Approximate Value Iteration and Variants

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Value Iteration

For $k = 1,\ 2, \ldots$

$$\forall s, a, \qquad Q_k(s,a) \leftarrow \underbrace{\boxed{R(s,a)}}_{\text{unknown}} + \gamma \sum_{s'} \underbrace{\boxed{P(s'|s,a)}}_{\text{unknown}} \max_{a'} Q_{k-1}(s', a')$$

Idea: In each iteration, use multiple samples to estimate the right-hand side.

Value Iteration with Samples

$s_1, a_1, r_1, s_2, a_2, r_2, \ldots s_H, a_H, r_H, s_{H+1}$

For $k = 1, 2, \ldots$

$(s_1, a_1, r_1, s_2)\ (s_2, a_2, r_2, s_3), \ldots\ (s_H, a_H, r_H, s_{H+1})\ (s, \ldots)$ $\sim P$

Obtain N samples $\{(s_i, a_i, r_i, s_i')\}_{i=1}^{N}$ where $\mathbb{E}[r_i] = R(s_i, a_i),\ s_i' \sim P(\cdot \mid s_i, a_i)$

Perform **regression** on $\{(s_i, a_i, r_i, s_i')\}_{i=1}^{N}$ to find Q_k such that

$$\forall s, a, \qquad Q_k(s, a) \approx R(s, a) + \gamma \sum_{s'} P(s' \mid s, a) \max_{a'} Q_{k-1}(s', a')$$

Perform one iteration of Value Iteration

$Q_\theta(s, a)$

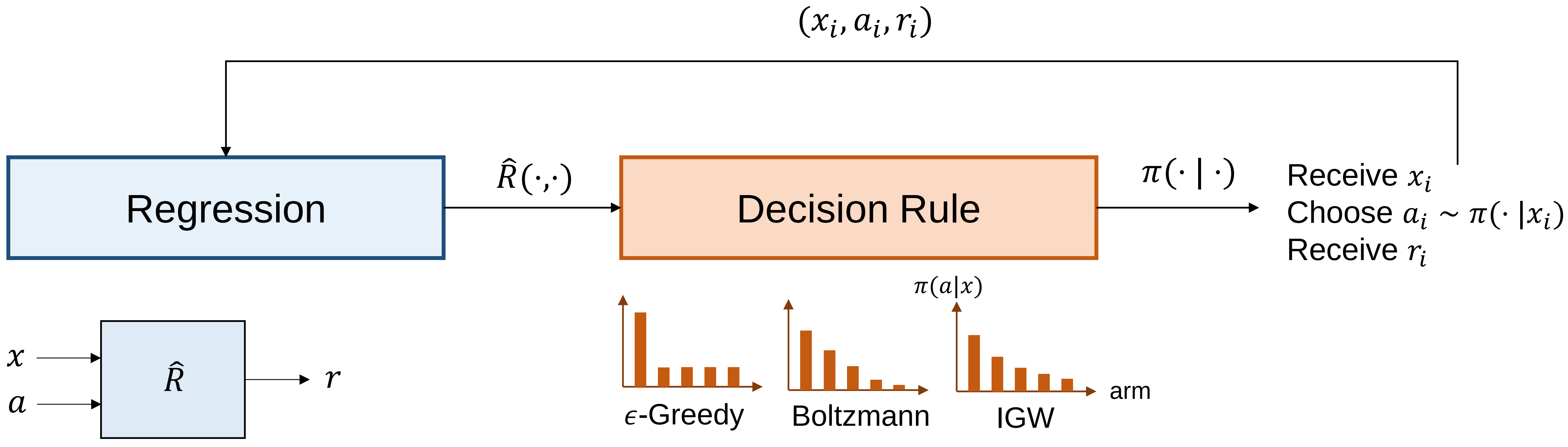
↑ parameter

Find θ_k that minimize

$$\theta_k = \arg\min_{\theta} \sum_{i=1}^{N} \left(Q_\theta(s_i, a_i) - \left(r_i + \gamma \max_{a'} Q_{\theta_{k-1}}(s_i', a') \right) \right)^2$$

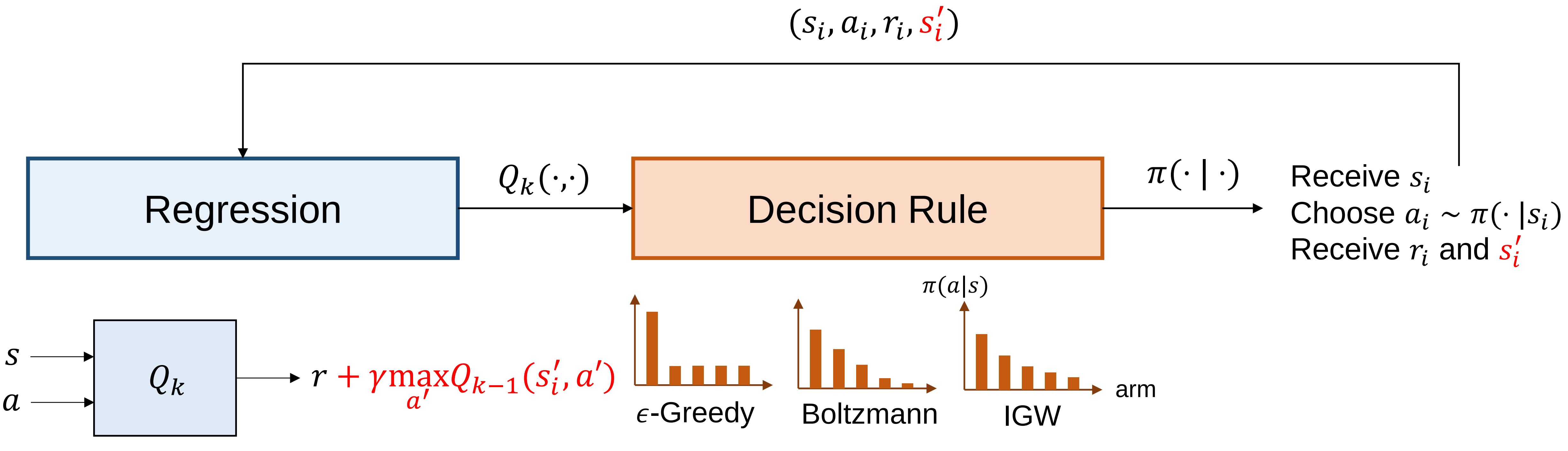
$$\mathbb{E}\left[\sim \mid s_i, a_i \right] = R(s_i, a_i) + \gamma \sum_{s'} P(s' \mid s, a) \max_{a'} Q_{\theta_{k-1}}(s', a')$$

Recall: Contextual Bandits with Regression



Train $\hat{R}$ such that $\hat{R}(x_i, a_i) \approx r_i$

Value Iteration with Regression



Train Q_k such that $Q_k(s_i, a_i) \approx r_i + \gamma \max_{a'} Q_{k-1}(s_i', a')$

This is just one iteration of Value Iteration

Value Iteration with Samples

For $k = 1,\ 2, \ldots$

For $i = 1, 2, \ldots, N$:

Choose action $a_i \sim \text{EG}(Q_{\theta_k}(s_i, \cdot))$

Receive reward $r_i \sim R(s_i, a_i)$ and $s_i' \sim P(\cdot\,|s_i, a_i)$

$s_{i+1} = s_i'$ if episode continues, $s_{i+1} \sim \rho$ if episode ends

$\theta \leftarrow \theta_k$

For $m = 1, 2, \ldots, M$:

Randomly pick an i (or a mini-batch) from $\{1, 2, \ldots, N\}$

$$\theta \leftarrow \theta - \alpha \nabla_\theta \left(Q_\theta(s_i, a_i) - r_i - \gamma \max_{a'} Q_{\theta_k}(s_i', a') \right)^2$$

$\theta_{k+1} \leftarrow \theta$

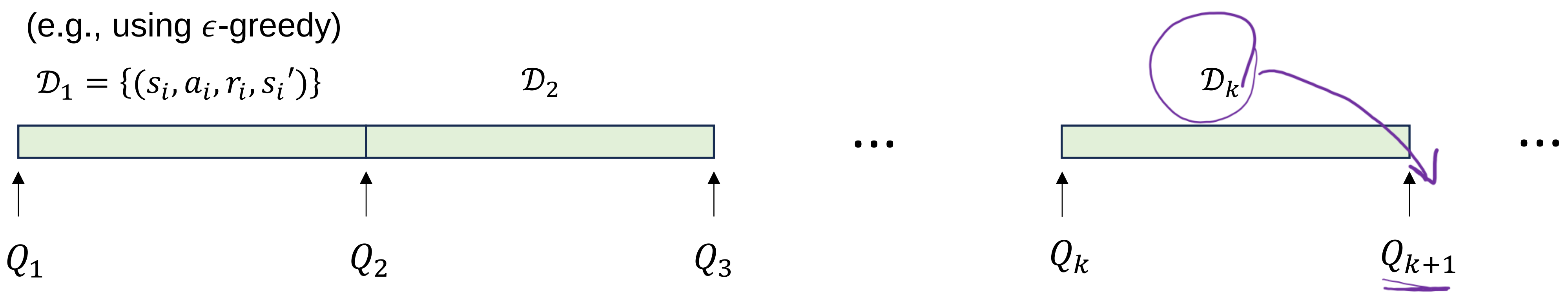
↑ Target network

Data collection

Perform one iteration of Value Iteration

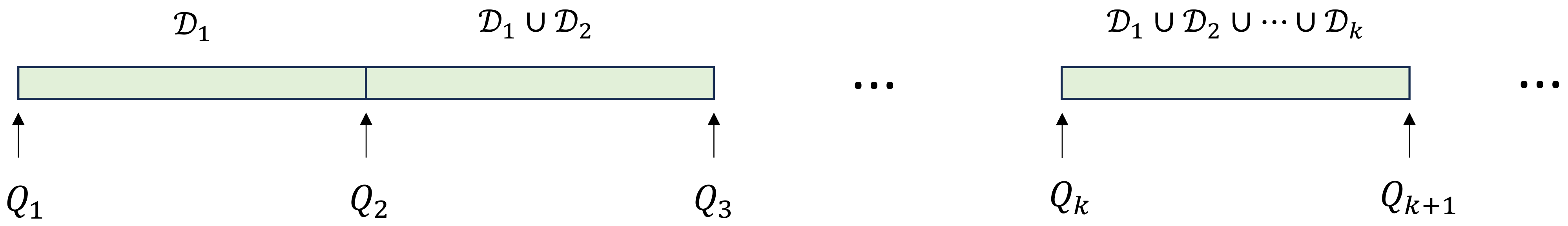
2nd for-loop: trying to find $\theta_{k+1} = \operatorname*{argmin}_\theta \sum_{i=1}^N \left(Q_\theta(s_i, a_i) - r_i - \gamma \max_{a'} Q_{\theta_k}(s_i', a') \right)^2$

It is Valid to Reuse Samples



The algorithm in the previous slide only use $\mathcal{D}_k$ to train θ_{k+1}.

However, as the reward function R and transition P remains unchanged, it is valid (actually, even better) to reuse samples:



Value Iteration with Reused Samples (= Deep Q-Learning or DQN)

HW4 task

Initialize $\mathcal{B} = \{\}$ ← Replay buffer

For $k = 1, 2, \ldots$

For $i = 1, 2, \ldots, N$:

Choose action $a_i \sim \text{EG}(Q_{\theta_k}(s_i, \cdot))$

Receive reward $r_i \sim R(s_i, a_i)$ and $s_i' \sim P(\cdot \mid s_i, a_i)$

$s_{i+1} = s_i'$ if episode continues, $s_{i+1} \sim \rho$ if episode ends

Insert (s_i, a_i, r_i, s_i') to $\mathcal{B}$

Data collection

$\theta \leftarrow \theta_k$

For $m = 1, 2, \ldots, M$:

Randomly pick an i (or a mini-batch) from $\mathcal{B}$

$$\theta \leftarrow \theta - \alpha \nabla_\theta \left(Q_\theta(s_i, a_i) - r_i - \gamma \max_{a'} Q_{\theta_k}(s_i', a') \right)^2$$

↑ Target network

$\theta_{k+1} \leftarrow \theta$

Perform one iteration of Value Iteration

Another Popular Implementation

HW4 task

Initialize $\mathcal{B} = \{\}$ $\leftarrow$ Replay buffer

For $k = 1,\ 2, \ldots$

For $i = 1, 2, \ldots, N$:

Choose action $a_i \sim \text{EG}(Q_\theta(s_i, \cdot))$

Receive reward $r_i \sim R(s_i, a_i)$ and $s_i' \sim P(\cdot\,|s_i, a_i)$

$s_{i+1} = s_i'$ if episode continues, $s_{i+1} \sim \rho$ if episode ends

Insert (s_i, a_i, r_i, s_i') to $\mathcal{B}$

For $m = 1, 2, \ldots, M$:

Randomly pick an i (or a mini-batch) from $\mathcal{B}$

$$\theta \leftarrow \theta - \nabla_\theta \left(Q_\theta(s_i, a_i) - r_i - \gamma \max_{a'} Q_{\overline{\theta}}(s_i', a')\right)^2$$

$\uparrow$ Target network

$$\overline{\theta} \leftarrow (1 - \tau)\overline{\theta} + \tau\theta$$

When Does DQN Succeed?

DQN tries to approximate **Value Iteration** by solving

$$\theta_{k+1} = \operatorname*{argmin}_{\theta} \sum_{i \in \mathcal{B}} \left(Q_\theta(s_i, a_i) - r_i - \gamma \max_{a'} Q_{\theta_k}(s_i', a') \right)^2 \qquad (1)$$

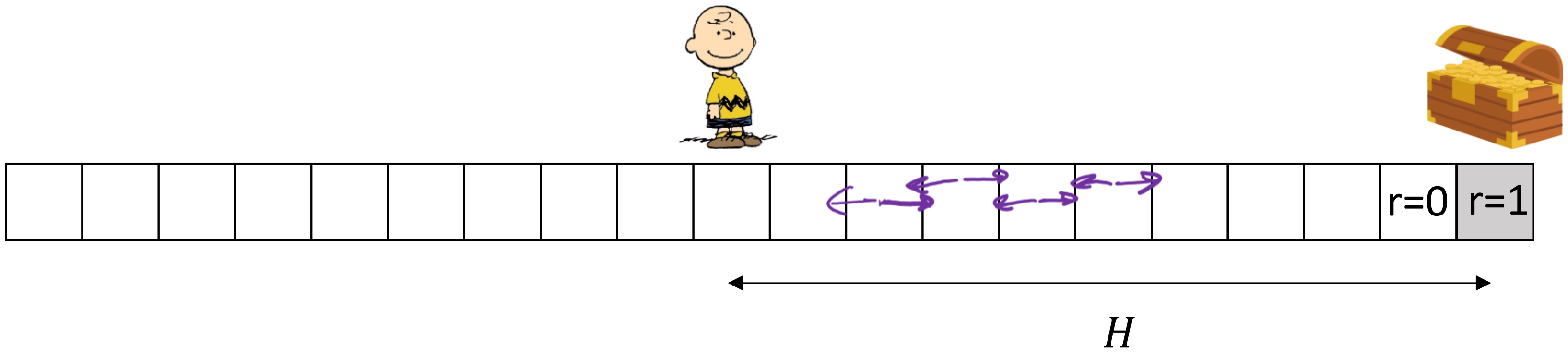
The true Value Iteration:

$$\forall \boldsymbol{s}, \boldsymbol{a}, \qquad Q_{k+1}(s, a) = R(s, a) + \gamma \sum_{s'} P(s' | s, a) \max_{a'} Q_k(s', a') \qquad (2)$$

Under what conditions can (1) well approximate (2)?

- $\mathcal{B}$ should contain a wide range of state-action pairs (a challenge of **exploration**)
- $Q_{\theta_{k+1}}(s, a)$ should recover $R(s, a) + \gamma \sum_{s'} P(s' | s, a) \max_{a'} Q_{\theta_k}(s', a')$ well for all state-actions (a challenge of **function approximation,** or **generalization**)

1. Exploration in MDPs (Not Easy)

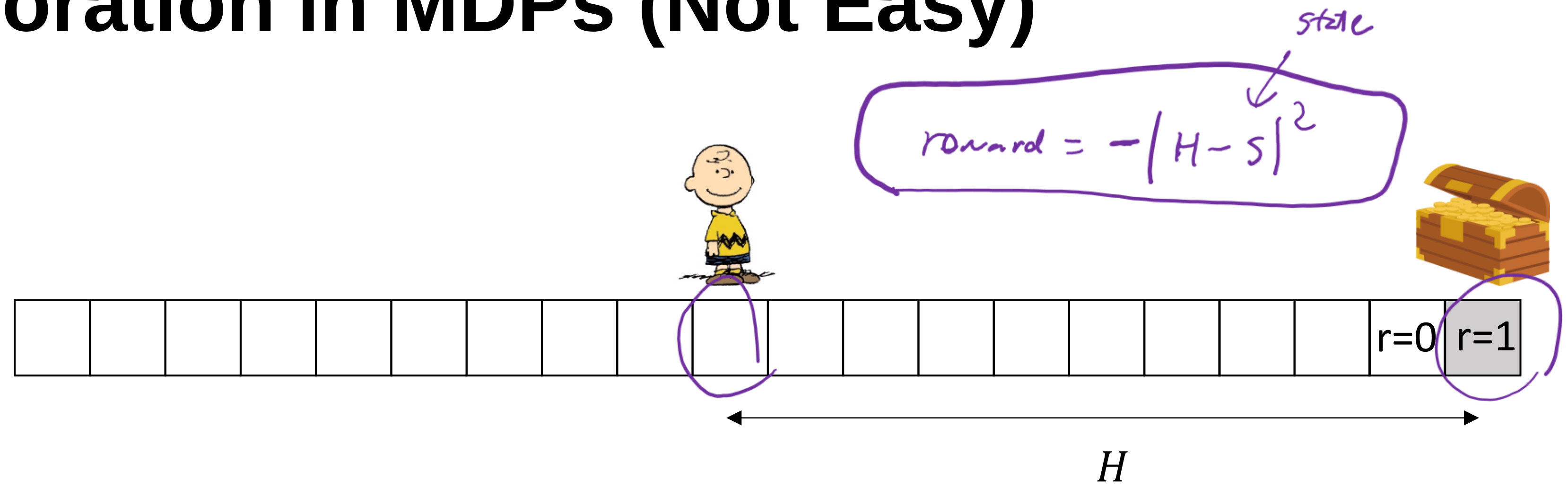


Environment:

- Fixed-horizon MDP with episode length H
- Initial state at 0
- A single rewarding state at state H
- Actions: Go LEFT or RIGHT

Suppose we perform DQN with ϵ-greedy with random initialization
⇒ On average, we need 2^H episodes to see the reward
(before that, we won't make any meaningful update and will just do random walk around state 0)

1. Exploration in MDPs (Not Easy)



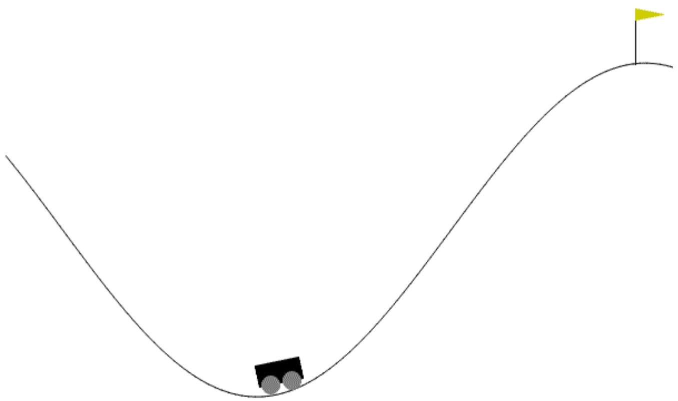
Key issue:

- The ϵ-greedy strategy (or BE, IGW) performs **action-space** exploration but not **state-space** exploration.
- This problem becomes more severe when the reward signal is **sparse**.
- To solve this, we usually require the exploration bonus (a form of reward shaping) technique – will be covered much later.

At this point (for the discussion of DQN), we pretend that EG, BE, or IGW will lead to sufficient exploration over the state space.

1. Exploration in MDPs (Not Easy)

Classic sparse-reward environments:



Mountain Car

Montezuma's Revenge

2. Function Approximation

To make DQN well approximate VI, we need

$$\forall s, a \qquad Q_{\theta_{k+1}}(s,a) \approx R(s,a) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q_{\theta_k}(s',a')$$

(ϵ-approximate) Bellman Completeness
an assumption both on the MDP and the function expressiveness

$$\forall \theta', \exists \theta \qquad \forall s, a, \qquad \left| Q_{\theta}(s,a) - \left(R(s,a) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q_{\theta'}(s',a') \right) \right| \leq \epsilon$$

This allows us to quantify the regression error in each iteration.

2. Function Approximation

In HW1 you have shown

ϵ-Greedy ensures

$$\text{Regret} \lesssim \epsilon T + \sqrt{\frac{AT \cdot \text{Err}}{\epsilon}}$$

Regression error

$$\text{Err} = \sum_{t=1}^{T} \left(\hat{R}_t(x_t, a_t) - R(x_t, a_t) \right)^2$$

In value-based contextual bandits, the requirement / assumption for function approximation is

$$\exists \theta \quad \forall x, a \qquad R_\theta(x, a) \approx R(x, a)$$

In value-based MDPs, the requirement / assumption for function approximation is

$$\forall \theta', \exists \theta \quad \forall s, a \qquad Q_\theta(s, a) \approx R(s, a) + \gamma \sum_{s'} P(s'|s, a) \max_{a'} Q_{\theta'}(s', a')$$

BC assumption

Analysis of DQN assuming sufficient exploration and Bellman Completeness

Recall the analysis for the exact Value Iteration:

1. Value Iteration will terminate.

$$|Q_k(s,a) - Q_{k-1}(s,a)| \le \epsilon \quad \forall s,a$$

$$\max_{s,a} |Q_k(s,a) - Q_{k-1}(s,a)| \le \gamma \max_{s,a} |Q_{k-1}(s,a) - Q_{k-2}(s,a)|$$

2. When it terminates, it holds that

$$|Q_k(s,a) - Q^\star(s,a)| \le \frac{\epsilon}{1-\gamma} \quad \forall s,a$$

ValueError $\le \frac{1}{1-\gamma}$ BellmanError

3. When it terminates, it holds that

$$V^\star(s) - V^{\hat{\pi}}(s) \le \frac{2\epsilon}{(1-\gamma)^2} \quad \forall s$$

where $\hat{\pi}(s) = \operatorname{argmax}_a Q_k(s,a)$

Suboptimality $\le \frac{1}{1-\gamma}$ ValueError

Completing the Analysis of VI (1st Step)

Analysis of DQN