# Approximate Value Iteration and Variants

Chen-Yu Wei

# Value Iteration

$$V^{(k)}(s) \leftarrow \max_a \Big\{ \underbrace{R(s,a) + \gamma \sum_{s'} P(s'|s,a) \underbrace{V^{(k-1)}(s')}_{\max_{a'} Q^{(k-1)}(s',a')}}_{Q^{(k)}(s,a)} \Big\}$$

For $k = 1,\ 2, \ldots$

$$\forall s, a, \qquad Q^{(k)}(s,a) \leftarrow \underbrace{\boxed{R(s,a)}}_{\text{unknown}} + \gamma \sum_{s'} \underbrace{\boxed{P(s'|s,a)}}_{\text{unknown}} \max_{a'} Q^{(k-1)}(s',a')$$

**Idea:** In each iteration, use multiple samples to estimate the right-hand side.

# Least-Square Value Iteration (LSVI)

For $k = 1,\ 2, \ldots$

We want these samples to be "exploratory"

Obtain $n$ samples $\mathcal{D}^{(k)} = \{(s_i, a_i, r_i, s_i')\}_{i=1}^{n}$ where $\mathbb{E}[r_i] = R(s_i, a_i),\ s_i' \sim P(\cdot\,|s_i, a_i)$

Perform **regression** on $\mathcal{D}^{(k)}$ to find $Q^{(k)}$ such that

$$Q^{(k)}(s,a) \approx R(s,a) + \gamma \mathbb{E}_{s'\sim P(\cdot|s,a)}\left[\max_{a'} Q^{(k-1)}(s',a')\right]$$

**Tabular** $\quad \forall s, a, \quad Q^{(k)}(s,a) = \dfrac{\sum_{i=1}^{n} \mathbb{I}\{(s_i,a_i) = (s,a)\}\left(r_i + \gamma \max_{a'} Q^{(k-1)}(s_i', a')\right)}{\sum_{i=1}^{n} \mathbb{I}\{(s_i,a_i) = (s,a)\}}$

$\overset{\mathbb{E}}{=} R(s,a) + \gamma \mathbb{E}_{s'\sim P(\cdot|s,a)}\left[\max_{a'} Q^{(k-1)}(s',a')\right]$

**General function approximation** $\quad \theta_k = \operatorname*{argmin}_{\theta} \sum_{i=1}^{n} \left(Q_\theta(s_i,a_i) - r_i - \gamma \max_{a'} Q_{\theta_{k-1}}(s_i',a')\right)^2$

**Linear function approximation** $\quad \theta_k = \left(\lambda I + \sum_{i=1}^{n\,(n^k)} \phi(s_i,a_i)\phi(s_i,a_i)^\top\right)^{-1} \left(\sum_{i=1}^{n\,(n_k)} \phi(s_i,a_i)\left(r_i + \gamma \max_{a'} \phi(s_i',a')^\top \theta_{k-1}\right)\right)$

# Comparison with Contextual Bandits

**Env**

$\xrightarrow{x_t}$

$\xleftarrow{a_t}$

$\xrightarrow{r_t}$

**Exploration**

$$p_t(a) \propto e^{\lambda \hat{R}(x_t, a)}$$

$$a_t = \underset{a}{\operatorname{argmax}} \left(\hat{R}(x_t, a) + b_t(a)\right)$$

$$\ldots$$

**Regression**

Fit $\hat{R}(x_i, a_i) \approx r_i$

**Env**

$\xrightarrow{s_t}$

$\xleftarrow{a_t}$

$\xrightarrow{r_t}$

**Exploration**

$$p_t(a) \propto e^{\lambda Q^{(k)}(s_t, a)}$$

$$a_t = \underset{a}{\operatorname{argmax}} \left(Q^{(k)}(s_t, a) + b_t(a)\right)$$

$$\ldots$$

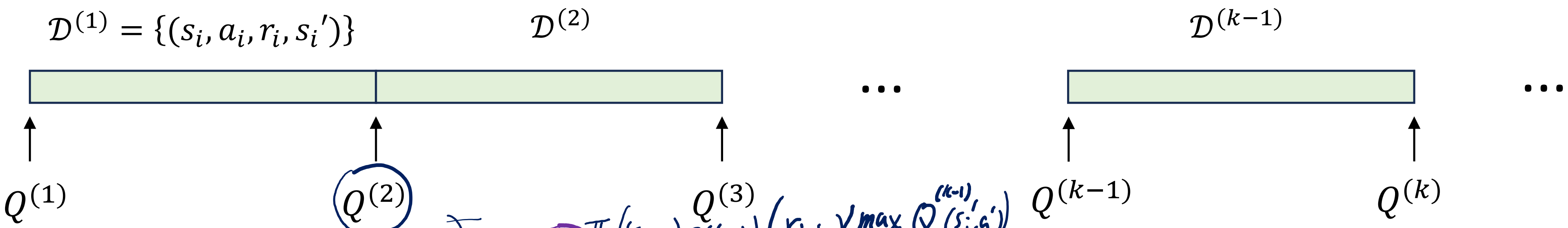
**Value Iteration + Regression**

For $k = 1, 2, \ldots$

Fit $Q^{(k)}(s_i, a_i) \approx r_i + \gamma \max_{a'} Q^{(k-1)}(s_i', a')$

# It is Valid to Reuse Samples

(e.g., using $\epsilon$-greedy)




$$Q^{(k)}(s,a) = \frac{\sum_{(s_i,a_i,r_i,s_i')\in \mathcal{D}^{(k-1)}} \mathbb{I}\left((s_i,a_i)=(s,a)\right)\left(r_i+\gamma \max_{a'} Q^{(k-1)}(s_i',a')\right)}{\sum_{(s_i,a_i,r_i,s_i')\in \mathcal{D}^{(k-1)}} \mathbb{I}\left((s_i,a_i)=(s,a)\right)}$$

$\mathcal{D}^{(1)} \cup \mathcal{D}^{(2)} \cup \dots \cup \mathcal{D}^{(k-1)}$

# LSVI that Reuses All Previous Samples

For $k = 1,\ 2, \ldots$

Obtain $n$ samples $\mathcal{D}^{(k)} = \{(s_i, a_i, r_i, s_i')\}_{i=1}^{n}$ where $\mathbb{E}[r_i] = R(s_i, a_i),\ s_i' \sim P(\cdot\,|s_i, a_i)$

Perform **regression** on $\mathcal{D}^{(1)} \cup \mathcal{D}^{(2)} \cup \cdots \cup \mathcal{D}^{(k)}$ to find $Q^{(k)}$ such that

$$Q^{(k)}(s,a) \approx R(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[\max_{a'} Q^{(k-1)}(s', a')\right]$$

In practice, we reuse “recent” data but not all previous data (discussed later).

# Analysis of LSVI under Certain Assumptions

To theoretically show that LSVI converges to the optimal value function, we will make some assumptions to ensure the following holds for all iteration $k$:

$$Q^{(k)}(s,a) \approx R(s,a) + \gamma\, \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \max_{a'} Q^{(k-1)}(s',a') \right]$$

Linear case:

$$\phi(s,a)^\top \theta_k \approx R(s,a) + \gamma\, \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \max_{a'} \phi(s',a')^\top \theta_{k-1} \right]$$

$d = S \cdot A$

$\phi(s,a) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ (s,a)-th entry

# Analysis of LSVI under Certain Assumptions

**1. Bellman Completeness Assumption:** For any $\theta \in \mathbb{R}^d$, there exists a $\theta' \in \mathbb{R}^d$ such that

$$\phi(s,a)^\top \theta' = R(s,a) + \gamma \, \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \max_{a'} \phi(s',a')^\top \theta \right] \qquad \forall s,a$$

This ensures that no matter what $\theta_{k-1}$ is, there always exists a $\theta_k^\star$ such that

$\forall s,a \quad \theta^\star_{k,s,a} \leftarrow \boxed{R(s,a) + \dots}$

$$\underbrace{\phi(s,a)^\top \theta_k^\star}_{\text{one-hot at (s,a) entry}} = R(s,a) + \gamma \, \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \max_{a'} \phi(s',a')^\top \theta_{k-1} \right]$$

$\boxed{\theta^\star_{k,s,a}}$

This is similar to the linear assumption $\phi(s,a)^\top \theta^\star = R(s,a)$ in contextual bandits, but is qualitatively stronger because the assumption require "for any $\theta$".

# Analysis of LSVI under Certain Assumptions

$D^{(1)} \cup \dots \cup D^{(k)}$

**2. Coverage Assumption:** The dataset $\cancel{\mathcal{D}^{(k)}}$ collected up to $k$-th iteration allows us to find $\theta_k$ so that for any $s, a$,

$$\left|\phi(s,a)^\top \theta_k - \phi(s,a)^\top \theta_k^\star\right| \le \epsilon_{\text{stat}}$$

(Similar to linear contextual bandits analysis) With

$$\theta_k = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^{n} \Big(\phi_i^\top \theta - \underbrace{\big(r_i + \gamma \max_{a'} \phi(s_i', a')^\top \theta_{k-1}\big)}_{\text{Expectation} = \phi_i^\top \theta_k^\star}\Big)^2 + \lambda \|\theta\|^2$$

we have $\left|\phi(s,a)^\top (\theta_k - \theta_k^\star)\right| \lesssim \sqrt{\beta} \|\phi(s,a)\|_{\Lambda^{-1}}$ where $\Lambda = \lambda I + \sum_{i=1}^{n} \phi_i \phi_i^\top$

In linear CB, we did not make such an assumption. What we did there is adding $\sqrt{\beta}\|\phi(s,a)\|_{\Lambda^{-1}}$ as **exploration bonus**, which encourages exploration and aims to make $\sqrt{\beta}\|\phi(s,a)\|_{\Lambda^{-1}}$ small for all $s, a$.

# Analysis of LSVI under Certain Assumptions (Recap)

**1. Bellman Completeness** (i.e., function approximation is sufficiently expressive)

$$\forall \theta_{k-1}, \exists \theta_k^\star \qquad \phi(s,a)^\top \theta_k^\star = R(s,a) + \gamma\, \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[\max_{a'} \phi(s',a')^\top \theta_{k-1}\right] \quad \forall s,a$$

$$\left( \forall \theta_{k-1}, \exists \theta_k^\star \qquad Q_{\theta_k^\star}(s,a) = R(s,a) + \gamma\, \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[\max_{a'} Q_{\theta_{k-1}}(s',a')\right] \quad \forall s,a \right)$$

**2. Coverage Assumption** (i.e., the collected data is sufficient and explores the state-action space) Regression over $\mathcal{D}^{(k)}$ allows us to find $\theta_k$ such that

$$\left|\phi(s,a)^\top \theta_k - \phi(s,a)^\top \theta_k^\star\right| \le \epsilon_{\text{stat}} \quad \forall s,a$$

$$\left( \left|Q_{\theta_k}(s,a) - Q_{\theta_k^\star}(s,a)\right| \le \epsilon_{\text{stat}} \quad \forall s,a \right)$$

The two assumptions jointly imply $Q_{\theta_k}(s,a) \approx R(s,a) + \gamma\, \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[\max_{a'} Q_{\theta_{k-1}}(s,a)\right]$

# Analysis of LSVI under Certain Assumptions

Under Bellman completeness and coverage assumptions, LSVI ensures

$$\left\|Q^{(k)} - Q^{\star}\right\|_{\infty} \leq O\left(\gamma^k \left\|Q^{(0)} - Q^{\star}\right\|_{\infty} + \frac{\epsilon_{\text{stat}}}{1-\gamma}\right)$$

where $\left\|Q^{(k)} - Q^{\star}\right\|_{\infty} \coloneqq \max_{s,a} \left|Q^{(k)}(s,a) - Q^{\star}(s,a)\right|$

Also, the greedy policy $\pi^{(k)}(s) = \operatorname{argmax}_a Q^{(k)}(s,a)$ satisfies for all $s$,

$$V^{\star}(s) - V^{\pi^{(k)}}(s) \leq O\left(\gamma^k \left\|Q^{(0)} - Q^{\star}\right\|_{\infty} + \frac{\epsilon_{\text{stat}}}{1-\gamma}\right)$$

$$\left| Q^{(k)}(s,a) - Q^*(s,a) \right| \leq \left| r(s,a) + \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)} \left( \max_{a'} Q^{(k-1)}(s',a') \right) - \underbrace{r(s,a) - \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)} \left( \max_{a'} Q^*(s',a') \right)}_{-Q^*(s,a)} \right| + \varepsilon_{stat}$$

$$Q^{(k)}(s,a) = \phi(s,a)^T \theta_k$$

Assumption 2 : $\left| Q^{(k)}(s,a) - r(s,a) - \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)} \left[ \max_{a'} Q^{(k-1)}(s',a') \right] \right| \leq \varepsilon_{stat}$

Bellman opt. eq. $Q^*(s,a) - r(s,a) - \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)} \left[ \max_{a'} Q^*(s',a') \right] = 0$

$$\leq \gamma \left| \mathbb{E}_{s' \sim p(\cdot|s,a)} \left[ \max_{a'} Q^{(k-1)}(s',a') - \max_{a'} Q^*(s',a') \right] \right| + \varepsilon_{stat}$$

$$\leq \gamma \left| \mathbb{E}_{s' \sim p(\cdot|s,a)} \max_{a'} \left| Q^{(k-1)}(s',a') - Q^*(s',a') \right| \right| + \varepsilon_{stat}$$

$$\left| \max_a f(a) - \max_a g(a) \right| \leq \max_a \left| f(a) - g(a) \right|$$

$$\leq \gamma \max_{s',a'} \left| Q^{(k-1)}(s',a') - Q^*(s',a') \right| + \varepsilon_{stat}$$

$$\Rightarrow \max_{s,a} \left| Q^{(k)}(s,a) - Q^*(s,a) \right| \leq \gamma \max_{s,a} \left| Q^{(k-1)}(s,a) - Q^*(s,a) \right| + \varepsilon_{stat}$$

$$\leq \gamma \left( \gamma \max_{s,a} \left| Q^{(k-2)}(s,a) - Q^*(s,a) \right| + \varepsilon_{stat} \right) + \varepsilon_{stat}$$

$$\leq \cdots \leq \gamma^k \max_{s,a} \left| Q^{(0)}(s,a) - Q^*(s,a) \right| + \varepsilon_{stat} \left( 1 + \gamma + \gamma^2 + \cdots \gamma^k \right) \leq \frac{1}{1-\gamma}$$

# Notes on Exploration in MDPs

# The Coverage Assumption

$$\left|\phi(s,a)^\top \theta_k - \phi(s,a)^\top \theta_k^\star\right| \le \epsilon_{\text{stat}} \quad \forall s,a$$

$\theta_k$: our regression solution

$\theta_k^\star$: ground truth

- Requires the state-action space to be explored
  - **Tabular case**: every state-action pair needs to be visited many times
  - **Linear case**: the feature space $\{\phi(s,a)\}_{s,a}$ needs to be explored in all directions

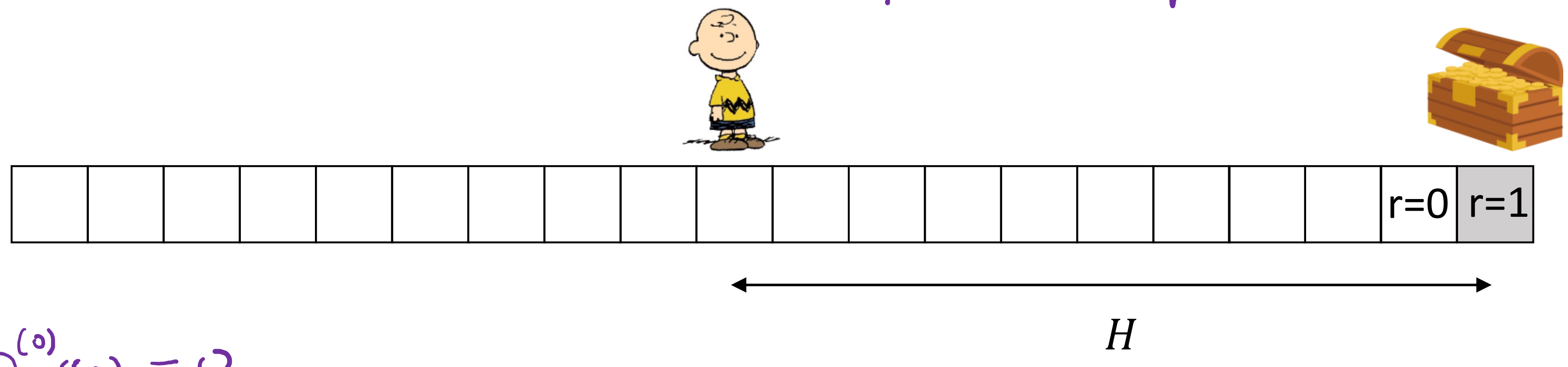
- In bandits, we focus on "action-space" exploration
  - Exploration bonus (UCB, Thompson Sampling) $a_t = \arg\max_a \{ \underline{\hat{R}(a)} + b_t(a) \}$
  - Randomization ($\epsilon$-greedy, Boltzmann exploration, inverse-gap weighting)

$p_t(a) \propto \exp(\lambda \hat{R}(a))$

- In MDPs, we further need "state-space" exploration

$$\begin{cases} a_1: \text{go right} \\ a_2: \text{go left} \end{cases}$$

Each episode has $H$ steps to execute




$Q^{(0)}(s,a) = 0$

If we do randomized exploration e.g. $p_t(a) \propto \exp(\lambda Q^{(k)}(s,a)) \longrightarrow$ Prob(reaching the $r=1$ state) $\simeq \frac{1}{2^H}$

$\varepsilon$-greedy

\# episodes needed to see signal $\gtrsim 2^H$

# Removing the Coverage Assumption

Use exploration bonus in LSVI:

**Tabular Case:** $\tilde{R}(s,a) = \hat{R}(s,a) + \frac{\text{const}}{\sqrt{n(s,a)}}$

**Linear MDP** (a class of MDPs that satisfies linear Bellman completeness)**:**
$\tilde{R}(s,a) = \phi(s,a)^\top \hat{\theta} + \text{const}\, \|\phi(s,a)\|_{\Lambda^{-1}}$ where $\Lambda = I + \sum_{i=1}^{t-1} \phi(s_i,a_i)\phi(s_i,a_i)^\top$

UCB in tabular MDP: Minimax regret bounds for reinforcement learning. 2017.

UCB in linear MDP: Provably efficient reinforcement learning with linear function approximation. 2019.

TS in tabular MDP: Near-optimal randomized exploration for tabular Markov decision processes. 2021.

TS in linear MDP: Frequentist regret bounds for randomized least-squares value iteration. 2020.

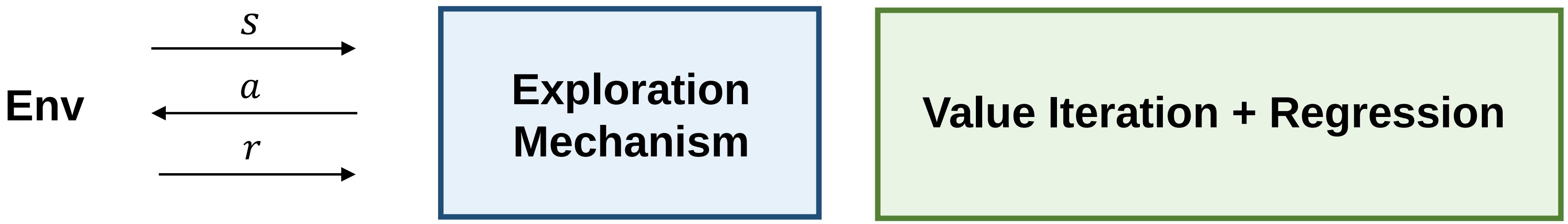
Exploration bonus for general function approximation (deep learning):

Unifying Count-Based Exploration and Intrinsic Motivation

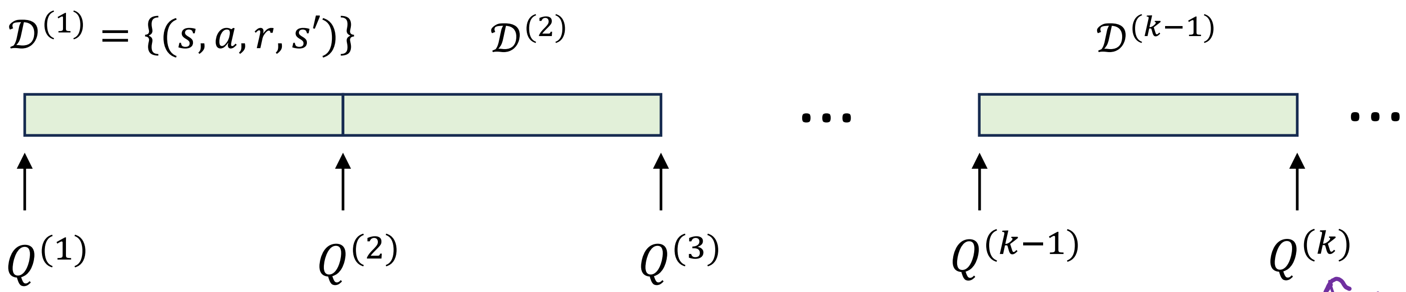
Curiosity-driven Exploration by Self-supervised Prediction

Exploration by Random Network Distillation

# Summary for LSVI




**Value Iteration + Regression**




$(\mathcal{D}^{(k-1)} \cup \dots \cup \mathcal{D}^{(1)}),\ Q^{(k-1)}$

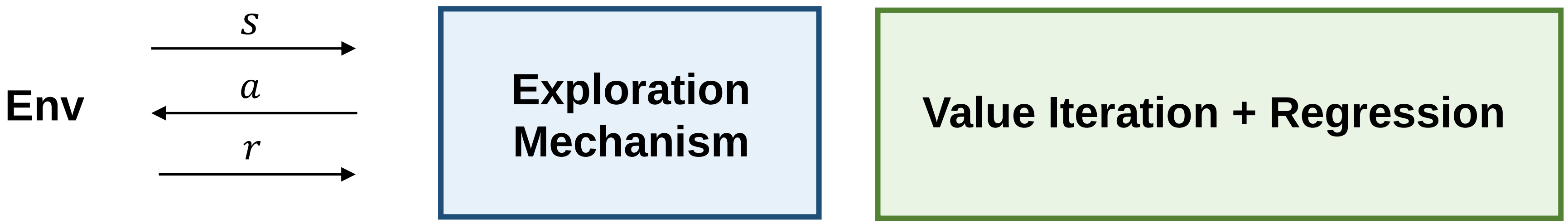
$$\theta_k = \operatorname*{argmin}_{\theta} \sum_{(s_i, a_i, r_i, s_i')} \left(Q_\theta(s_i, a_i) - r_i - \gamma \max_{a'} Q_{\theta_{k-1}}(s_i', a')\right)^2$$

not reuse sample (use $\mathcal{D}^{(k-1)}$) or
reuse sample (use $\mathcal{D}^{(1)} \cup \cdots \cup \mathcal{D}^{(k-1)}$)

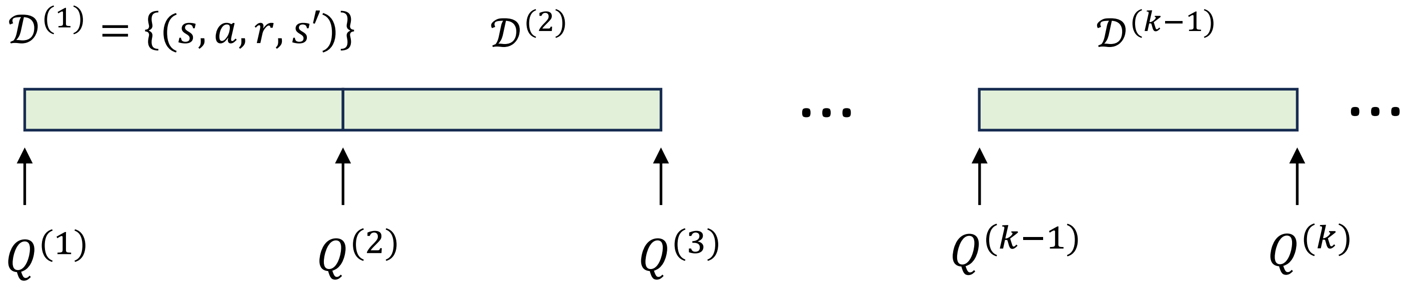
*cf.* Contextual bandits (only regression)

$$\theta_k = \operatorname*{argmin}_{\theta} \sum_{(x_i, a_i, r_i)} \left(R_\theta(x_i, a_i) - r_i\right)^2$$

# Summary for LSVI



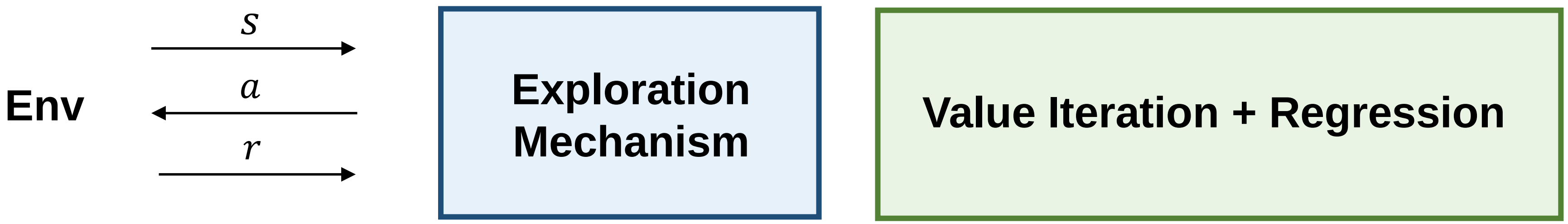

**Value Iteration + Regression**




Bellman completeness assumption $\Rightarrow$ $\exists \theta_k^\star, \quad \forall s, a, \quad Q_{\theta_k^\star}(s,a) = R(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[\max_{a'} Q_{\theta_{k-1}}(s', a')\right]$
(function expressiveness assumption)

Coverage assumption $\Rightarrow$ $\forall s, a, \quad \left|Q_{\theta_k}(s,a) - Q_{\theta_k^\star}(s,a)\right| \le \epsilon_{\text{stat}}$
(exploration assumption)

# Summary for LSVI




**Exploration Mechanism**

1. Randomized policies ($\epsilon$-Greedy, Boltzmann exploration, inverse-gap weighting)
   – perform local exploration

2. Exploration bonus (UCB) / Randomized values (TS)
   – can give rigorous regret bounds for tabular MDPs and MDPs with linear Bellman completeness
   – perform wider state space exploration

**Other names for LSVI**: Fitted Q Iteration, Least-square Q Iteration

# Q-Learning

# Q-Learning (Watkins, 1992)

$\hat{R}^{(i)}(a) = (1-\alpha)\hat{R}^{(i-1)}(a) + \alpha\, r_i(a)$

$= (1-\alpha)\left((1-\alpha)\hat{R}^{(i-2)}(a) + \alpha r_{i-1}(a)\right) + \alpha r_i(a)$

$\Rightarrow \hat{R}^{(i)}(a) = \sum_{j=1}^{i} \alpha(1-\alpha)^{i-j} r_j(a)$

For $i = 1,\ 2, \ldots$

Obtain sample $(s_i, a_i, r_i, s_i')$

$$Q^{(i)}(s_i, a_i) \leftarrow (1-\alpha_i)Q^{(i-1)}(s_i, a_i) + \alpha_i\left(r_i + \gamma \max_a Q^{(i-1)}(s_i', a)\right)$$

$$Q^{(i)}(s, a) \leftarrow Q^{(i-1)}(s, a) \qquad \forall (s, a) \neq (s_i, a_i)$$

Function approximation : $Q_\theta(s,a)$

*cf.* LSVI:

$$\forall s, a, \qquad Q^{(k)}(s, a) \leftarrow \frac{\sum_{i=1}^{n_k} \mathbb{I}\{(s_i, a_i) = (s, a)\}\left(r_i + \gamma \max_{a'} Q^{(k-1)}(s_i', a')\right)}{\sum_{i=1}^{n_k} \mathbb{I}\{(s_i, a_i) = (s, a)\}}$$

# Q-Learning (Watkins, 1992)

Fixed an $(s,a)$. Let's see what $\underline{Q^{(k)}(s,a)}$

Assume that before iteration $k$, $(s,a)$ has been visited in iteration $j_1, j_2, \cdots, j_\tau < k$

$$Q^{(k)}(s,a) = \sum_{i=1}^{\tau} \alpha(1-\alpha)^{\tau-i} \Big( \underset{\uparrow \atop R(s,a)}{r_{j_i}} + \gamma \max_{a'} Q^{(j_i)}(s'_{j_i}, a') \Big)$$

# Q-Learning (Watkins, 1992)

Suppose that $\alpha_i = \frac{1}{i^\beta}$ for some $\frac{1}{2} < \beta \leq 1$, and every state-action pair is visited infinitely often. Then

$$Q^{(i)}(s,a) \to Q^\star(s,a) \quad \forall s,a.$$

Gen Li, Yuting Wei, Yuejie Chi, Yuantao Gu, Yuxin Chen. Sample Complexity of Asynchronous Q-Learning: Sharper Analysis and Variance Reduction. 2020.

# Watkins's Q-Learning + Linear Function Approximation

For $i = 1,\ 2, \ldots$

Obtain sample $(s_i, a_i, r_i, s_i')$

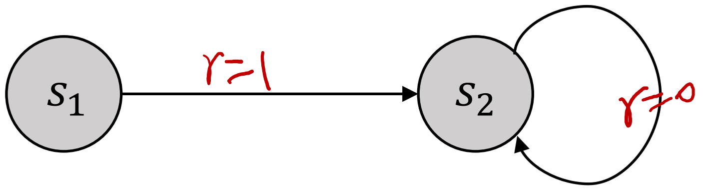
$$\theta_i \leftarrow \theta_{i-1} - \alpha \nabla_\theta \left( \phi(s_i, a_i)^\top \theta - r_i - \gamma \max_a \phi(s_i', a)^\top \theta_{i-1} \right)^2 \Bigg|_{\theta = \theta_{i-1}}$$

$$= \theta_{i-1} - 2\alpha \left( \phi(s_i, a_i)^\top \theta_{i-1} - r_i - \gamma \max_a \phi(s_i', a)^\top \theta_{i-1} \right) \phi(s_i, a_i)$$

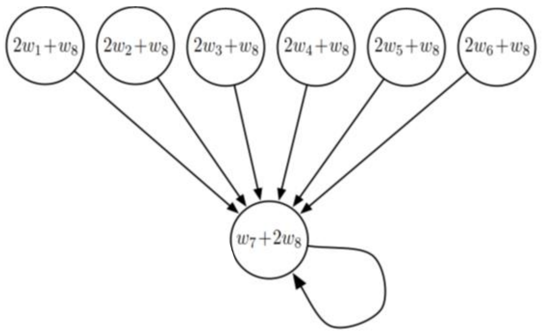
*c.f.* LSVI: $$\theta_k = \operatorname*{argmin}_\theta \sum_{i=1}^{n_k} \Big( \underbrace{\phi(s_i, a_i)^\top \theta}_{Q_\theta(s_i, a_i)} - r_i - \gamma \max_{a'} \phi(s_i', a')^\top \theta_{k-1} \Big)^2$$

# Watkins's Q-Learning + LFA Does Not Converge

Even when Bellman completeness and coverage assumptions hold




$\phi(s_1, a) = (1,0)$ $\qquad$ $\phi(s_2, a) = (2,1)$




Simplified from the "Baird's counterexample"
(see Sutton and Barto Section 11.2)

Bellman completeness assumption

For any $\theta' \in \mathbb{R}^2$, there exists a $\theta \in \mathbb{R}^2$ such that

$$\phi(s,a)^T\theta = R(s,a) + \gamma \mathop{\mathbb{E}}_{s'\sim P(\cdot|s,a)}\left[\max_{a'} \phi(s',a')^T\theta'\right] \quad \forall s,a.$$

$$\begin{cases}\phi(s_1,a)^T\theta = R(s_1,a)+\gamma\phi(s_2,a)^T\theta' \\ \phi(s_2,a)\,\theta = R(s_2,a)+\gamma\phi(s_2,a)^T\theta'\end{cases}$$

two variables $(\theta_1, \theta_2)$ with two linearly independent constraints

# The Effect of Fixing the Target

$Kn = 100000$

For $k = 1,\ 2, \ldots$ $K$

$\theta_{k-1} \leftarrow \theta$

when $n$ is large $\Rightarrow \theta \approx \underset{\theta}{\operatorname{argmin}} \left\{ \frac{1}{2}\left(\phi(s_1,a)^\top\theta - 1 - \gamma\phi(s_2,a)^\top\theta_{k-1}\right)^2 + \frac{1}{2}\left(\phi(s_2,a)^\top\theta - 0 - \gamma\phi(s_2,a)^\top\theta_{k-1}\right)^2 \right\}$

For $i = 1, \ldots, n$:

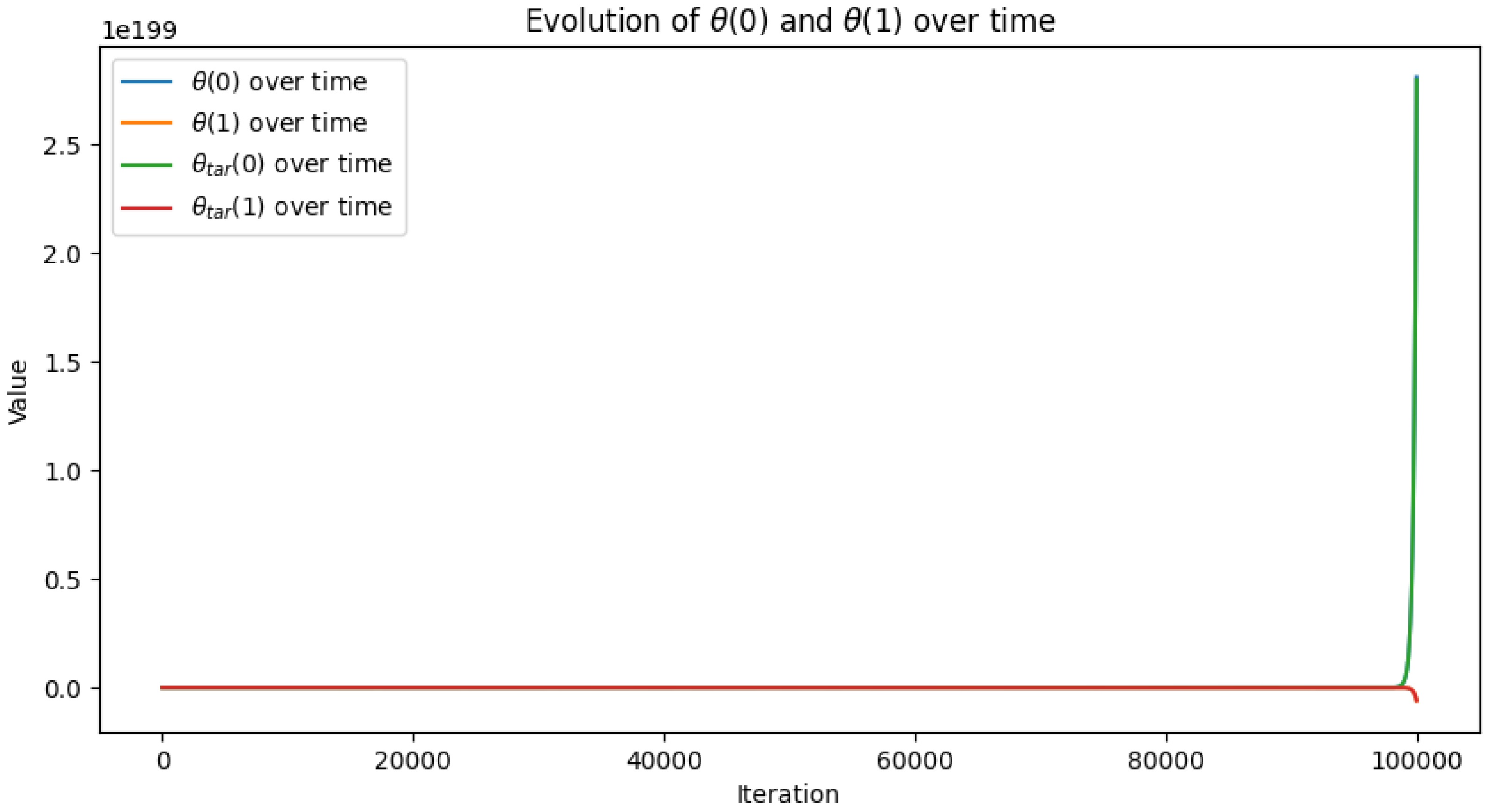
Sample $(s, a, r, s') \sim \text{Uniform}\{(s_1, a, 1, s_2),\ (s_2, a, 0, s_2)\}$

$$\theta \leftarrow \theta - \alpha\left(\phi(s,a)^\top\theta - r - \gamma\phi\left(s', a\right)^\top \theta_{k-1}\right)\phi(s,a)$$

$\theta_k \leftarrow \theta$

# The Effect of Fixing the Target

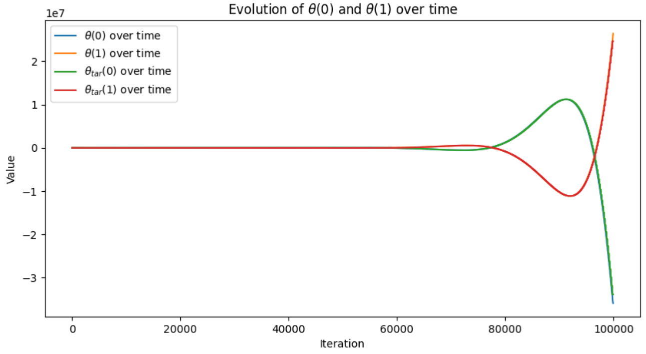
$n=1$



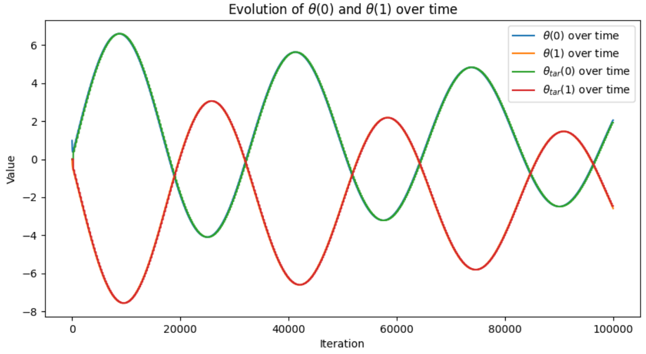
\# of iterations in outer loop

$$\downarrow$$

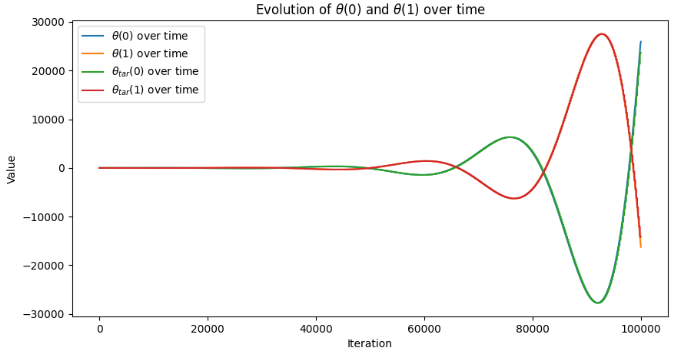
$$K = \frac{100000}{n}$$



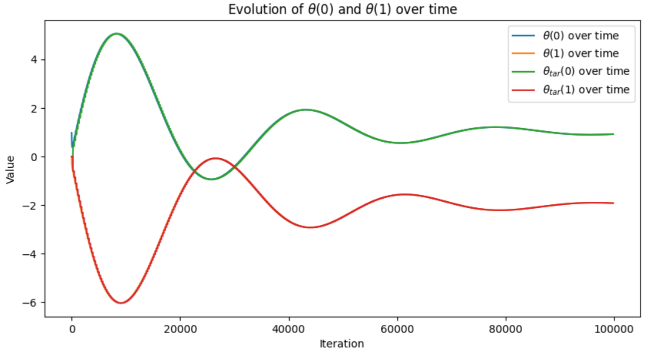

n=150



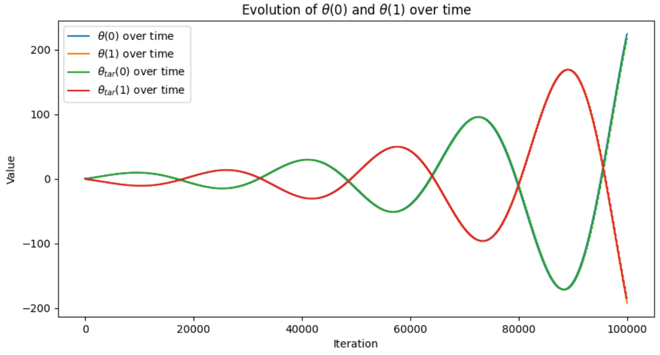

n=210



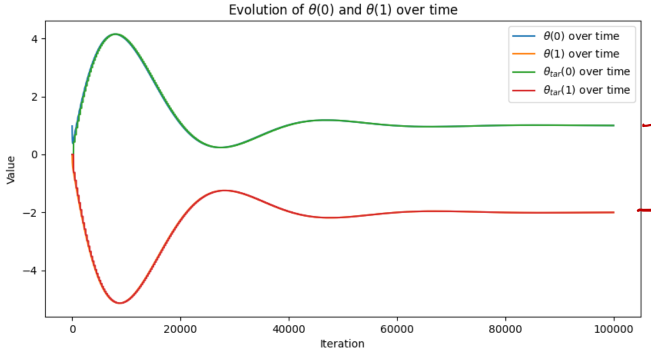

n=170




n=230



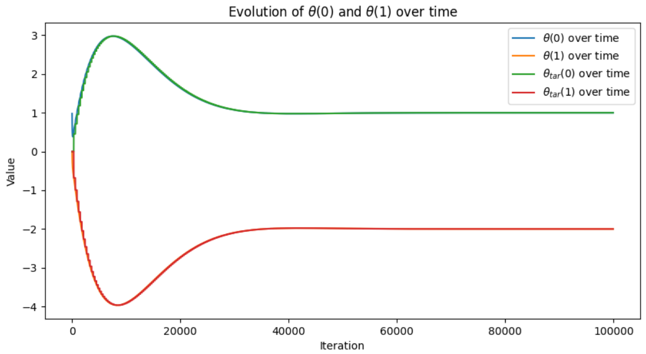

n=190



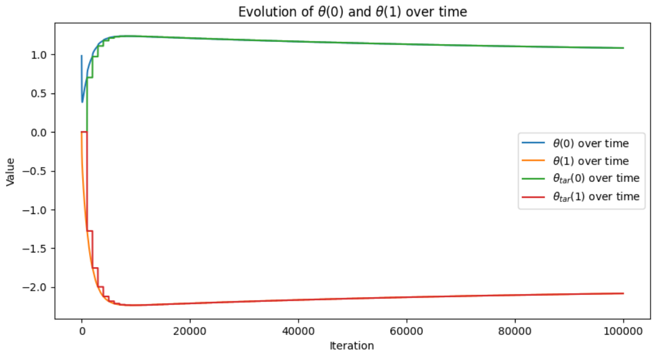

$\rightarrow 1$

$\rightarrow -2$

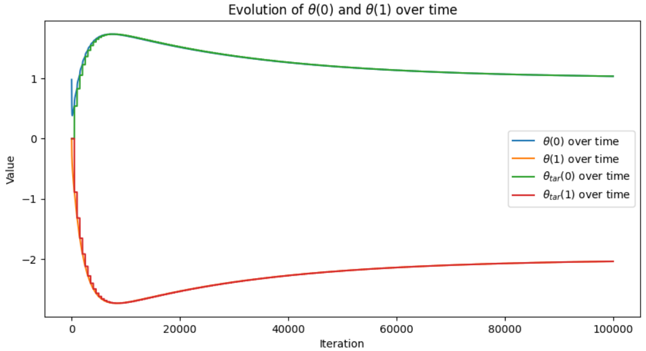
n=250



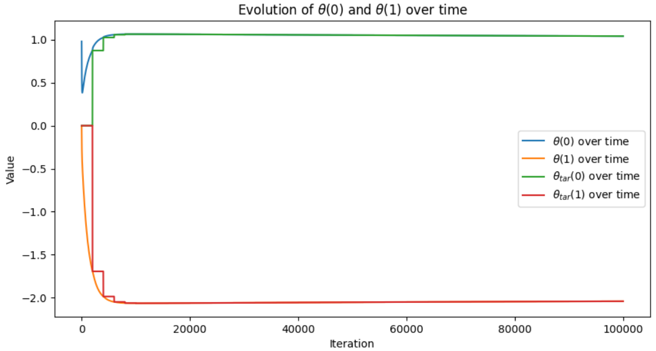

n=300



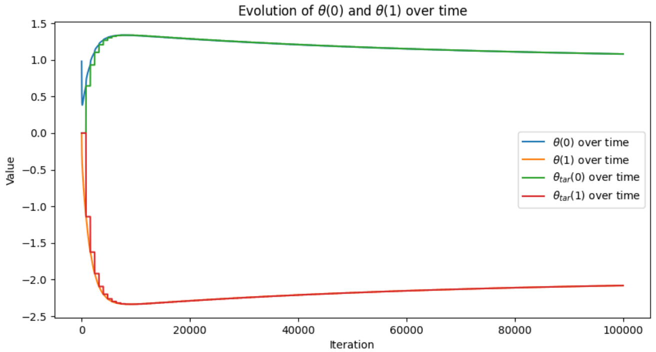

n=1000



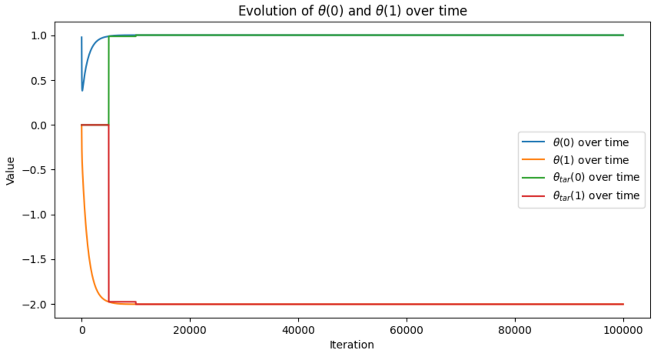

n=500




n=2000




n=800




n=5000

# Watkins's Q-Learning vs. LSVI

Under coverage assumption
(i.e., the data $\{(s_i, a_i, r_i, s_i')\}$ sufficiently cover every state-action pair / feature space)

| | **LSVI** | **Watkins's Q-Learning** |
|---|---|---|
| Convergence in the tabular case | $Q^{(k)} \to Q^\star$ | $Q^{(k)} \to Q^\star$ |
| Convergence under function approximation | $Q^{(k)} \to Q^\star$ under BC | Diverges even with BC |
| Update style | Two time-scale | Single time-scale |

# Techniques for Function Approximation (Deep Q-Learning)

# Use LSVI Updates

For $k = 1,\ 2, \ldots$

Collect samples $\mathcal{D}^{(k)}$ (consisting of $(s, a, r, s')$ tuples) using some exploratory policy

Perform regression over dataset $\mathcal{D}^{(1)} \cup \mathcal{D}^{(2)} \cup \cdots \cup \mathcal{D}^{(k)}$:

$$\theta_k = \underset{\theta}{\operatorname{argmin}} \sum_{(s,a,r,s') \in \mathcal{D}} \left( Q_\theta(s, a) - r + \gamma \max_{a'} Q_{\theta_{k-1}}(s', a') \right)^2$$

**Regression**

# Implement Regression with SGD

For $k = 1,\ 2, \ldots$

Collect samples $\mathcal{D}^{(k)}$ (consisting of $(s, a, r, s')$ tuples) using some exploratory policy

$\theta_{k-1} \leftarrow \theta$

For $i = 1,\ 2, \ldots, n$:

Randomly draw a minibatch $\{(s_i, a_i, r_i, s_i')\}_{i=1}^{b}$ from $\mathcal{D}^{(1)} \cup \mathcal{D}^{(2)} \cup \cdots \cup \mathcal{D}^{(k)}$

$$\theta \leftarrow \theta - \alpha \sum_{i=1}^{b} \nabla_\theta \left( Q_\theta(s_i, a_i) - r_i + \gamma \max_{a'} Q_{\theta_{k-1}}(s_i', a') \right)^2$$

# Typical Implementation of Deep Q-Learning

Interleaving data collection and SGD

For $i = 1,\ 2, \ldots$

Obtain a new sample $(s, a, r, s')$ and insert it to a **replay buffer** $\mathcal{B}$

Randomly draw a minibatch $\{(s_i, a_i, r_i, s_i')\}_{i=1}^{b}$ from $\mathcal{B}$ and perform

$$\theta \leftarrow \theta - \alpha \sum_{i=1}^{b} \nabla_\theta \left( Q_\theta(s_i, a_i) - r_i + \gamma \max_{a'} Q_{\theta_{\text{tar}}}(s_i', a') \right)^2$$

**// Option 1**

If $i \bmod n = 0$:

$\theta_{\text{tar}} \leftarrow \theta$

**// Option 2** $\tau = 0.999$

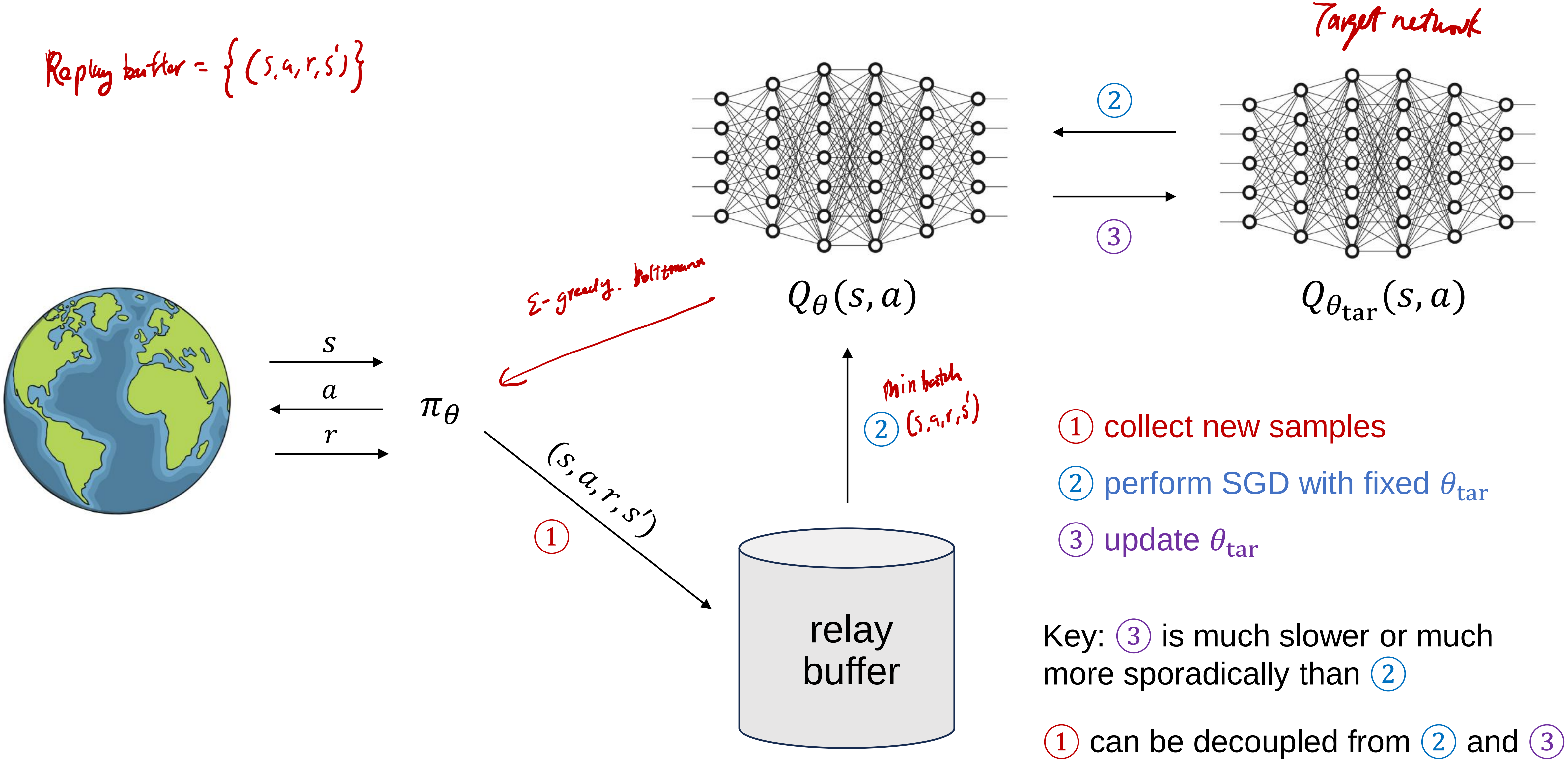
$\theta_{\text{tar}} \leftarrow \tau\theta_{\text{tar}} + (1 - \tau)\theta$

The following update converges but to the wrong place when the transition is non-deterministic:

$$\theta \leftarrow \theta - \alpha \sum_{i=1}^{b} \nabla_\theta \left( Q_\theta(s_i, a_i) - r_i + \gamma \max_{a'} Q_\theta(s_i', a') \right)^2$$

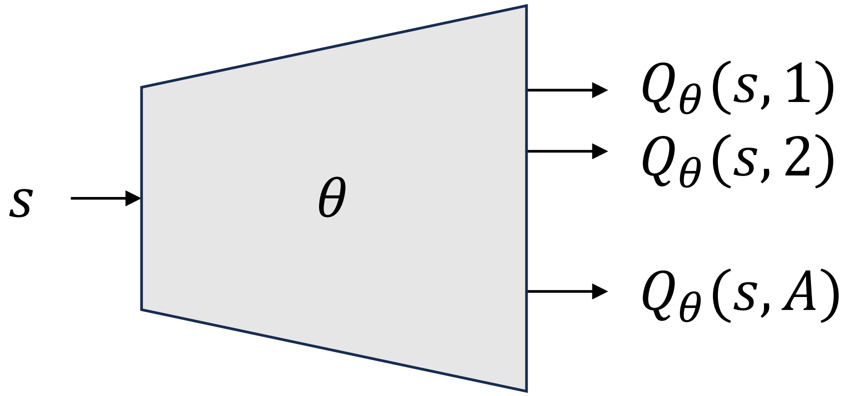
See Sutton & Barto Section 11.5 or Nan Jiang's lecture note (P.17 bellman error minimization)

# Target Network and Replay Buffer



# Q-Network Design

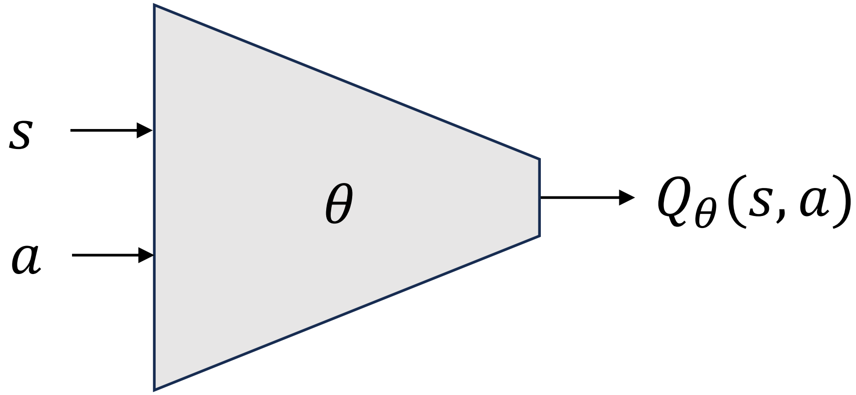
$Q_\theta(s,a)$




$\max_a Q_\theta(s,a)$ is straightforward

only handles discrete action case

**Deep Q-Network**



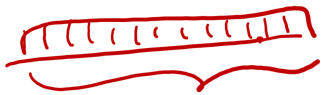

$\max_a Q_\theta(s,a)$ requires run $a \leftarrow a + \eta \frac{\partial\, Q_\theta(s,a)}{\partial a}$ iteratively

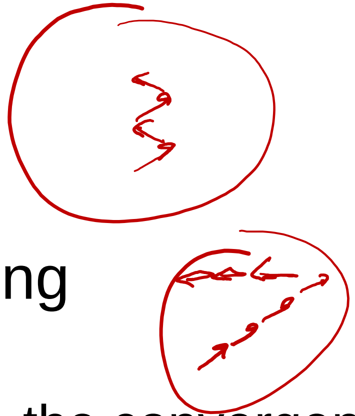
can handle continuous action case

**Deep Deterministic Policy Gradient**

(covered later in the semester)

# Replay Buffer and Sampling





**Standard implementation:** First-in-first-out queue + Uniform sampling

- The data collected from $\pi_\theta$ is not i.i.d.
- Uniform sampling from a large pool makes the data more similar to i.i.d. – the convergence of SGD requires samples to be i.i.d.

**Prioritized replay:** priority queue + prioritized sampling + importance weight

- Priority queue with priority proportional to $|\delta_i|$, where $\delta_i = Q_\theta(s_i, a_i) - r_i - \gamma \max_{a'} Q_{\theta_{\text{tar}}}(s_i', a')$
- Sample from the buffer with probability $P_i \propto |\delta_i|^\alpha$
- Perform SGD with importance weight $w_i = \left(\frac{P_i}{\max_j P_j}\right)^{-\beta}$, i.e.,

$$\theta \leftarrow \theta - \alpha\, w_i \nabla_\theta \left(Q_\theta(s_i, a_i) - r_i + \gamma \max_{a'} Q_{\theta_{\text{tar}}}(s_i', a')\right)^2$$

Schaul, Quan, Antonoglou, Silver. [Prioritized Experience Replay](). 2015.