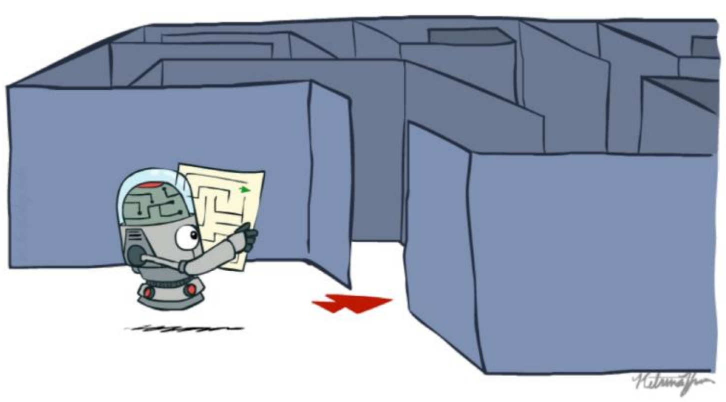
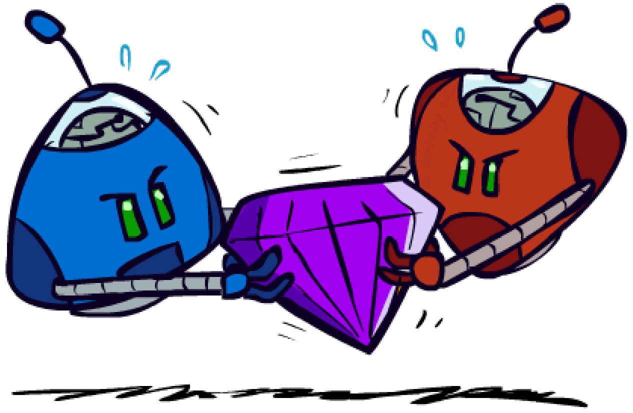
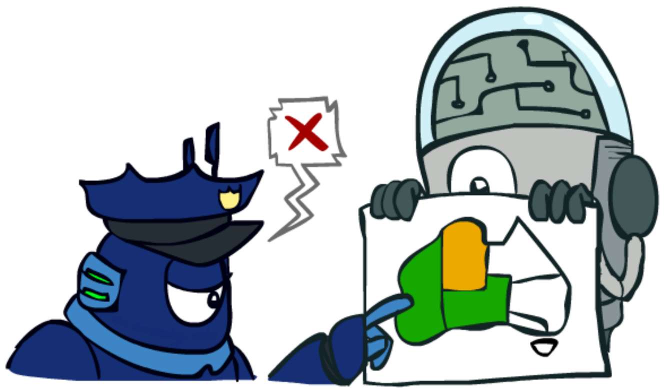


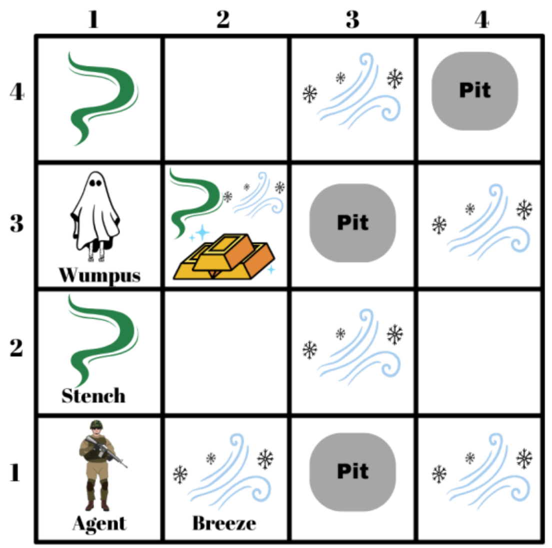
Search



Adversarial Search



Constraint Satisfaction

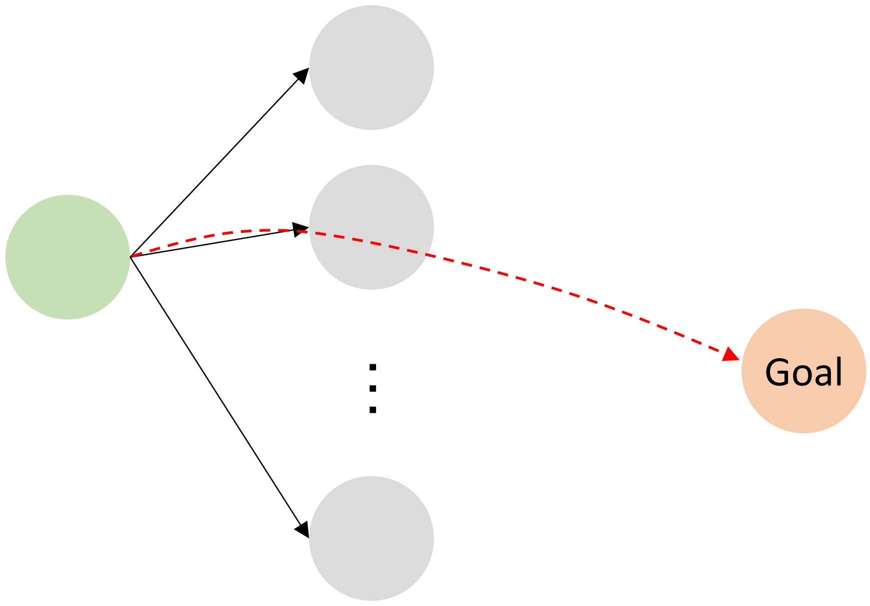


Logic

State of a game

Assignment for variables

Knowledge base

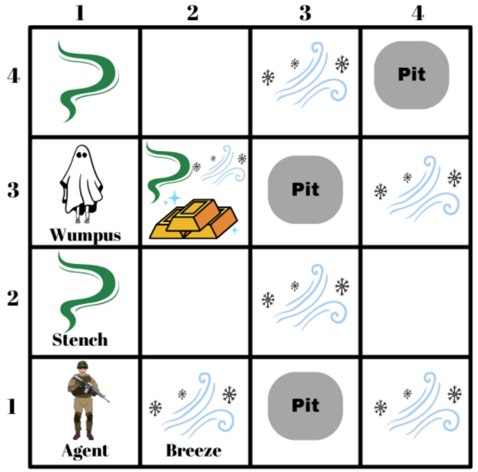


pruning, decision ordering

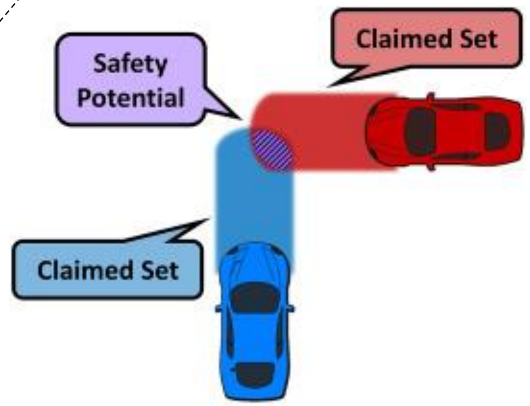
The techniques we learned help the computer to more efficiently search in a (exponentially) large state space.

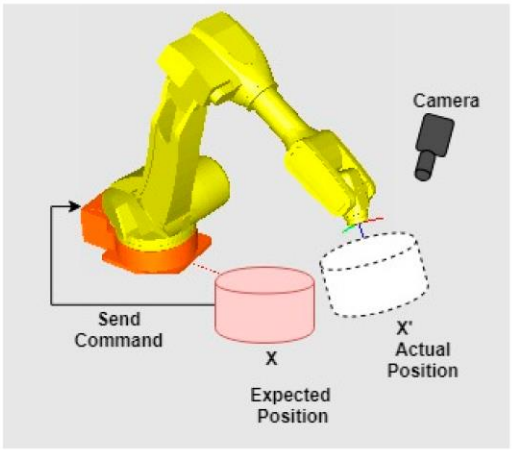
The problems we have dealt with are overall complex (large state space), but the **rules** are usually **deterministic** and **known** and **simple.**

Deterministic vs. Random / Uncertain



Breeze ⇒ Pit in adjacent squares





Toothache ⇒ Cavity ∨ Gum Problem ∨ Abscess …

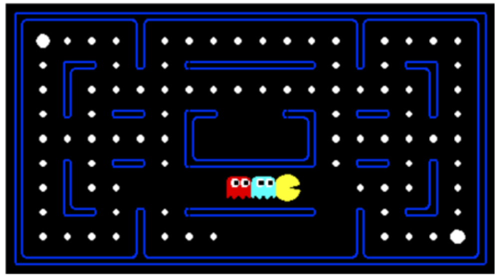
0.3 0.1 0.05

Randomness / uncertainty usually come from **ignorance**: Not all the necessary information has been or can be gathered

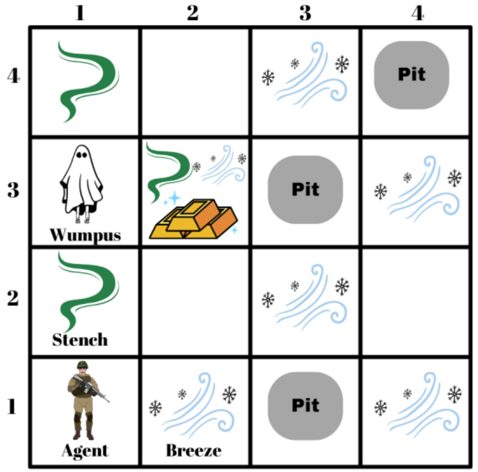
→ Probabilistic modeling

Known vs. Unknown

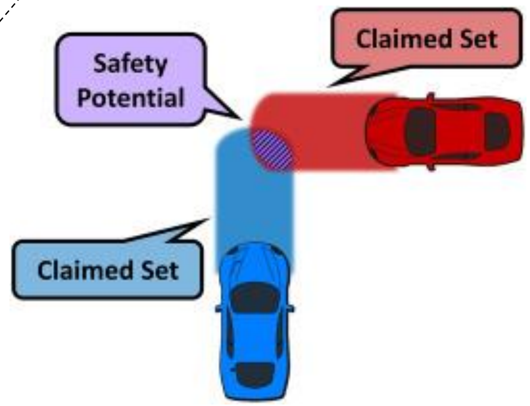
What is the **state distribution** if taking a certain action?

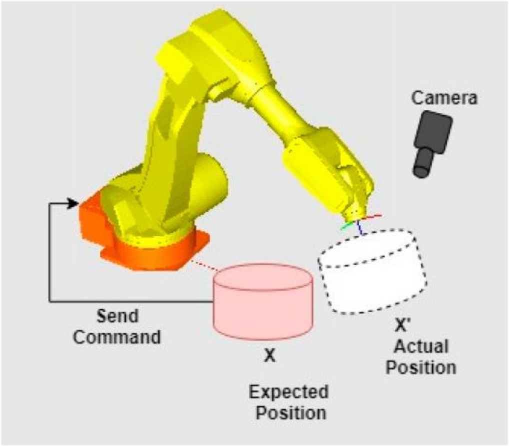


Ghost takes uniformly random actions



Breeze ⇒ Pit in adjacent squares





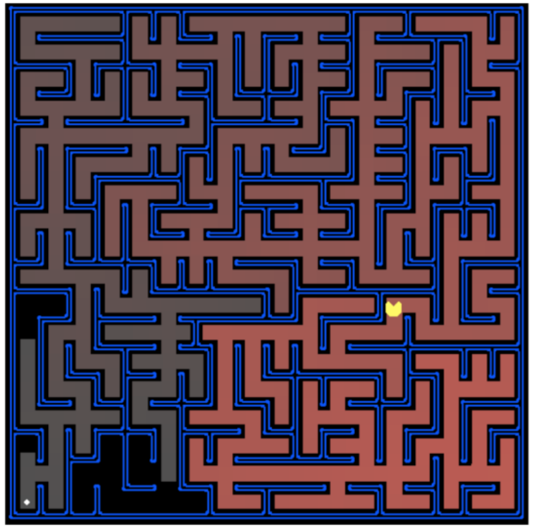
Toothache ⇒ Cavity ∨ Gum Problem ∨ Abscess …

0.3 0.1 0.05

Often times, the outcome of an action or the underlying state given observations has to be **learned from experiences**

→ Machine Learning

Simple (Easily Explainable) vs. Complicated

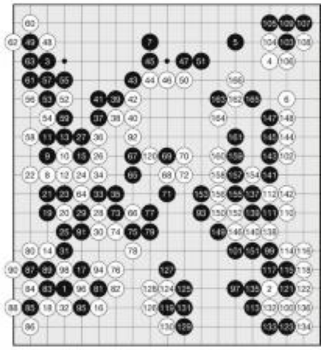


State = Pacman position

Action = NSEW

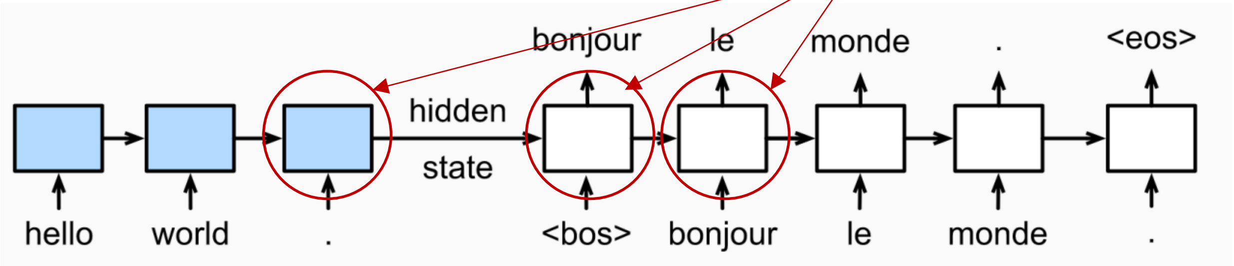
Next state = State applying Action

Heuristic = Distance to goal



Heuristic = ?

States that summarize the “meaning” of the English sentence, and the current progress of the translation



To perform the task well, we may need a good way to **encode** states (instead of its original form) and/or actions.

→ Human designed features, or

→ Representation Learning (Deep Learning)

Roadmap

- Search in deterministic models (finished) (Most techniques were developed before 1990)
- Probabilistic modeling
- Machine learning / deep learning: learning the model parameters and state representations from data (Most techniques were developed after 1990)
- Reinforcement learning ≈ performing **search and learning** simultaneously or interleavingly
- **Reminder:** this course is unable to give you a full picture of ML/DL/RL. If you're interested in any of them, you should take dedicated courses in the future.

Probability

Chen-Yu Wei

Uncertainty

- General situation:
  - **Observed variables (evidence)**: Agent knows certain things about the state of the world (toothache)
  - **Unobserved variables**: Agent needs to reason about other aspects (condition?)
  - **Model**: Agent knows something about how the known variables relate to the unknown variables (the probability of cavity given toochache)

- Uncertainty modeling is a way to incorporate our beliefs and knowledge
  - Can generalize CSP and logic that we discussed before

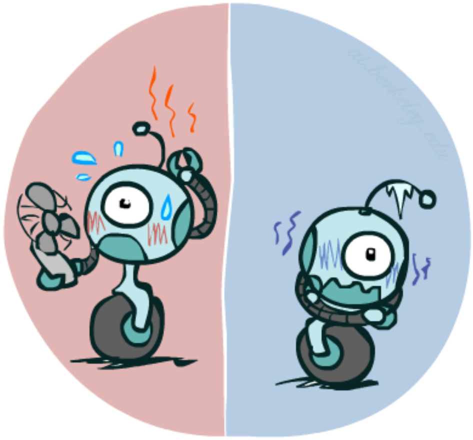
Random Variables

- A random variable is some aspect of the world which we (may) have uncertainty about
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- Like variables in a CSP, random variables have domains
  - R in {true, false}   (often write as {+r, -r})
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), …}

Probability Distributions

- Associate a probability with each value

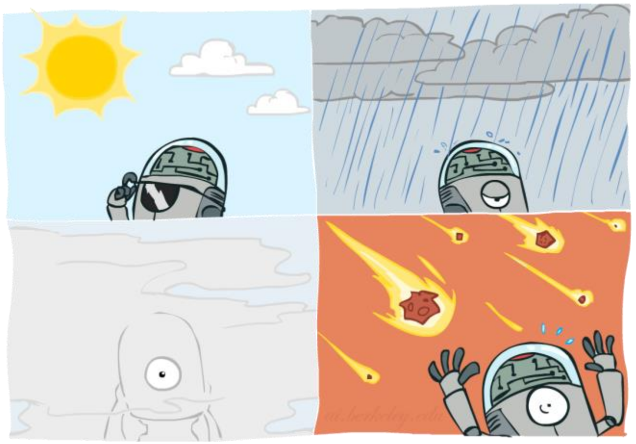
  - Temperature:



$$P(T)$$

T	P
hot	0.5
cold	0.5

  - Weather:



$$P(W)$$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Shorthand notation:

$$P(hot) = P(T = hot),$$
$$P(cold) = P(T = cold),$$
$$P(rain) = P(W = rain),$$
$$\dots$$

OK *if* all domain entries are unique

- A distribution is a TABLE of probabilities
- A probability (lower case value) is a single number $P(W = rain) = 0.1$
- Must have: $\forall x \;\; P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

- Must obey:

$$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Models

- Probabilistic models (a joint distribution):
  - Random variables with domains
  - Joint distributions: say whether assignments (outcomes) are likely

- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: specify whether assignments are possible

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Constraint over T,W

T	W	P
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T

Events

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny? 0.4
  - Probability that it's hot? 0.5
  - Probability that it's hot OR sunny? 0.7
- Typically, the events we care about are *partial assignments*, like P(T=hot)

Quiz: Events

- P(+x, +y) ? 0.2

- P(+x) ? 0.2+0.3 =0.5

- P(-y OR +x) ? 0.2+0.3+0.1 =0.6

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t,s)$$

$$P(s) = \sum_t P(t,s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.4

Quiz: Marginal Distributions

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$$P(X)$$

X	P
+x	0.5
-x	0.5

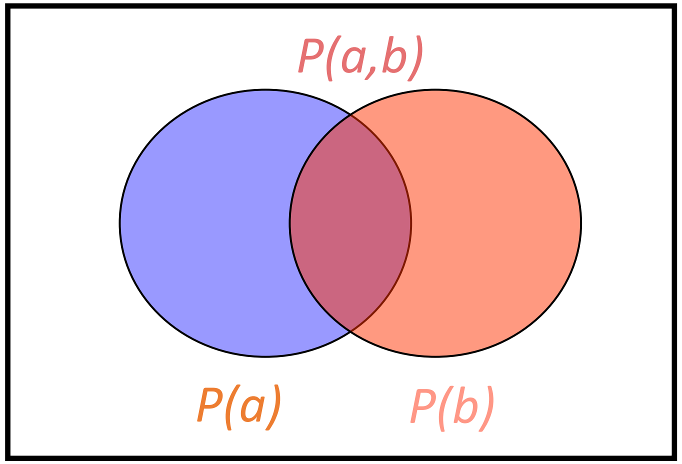
$$P(Y)$$

Y	P
+y	0.6
-y	0.4

Conditional Probabilities

- Relation between joint and conditional probabilities

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$
$$= 0.2 + 0.3 = 0.5$$

Quiz: Conditional Probabilities

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) = ? $\frac{P(+x, +y)}{P(+y)} = \frac{0.2}{0.6}$
- P(-x | +y) = ? $\frac{P(-x, +y)}{P(+y)} = \frac{0.4}{0.6}$
- P(-y | +x) = ? $\frac{P(+x, -y)}{P(+x)} = \frac{0.3}{0.5}$

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$$P(W|T = hot)$$

W	P
sun	0.8
rain	0.2

$$P(W|T = cold)$$

W	P
sun	0.4
rain	0.6

Joint Distribution

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Distributions

$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned} P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\ &= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.2}{0.2+0.3} = 0.4 \end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned} P(W=r|T=c) &= \frac{P(W=r,T=c)}{P(T=c)} \\ &= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.3}{0.2+0.3} = 0.6 \end{aligned}$$

$$P(W|T=c)$$

W	P
sun	0.4
rain	0.6

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

$$\frac{0.2}{0.2+0.3} = 0.4$$

$$\frac{0.3}{0.2+0.3} = 0.6$$

NORMALIZE the selection (make it sum to one)

$P(W|T = c)$

W	P
sun	0.4
rain	0.6

Quiz: Normalization Trick

- P(X | Y=-y) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence

X	Y	P
+x	-y	0.3
-x	-y	0.1

NORMALIZE the selection (make it sum to one)

X	P
+x	3/4
-x	1/4

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities

- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's *beliefs* given the evidence

- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes *beliefs to be updated*

Inference by Enumeration

** Works fine with multiple query variables, too*

- General case:
  - Evidence variables: $E_1 \ldots E_k = e_1 \ldots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \ldots H_r$

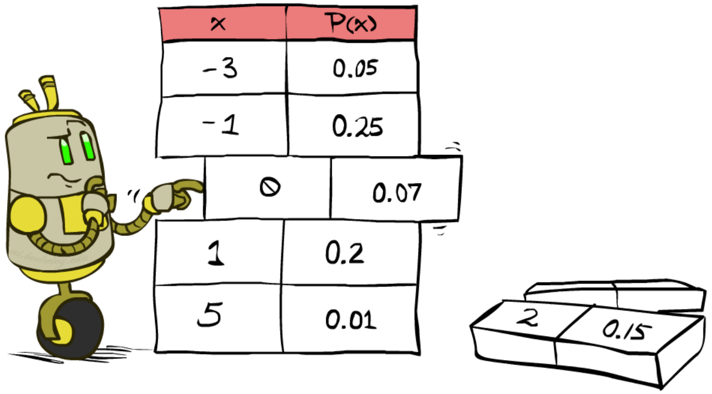
$X_1, X_2, \ldots X_n$

All variables

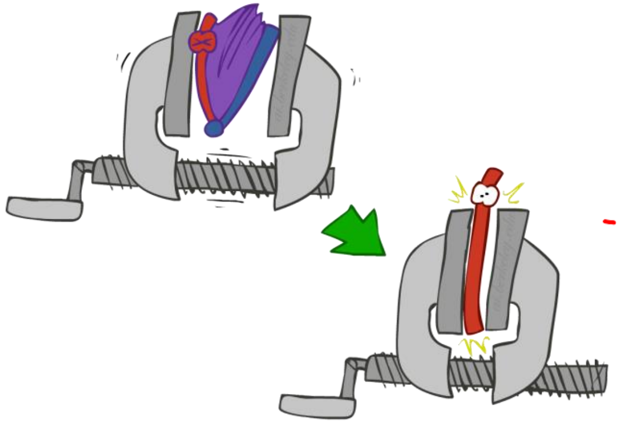
- We want:

$$P(Q|e_1 \ldots e_k)$$

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \ldots e_k) = \sum_{h_1 \ldots h_r} \underbrace{P(Q, h_1 \ldots h_r, e_1 \ldots e_k)}_{X_1, X_2, \ldots X_n}$$

- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration

- P(W)?

W	
s	0.65
r	0.35

- P(W | winter)?

W	P
s	0.25
r	0.25

Normalize $\longrightarrow$

W	P
s	0.5
r	0.5

- P(W | winter, hot)?

W	P
s	2/3
r	1/3

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

size of domain $= d$

\# of variable $= n$

$d \times d \times \cdots \times d = d^n$

- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x,y) \quad \Longleftrightarrow \quad P(x|y) = \frac{P(x,y)}{P(y)}$$

The Product Rule

$$P(y)P(x|y) = P(x, y)$$

$$\underbrace{P(\text{sun})}_{0.8}\ \underbrace{P(\text{wet}|\text{sun})}_{0.1} = P(\text{wet}, \text{sun})$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D, W)$

D	W	P
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this always true?

$$p(\cancel{x_1}) \cdot \frac{P(x_1, \cancel{x_2})}{\cancel{P(x_1)}} \cdot \frac{p(x_1, x_2, x_3)}{p(\cancel{x_1, x_2})}$$

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later

Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\underbrace{\text{cause}}_{\text{cavity}}|\underbrace{\text{effect}}_{\text{toothache}}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})} = \sum_{c'} \underline{P(\text{effect}|c')}\,\underline{P(c')}$$

Quiz: Bayes' Rule

- Given:

$P(W)$

~~D~~ W	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

$p(W|dry)$

W	P
sun	
rain	

$\frac{0.72}{0.78}$ $\quad$ $12/13$

$\frac{0.06}{0.78}$ $\quad$ $1/13$

$$p(sun|dry) = \frac{\boxed{p(dry|sun)\,p(sun)}}{p(dry)} \qquad 0.9 \times 0.8 = 0.72$$

$$p(r|dry) = \frac{\boxed{p(dry|r)\,p(r)}}{p(dry)} \qquad 0.3 \times 0.2 = 0.06$$

Independence

- Two variables are *independent* in a joint distribution if:

$$P(X,Y) = P(X)P(Y)$$
$$\forall x, y\, P(x,y) = P(x)P(y)$$

$$X \perp\!\!\!\perp Y$$

  - Says the joint distribution *factors* into a product of two simple ones
  - Usually variables aren't independent!

- Can use independence as a *modeling assumption*
  - Independence can be a simplifying assumption
  - *Empirical* joint distributions: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity}?

- Independence is like something from CSPs: what?

Example: Independence?

$$P(T)$$

T	P
hot	0.5
cold	0.5

$$P_1(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P_2(T, W) = P(T)P(W)$$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$$P(W)$$

W	P
sun	0.6
rain	0.4

Example: Independence

- N fair, independent coin flips:

$P(X_1)$

H	0.5
T	0.5

$P(X_2)$

H	0.5
T	0.5

$\ldots$

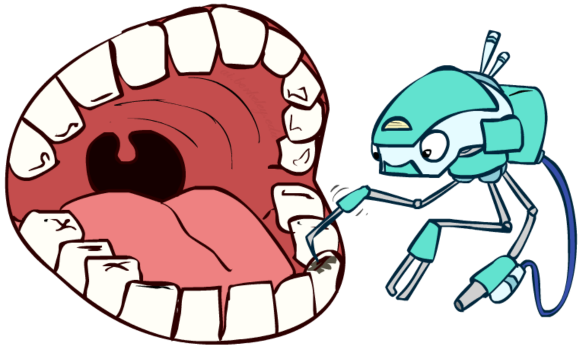
$P(X_n)$

H	0.5
T	0.5

$P(X_1, X_2, \ldots X_n)$

2^n

Conditional Independence



- P(Toothache, Cavity, Catch)

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)

- The same independence holds if I don't have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)

- Catch is *conditionally independent* of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)
- Equivalent statements:
  - P(Toothache | Catch , Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily

Conditional Independence

- Unconditional (absolute) independence very rare (why?)

- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.

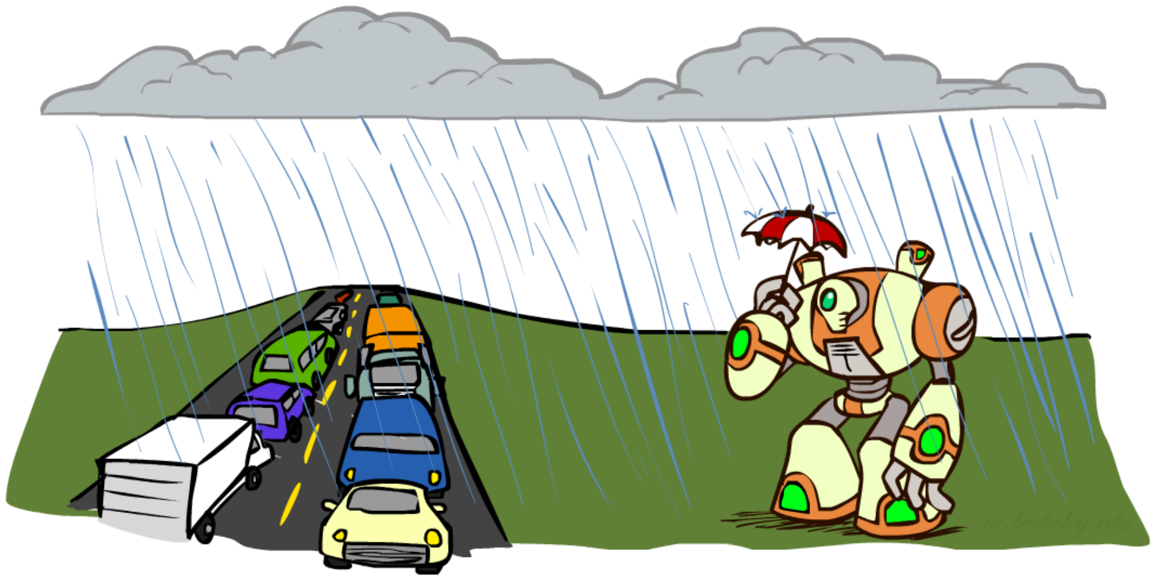
- X is conditionally independent of Y given Z $\quad X \perp\!\!\!\perp Y | Z$

if and only if: $\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$

or, equivalently, if and only if $\forall x, y, z : P(x|z, y) = P(x|z)$

Conditional Independence

- What about this domain:
  - Traffic
  - Umbrella
  - Raining



Conditional Independence

- What about this domain:
  - Fire
  - Smoke
  - Alarm