Homework 1

6771 Reinforcement Learning (Fall 2025)

Deadline: 11:59pm, September 28, 2025

You may type or handwrite your solution. If handwritten, take photos and include them as figures in latex. Submit the answer as a .pdf and the code (for Problem 3) as a .py file to Gradescope.

The latex template can be accessed at https://www.overleaf.com/read/xhpccjwhbcjs#f9dad9

1 ϵ -greedy for Contextual Bandits

In this problem, we will derive the regret bound of ϵ -greedy in contextual bandits with a regression oracle (Page 41 here). Consider the algorithm below.

Algorithm 1 ϵ -Greedy for contextual bandits

Parameter: $\epsilon \in [0, 1]$, A (number of actions)

Given: A regression oracle for t = 1, 2, ..., T do

Receive x_t , and obtain \hat{R}_t from the regression oracle.

Define

$$\pi_t(a) = \begin{cases} 1 - \epsilon + \frac{\epsilon}{A} & \text{if } a = \operatorname{argmax}_{a'} \hat{R}_t(x_t, a') \\ \frac{\epsilon}{A} & \text{otherwise} \end{cases}$$

Sample $a_t \sim \pi_t$, and receive $r_t = R(x_t, a_t) + w_t$, where R is the underlying true reward function and w_t is a zero-mean noise.

Define $a_t^{\star} = \operatorname{argmax}_a R(x_t, a)$. Assume that $R(x, a) \in [0, 1]$ and $\hat{R}_t(x, a) \in [0, 1]$ for any x, a, t.

(a) (5%) Show the following inequality. Note that the left-hand side is the expected regret at round t, and the right-hand side is ϵ plus the estimation error of the regression oracle.

$$R(x_t, a_t^\star) - \mathbb{E}_{a \sim \pi_t} \left[R(x_t, a) \right] \leq \epsilon + \underbrace{\mathbb{E}_{a \sim \pi_t} \left[\hat{R}_t(x_t, a) - R(x_t, a) \right]}_{\text{estimation error on } a_t} + \underbrace{R(x_t, a_t^\star) - \hat{R}_t(x_t, a_t^\star)}_{\text{estimation error on } a_t^\star}.$$

Proof.

Right hand side — Left hand side

$$= \epsilon + \mathbb{E}_{a \sim \pi_t} [\hat{R}_t(x_t, a)] - \hat{R}_t(x_t, a_t^*)$$

$$= \epsilon + (1 - \epsilon) \max_{a'} \hat{R}_t(x_t, a') + \frac{\epsilon}{A} \sum_{a=1}^A \hat{R}_t(x_t, a) - \hat{R}_t(x_t, a_t^*)$$
 (by the definition of π_t)

$$\geq \epsilon + \frac{\epsilon}{A} \sum_{a=1}^{A} \hat{R}_t(x_t, a) - \epsilon \hat{R}_t(x_t, a_t^*)$$

$$\geq 0.$$

$$(\max_{a'} \hat{R}_t(x_t, a') \geq \hat{R}_t(x_t, a_t^*))$$

$$\geq 0.$$

$$(0 \leq \hat{R}_t(x, a) \leq 1)$$

(b) (5%) Show that the two estimation error terms in (a) can be bounded as

$$\mathbb{E}_{a \sim \pi_t} \left[\hat{R}_t(x_t, a) - R(x_t, a) \right] \le \sqrt{\mathbb{E}_{a \sim \pi_t} \left[\left(\hat{R}_t(x_t, a) - R(x_t, a) \right)^2 \right]},$$

$$R(x_t, a_t^*) - \hat{R}_t(x_t, a_t^*) \le \sqrt{\frac{1}{\pi_t(a_t^*)}} \mathbb{E}_{a \sim \pi_t} \left[\left(\hat{R}_t(x_t, a) - R(x_t, a) \right)^2 \right],$$

respectively.

Proof. First inequality is by the well-known fact that $\mathbb{E}[X] \leq \sqrt{\mathbb{E}[X^2]}$ for any random variable X. Second inequality:

$$R(x_{t}, a_{t}^{\star}) - \hat{R}_{t}(x_{t}, a_{t}^{\star}) \leq \sqrt{\left(R(x_{t}, a_{t}^{\star}) - \hat{R}_{t}(x_{t}, a_{t}^{\star})\right)^{2}}$$

$$= \sqrt{\frac{1}{\pi_{t}(a^{\star})} \pi_{t}(a^{\star}) \left(R(x_{t}, a_{t}^{\star}) - \hat{R}_{t}(x_{t}, a_{t}^{\star})\right)^{2}}$$

$$\leq \sqrt{\frac{1}{\pi_{t}(a^{\star})} \sum_{a \in \mathcal{A}} \pi_{t}(a) \left(R(x_{t}, a) - \hat{R}_{t}(x_{t}, a)\right)^{2}}.$$

(c) (5%) Combining (a) and (b), show that the one-step expected regret at round t can be upper bounded as

$$R(x_t, a_t^{\star}) - \mathbb{E}_{a \sim \pi_t} \left[R(x_t, a) \right] \le \epsilon + 2 \sqrt{\frac{A}{\epsilon} \cdot \mathbb{E}_{a \sim \pi_t} \left[\left(\hat{R}_t(x_t, a) - R(x_t, a) \right)^2 \right]}.$$

(Hint: note that $\pi_t(a_t^{\star}) \geq \frac{\epsilon}{A}$)

Proof. Combining the bounds in (a) and (b), and upper bound $1 \leq \frac{1}{\pi_t(a^*)}$ we get

$$R(x_t, a_t^{\star}) - \mathbb{E}_{a \sim \pi_t} \left[R(x_t, a) \right] \le \epsilon + 2\sqrt{\frac{1}{\pi_t(a^{\star})}} \mathbb{E}_{a \sim \pi_t} \left[\left(R(x_t, a) - \hat{R}_t(x_t, a) \right)^2 \right].$$

Then use the hint.

(d) (5%) Show that expected total regret $\mathbb{E}\left[\sum_{t=1}^{T}\left(R(x_t, a_t^{\star}) - R(x_t, a_t)\right)\right]$ can be upper bounded by

$$\epsilon T + 2\sqrt{\frac{AT\mathbb{E}\left[\mathsf{Err}\right]}{\epsilon}},$$

where Err is the total regression error defined as

$$\operatorname{Err} = \sum_{t=1}^{T} \mathbb{E}_{a \sim \pi_t} \left[\left(\hat{R}_t(x_t, a) - R(x_t, a) \right)^2 \right].$$

Proof. For simplicity, denote $\operatorname{reg}_t = R(x_t, a_t^\star) - \mathbb{E}_{a \sim \pi_t} \left[R(x_t, a) \right]$ and $e_t = \mathbb{E}_{a \sim \pi_t} \left[\left(\hat{R}_t(x_t, a) - R(x_t, a) \right)^2 \right]$.

From (c) we have $\operatorname{reg}_t \leq \epsilon + 2\sqrt{\frac{A}{\epsilon}}e_t$. Summing this over t and use Cauchy-Schwarz inequality we get

$$\begin{split} \sum_{t=1}^{T} \text{reg}_t & \leq \epsilon T + 2 \sum_{t=1}^{T} \sqrt{\frac{A}{\epsilon}} e_t \\ & \leq \epsilon T + 2 \sqrt{\frac{AT}{\epsilon} \sum_{t=1}^{T} e_t} \\ & = \epsilon T + 2 \sqrt{\frac{AT}{\epsilon}} \text{Err.} \end{split}$$

Finally, taking expectation on two sides, and using the fact $\mathbb{E}[\sqrt{X}] \leq \sqrt{\mathbb{E}[X]}$ finishes the proof.

2 $\tilde{O}(\sqrt{AT})$ Regret Bound for UCB

In this problem, we will show that UCB for multi-armed bandits ensures a $\tilde{O}(\sqrt{AT})$ regret (Page 49 here). We first define the following quantities: Let $R(a) \in [0,1]$ be the true mean of the reward of arm a. Define $a^* \triangleq \operatorname{argmax}_{a \in [A]} R(a)$.

We consider the UCB algorithm described in Algorithm 2. Assume that the number of arms A is less or equal to the number of rounds T, and assume w_t is zero-mean and 1-sub-Gaussian.

Algorithm 2 UCB for multi-armed bandits

Input: \overline{A} (number of arms), T (total number of rounds), δ (failure probability).

for $t = 1, \ldots, A$ do

for $t = A + 1, \ldots, T$ do

Define

$$N_t(a) = \sum_{s=1}^{t-1} \mathbb{I}\{a_s = a\}, \quad \hat{R}_t(a) = \frac{\sum_{s=1}^{t-1} \mathbb{I}\{a_s = a\}r_s}{N_t(a)}, \quad conf_t(a) = \sqrt{\frac{2\log(2/\delta)}{N_t(a)}}, \quad \tilde{R}_t(a) = \hat{R}_t(a) + conf_t(a).$$

Draw $a_t = \operatorname{argmax}_a \tilde{R}_t(a)$ and observe $r_t = R(a_t) + w_t$.

Recall that the regret is defined as Regret = $TR(a^*) - \sum_{t=1}^{T} R(a_t)$.

(a) (5%) Use Theorem 1 to show that with probability $1 - AT\delta$, $|\hat{R}_t(a) - R(a)| \leq conf_t(a)$ for all time t and arm a.

Proof. By definition,

$$\left| \hat{R}_t(a) - R(a) \right| = \left| \frac{1}{N_t(a)} \sum_{s=1}^{t-1} \mathbb{I}\{a_s = a\} r_s - R(a) \right|.$$
 (1)

For each fixed arm a and count $N_t(a)$, the terms inside the sum have mean R(a) because whenever $a_s = a$ we observe $r_s = R(a) + w_s$ with $\mathbb{E}[w_s] = 0$. Applying Theorem 1 (with $\sigma = 1$) to this average yields, for the fixed a and $N_t(a)$,

$$\left| \hat{R}_t(a) - R(a) \right| \leq \sqrt{\frac{2 \log(2/\delta)}{N_t(a)}} \ = \ \operatorname{conf}_t(a)$$

with probability at least $1 - \delta$. Finally, a union bound over all $a \in [A]$ and all $t \le T$ (equivalently, over all possible values of $N_t(a)$) shows that (1) holds simultaneously for every a and t with probability at least $1 - AT\delta$.

(b) (5%) Assume the inequality in (a) holds for all t and a. Prove that for all t > A:

$$R(a^{\star}) - R(a_t) \leq 2conf_t(a_t).$$

(Hint 1: Decompose the left-hand side as $(\tilde{R}_t(a^*) - \tilde{R}_t(a_t)) + (\tilde{R}_t(a_t) - R(a_t)) + (R(a^*) - \tilde{R}_t(a^*))$ (Hint 2: Bound the second and the third parts in Hint 1 with the help of (a))

Proof.

Assume (2a):
$$\forall t, a, \ |\hat{R}_t(a) - R(a)| \le \operatorname{conf}_t(a)$$
.
 $R(a^*) - R(a_t) = (\tilde{R}_t(a^*) - \tilde{R}_t(a_t)) + (\tilde{R}_t(a_t) - R(a_t)) + (R(a^*) - \tilde{R}_t(a^*))$
 $\le 0 + (\hat{R}_t(a_t) + \operatorname{conf}_t(a_t) - R(a_t)) + 0$
 $\le (|\hat{R}_t(a_t) - R(a_t)| + \operatorname{conf}_t(a_t))$
 $\le 2 \operatorname{conf}_t(a_t)$.

The **first term** is ≤ 0 because $a_t \in \arg\max_a \tilde{R}_t(a)$, so $\tilde{R}_t(a_t) \geq \tilde{R}_t(b)$ for any b, in particular $b = a^*$; this uses only the argmax property, not the specific form of \tilde{R}_t . For the **third term**, assumption (2a) implies $\tilde{R}_t(a^*) = \hat{R}_t(a^*) + \cosh(a^*) \geq R(a^*)$, hence $R(a^*) - \tilde{R}_t(a^*) \leq 0$. For the **second term**, (2a) gives $|\hat{R}_t(a_t) - R(a_t)| \leq \cosh(a_t)$, yielding the final bound.

(c) (5%) Show the inequality:

$$\sum_{t=A+1}^{T} conf_t(a_t) \le 2\sqrt{2\log(2/\delta)AT}.$$

(Hint 1: Write the left -hand side as $\sum_{a=1}^{A}\sum_{t=A+1}^{T}\mathbb{I}\{a_t=a\}conf_t(a)$)

(Hint 2: Use the inequality $1 + \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} + \cdots + \sqrt{\frac{1}{m}} \le 2\sqrt{m}$)

(Hint 3: Use the Cauchy-Schwarz inequality $\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_A} \le \sqrt{A(m_1 + m_2 + \cdots + m_A)}$)

Proof. Let M_a be the number of **post-initialization** pulls of arm a (so $\sum_{a=1}^{A} M_a = T - A$). Consider arm a alone. After initialization, the **first** time we pull a, the confidence is

$$\sqrt{\frac{2\log(2/\delta)}{1}}$$

the **second** time we pull a, it is

$$\sqrt{\frac{2\log(2/\delta)}{2}},$$

and in general, on the j-th post-init pull of a the confidence is

$$\sqrt{\frac{2\log(2/\delta)}{j}}, \quad j = 1, \dots, M_a.$$

Therefore the total confidence spent is exactly

$$\sum_{t=A+1}^{T} \operatorname{conf}_{t}(a_{t}) = \sqrt{2 \log_{\delta}^{2}} \sum_{a=1}^{A} \sum_{j=1}^{M_{a}} \frac{1}{\sqrt{j}}.$$

Use $\sum_{j=1}^{m} j^{-1/2} \le 2\sqrt{m}$ and then Cauchy–Schwarz:

$$\sum_{t=A+1}^{T} \operatorname{conf}_{t}(a_{t}) \leq 2\sqrt{2\log\frac{2}{\delta}} \sum_{a=1}^{A} \sqrt{M_{a}} \leq 2\sqrt{2\log\frac{2}{\delta}} \sqrt{A\sum_{a=1}^{A} M_{a}} \leq 2\sqrt{2\log\frac{2}{\delta} AT}.$$

This is the claimed bound.

(d) (5%) Conclude that with probability at least $1 - AT\delta$, Regret $\leq A + 4\sqrt{2\log(2/\delta)AT}$.

Proof. From part (b), on the event of (a) we have for all t > A,

$$R(a^{\star}) - R(a_t) \leq 2 \operatorname{conf}_t(a_t).$$

Thus the regret decomposes as

Regret =
$$\sum_{t=1}^{T} (R(a^*) - R(a_t)) \le \underbrace{\sum_{t=1}^{A} (R(a^*) - R(a_t))}_{\le A} + 2 \underbrace{\sum_{t=A+1}^{T} \operatorname{conf}_t(a_t)}_{\le A}$$
.

Use the bound from part (c):

$$2\sum_{t=A+1}^{T} \operatorname{conf}_{t}(a_{t}) \leq 2 \cdot 2\sqrt{2\log(2/\delta)} \, AT = 4\sqrt{2\log(2/\delta)} \, AT.$$

Therefore, with probability at least $1 - AT\delta$,

Regret
$$\leq A + 4\sqrt{2\log(2/\delta)} AT$$
.

3 Implementing Contextual Bandit Algorithms

In this homework, we will implement value-based contextual bandit algorithms covered in the lecture. Specifically, we will implement

• Algorithms based on regression oracle and exploration mechanisms including ϵ -greedy (EG), Boltzmann exploration (BE), and inverse gap weighting (IGW).

The starter code can be accessed at http://bahh723.github.io/rl2025fa_files/bandits.py.

Note: Please maintain your code nicely, as we will implement more algorithms in the same python file in HW2.

3.1 Data

To simulate a contextual bandit environment, we use existing classification dataset for supervised learning. Specifically, in this homework, we use the mnist dataset with some pruning and modification. Originally, it is a classification dataset where features are $28 \text{ pixel} \times 28 \text{ pixel}$ gray-scale images of digits, and the classes correspond to 10 digits.

3.2 Data Conversion

For simplicity, we trim the dataset so that it only contains 4 classes corresponding to digits '0', '1', '2', and '3', each having 2500 samples, summing up to 10000 samples. In each round, the environment will reveal the image (context), and the learner has to pick one of the classes (action). We artificially make the reward function change once in the middle. Specifically, in our case the time horizon is T = 10000. For $t \le 5000$ (Phase 1), the reward function is the following:

$$R(x,a) = \begin{cases} 0.5 & \text{if } a \text{ is the correct digit of image } x \\ 0 & \text{otherwise} \end{cases}$$

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For t > 5000 (Phase 2), the reward is the following:

$$R(x,a) = \begin{cases} 0.5 & \text{if } a \text{ is the correct digit of image } x \\ 1 & \text{if } (a-1) \bmod 4 \text{ is the is the correct digit of image } x \\ 0 & \text{otherwise} \end{cases}$$

For example, in Phase 2, if the learner chooses action 3 when seeing an image of digit 2, then it receives a reward of 1. We assume that the learner sees the true noiseless reward $R(x_t, a_t)$.

This conversion is already implemented in the starter code, so you don't need to do anything here.

3.3 CB Algorithm based on regression oracle

In this part, we will implement the simple value-based algorithm based on regression on the reward function. In the class, we have introduce three common exploration mechanism in this case: ϵ -Greedy, Boltzmann Exploration, and Inverse Gap Weighting. They share the same pseudo-code outlined in Algorithm 3.

Algorithm 3 An algorithm based on regression

- 1 Randomly initialize a reward network R_{θ} that takes the context as input and outputs the reward of each action.
- 2 Let θ_1 be the initial weights for the reward network.

In Algorithm 3, $f: \mathbb{R}^A \to \Delta_A$ is a link function that maps an A-dimensional real vector to a distribution over A actions. The three exploration mechanisms correspond to the following three choices of f:

- ϵ -Greedy: $[f(v)]_a = \frac{\epsilon}{A} + (1 \epsilon)\mathbb{I}\{a = \operatorname{argmax}_{a'} v(a')\}.$
- • Boltzmann Exploration: $[f(v)]_a = \frac{e^{\lambda v(a)}}{\sum_{a'} e^{\lambda v(a')}}.$
- Inverse Gap Weighting: $[f(v)]_a = \begin{cases} \frac{1}{A + \lambda(\max_{a'} v(a') v(a))} & \text{if } a \neq \operatorname{argmax}_{a'} v(a') \\ 1 \sum_{a' \neq a} [f(v)]_{a'} & \text{if } a = \operatorname{argmax}_{a'} v(a') \end{cases}.$

Algorithm 3 may be slightly different from the standard way of implementing such algorithm — usually, in Eq. (1) we may reuse data collected in the past by having a *replay buffer* that stores old data. But since our artificial problem has a non-stationary reward function, reusing data from the past becomes not a great idea.

Please complete the following tasks in (a)-(g). Note that the starter code already calculate the average reward in Phase 1, Phase 2, and overall horizon, respectively. The starter code also already implement the regression oracle (1). A figure of running average is also generated for a single parameter. The major task is to code up the three link functions above, run the experiments, and combine the figures for multiple parameters.

Note that you are allowed to change the default hyperparameters in the starter code (N, M), optimizer, learning rates, etc.). In the table below, you may also change the values of hyperparameters or add additional ones if you feel that the given values cannot reflect the trend.

The starter code already implements two baseline methods: one uniformly randomly chooses an action, and the other greedily chooses an action with the highest estimated reward. Run them with the command line:

```
python bandits.py --algorithm Rand
python bandits.py --algorithm Greedy
```

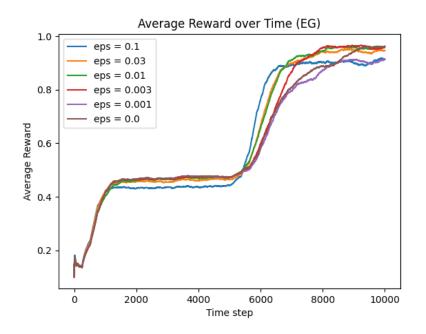
For each method you implement, a correct implementation must achieve an "overall reward" of at least 0.6 under the best choice of the hyperparameter. Meeting this requirement is necessary to get full credit.

We provide our implementation in here, and report our results below. We set N=M=1 in all experiments.

(a) (5%) Implement ϵ -Greedy and, for different values of ϵ , record in the table below the average reward in Phase 1, Phase 2, and over the entire horizon.

ϵ	Phase 1	Phase 2	Overall
0.1	0.4140	0.8480	0.6310
0.03	0.4327	0.8549	0.6438
0.01	0.4350	0.8615	0.6483
0.003	0.4382	0.8424	0.6403
0.001	0.4401	0.7917	0.6159
0	0.4401	0.8147	0.6274

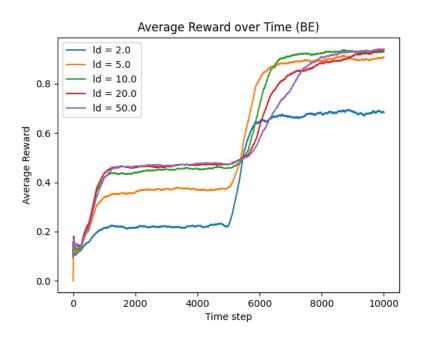
(b) (5%) Plot the running average reward over time for every parameter setting from part (a) on the same figure, following the example provided in the appendix.



(c) (5%) Implement Boltzmann Exploration and, for different values of λ , record in the table below the average reward in Phase 1, Phase 2, and over the entire horizon.

λ	Phase 1	Phase 2	Overall
2	0.2135	0.6588	0.4362
5	0.3489	0.8547	0.6018
10	0.4244	0.8565	0.6405
20	0.4406	0.8145	0.6276
50	0.4392	0.8040	0.6216

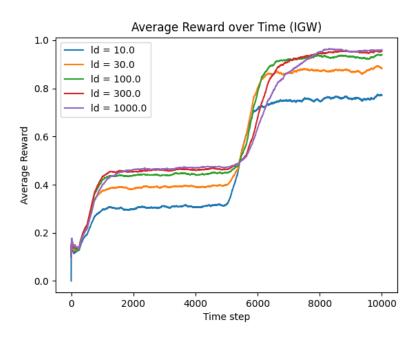
(d) (5%) Plot the running average reward over time for every parameter setting from part (c) on the same figure.



(e) (5%) Implement Inverse Gap Weighting and, for different values of λ , record in the table below the average reward in Phase 1, Phase 2, and over the entire horizon.

λ	Phase 1	Phase 2	Overall
10	0.2945	0.7304	0.5125
30	0.3726	0.8327	0.6027
100	0.4185	0.8717	0.6451
300	0.4346	0.8640	0.6493
1000	0.4339	0.8488	0.6414

(f) (5%) Plot the running average reward over time for every parameter setting from part (e) on the same figure.



(g) (5%) Have you noticed any trends in the experiments from (a)–(f)? In particular, how does the parameter influence the average reward in Phase 1 and Phase 2, respectively? What is the underlying explanation for this effect?

Suggested Answer When decreasing ϵ in ϵ -Greedy, or increasing λ in Boltzmann exploration or inverse gap weighting, there will be *decreasing* amount of exploration and *increasing* amount of exploitation. In Phase 1, the more the exploration, the worse the average performance. This is because the environment is so simple and thus explicit exploration is almost not needed. However, higher exploration leads to *quicker discovery* of the change point around t=5000 (observe the timing when the curves rise around t=5000). Like in Phase 1, higher exploration leads to worse performance at the end of Phase 2. Therefore, there is a tradeoff between exploitation and exploration in Phase 2, and the best performed parameter is usually not the extreme ones.

4 Survey

(a) (5%) How much time did you use to complete each of the problems in this homework? Do you have any suggestion for the course?

Appendix

A Inequality

Theorem 1 (Hoeffding's Inequality). Let X_1, X_2, \ldots, X_N be i.i.d. σ -sub-Gaussian random variables with mean μ . Then with probability at least $1 - \delta$,

$$\left| \frac{1}{N} \sum_{i=1}^{N} X_i - \mu \right| \le \sigma \sqrt{\frac{2 \log(2/\delta)}{N}}.$$

B Plot

In the starter code, we plot the learning curve for a single parameter. You need to plot the learning curves for multiple parameter values on the same figure. Below is an example that shows how to plot two curves, although in usually need to plot 5-6 curves in the problems above.

