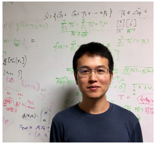
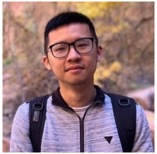
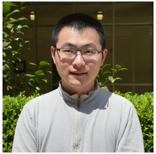
Linear Last-iterate Convergence in Constrained Saddle-point Optimization

Chung-Wei Lee

joint with **Haipeng Luo**, **Chen-Yu Wei** and **Mengxiao Zhang**







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A One-sentence (Informal) Summary

We prove that the last-iterate of

Optimistic Gradient Descent Ascent (OGDA) and

Optimistic Multiplicative Weights Update (OMWU)

converges to the Nash equilibrium **exponentially** fast,

in various constrained settings including matrix games and

strongly-convex-strongly-concave functions.

Saddle-point Optimization

- Consider constrained saddle-point optimization in the form

$$\min_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{y}\in\mathcal{Y}} f(\boldsymbol{x}, \boldsymbol{y}),$$

where $\mathcal{X}$ and $\mathcal{Y}$ are closed convex sets, and f is a continuous differentiable function that is convex in $\boldsymbol{x}$ and concave in $\boldsymbol{y}$.

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- Goal: find a *Nash equilibrium* $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in \mathcal{X}^* \times \mathcal{Y}^*$ satisfying

$$f(\boldsymbol{x}^*, \boldsymbol{y}) \leq f(\boldsymbol{x}^*, \boldsymbol{y}^*) \leq f(\boldsymbol{x}, \boldsymbol{y}^*)$$

for any $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$.

"No-Regret" Algorithms

- (Projected) Gradient Descent Ascent (GDA):

$$\boldsymbol{x}_{t+1} = \Pi_{\mathcal{X}}\left(\boldsymbol{x}_t - \eta \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_t, \boldsymbol{y}_t)\right), \quad \boldsymbol{y}_{t+1} = \Pi_{\mathcal{Y}}\left(\boldsymbol{y}_t + \eta \nabla_{\boldsymbol{y}} f(\boldsymbol{x}_t, \boldsymbol{y}_t)\right)$$

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- Multiplicative Weights Update (MWU):

$$\boldsymbol{x}_{t+1} \propto \left(\boldsymbol{x}_t \odot \exp(-\eta \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_t, \boldsymbol{y}_t))\right), \quad \boldsymbol{y}_{t+1} \propto \left(\boldsymbol{y}_t \odot \exp(\eta \nabla_{\boldsymbol{y}} f(\boldsymbol{x}_t, \boldsymbol{y}_t))\right),$$

when $\mathcal{X}$ and $\mathcal{Y}$ are simplex, and $\odot$ denotes the element-wise product.

Optimistic No-Regret Algorithms

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- However, averaging large neural networks is usually prohibited.
- This motivates us to consider the last-iterate $(\boldsymbol{x}_T, \boldsymbol{y}_T)$ convergence.

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Question: Whether OGDA and OMWU can achieve last-iterate convergence in **constrained** saddle-point optimization with **concrete** convergence rate?

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- For OGDA, we get more general results: under a sufficient condition called SP-MS, OGDA with constant learning rate converges exponentially fast.
- The SP-MS condition includes many settings such as bilinear games over any polytope and strongly-convex-strongly-concave functions without uniqueness assumption.

Experiments

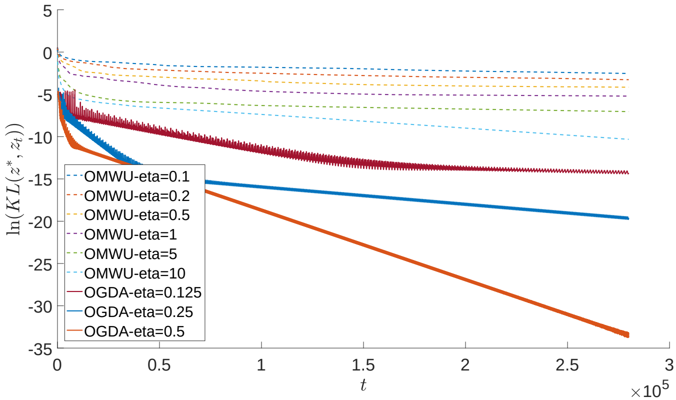

Figure: Experiments of OGDA and OMWU with different learning rates for a matrix game.

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- It is also interesting to generalize the results to Markov/Stochastic Games.
- For this direction, see our new paper *Last-iterate Convergence of Decentralized Optimistic Gradient Descent/Ascent in Infinite-horizon Competitive Markov Games* on arXiv.