langle p_t, r_t \rangle$$

Strategies?

- Follow the leader

$$a_t = \max_{a\in\mathcal{A}} \left\{\sum_{i=1}^{t-1} r_i(a)\right\}$$

time	1	2	3	4	5	· ·
action 1	1/2	1	0	1	0	. . . .
action 2	1	0	1	0	1	. . . .

Learner total reward ≤ 1

Total reward of best action $\approx \frac{T}{2}$

Incremental Updates

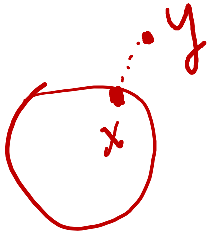
Projected gradient ascent:

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

Exponential weight updates:

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a')\exp(\eta r_t(a'))}$$

learner's expected reward (p) = $\langle p, r_t \rangle$



$$\Pi_\Delta(y) = \underset{x \in \Delta}{\operatorname{argmin}} \|x-y\|_2$$

Equivalent Forms of EWU

$$p_{t+1}(a) \propto \underline{p_t(a)} \exp(\eta r_t(a)) \propto p_{t-1}(a) \exp(\eta r_{t-1}(a)) \exp(\eta r_t(a)) \dots$$

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a') \exp(\eta r_t(a'))}$$

$$p_{t+1}(a) = \frac{\exp\left(\eta \sum_{i=1}^{t} r_i(a)\right)}{\sum_{a' \in \mathcal{A}} \exp\left(\eta \sum_{i=1}^{t} r_i(a')\right)}$$

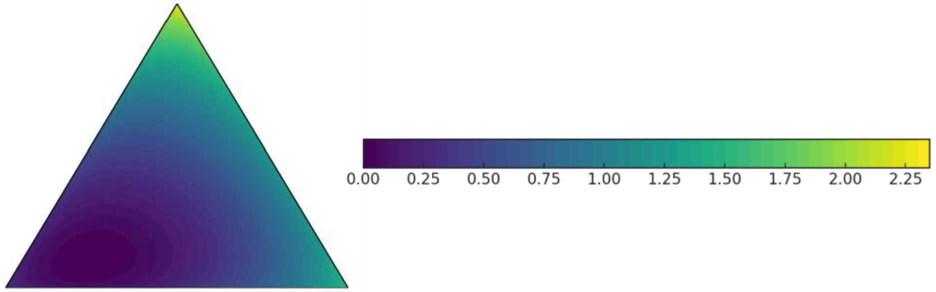
$$\to \underline{\eta t \cdot \hat{r}_t(a)}$$

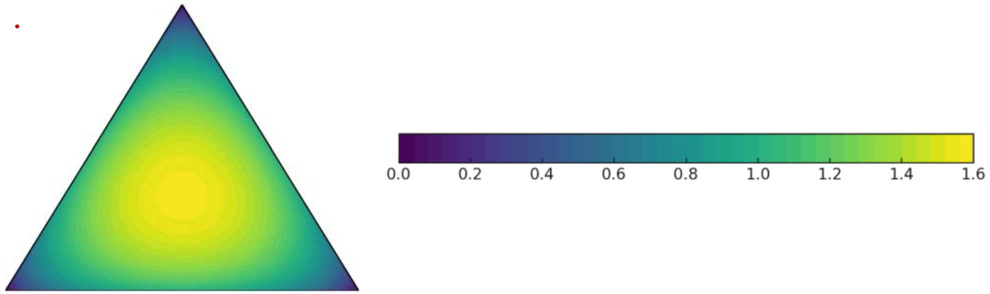
$$p_{t+1} = \operatorname*{argmax}_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

$$\mathrm{KL}(p, q) := \sum_{a=1}^{A} p(a) \ln \frac{p(a)}{q(a)}$$ **(KL divergence)**

$$p_{t+1} = \operatorname*{argmax}_{p \in \Delta_{\mathcal{A}}} \left\{ \left\langle p, \sum_{i=1}^{t} r_i \right\rangle + \frac{1}{\eta} H(p) \right\}$$

$$H(p) := \sum_{a=1}^{A} p(a) \ln \frac{1}{p(a)}$$ **(Shannon entropy)**





Regret Bound for Exponential Weight Updates

Theorem.

Assume that $\eta r_t(a) \le 1$ for all t, a. Then EWU

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a')\exp(\eta r_t(a'))}$$

ensures

$$\text{Regret} = \max_{a^\star} \sum_{t=1}^{T} \left(r_t(a^\star) - \langle p_t, r_t \rangle\right) \le \frac{\ln A}{\eta} + \eta \sum_{t=1}^{T} \sum_{a=1}^{A} p_t(a) \underline{r_t(a)^2}$$

$$|r_t(a)| \le R_{max}$$

$$\le \frac{\ln A}{\eta} + \eta \sum_t \underbrace{\boxed{\sum_a p_t(a)}}_{=1} R_{max}^2$$

$$= \frac{\ln A}{\eta} + \eta T \cdot R_{max}$$

$$\Big\downarrow \text{optimal } \eta$$

$$R_{max}\sqrt{T \ln A}$$

Regret Bound Analysis

$$p_1(a) = \frac{1}{A}$$

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a'} p_t(a')\exp(\eta r_t(a'))}$$

$$\Rightarrow \log\frac{p_{t+1}(a^*)}{p_t(a^*)} = \log\frac{\exp(\eta r_t(a^*))}{\sum_a p_t(a)\exp(\eta r_t(a))} = \eta r_t(a^*) - \log\left(\sum_a p_t(a)\exp(\eta r_t(a))\right)$$

$$\Rightarrow \underline{r_t(a^*) - \langle p_t, r_t\rangle} = \frac{1}{\eta}\log\frac{p_{t+1}(a^*)}{p_t(a^*)} + \underbrace{\frac{1}{\eta}\log\left(\sum_a p_t(a)\exp(\eta r_t(a))\right) - \langle p_t, r_t\rangle}_{\le\ \eta\sum_a p_t(a)\, r_t(a)^2}$$

$$\Rightarrow \text{Regret} = \sum_{t=1}^{T}\left(r_t(a^*) - \langle p_t, r_t\rangle\right)$$

$$= \frac{1}{\eta}\log\frac{\overset{\le 1}{p_{T+1}(a^*)}}{\underset{\frac{1}{A}}{p_1(a^*)}} + \eta\sum_t\sum_a p_t(a)\, r_t(a)^2$$

$$\le \frac{1}{\eta}\log A + \eta\sum_{t,a} p_t(a)\, r_t(a)^2$$

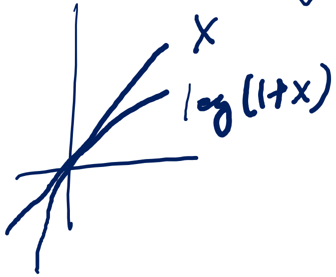
$$\frac{1}{\eta}\log\left(\sum_a p_t(a)\underbrace{\exp\left(\eta r_t(a)\right)}\right) - \langle p_t, r_t\rangle$$

$$\le \frac{1}{\eta}\log\left(\sum_a p_t(a)\underbrace{\left(1+\eta r_t(a)+\eta^2 r_t(a)^2\right)}\right) - \langle p_t, r_t\rangle \qquad e^x \le 1+x+x^2 \text{ for } x \le 1$$

$$= \frac{1}{\eta}\log\left(\underbrace{\sum_a p_t(a)}_{1} + \eta\sum_a p_t(a) r_t(a) + \eta^2\sum_a p_t(a) r_t(a)^2\right) - \langle p_t, r_t\rangle$$

$$\le \frac{1}{\eta}\left(\eta \underline{\sum_a p_t(a) r_t(a)} + \eta^2\sum_a p_t(a) r_t(a)^2\right) \underline{- \langle p_t, r_t\rangle} \qquad \left[\underline{\log(1+x) \le x}\right] \quad \forall x > -1$$

$$= \eta\sum_a p_t(a) r_t(a)^2$$



Exponential Weight Updates

$$\Delta_{\mathcal{A}} = \left\{ x \in \mathbb{R}^{A} : \underline{\sum_a x(a) = 1}, \ \underline{x(a) \geq 0} \right\}$$

Exponential Weight Updates = KL divergence regularized policy updates

$$p_{t+1}(a) = \frac{p_t(a)\exp(\eta r_t(a))}{\sum_{a' \in \mathcal{A}} p_t(a')\exp(\eta r_t(a'))} = \quad p_{t+1} \overset{\emptyset}{=} \operatorname*{argmax}_{p \in \Delta_{\mathcal{A}}} \left\{ \underbrace{\langle p, r_t \rangle - \frac{1}{\eta}\mathrm{KL}(p, p_t)}_{F(p)} \right\}$$

$$\mathrm{KL}(p, p_t) = \sum_a p(a) \log \frac{p(a)}{p_t(a)}$$

$$\frac{\partial}{\partial p(a)} = \log \frac{p(a)}{p_t(a)} + 1$$

$$\mathcal{L}(p) = \underline{F(p)} + \underline{\lambda\left(\sum_a p(a) - 1\right)} + \underline{\sum_a \mu(a) p(a)}, \quad \mu(a) \geq 0$$

$$\underline{\mu(a) p(a) = 0} \quad \forall a$$

$$\frac{\partial}{\partial p(a)} = r_t(a) - \frac{1}{\eta}\left(\log \frac{p(a)}{p_t(a)} + 1\right) + \lambda + \mu(a) = 0$$

$$\Rightarrow \quad p(a) = p_t(a) \exp\left(\eta r_t(a) + \eta \lambda + \right)$$

KL divergence regularized policy updates is the basis of many RL algorithms (e.g., PPO, SAC).

Projected Gradient Descent

Projected Gradient Descent = Euclidean norm regularized policy updates

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

$$=$$

$$p_{t+1} = \underset{p \in \Delta_{\mathcal{A}}}{\operatorname{argmax}} \left\{ \langle p, r_t \rangle - \frac{1}{2\eta} \| p - p_t \|_2^2 \right\}$$

Recap

- We considered **full-information**, **policy-based** learning algorithms for the expert problem (finite action decision-making problem).
- We showed that the **exponential weight algorithm** achieves sub-linear regret even when the reward is adversarial.
- Actually, **projected gradient descent** also works, though we haven’t provided a proof (will do it today).
- Exponential weight updates = KL divergence regularized gradient updates
- Projected gradient descent = Euclidean norm regularized gradient updates

Distance Regularized Updates

Projected Gradient Descent

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{2\eta} \|p - p_t\|_2^2 \right\}$$

Exponential Weight Updates

$$p_{t+1}(a) \propto p_t(a) \exp(\eta r_t(a))$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

- Adversarial reward
- Stochastic reward
- For non-linear functions, gradient only approximates the function locally

General Framework: Mirror Descent

Projected Gradient Descent

$$p_{t+1} = \Pi_{\Delta_{\mathcal{A}}}(p_t + \eta r_t)$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{2\eta} \|p - p_t\|_2^2 \right\}$$

Exponential Weight Updates

$$p_{t+1}(a) \propto p_t(a) \exp(\eta r_t(a))$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

$$\psi(p) = \frac{1}{2}\|p\|_2^2$$

$$\psi(p) = \sum_{a=1}^{A} p(a) \ln p(a)$$

Mirror Descent

$$p_{t+1} = \max_{p \in \Omega} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} D_\psi(p, p_t) \right\}$$

$$D_\psi(p, q) := \psi(p) - \psi(q) - \langle \nabla\psi(q), p - q \rangle$$

(Bregman divergence w.r.t. the potential function / regularizer ψ)

Online Linear Optimization and Projected Gradient Descent

Given: Convex feasible set $\Omega \subseteq \mathbb{R}^d$

For time $t = 1, 2, \dots, T$:

Learner chooses a point $w_t \in \Omega$

Environment reveals a reward vector $r_t \in \mathbb{R}^d$

$$\text{Regret} = \max_{w \in \Omega} \sum_{t=1}^{T} \langle w, r_t \rangle - \sum_{t=1}^{T} \langle w_t, r_t \rangle$$

Projected Gradient Descent

Arbitrary $w_1 \in \Omega$

$w_{t+1} = \Pi_\Omega(w_t + \eta r_t)$

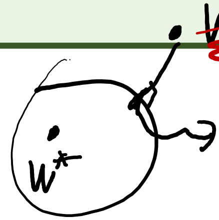
Regret Bound of Projected Gradient Descent

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$= \underline{DG\sqrt{T}}$$

Theorem. Projected Gradient Descent ensures

$$\sum_{t=1}^{T}\langle w^\star, r_t\rangle - \sum_{t=1}^{T}\langle w_t, r_t\rangle \le \frac{\|w^\star - w_1\|_2^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\|r_t\|_2^2 \le \underbrace{\frac{D^2}{2\eta}}_{a} + \underbrace{\frac{\eta G^2 T}{2}}_{b} \overset{\text{optimal}}{=} DG\sqrt{T}$$



~~$w_t+\eta r_t$~~ u

~~$\Pi(w_t+\eta r_t)$~~ $\Pi(u)$

$$\|w^* - w_{t+1}\|^2 = \|w^* - \Pi_\Omega(w_t+\eta r_t)\|^2$$

$$\le \|w^* - (w_t+\eta r_t)\|^2$$

$$= \|w^* - w_t\|^2 - 2\eta\langle w^*-w_t, r_t\rangle + \eta^2\|r_t\|^2$$

$$\|a+b\|^2 = \|a\|^2 + 2\langle a,b\rangle + \|b\|^2$$

$$\Rightarrow \sum_{t=1}^{T}\langle w^*-w_t, r_t\rangle \le \sum_{t=1}^{T}\frac{1}{2\eta}\left(\|w^*-w_t\|^2 - \|w^*-w_{t+1}\|^2\right) + \sum_{t=1}^{T}\frac{\eta}{2}\|r_t\|^2$$

$$\frac{1}{2\eta}\left(\|w^*-w_1\|^2 - \|w^*-w_{T+1}\|^2\right)$$

Summary

$\boxed{|r_t(a)| \le 1}$, $\forall a \Rightarrow \|r_t\| \doteq \sqrt{\sum_a (r_t(a))^2}$

$\approx \sqrt{A}$

Projected Gradient Descent

$$w_{t+1} = \Pi_{\Omega}(w_t + \eta r_t)$$

$$w_{t+1} = \max_{w \in \Omega} \left\{ \langle w, r_t \rangle - \frac{1}{2\eta} \|w - w_t\|_2^2 \right\}$$

$$\text{Regret} \le O\left(DG\sqrt{T}\right)$$

$$= O\left(\sqrt{AT}\right)$$ (in the expert setting)

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$D = \max_{x,y \in \Omega} \|x-y\|_2$, $G = \max_t \|r_t\|_2$

Exponential Weight Updates

$$p_{t+1}(a) \propto p_t(a) \exp(\eta r_t(a))$$

$$p_{t+1} = \max_{p \in \Delta_{\mathcal{A}}} \left\{ \langle p, r_t \rangle - \frac{1}{\eta} \mathrm{KL}(p, p_t) \right\}$$

$$\text{Regret} \le O\left(\sqrt{T \log A}\right)$$

$\frac{D^2}{2\eta} + \frac{\eta}{2} G^2 T \underset{\substack{\uparrow \\ \text{optimal } \eta}}{=\!=} DG\sqrt{T}$