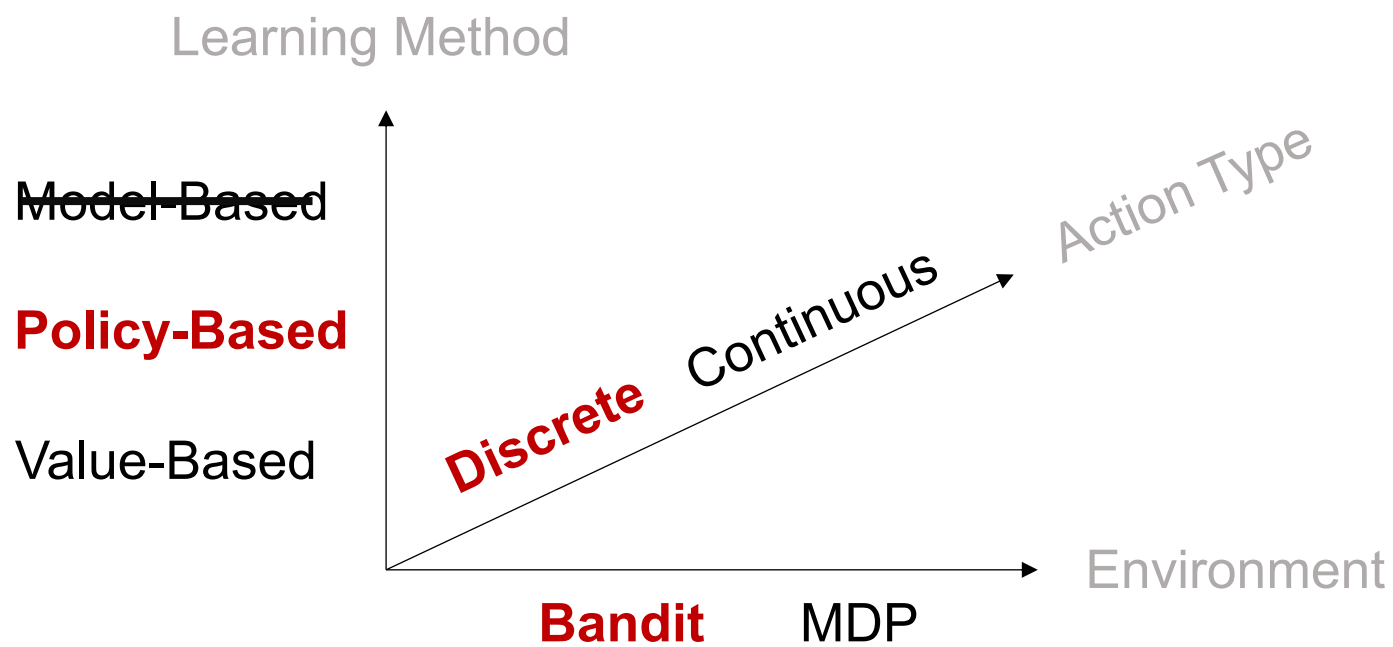
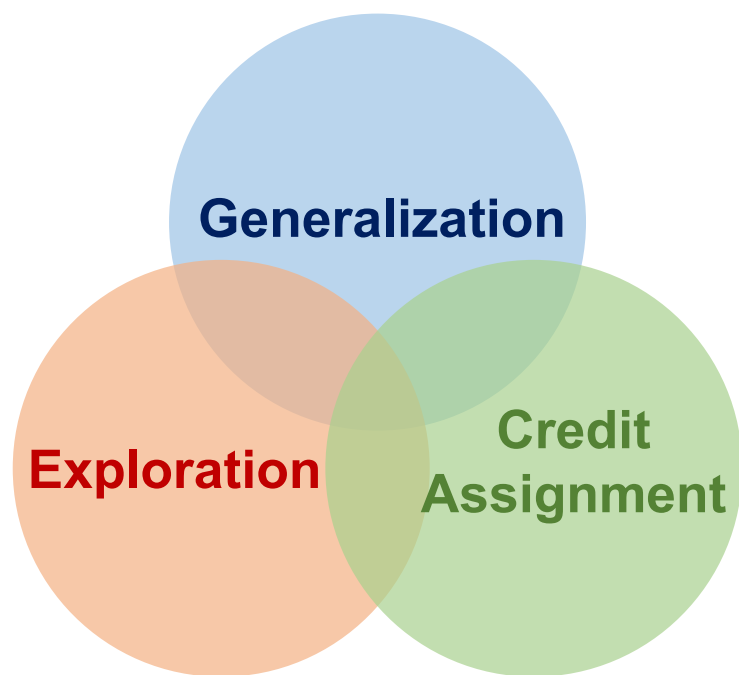


Bandits 2

Chen-Yu Wei

Roadmap



Policy-Based Bandits

- Key challenges: **Exploration** and **Generalization (if there are contexts)**
- Algorithms we will discuss:
 - KL-regularized policy updates (PPO, NPG)
 - Policy gradient (REINFORCE)
- We will add a little discussion on “time-varying” reward functions to motivate the algorithm design

The Full-Information MAB

Given: set of actions $\mathcal{A} = \{1, \dots, A\}$

For time $t = 1, 2, \dots, T$:

Environment decides the reward of all actions $r_t(1), r_t(2), \dots, r_t(A)$ without revealing

The learner chooses an action a_t

Environment reveals the reward $r_t(a)$ **of all actions**

$$\text{Regret} = \max_a \sum_{t=1}^T r_t(a) - \sum_{t=1}^T r_t(a_t)$$

KL-Regularized Policy Updates

$$r_t = \begin{bmatrix} r_t(i) \\ \vdots \\ r_t(A) \end{bmatrix} \quad \pi = \begin{bmatrix} \pi(i) \\ \vdots \\ \pi(A) \end{bmatrix}$$

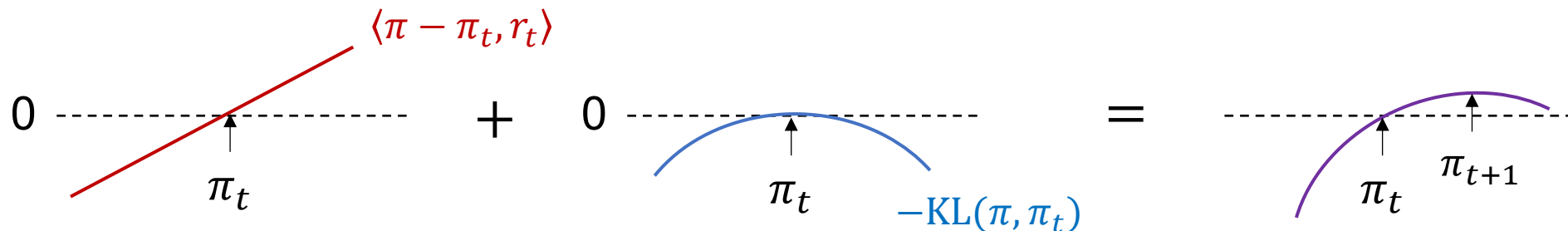
$\langle \pi, r_t \rangle$

$$\pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \langle \pi - \pi_t, r_t \rangle - \frac{1}{\eta} \text{KL}(\pi, \pi_t) \right\}$$

$$= \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \underbrace{\sum_a (\pi(a) - \pi_t(a)) r_t(a)}_{\text{The Improvement of } \pi \text{ over } \pi_t \text{ on } r_t} - \underbrace{\frac{1}{\eta} \sum_a \pi(a) \log \frac{\pi(a)}{\pi_t(a)}}_{\text{Distance between } \pi \text{ and } \pi_t} \right\}$$

The Improvement of π over π_t on r_t

Distance between π and π_t



Why regularizing the update?

Why KL-Regularized Policy Updates?

1. Maintaining **stability** for adversarial environments

Time	1	2	3	4	5	6	...
$R_t(1)$	0.5	0	1	0	1	0	...
$R_t(2)$	0	1	0	1	0	1	...

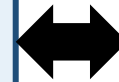
Follow the leader:
$$a_t = \max_{a \in \mathcal{A}} \left\{ \sum_{i=1}^{t-1} r_i(a) \right\}$$

2. When combining the algorithm with function approximation, the gradient only approximates the **local** reward landscape.

KL-Regularized Policy Updates

Exponential weight updates

$$\pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \langle \pi - \pi_t, r_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi, \pi_t) \right\}$$



$$\pi_{t+1}(a) = \frac{\pi_t(a) e^{\eta r_t(a)}}{\sum_{b \in \mathcal{A}} \pi_t(b) e^{\eta r_t(b)}}$$

Solving the optimization

Regret Bound for Exponential Weight Updates

Theorem.

Will be proven in HW2

Assume that $\eta r_t(a) \leq 1$ for all t, a . Then EWU

$$\pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \langle \pi - \pi_t, r_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi, \pi_t) \right\}$$

ensures for any $a^* \in \mathcal{A}$,

$$\sum_{t=1}^T (r_t(a^*) - \langle \pi_t, r_t \rangle) \leq \frac{\log A}{\eta} + \eta \sum_{t=1}^T \sum_{a=1}^A \pi_t(a) r_t(a)^2$$

Handwritten notes: A red box around the first term of the sum, and a red circle around the second term. A red arrow points from the first term to the second term. A red '1' is written above the second term, and a red '≤ 1' is written to its right.

If $|r_t(a)| \leq 1$ and $\eta \leq 1 \Rightarrow$

$$\sum_{t=1}^T (r_t(a^*) - r_t(a_t)) \leq \frac{\log A}{\eta} + \eta T \approx \sqrt{(\log A)T}$$

Questions and Discussions

- Why do we care about regret against a **fixed** action when the reward function is changing?
 - Environments where reward function is manipulated by an adversary
 - For MDPs, the “long-term reward” changes over time (because the learner’s policy in later steps changes over time).
 - A lot of other applications: game theory, constrained optimization, boosting, etc.

Exponential Weight Update \in Mirror Ascent

General form of **Mirror Ascent**:

$$f_t(\pi) = \langle \pi, r_t \rangle$$

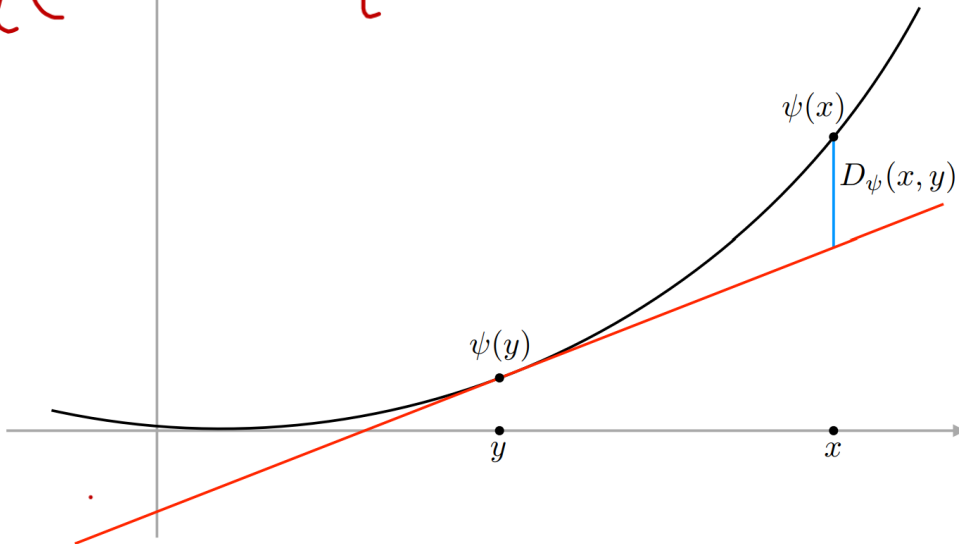
$$\nabla f_t(\pi) = r_t$$

$$x_{t+1} = \operatorname{argmax}_{x \in \Omega} \left\{ \langle x - x_t, r_t \rangle - \frac{1}{\eta} \underbrace{D_\psi(x, x_t)} \right\}$$

Usually, $r_t = \nabla f_t(x_t)$ for some function f_t that we want to maximize

Bregman divergence with respect to a convex function ψ

$$D_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle$$



$$\sum_a x(a) \log \frac{1}{x(a)}$$

$$P_{\Omega}(x) = \operatorname{argmin}_{y \in \Omega} \|x - y\|_2$$

Exponential Weight Update \in Mirror Ascent



Special cases of **Mirror Ascent**: $x_{t+1} = \operatorname{argmax}_{x \in \Omega} \left\{ \langle x - x_t, r_t \rangle - \frac{1}{\eta} D_{\psi}(x, x_t) \right\}$

$\psi(x)$	$D_{\psi}(x, y)$	Update Rule $\nabla f_t(x_t)$	Ω
$\frac{1}{2} \ x\ _2^2$	$\frac{1}{2} \ x - y\ _2^2$	$x_{t+1} = \mathcal{P}_{\Omega}(x_t + \eta r_t)$ (Projected) Gradient Ascent	Any convex set
$\sum_a x(a) \log x(a)$ Negative entropy	$\sum_a x(a) \log \frac{x(a)}{y(a)}$	$x_{t+1}(a) = \frac{x_t(a) e^{\eta r_t(a)}}{\sum_b x_t(b) e^{\eta r_t(b)}}$	Probability space
$\sum_a \log \frac{1}{x(a)}$	$\sum_a \left(\frac{x(a)}{y(a)} - \log \frac{x(a)}{y(a)} - 1 \right)$	$\frac{1}{x_{t+1}(a)} = \frac{1}{x_t(a)} - \eta r_t(a) + \gamma_t$	Probability space
$-2 \sum_a \sqrt{x(a)}$	$\sum_a \frac{(\sqrt{x(a)} - \sqrt{y(a)})^2}{2\sqrt{y(a)}}$	$\frac{1}{\sqrt{x_{t+1}(a)}} = \frac{1}{\sqrt{x_t(a)}} - \eta r_t(a) + \gamma_t$	Probability space
$\frac{1}{2} \ x\ _M^2 = \frac{1}{2} x^{\top} M x$	$\frac{1}{2} \ x - y\ _M^2$	$x_{t+1} = \mathcal{P}_{\Omega}(x_t + \eta M^{-1} r_t)$	Any convex set

Multi-Armed Bandits

Adversarial Multi-Armed Bandits

Given: set of arms $\mathcal{A} = \{1, \dots, A\}$

For time $t = 1, 2, \dots, T$:

Environment decides the reward vector $r_t = (r_t(1), \dots, r_t(A))$ (not revealing)

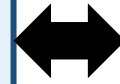
Learner chooses an arm $a_t \in \mathcal{A}$

Learner observes $r_t(a_t)$

$$\text{Regret} = \max_{a \in \mathcal{A}} \sum_{t=1}^T r_t(a) - \sum_{t=1}^T r_t(a_t)$$

Recall: Exponential Weight Updates

$$\pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \langle \pi - \pi_t, r_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi, \pi_t) \right\}$$



$$\pi_{t+1}(a) = \frac{\pi_t(a) e^{\eta r_t(a)}}{\sum_{b \in \mathcal{A}} \pi_t(b) e^{\eta r_t(b)}}$$

Exponential Weight Updates for Bandits?

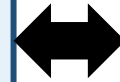
$$\pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \langle \pi - \pi_t, \mathbf{r}_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi, \pi_t) \right\} \iff \pi_{t+1}(a) = \frac{\pi_t(a) e^{\eta \mathbf{r}_t(a)}}{\sum_{b \in \mathcal{A}} \pi_t(b) e^{\eta \mathbf{r}_t(b)}}$$

No longer observable

Only update the arm that we choose?

Exponential Weight Updates for Bandits?

$$\pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \langle \pi - \pi_t, \hat{r}_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi, \pi_t) \right\}$$



$$\pi_{t+1}(a) = \frac{\pi_t(a) e^{\eta \hat{r}_t(a)}}{\sum_{b \in \mathcal{A}} \pi_t(b) e^{\eta \hat{r}_t(b)}}$$

- $\hat{r}_t(a)$ is an “**estimator**” for $r_t(a)$
- But we can only observe the reward of one arm
- Furthermore, $r_t(a)$ is different in every round (If we do not sample arm a in round t , we’ll never be able to estimate $r_t(a)$ in the future)

Unbiased Reward / Gradient Estimator

$$\hookrightarrow \mathbb{E}_{a_t} [\hat{r}_t(a)] = r_t(a) \quad \forall a$$

proof: Fix an a .

$$\mathbb{E}_{a_t} [\hat{r}_t(a)] = \overset{\text{Pr(choosing } a)}{\pi_t(a)} \cdot \frac{r_t(a)}{\pi_t(a)} + \overset{\text{Pr(not choosing } a)}{(1 - \pi_t(a))} \cdot 0 = r_t(a)$$

Weight a sample by **the inverse of the probability we observe it**

$$\hat{r}_t(a) = \frac{r_t(a)}{\pi_t(a)} \mathbb{I}\{a_t = a\} = \begin{cases} \frac{r_t(a)}{\pi_t(a)} & \text{if } a_t = a \quad \checkmark \\ 0 & \text{otherwise} \quad \checkmark \end{cases}$$

Inverse Propensity Weighting / Inverse Probability Weighting / Importance Weighting

Directly Applying Exponential Weights

$\pi_1(a) = 1/A$ for all a

For $t = 1, 2, \dots, T$:

Sample $a_t \sim \pi_t$, and observe $r_t(a_t)$

Define for all a :

$$\hat{r}_t(a) = \frac{r_t(a)}{\pi_t(a)} \mathbb{I}\{a_t = a\}$$

Update policy:

$$\pi_{t+1}(a) = \frac{\pi_t(a) \exp(\eta \hat{r}_t(a))}{\sum_{a' \in \mathcal{A}} \pi_t(a') \exp(\eta \hat{r}_t(a'))}$$

If we choose action a_t

then $\hat{r}_t(a_t) \geq 0$

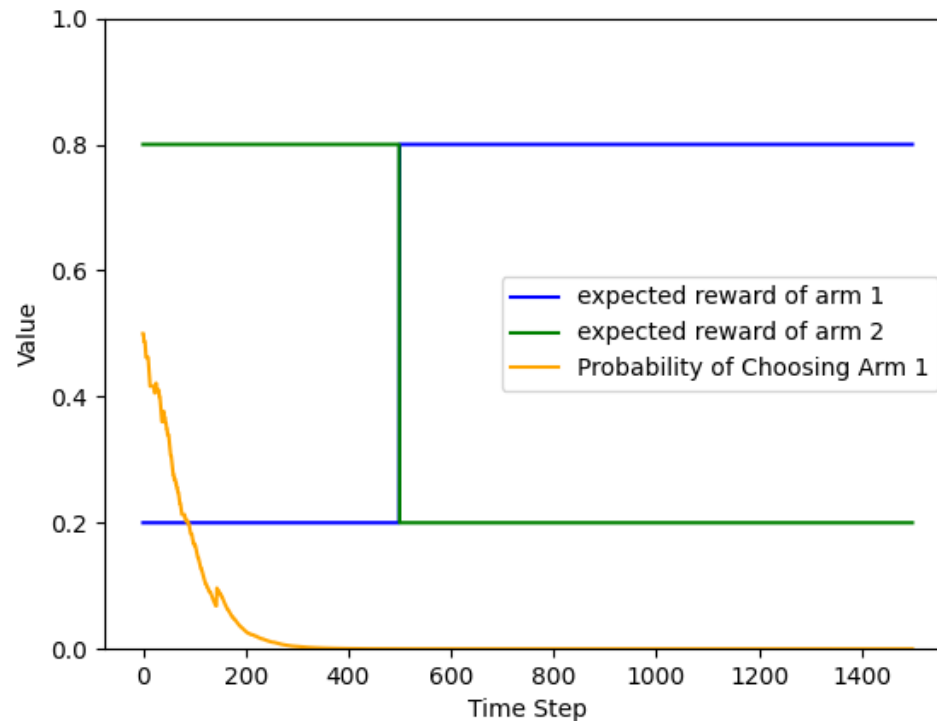
while $r_t(a) = 0 \quad \forall a \neq a_t$

\Rightarrow

$$\begin{cases} \pi_{t+1}(a_t) \geq \pi_t(a_t) \\ \pi_{t+1}(a) \leq \pi_t(a) \quad \forall a \neq a_t \end{cases}$$

Simple Experiment

- $A = 2$, $T = 1500$, $\eta = 1/\sqrt{T}$
- For $t \leq 500$, $r_t = [\text{Bernoulli}(0.2), \text{Bernoulli}(0.8)]$
- For $500 < t \leq 1500$, $r_t = [\text{Bernoulli}(0.8), \text{Bernoulli}(0.2)]$
- [code](#)



Recall the Theorem

$$\frac{\ln A}{\eta} + 2AT = \sqrt{AT \ln A} = \tilde{O}(\sqrt{AT})$$

Theorem. Does this still hold? ❌

Assume that $\eta \hat{r}_t(a) \leq 1$ for all t, a . Then EWU

$$\pi_{t+1}(a) = \frac{\pi_t(a) \exp(\eta \hat{r}_t(a))}{\sum_{a' \in \mathcal{A}} \pi_t(a') \exp(\eta \hat{r}_t(a'))}$$

ensures for any a^* ,

$$\sum_{t=1}^T (\hat{r}_t(a^*) - \langle \pi_t, \hat{r}_t \rangle) \leq \frac{\ln A}{\eta} + \eta \underbrace{\sum_{t=1}^T \sum_{a=1}^A \pi_t(a) \hat{r}_t(a)^2}_{\text{circled in red}}$$

$$\mathbb{E}[\sim] \leq AT$$

(cf: $\mathbb{E}[\sim] \leq T$)
in full-info case

Is this still related to regret? ○

Is this still well-bounded? ○

$$\hat{\sigma}_t(a) =$$



if $a_t = a$

otherwise

$\lesssim 1$

$\gamma_t(a) \in (0, 1]$

$$\gamma = \frac{1}{\sqrt{T}}$$

Solution 1: Adding Extra Exploration

- **Idea:** use at least η probability to choose each arm
- Instead of sampling a_t according to π_t , use

$$\pi'_t(a) = (1 - A\eta)\pi_t(a) + \eta$$

Then the unbiased reward estimator becomes

$$\hat{r}_t(a) = \frac{r_t(a)}{\pi'_t(a)} \mathbb{I}\{a_t = a\} = \frac{r_t(a)}{(1 - A\eta)\pi_t(a) + \eta} \mathbb{I}\{a_t = a\} \leq \frac{r_t(\leq)}{?} \mathbb{I}\{a_t = a\}$$

$$? \hat{r}_t(a) \leq \underline{r_t(a)} \mathbb{I}\{a_t = a\} \leq 1$$

Applying Solution 1

$\pi_1(a) = 1/A$ for all a

For $t = 1, 2, \dots, T$:

Sample a_t from $\pi'_t = (1 - A\eta)\pi_t + A\eta \text{ uniform}(\mathcal{A})$, and observe $r_t(a_t)$

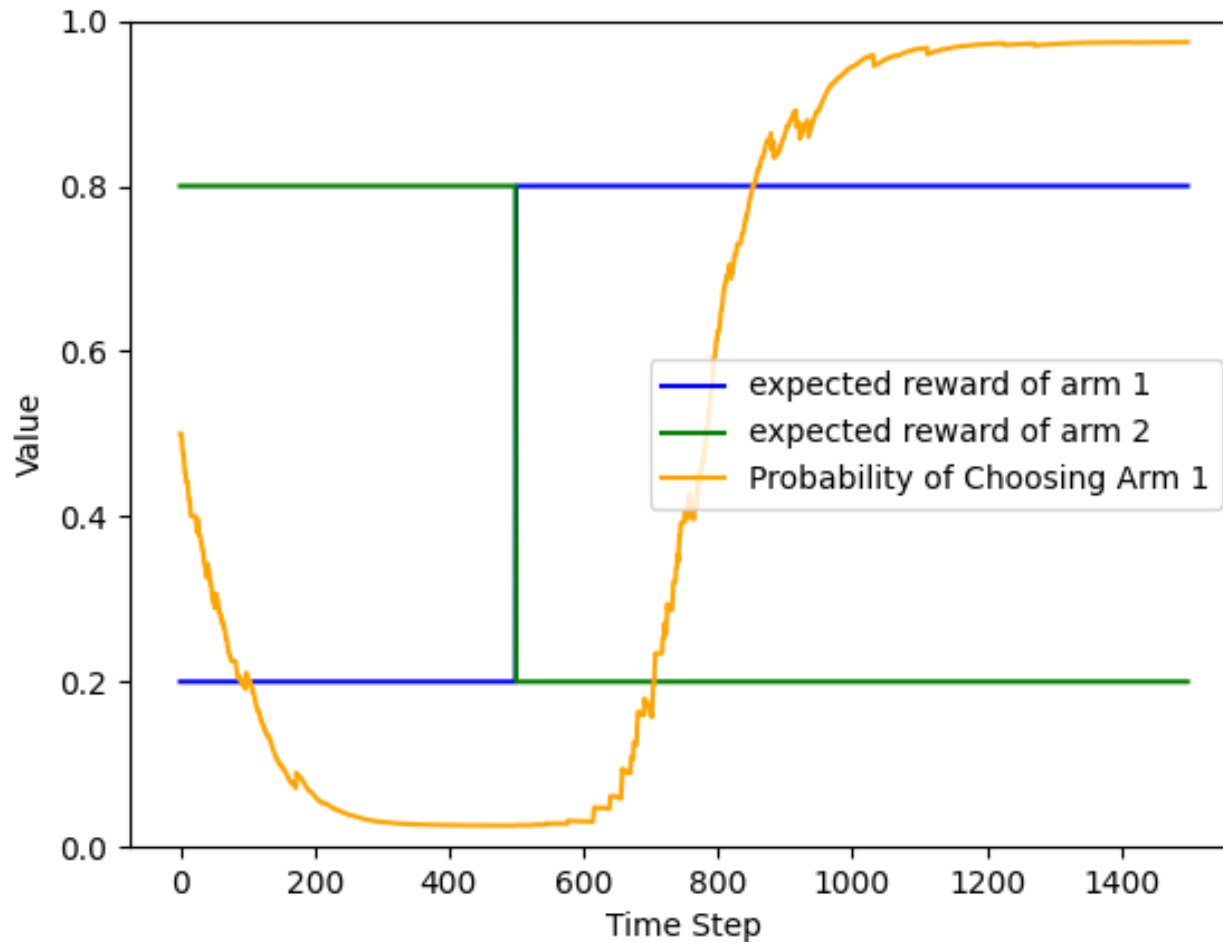
Define for all a :

$$\hat{r}_t(a) = \frac{r_t(a)}{\pi'_t(a)} \mathbb{I}\{a_t = a\}$$

Update policy:

$$\pi_{t+1}(a) = \frac{\pi_t(a) \exp(\eta \hat{r}_t(a))}{\sum_{a' \in \mathcal{A}} \pi_t(a') \exp(\eta \hat{r}_t(a'))}$$

Solution 1: Adding Extra Exploration



Regret Bound for Solution 1

Theorem. Exponential weights with Solution 1 ensures

$$\max_{a^*} \mathbb{E} \left[\sum_{t=1}^T (r_t(a^*) - r_t(a_t)) \right] \leq O \left(\frac{\ln A}{\eta} + \eta AT \right) \approx \sqrt{AT \ln A}.$$

Solution 2: Reward Estimator with a Baseline

$$\hat{r}_t(a)$$

- The condition only requires $\eta \hat{r}_t(a) \leq 1$. The reward estimator is allowed to be **very negative**!

The fact that mirror ascent **cannot handle** very positive unbiased reward estimator but **can handle** a negative one is somewhat technical in the proof.

✓ **High-level implication:** you should not surprise mirror ascent by a very good reward on an action sampled with very low probability.

- Still sample a_t from π_t , but construct the reward estimator as

$$\hat{r}_t(a) = \frac{r_t(a) - 1}{\pi_t(a)} \mathbb{I}\{a_t = a\} + 1 \leq 1$$

- Why this resolves the issue?

$$\begin{aligned} & \mathbb{E}_{a_t} \left[\frac{r_t(a) - 1}{\pi_t(a)} \mathbb{I}\{a_t = a\} + 1 \right] \\ &= (r_t(a) - 1) + 1 = r_t(a) \end{aligned}$$

Applying Solution 2

$$\begin{cases} \hat{r}_t(a_t) \leq 1 \\ \hat{r}_t(a) = 1 \quad \forall a \neq a_t \end{cases}$$

$$\pi_1(a) = 1/A \text{ for all } a$$

For $t = 1, 2, \dots, T$:

Sample a_t from π_t , and observe $r_t(a_t)$

Define for all a :

$$\hat{r}_t(a) = \frac{r_t(a) - 1}{\pi_t(a)} \mathbb{I}\{a_t = a\} + 1 \text{ or equivalently } \hat{r}_t(a) = \frac{r_t(a) - \text{baseline}}{\pi_t(a)} \mathbb{I}\{a_t = a\}$$

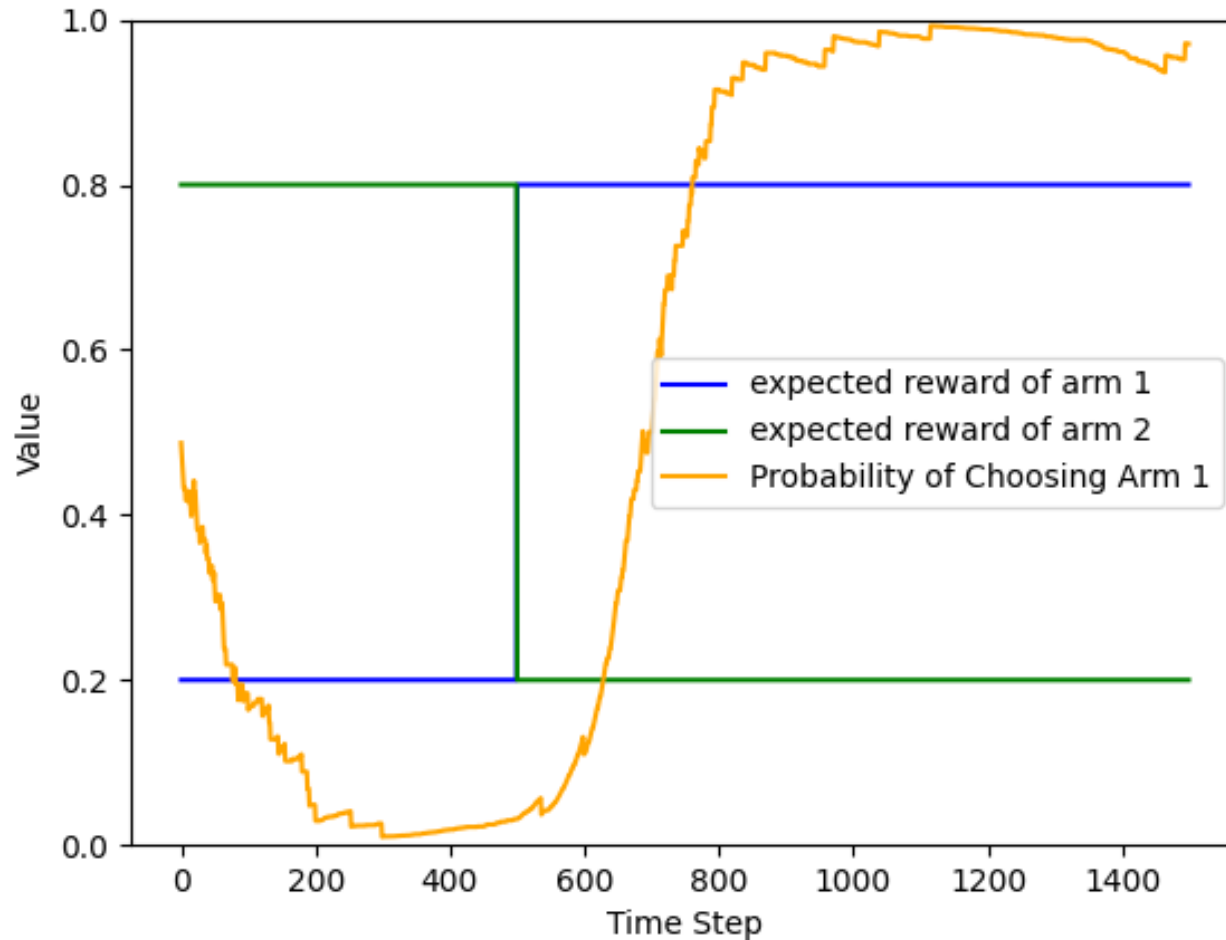
Update policy:

$$\pi_{t+1}(a) = \frac{\pi_t(a) \exp(\eta \hat{r}_t(a))}{\sum_{a' \in \mathcal{A}} \pi_t(a') \exp(\eta \hat{r}_t(a'))} = \frac{\pi_t(a) \exp(\eta (\hat{r}_t(a) + c))}{\sum_{a'} \pi_t(a') \exp(\eta (\hat{r}_t(a') + c))}$$

$$\begin{cases} \pi_{t+1}(a_t) \leq \pi_t(a_t) \\ \pi_{t+1}(a) \geq \pi_t(a) \quad \forall a \neq a_t \end{cases}$$

$$\exp(\eta \hat{r}_t(a)) \exp(\eta c)$$

Solution 2: Reward Estimator with a Baseline



Regret Bound for Solution 2

Theorem. Exponential weights with Solution 2 ensures

$$\max_{a^*} \mathbb{E} \left[\sum_{t=1}^T (r_t(a^*) - r_t(a_t)) \right] \leq O \left(\frac{\ln A}{\eta} + \eta AT \right)$$

EXP3 Algorithm

“**Ex**ponential weight algorithm for **Ex**ploration and **Ex**ploitation”

- Exponential weights + either of the two solutions

Peter Auer, Nicolò Cesa-Bianchi, Yoav Freund, Robert Schapire.
The Nonstochastic Multiarmed Bandit Problem. 2002.

EXP3-IX

$\pi_1(a) = 1/A$ for all a

For $t = 1, 2, \dots, T$:

Sample a_t from π_t and observe $r_t(a_t)$

Define for all a :

$$\hat{r}_t(a) = \frac{r_t(a) - 1}{\pi_t(a) + \eta} \mathbb{I}\{a_t = a\}$$

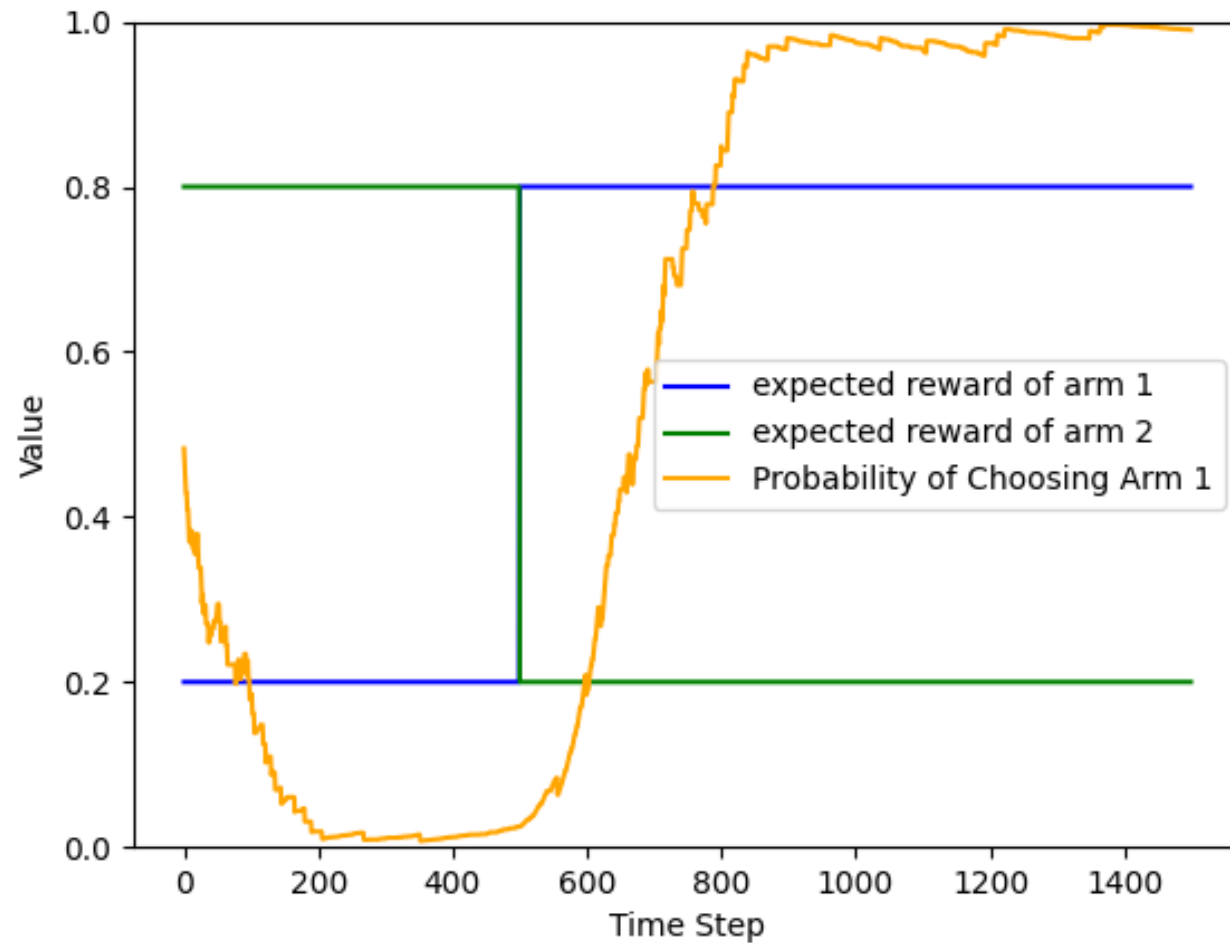
$$\underline{\mathbb{E}[\hat{r}_t(a)] \neq r_t(a)}$$

Update policy:

$$\pi_{t+1}(a) = \frac{\pi_t(a) \exp(\eta \hat{r}_t(a))}{\sum_{a' \in \mathcal{A}} \pi_t(a') \exp(\eta \hat{r}_t(a'))}$$

EXP3-IX

$$\hat{r}_t(a) = \frac{r_t(a) - 1}{\pi_t(a) + \eta} \mathbb{I}\{a_t = a\}$$



Regret Bound for EXP3-IX

Theorem. EXP3-IX ensures **with high probability**,

$$\max_{a^*} \sum_{t=1}^T (r_t(a^*) - r_t(a_t)) \leq \tilde{O} \left(\frac{\ln A}{\eta} + \eta AT \right)$$

Gergely Neu. Explore no more: Improved high-probability regret bounds for non-stochastic bandits. 2015.

The Role of Baseline

$$\hat{r}_t(a) = \frac{r_t(a) - b_t}{\pi_t(a)} \mathbb{I}\{a_t = a\}$$
$$\pi_{t+1}(a) = \frac{\pi_t(a) \exp(\eta \hat{r}_t(a))}{\sum_{a' \in \mathcal{A}} \pi_t(a') \exp(\eta \hat{r}_t(a'))} \quad \text{or} \quad \pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \langle \pi, \hat{r}_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi, \pi_t) \right\}$$

Larger b_t : More exploratory (tends to decrease the probability of the action just chosen)
– needed to detect changes in the environment.

In fixed reward function setting (non-adversarial), we usually set b_t to be close to the recent performance level of the learner itself

- When finding an action better than the learner itself, increase its probability
- Otherwise, decrease its probability

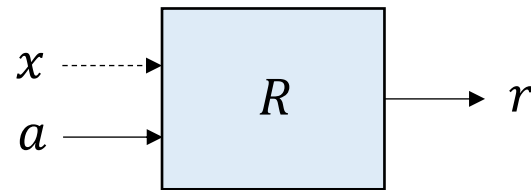
Summary

- Exponential weight update (EWU) is an effective algorithm for full-information setting. It guarantees sublinear regret even when the environment changes over time.
- Extending EWU to bandit with naïve unbiased reward estimator does not work (lack of exploration). Two ways to fix it:
 - Adding **extra uniform exploration** with probability $\geq A\eta$
 - Adding a **baseline** in the reward estimator to encourage exploration
- High-probability bounds can be achieved by adding **baseline** and **bias** (EXP3-IX).

Review: Exploration Strategies for Bandits

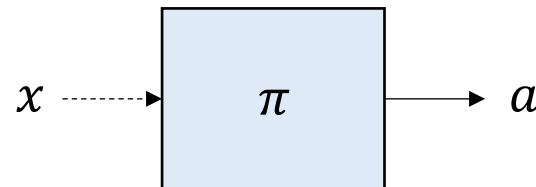
x : context, a : action, r : reward

Value-based



(context, action) to reward

Policy-based



context to action distribution

MAB

CB

Mean estimation
+
EG, BE, IGW

Regression
+
EG, BE, IGW

Uncertainty as bonus

KL-regularized update
with reward estimators
(EXP3)

+

baseline, bias,
uniform exploration

Next

Contextual Bandits

Contextual Bandits

For time $t = 1, 2, \dots, T$:

Environment generates a context $x_t \in \mathcal{X}$

Learner chooses an action $a_t \in \mathcal{A}$

Learner observes $r_t(x_t, a_t)$

KL-Regularized Policy Updates

$$\pi_{t+1} = \operatorname{argmax}_{\pi \in \Delta(\mathcal{A})} \left\{ \sum_a \pi(a) \hat{r}_t(a) - \frac{1}{\eta} \sum_a \pi(a) \log \frac{\pi(a)}{\pi_t(a)} \right\}$$

$$\hat{r}_t(a) = \frac{r_t(a) - b_t}{\pi_t(a)} \mathbb{I}\{a_t = a\}$$

$$\theta_{t+1} = \operatorname{argmax}_{\theta} \left\{ \sum_a \pi_{\theta}(a|x_t) \hat{r}_t(x_t, a) - \frac{1}{\eta} \sum_a \pi_{\theta}(a|x_t) \log \frac{\pi_{\theta}(a|x_t)}{\pi_{\theta_t}(a|x_t)} \right\}$$

$$\hat{r}_t(x_t, a) = \frac{r_t(x_t, a) - b_t(x_t)}{\pi_{\theta_t}(a|x_t)} \mathbb{I}\{a_t = a\}$$

KL-Regularized Policy Updates

For $t = 1, 2, \dots, T$:

Receive context x_t

Take action $a_t \sim \pi_{\theta_t}(\cdot|x_t)$ and receive reward $r_t(x_t, a_t)$

Create reward estimator $\hat{r}_t(x_t, a) = \frac{r_t(x_t, a) - b_t(x_t)}{\pi_{\theta_t}(a|x_t)} \mathbb{I}\{a_t = a\}$

Update

$$\theta_{t+1} = \operatorname{argmax}_{\theta} \left\{ \sum_a \pi_{\theta}(a|x_t) \hat{r}_t(x_t, a) - \frac{1}{\eta} \sum_a \pi_{\theta}(a|x_t) \log \frac{\pi_{\theta}(a|x_t)}{\pi_{\theta_t}(a|x_t)} \right\}$$

Proximal Policy Optimization (PPO) for CB

For $t = 1, 2, \dots, T$:

For $i = 1, \dots, N$:

Receive context x_i

Take action $a_i \sim \pi_{\theta_t}(\cdot|x_i)$ and receive reward $r_i(x_i, a_i)$

Create reward estimator $\hat{r}_i(x_i, a) = \frac{r_i(x_i, a) - b_t(x_i)}{\pi_{\theta_t}(a|x_i)} \mathbb{I}\{a_i = a\}$

For $j = 1, \dots, M$:

one iteration of mirror ascent

For minibatch $\mathcal{B} \subset \{1, 2, \dots, N\}$ of size B :

$$\begin{aligned}\theta &\leftarrow \theta + \nabla_{\theta} \frac{1}{B} \sum_{i \in \mathcal{B}} \left(\sum_a \pi_{\theta}(a|x_i) \hat{r}_i(x_i, a) - \frac{1}{\eta} \sum_a \pi_{\theta}(a|x_i) \log \frac{\pi_{\theta}(a|x_i)}{\pi_{\theta_t}(a|x_i)} \right) \\ &= \theta + \nabla_{\theta} \frac{1}{B} \sum_{i \in \mathcal{B}} \left(\frac{\pi_{\theta}(a_i|x_i)}{\pi_{\theta_t}(a_i|x_i)} (r_i(x_i, a_i) - b_t(x_i)) - \frac{1}{\eta} \sum_a \pi_{\theta}(a|x_i) \log \frac{\pi_{\theta}(a|x_i)}{\pi_{\theta_t}(a|x_i)} \right)\end{aligned}$$

$\theta_{t+1} \leftarrow \theta$

Proximal Policy Optimization (PPO) for CB

$$\theta \leftarrow \theta + \nabla_{\theta} \frac{1}{B} \sum_{i \in \mathcal{B}} \left(\frac{\pi_{\theta}(a_i | x_i)}{\pi_{\theta_t}(a_i | x_i)} (r_i(x_i, a_i) - b_t(x_i)) - \underbrace{\frac{1}{\eta} \sum_a \pi_{\theta}(a | x_i) \log \frac{\pi_{\theta}(a | x_i)}{\pi_{\theta_t}(a | x_i)}}_{\text{KL}(\pi_{\theta}(\cdot | x_i), \pi_{\theta_t}(\cdot | x_i))} \right)$$

- May replace $\text{KL}(\pi_{\theta}(\cdot | x_i), \pi_{\theta_t}(\cdot | x_i))$ by $\text{KL}(\pi_{\theta_t}(\cdot | x_i), \pi_{\theta}(\cdot | x_i))$. The latter is easier to construct unbiased estimator.
- Although this term can be calculated exactly, we often use samples to estimate it (so we do not need to sum over a)

Estimating KL by Samples

<http://joschu.net/blog/kl-approx.html>

Sample $a_i \sim \pi_{\theta_t}(\cdot | x_i)$ and define $kl_i(\theta_t, \theta) = \frac{\pi_{\theta}(a_i | x_i)}{\pi_{\theta_t}(a_i | x_i)} - 1 - \log \frac{\pi_{\theta}(a_i | x_i)}{\pi_{\theta_t}(a_i | x_i)}$

Then $\mathbb{E}_{a_i \sim \pi_{\theta_t}(\cdot | x_i)}[kl_i(\theta_t, \theta)] = \text{KL}(\pi_{\theta_t}(\cdot | x_i), \pi_{\theta}(\cdot | x_i))$ Just need one sample of a_i

As we see before, the ways to construct an unbiased estimator are not unique.
This is a good one with low variance.

PPO with KL Estimator

For $t = 1, 2, \dots, T$:

For $i = 1, \dots, N$:

Receive context x_i

Take action $a_i \sim \pi_{\theta_t}(\cdot|x_i)$ and receive reward $r_i(x_i, a_i)$

Create reward estimator $\hat{r}_i(x_i, a) = \frac{r_i(x_i, a) - b_t(x_i)}{\pi_{\theta_t}(a|x_i)} \mathbb{I}\{a_i = a\}$

For $j = 1, \dots, M$:

For minibatch $\mathcal{B} \subset \{1, 2, \dots, N\}$ of size B :

$$\theta \leftarrow \theta + \nabla_{\theta} \frac{1}{B} \sum_{i \in \mathcal{B}} \left(\frac{\pi_{\theta}(a_i|x_i)}{\pi_{\theta_t}(a_i|x_i)} (r_i(x_i, a_i) - b_t(x_i)) - \frac{1}{\eta} \textcolor{red}{kl}_i(\theta_t, \theta) \right)$$

$\theta_{t+1} \leftarrow \theta$

$$kl_i(\theta_t, \theta) = \frac{\pi_{\theta}(a_i|x_i)}{\pi_{\theta_t}(a_i|x_i)} - 1 - \log \frac{\pi_{\theta}(a_i|x_i)}{\pi_{\theta_t}(a_i|x_i)}$$

Summary: PPO

- PPO-CB can be viewed as an extension of EXP3 to contextual bandits. The central idea is KL-regularized policy updates
- Common techniques: baselines, avoiding **overly positive** reward estimator. These techniques prevent over exploitation
- PPO additional uses batching, reversed KL divergence, and KL estimators for computational efficiency

NPG and PG

Natural Policy Gradient

$$\textbf{(PPO)} \quad \theta_{t+1} = \operatorname{argmax}_{\theta} \mathbb{E}_x \left[\sum_a (\pi_{\theta}(a|x) - \pi_{\theta_t}(a|x)) \hat{r}_t(x, a) - \frac{1}{\eta} \sum_a \pi_{\theta}(a|x) \log \frac{\pi_{\theta}(a|x)}{\pi_{\theta_t}(a|x)} \right]$$

η close to zero

$$\textbf{(NPG)} \quad \theta_{t+1} = \theta_t + \eta F_t^{-1} \mathbb{E}_x \left[\sum_a \nabla_{\theta} \pi_{\theta}(a|x) \hat{r}_t(x, a) \right] \Big|_{\theta=\theta_t}$$

where $F_{\theta_t} = \mathbb{E}_x \mathbb{E}_{a \sim \pi_{\theta_t}(\cdot|x)} \left[(\nabla_{\theta} \log \pi_{\theta}(a|x)) (\nabla_{\theta} \log \pi_{\theta}(a|x))^{\top} \right] \Big|_{\theta=\theta_t}$ **Fisher information matrix**

Natural Policy Gradient (w/o context + full-info)

(PPO)
$$\theta_{t+1} = \operatorname{argmax}_{\theta} \sum_a \left(\pi_{\theta}(a) - \pi_{\theta_t}(a) \right) r_t(a) - \frac{1}{\eta} \sum_a \pi_{\theta}(a) \log \frac{\pi_{\theta}(a)}{\pi_{\theta_t}(a)}$$

η close to zero

(NPG)
$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \sum_a \nabla_{\theta} \pi_{\theta}(a) r_t(a) \Big|_{\theta=\theta_t}$$

where $F_{\theta_t} = \mathbb{E}_{a \sim \pi_{\theta_t}} [(\nabla_{\theta} \log \pi_{\theta}(a)) (\nabla_{\theta} \log \pi_{\theta}(a))^{\top}] \Big|_{\theta=\theta_t}$ **Fisher information matrix**

Proof Sketch

$$f(\theta) \approx f(\theta_t) + (\theta - \theta_t)^\top [\nabla_\theta f(\theta)]_{\theta=\theta_t} + \frac{1}{2} (\theta - \theta_t)^\top [\nabla_\theta^2 f(\theta)]_{\theta=\theta_t} (\theta - \theta_t)$$

PPO

$$\theta_{t+1} = \operatorname{argmax}_\theta \left\{ \langle \pi_\theta - \pi_{\theta_t}, r_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi_\theta, \pi_{\theta_t}) \right\}$$

$$\begin{aligned} \langle \pi_\theta - \pi_{\theta_t}, r_t \rangle &= \sum_a (\pi_\theta(a) - \pi_{\theta_t}(a)) r_t(a) \\ &\approx (\theta - \theta_t)^\top \sum_a [\nabla_\theta \pi_\theta(a)]_{\theta=\theta_t} r_t(a) \end{aligned}$$

$$F_{\theta_t} = [\nabla_\theta^2 \operatorname{KL}(\pi_\theta, \pi_{\theta_t})]_{\theta=\theta_t} \quad \textbf{(exercise)}$$

$$\operatorname{KL}(\pi_\theta, \pi_{\theta_t}) \approx \frac{1}{2} (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t)$$

$$\begin{aligned} \theta_{t+1} &\approx \operatorname{argmax}_\theta \left\{ (\theta - \theta_t)^\top g_t - \frac{1}{2\eta} (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \right\} \\ &= \theta_t + \eta F_{\theta_t}^{-1} g_t \quad \textbf{NPG} \end{aligned}$$

NPG vs. PG

NPG

$$\theta_{t+1} = \theta_t + \eta F_t^{-1} \sum_a \nabla_{\theta} \pi_{\theta}(a) r_t(a) \Big|_{\theta=\theta_t}$$

(Vanilla) PG

$$\theta_{t+1} = \theta_t + \eta \sum_a \nabla_{\theta} \pi_{\theta}(a) r_t(a) \Big|_{\theta=\theta_t}$$

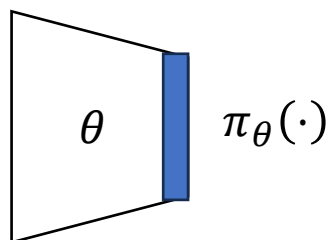
NPG vs. PG

NPG

$$\theta_{t+1} = \theta_t + \eta F_t^{-1} \sum_a \nabla_{\theta} \pi_{\theta}(a) r_t(a) \Big|_{\theta=\theta_t}$$



$$\theta_{t+1} = \operatorname{argmax}_{\theta} \langle \pi_{\theta} - \pi_{\theta_t}, r_t \rangle - \frac{1}{\eta} \text{KL}(\pi_{\theta}, \pi_{\theta_t})$$

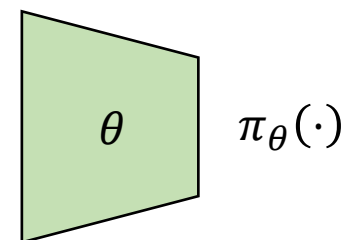


PG

$$\theta_{t+1} = \theta_t + \eta \sum_a \nabla_{\theta} \pi_{\theta}(a) r_t(a) \Big|_{\theta=\theta_t}$$



$$\theta_{t+1} = \operatorname{argmax}_{\theta} \langle \pi_{\theta} - \pi_{\theta_t}, r_t \rangle - \frac{1}{2\eta} \|\theta - \theta_t\|^2$$



Example: NPG vs. PG with softmax policy

Consider multi-armed bandits with **softmax policy** $\pi_{\theta}(a) = \frac{e^{\theta(a)}}{\sum_{a'} e^{\theta(a')}}$
parameterized by $\theta(1), \theta(2), \dots, \theta(A)$

NPG (= Exponential Weight, without requiring $\eta \approx 0$ assumption)

For $t = 1, 2, \dots$

$$\theta_{t+1}(a) \leftarrow \theta_t(a) + \eta r_t(a)$$

Check the equivalence (exercise)

NPG can also be written as

$$\theta_{t+1}(a) \leftarrow \theta_t(a) + \eta \tilde{r}_t(a)$$

$$\tilde{r}_t(a) = r_t(a) - \sum_{a'} \pi_{\theta_t}(a') r_t(a')$$

PG

For $k = 1, 2, \dots$

$$\theta_{t+1}(a) \leftarrow \theta_t(a) + \eta \pi_{\theta_t}(a) \tilde{r}_t(a)$$

NPG (EW) vs. PG

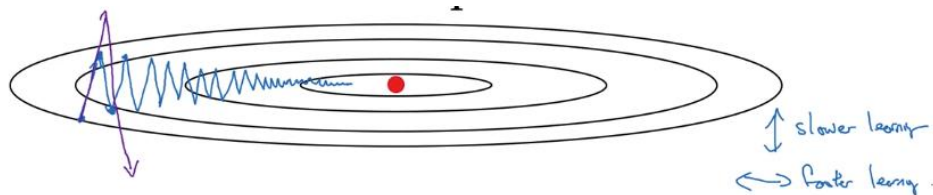
EW: $\theta_{t+1}(a) \leftarrow \theta_t(a) + \eta \tilde{r}_t(a)$

PG: $\theta_{t+1}(a) \leftarrow \theta_t(a) + \eta \pi_{\theta_t}(a) \tilde{r}_t(a)$

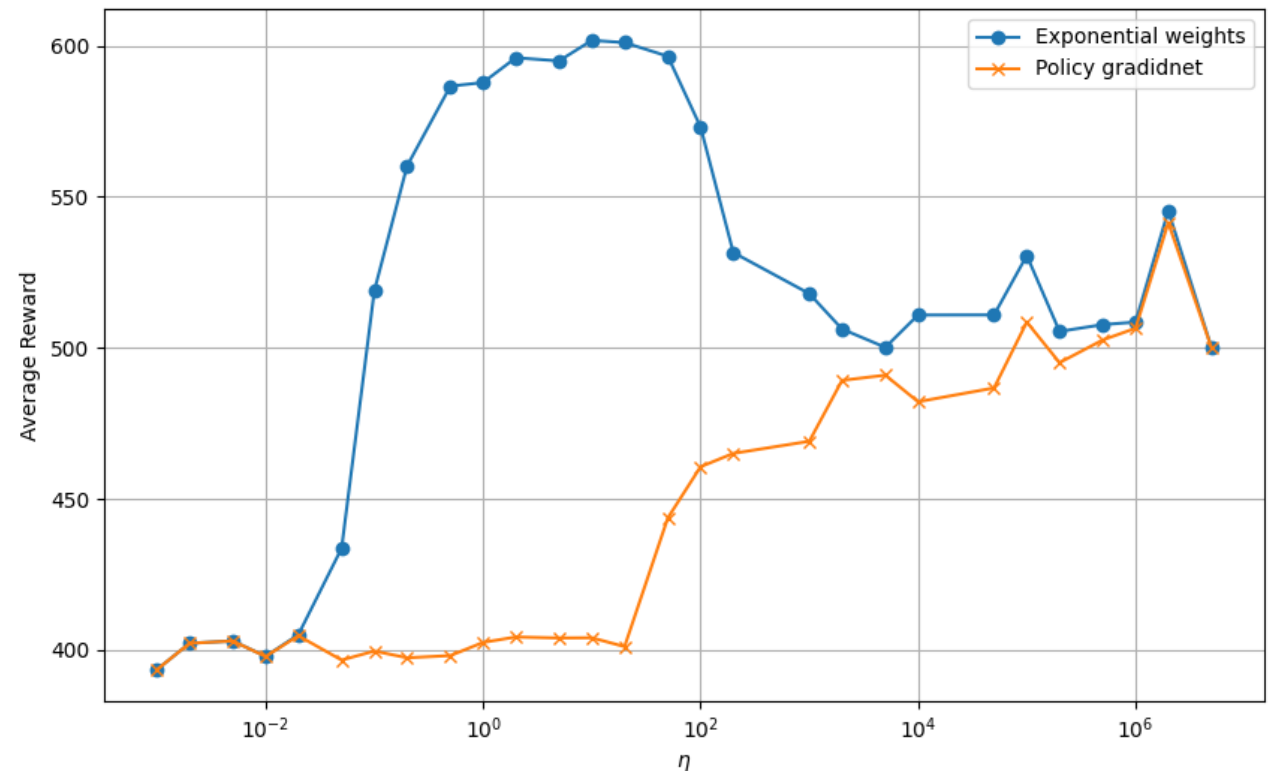
Reward = [Ber(0.6), Ber(0.4)]

Initial policy $\pi = [0.0001, 0.9999]$

Plot total reward in 1000 rounds



<https://math.stackexchange.com/questions/2285282/relating-condition-number-of-hessian-to-the-rate-of-convergence>



NPG and PG with bandit feedback

$$\theta_{t+1} = \theta_t + \eta F_t^{-1} \sum_a \nabla_{\theta} \pi_{\theta}(a) \hat{r}_t(a) \Big|_{\theta=\theta_t}$$

$$\theta_{t+1} = \theta_t + \eta \sum_a \nabla_{\theta} \pi_{\theta}(a) \hat{r}_t(a) \Big|_{\theta=\theta_t}$$

PG (REINFORCE) for contextual bandits

For $t = 1, 2, \dots, T$:

Receive context x_t

Take action $a_t \sim \pi_{\theta_t}(\cdot|x_t)$ and receive reward $r_t(x_t, a_t)$

Update

$$\theta_{t+1} \leftarrow \theta_t + \eta [\nabla_{\theta} \log \pi_{\theta}(a_t|x_t)]_{\theta=\theta_t} (r_t(x_t, a_t) - b_t(x_t))$$

Or simply written as

$$\theta \leftarrow \theta + \eta \underbrace{\nabla_{\theta} \log \pi_{\theta}(a_t|x_t)}_{\text{Coming from inverse propensity weighting / importance weighting}} (r_t(x_t, a_t) - b_t(x_t))$$

Coming from inverse propensity weighting / importance weighting

Verify (again) that reward offset does not affect the algorithm

Natural Policy Gradient

For $t = 1, 2, \dots, T$:

Receive context x_t

Take action $a_t \sim \pi_{\theta_t}(\cdot|x_t)$ and receive reward $r_t(x_t, a_t)$

Update

$$\theta_{t+1} \leftarrow \theta_t + \eta F_{\theta_t}^{-1} [\nabla_{\theta} \log \pi_{\theta}(a_t|x_t)]_{\theta=\theta_t} (r_t(x_t, a_t) - b_t(x_t))$$

A naïve calculation of $F_{\theta_t}^{-1}$ will take $O(d^3)$ time

Sample-Based NPG*

A naïve calculation of $F_{\theta_t}^{-1}$ will take $O(d^3)$ time

But we can actually view $h_t := F_{\theta_t}^{-1} g_t$ as a solution of a linear regression problem

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \mathbb{E}_{a \sim \pi_{\theta_t}} [(\nabla_{\theta} \log \pi_{\theta_t}(a)) r_t(a)]$$



$$\text{where } F_{\theta_t} = \mathbb{E}_{a \sim \pi_{\theta_t}} [(\nabla_{\theta} \log \pi_{\theta_t}(a)) (\nabla_{\theta} \log \pi_{\theta_t}(a))^{\top}]$$

$$h_t = \left(\mathbb{E}_{a \sim \pi_{\theta_t}} [\phi_t(a) \phi_t(a)^{\top}] \right)^{-1} \mathbb{E}_{a \sim \pi_{\theta_t}} [\phi_t(a) r_t(a)]$$

$$= \underset{h}{\operatorname{argmin}} \mathbb{E}_{a \sim \pi_{\theta_t}} [(\phi_t(a)^{\top} h - r_t(a))^2]$$

$$\phi_t(a) = \nabla_{\theta} \log \pi_{\theta_t}(a)$$

Summary: Policy Learning in Bandits

PG	PPO / NPG
$\theta_{t+1} = \operatorname{argmax}_{\theta} \langle \pi_{\theta} - \pi_{\theta_t}, \hat{r}_t \rangle - \frac{1}{2\eta} \ \theta - \theta_t\ ^2$	$\theta_{t+1} = \operatorname{argmax}_{\theta} \langle \pi_{\theta} - \pi_{\theta_t}, \hat{r}_t \rangle - \frac{1}{\eta} \operatorname{KL}(\pi_{\theta}, \pi_{\theta_t})$
$\theta \leftarrow \theta + \eta \nabla_{\theta} \langle \pi_{\theta}, \hat{r}_t \rangle$	$\theta \leftarrow \theta + \eta F_{\theta}^{-1} \nabla_{\theta} \langle \pi_{\theta}, \hat{r}_t \rangle$
<div style="text-align: center;">  $\hat{r}_t(a) = \frac{r_t(a) - b_t}{\pi_{\theta_t}(a)} \mathbb{I}\{a = a_t\}$ </div> $\theta \leftarrow \theta + \eta \nabla_{\theta} \log \pi_{\theta}(a_t) (r_t(a_t) - b_t)$	<div style="text-align: center;">  </div> $\theta \leftarrow \theta + \eta F_{\theta}^{-1} \nabla_{\theta} \log \pi_{\theta}(a_t) (r_t(a_t) - b_t)$

$$F_{\theta} = \mathbb{E}_{a \sim \pi_{\theta}} [(\nabla_{\theta} \log \pi_{\theta}(a)) (\nabla_{\theta} \log \pi_{\theta}(a))^{\top}]$$