Logic

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Wumpus World

Performance

Gold +1000, death -1000, -1 per step, -10 for using the arrow

Environment

Perceive stench if adjacent to wumpus

Perceive breeze if adjacent to pit

Perceive glitter if in the square of gold

Can grab gold if in the square of gold

Can shoot and kill wumpus if you're facing it (shooting uses up the only arrow)

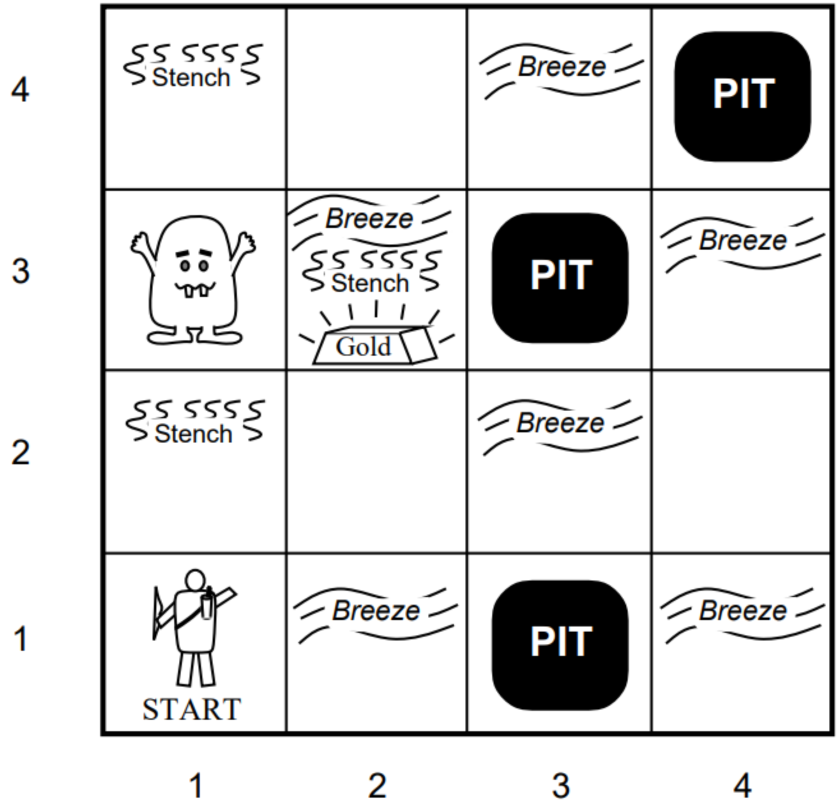
Die if entering a square with pit or living wumpus

Actions

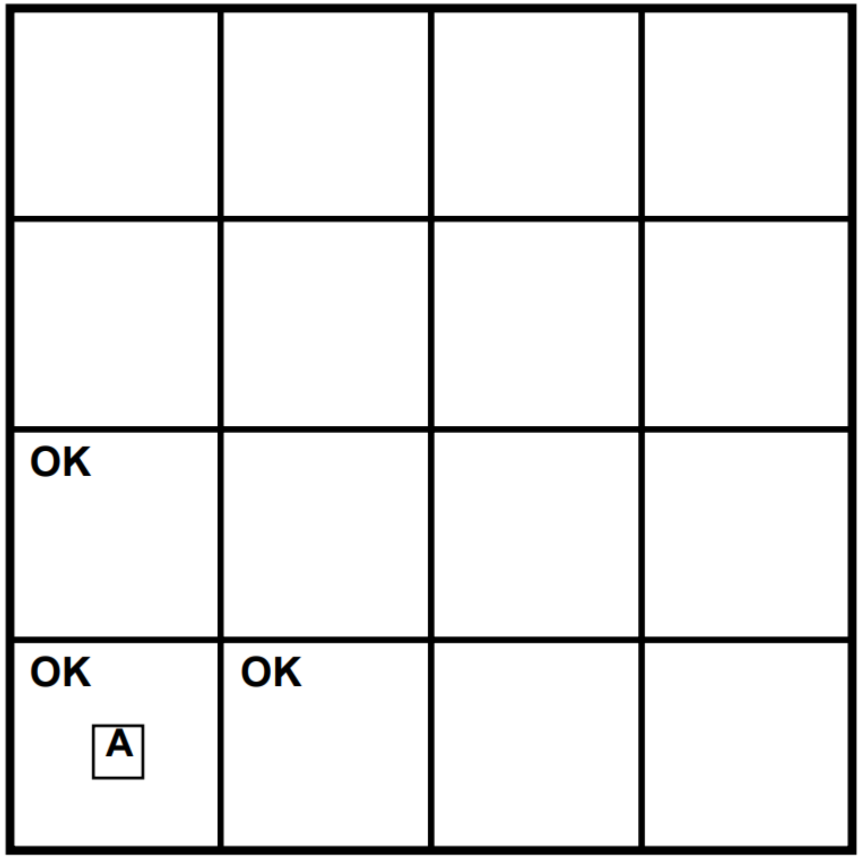
Left turn, right turn, forward, grab, shoot

Sensors

Breeze, glitter, smell



Exploring a wumpus world



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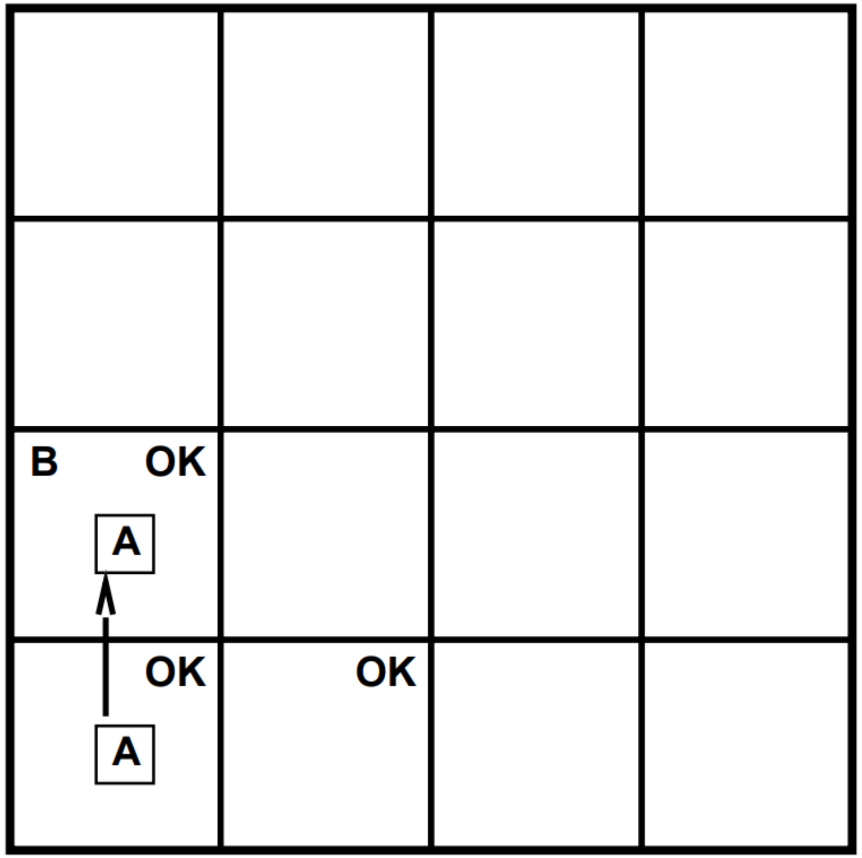
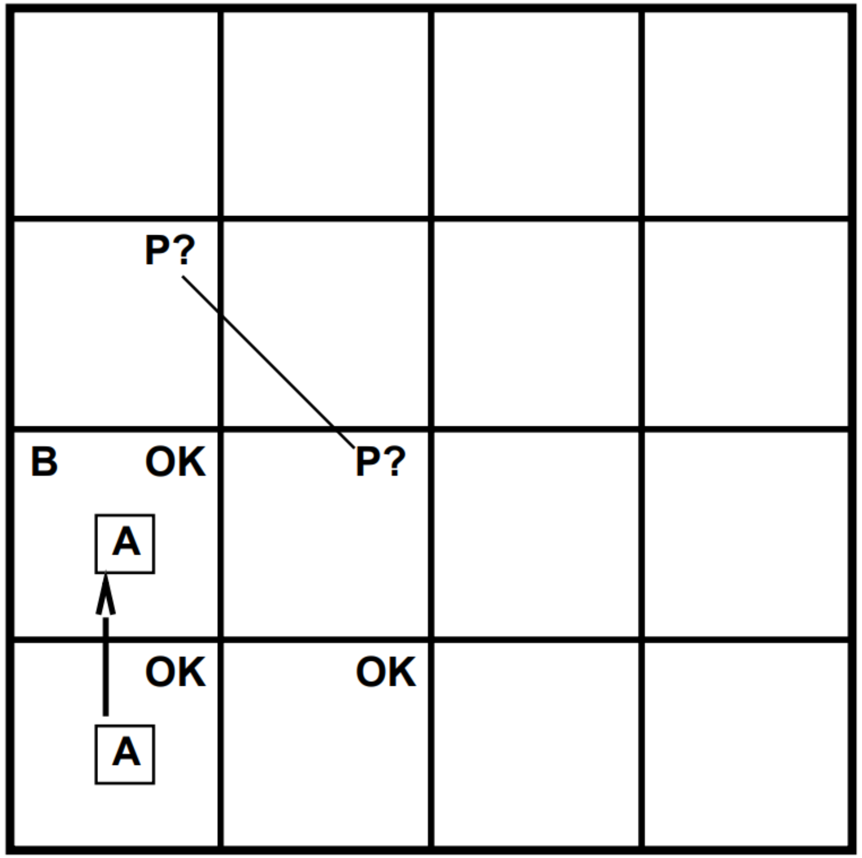


Exploring a wumpus world



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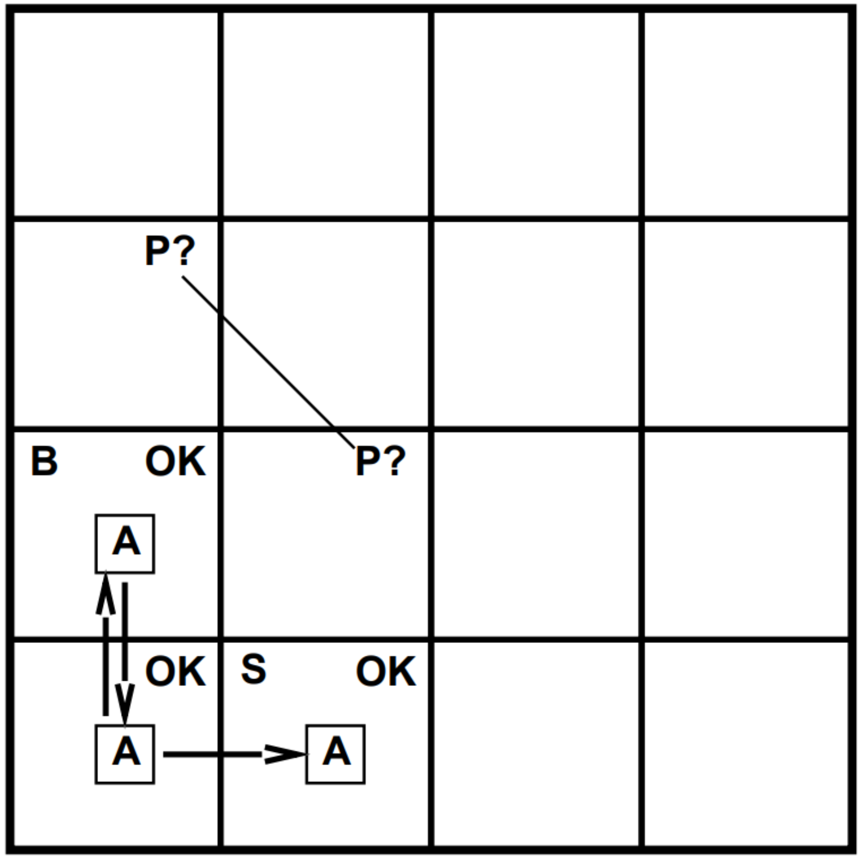

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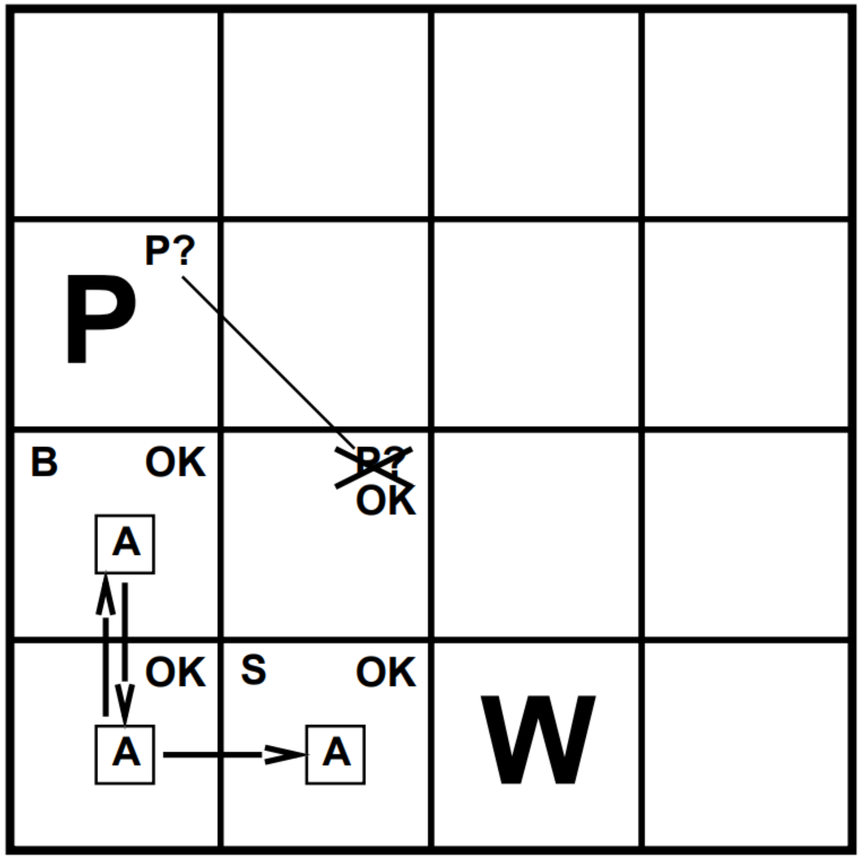

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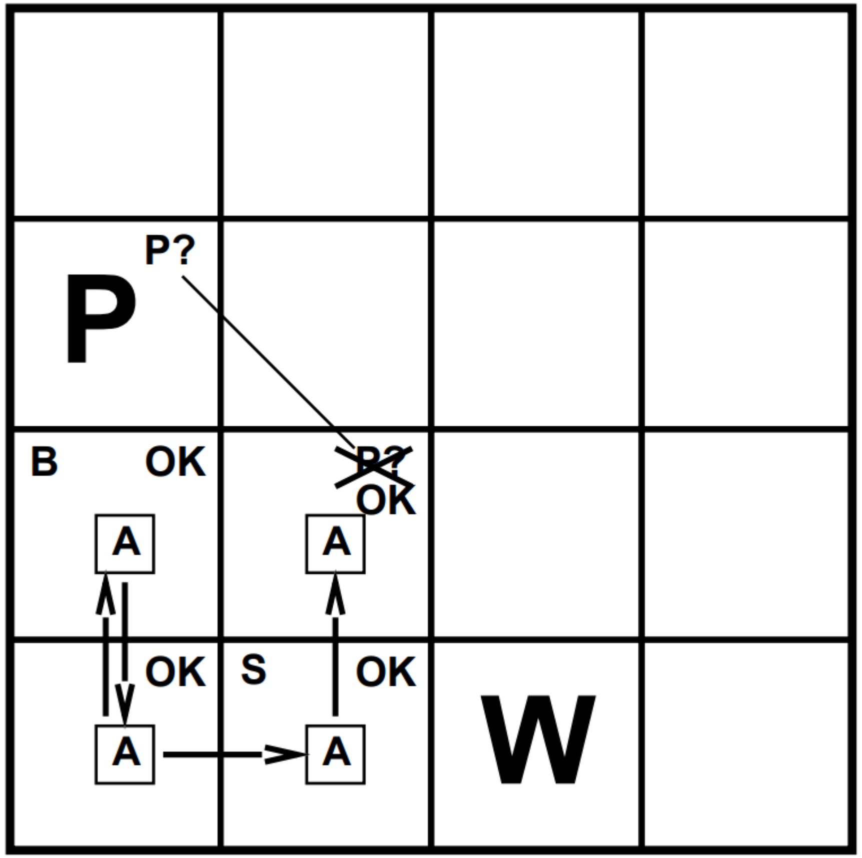
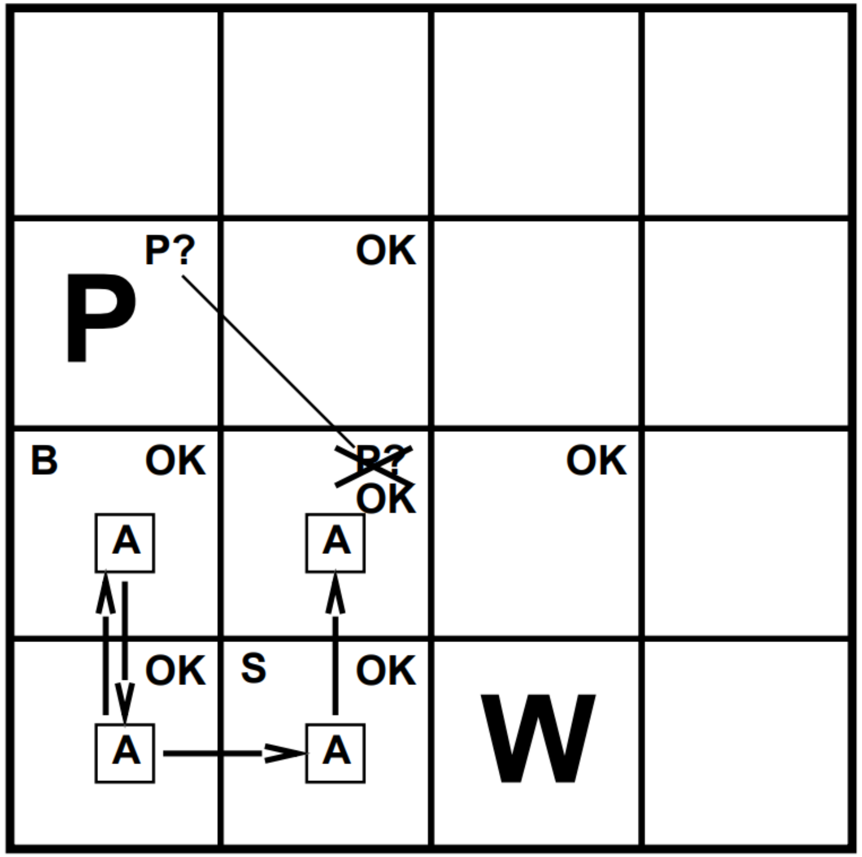
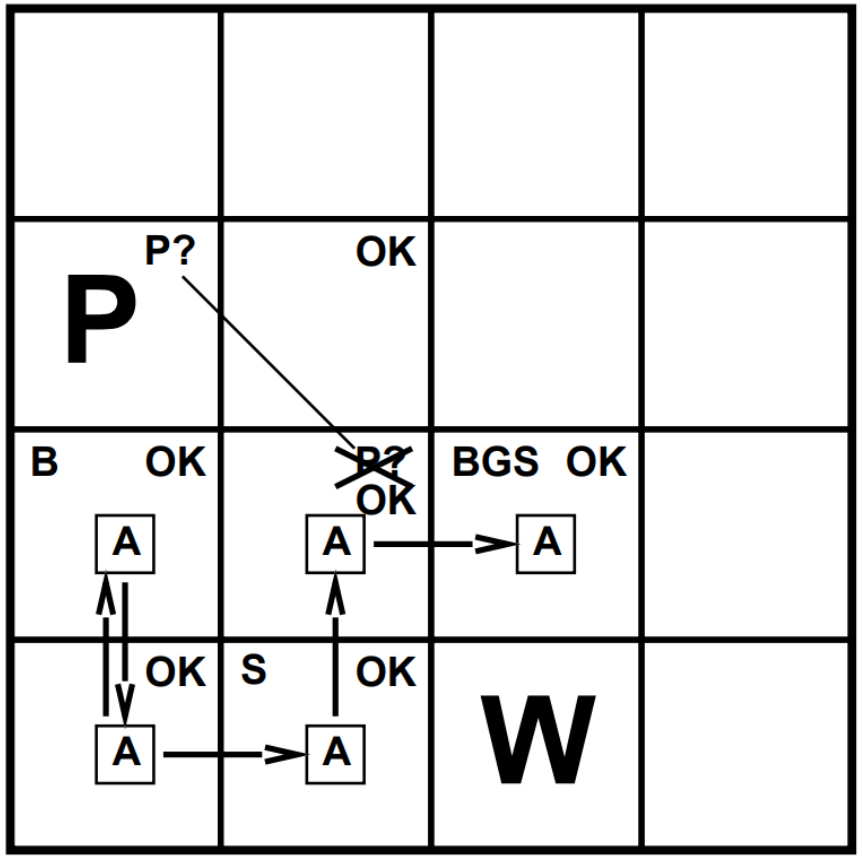


Exploring a wumpus world



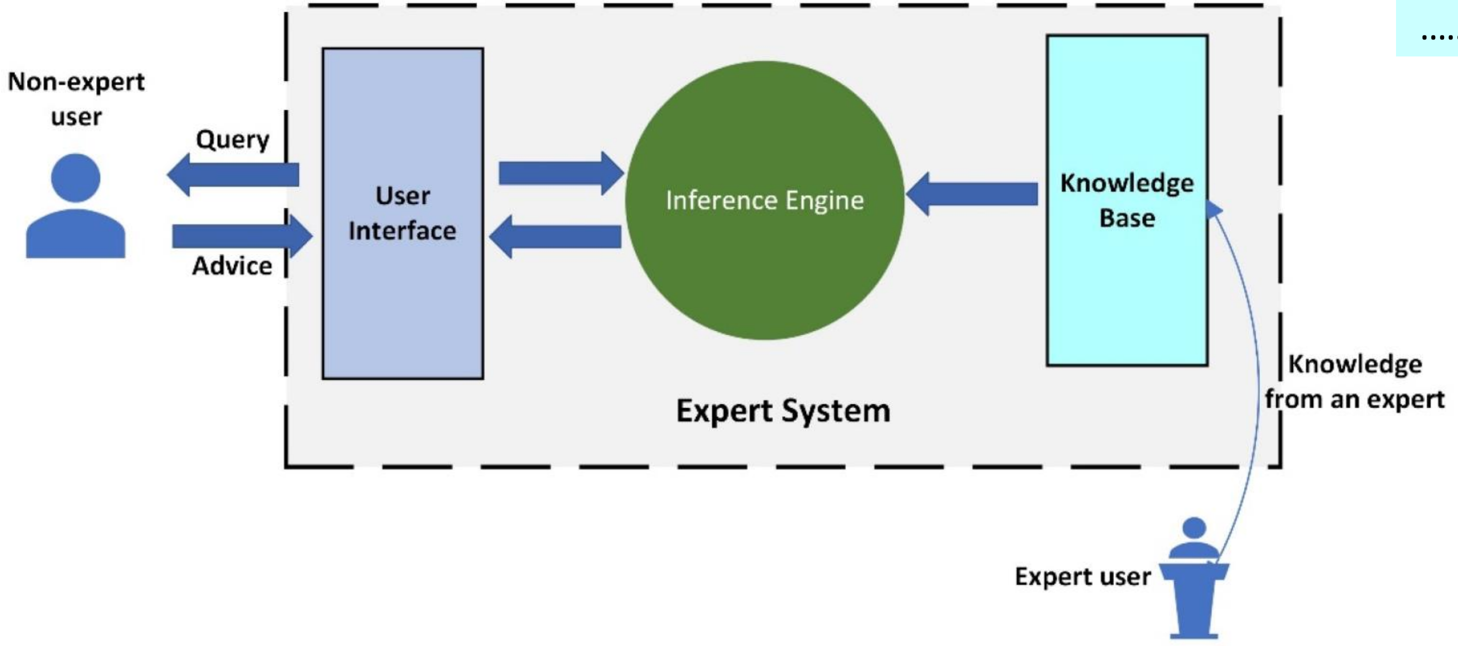
Exploring a wumpus world



Systems with Logical Reasoning

- Knowledge base
  - Consists of some prior knowledge
- Inference engine
  - Derive new knowledge or make some claims
- User Interaction
  - **Tell** information
  - **Ask** question

Example: Expert System



Knowledge base

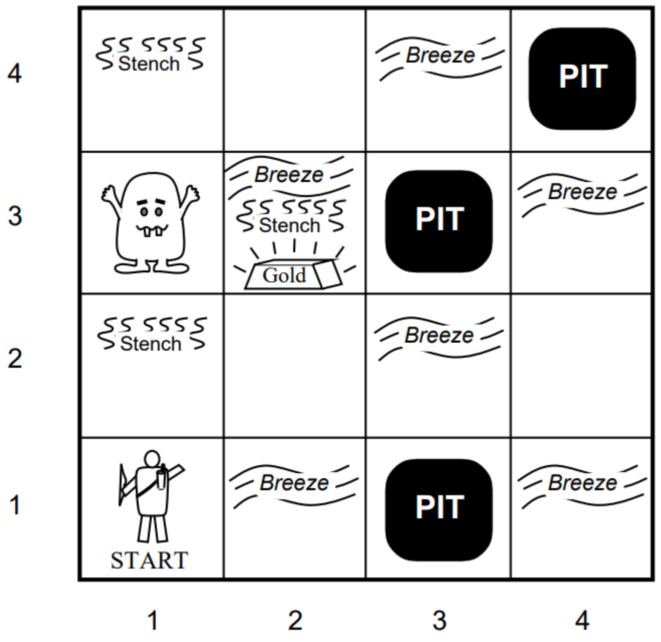
If **`has_hair`**, then **`mammal`**.
If **`mammal`** and **`has_hooves`**, then **`ungulate`**.
If **`has_feathers`**, then **`bird`**.
If **`mammal`** and **`carnivore`** and **`has_dark_spots`**, then **`cheetah`**.
If **`mammal`** and **`carnivore`** and **`has_black_stripes`**, then **`tiger`**.
If **`bird`** and **`does_not_fly`** and **`has_long_neck`**, then **`ostrich`**.
……

User interaction

File Edit Settings Run Debug Help

```
Welcome to SWI-Prolog (threaded, 64 bits, version 9.2.6)
SWI-Prolog comes with ABSOLUTELY NO WARRANTY. This is free software.
Please run ?- license. for legal details.

For online help and background, visit https://www.swi-prolog.org
For built-in help, use ?- help(Topic). or ?- apropos(Word).

?- go.
Does the animal have hair? yes.

Does the animal eat meat? |: no.

Does the animal have pointed teeth? |: no.

Does the animal have hooves? |: yes.

Does the animal have a long neck? |: yes.

Does the animal have long legs? |: yes.

I guess that the animal is: giraffe
true.

?-
```

Example: wumpus world



Knowledge base

Perceive stench if adjacent to wumpus

Perceive breeze if adjacent to pit

Perceive glitter if in the square of gold

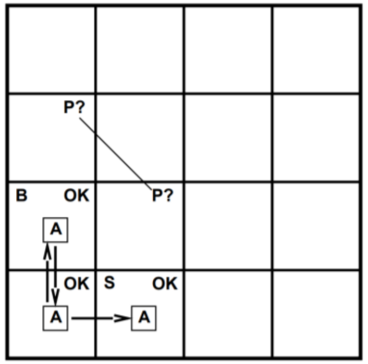
…

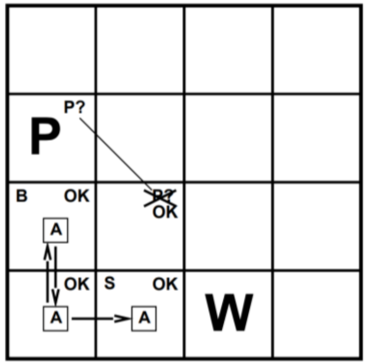
User interaction

Tell the logic system whether stench, breeze, glitter is perceived

Ask for the next action

Inference Engine





Ingredients of Propositional Logic

Sentence

Knowledge base consists of “sentences”

Inference algorithm derives new “sentences” and add them to the knowledge base

Example:

KB = { “Rain→Wet”, “Rain” }

Inference algorithm derives a new sentence “Wet” based on KB

Now KB becomes

KB = {“Rain→Wet”, “Rain”, “Wet” }

Ingredients of Logic – Syntax

Define what are valid sentences.

E.g., syntax in **python**:

“ for x in range(10): ” Valid

“ for x range(10): ” Invalid (the python interpreter cannot understand)

E.g. syntax in **math**:

“ x + y = 5” Valid

“ x 5 = y + ” Invalid

Ingredients of Logic – Syntax

Syntax in **propositional logic**:

- A proposition symbols X is a sentence
  (a propositional symbol is a Boolean variable)
- If α is a sentence then $\neg\alpha$ is a sentence
- If α and β are sentences then $\alpha \wedge \beta$ is a sentence
- If α and β are sentences then $\alpha \vee \beta$ is a sentence
- If α and β are sentences then $\alpha \Rightarrow \beta$ is a sentence
- If α and β are sentences then $\alpha \Leftrightarrow \beta$ is a sentence

The $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ symbols have no meaning here. Their meanings are specified by the "semantics" of logic (discussed next).

Ingredients of Logic – Semantics

Let’s first define “models”. A model is a configuration of the world.

In propositional logic, a model is an **assignment of truth values** to propositional symbols.

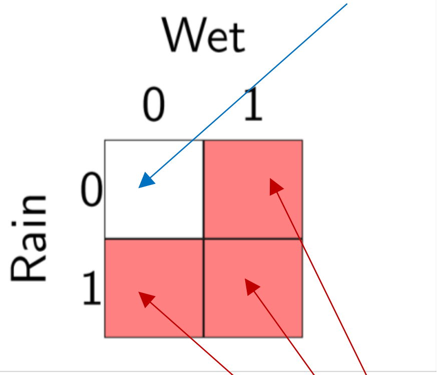
E.g., There are four possible models in the raining example:

		Wet	
		0	1
Rain	0		
	1		

Ingredients of Logic – Semantics

$f = \text{Rain} \lor \text{Wet}$

models where the sentence f is false



P	Q	(P v Q)
T	T	T
T	F	T
F	T	T
F	F	F

models where the sentence f is true

Ingredients of Logic – Semantics

P	~P
T	F
F	T

P	Q	(P ^ Q)
T	T	T
T	F	F
F	T	F
F	F	F

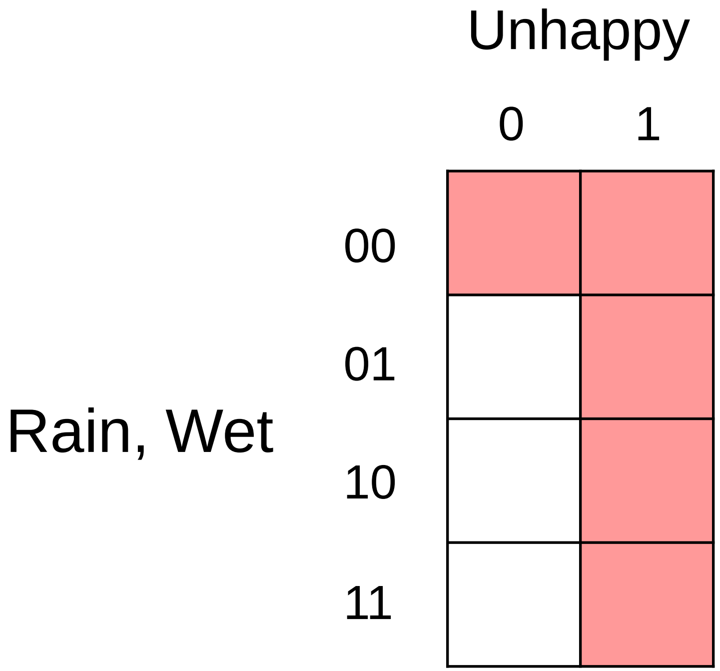
P	Q	(P v Q)
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	(P =>Q)
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	(P ⇔Q)
T	T	T
T	F	F
F	T	F
F	F	T

Ingredients of Logic – Semantics

f: (Rain ∨ Wet) ⇒ Unhappy

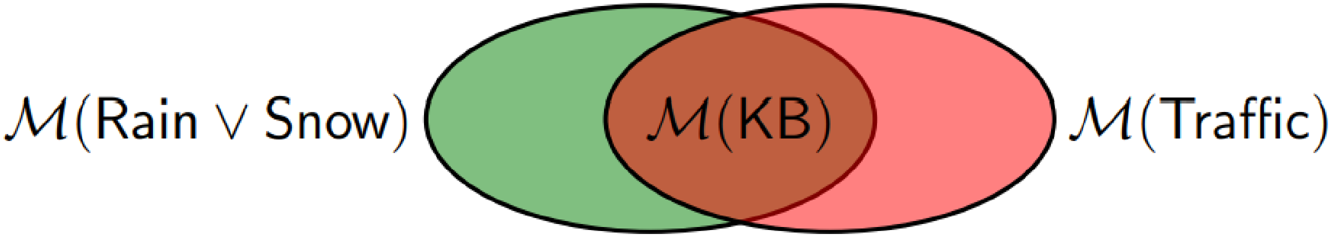


$\mathcal{M}(f)$: the set of models where sentence f is true.

Ingredients of Logic – Knowledge Base

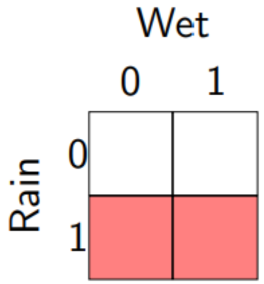
Knowledge base = a collection of sentences

Let $\mathsf{KB} = \{\mathsf{Rain} \vee \mathsf{Snow}, \mathsf{Traffic}\}$.

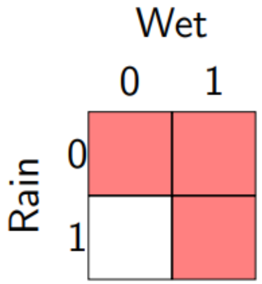


Ingredients of Logic – Knowledge Base

$\mathcal{M}(\text{Rain})$

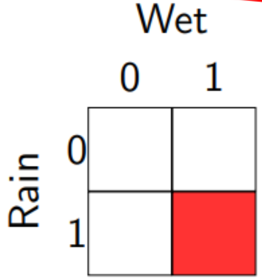


$\mathcal{M}(\text{Rain} \to \text{Wet})$



KB

$\mathcal{M}(\{\text{Rain}, \text{Rain} \to \text{Wet}\})$



$\alpha = \text{Wet}$

Adding more formulas to the knowledge base:

$$\text{KB} \longrightarrow \text{KB} \cup \{f\}$$

Shrinks the set of models:

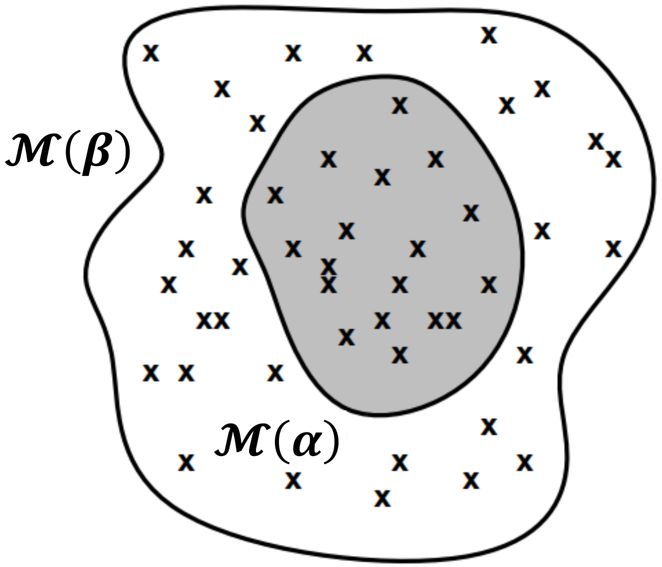
$$\mathcal{M}(\text{KB}) \longrightarrow \mathcal{M}(\text{KB}) \cap \mathcal{M}(f)$$

Recap: Propositional Logic

- **Sentence:** propositional symbols, or their negations ($\neg$), or their combinations through $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.
- **Models:** An assignment of truth values to propositional symbols.
- **Knowledge base:** a set of sentences
- $\mathcal{M}(f)$: the set of models where sentence f is true.

Entailment

- Sentence α **entails** sentence β means that (in high level) sentence β follows logically from sentence α
- Denoted as $\alpha \vDash \beta$
- $\alpha \vDash \beta$ if and only if $\mathcal{M}(\alpha) \subset \mathcal{M}(\beta)$
- **Example:** Rain ∧ Snow ⊨ Snow



Inference Algorithms

- Given KB and α, the algorithm tries to derive sentence α.
- If an algorithm $\mathcal{A}$ is able to derive α from KB, we write KB $\vdash_{\mathcal{A}} \alpha$
  - This is different from KB $\vDash \alpha$,
- Soundness (correctness)
  - The algorithm can only derive α when α is entailed by KB.
  - In other words: If KB $\vdash_{\mathcal{A}} \alpha$, then KB $\vDash \alpha$
- Completeness
  - For any α that KB entails, the algorithm is able to derive α.
  - If other words: If KB $\vDash \alpha$, then if KB $\vdash_{\mathcal{A}} \alpha$

A (Simple) Inference Algorithm: Model Checking

```
function TT-ENTAILS?(KB, α) returns true or false
  inputs: KB, the knowledge base, a sentence in propositional logic
          α, the query, a sentence in propositional logic

  symbols ← a list of the proposition symbols in KB and α
  return TT-CHECK-ALL(KB, α, symbols, { })


function TT-CHECK-ALL(KB, α, symbols, model) returns true or false
  if EMPTY?(symbols) then
     if PL-TRUE?(KB, model) then return PL-TRUE?(α, model)
     else return true        //  when KB is false, always return true
  else
     P ← FIRST(symbols)
     rest ← REST(symbols)
     return (TT-CHECK-ALL(KB, α, rest, model ∪ {P = true})
             and
             TT-CHECK-ALL(KB, α, rest, model ∪ {P = false }))
```

A (Simple) Inference Algorithm: Model Checking

Model Checking (KB, α):

Let $\mathcal{M}$ be the set of all possible models
($|\mathcal{M}| = 2^N$ if there are N propositional symbols in KB ∪ $\{\alpha\}$)

For $m \in \mathcal{M}$:

If KB is True in m and α is False in m : **return** False

return True

Theorem Proving

Idea: Instead of checking all models, will just perform manipulations on the sentence level.

Inference Rules

$\alpha \to \beta \equiv \neg \alpha \vee \beta$

$\neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$

$\equiv (\neg \alpha_1) \vee (\quad) \cdots \vee (\neg \alpha_k)$

- Modus Ponens (Latin for *mode the affirms*)

$$\frac{\alpha_1, \alpha_2, \dots, \alpha_k, \quad (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k) \Rightarrow \beta}{\beta}$$

← premises

← conclusion

or $$\frac{\alpha_1, \alpha_2, \dots, \alpha_k, \quad (\neg\alpha_1 \vee \neg\alpha_2 \vee \cdots \vee \neg\alpha_k \vee \beta)}{\beta}$$

- And Eliminations

$$\frac{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k}{\alpha_i}$$

Standard Logical Equivalence

(can be applied in any steps in the inference algorithm)

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) \quad \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) \quad \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) \quad \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) \quad \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) \quad \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) \quad \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \quad \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

Inference Rules

Example: KB = {Rain ⇒ Wet, Wet ⇒ Unhappy, Rain}, α = Unhappy.

Applying Modus Ponens on KB
(i.e., try to **match** sentences in KB with premises α and β)

$$\frac{\text{Rain, Rain} \Rightarrow \text{Wet}}{\text{Wet}}$$

Modus Ponens:

$$\frac{\alpha_1, \dots, \alpha_k, \quad (\alpha_1 \wedge \cdots \wedge \alpha_k) \Rightarrow \beta}{\beta}$$

KB = {Rain ⇒ Wet, Wet ⇒ Unhappy, Rain, Wet}
Applying Modus Ponens on KB

$$\frac{\text{Wet}, \quad \text{Wet} \Rightarrow \text{Unhappy}}{\text{Unhappy}}$$

Forward Inference

Input: KB, α, $\mathfrak{I} =$ a set of inference rule

If $\alpha \in$ KB: **return** True

Repeat:

Choose a set of sentences $\alpha_1, \dots, \alpha_k \in$ KB such that

$$\frac{\alpha_1, \alpha_2, \dots, \alpha_k}{\beta}$$

matches a rule in $\mathfrak{I}$, and $\beta \notin$ KB.

If $\beta = \alpha$: **return** True

If such $(\alpha_1, \alpha_2, \dots, \alpha_k, \beta)$ does not exist: **return** False

Add β to KB.

Forward Inference

- Forward inference is a search problem
  - What are the states, actions, successor function, and goal test?
  - Algorithms introduced for search problems can be applied here.

- Is the forward inference algorithm sound?
  - Yes, as long as all inference rules you use are sound

- Is forward inference complete?

Forward Inference

Example:

KB = {Rain ⇒ Wet,   Rain v Shine,   Wet v Shine ⇒ Happy}

α = Happy

Use Forward Inference algorithm with $\mathfrak{I} =$ {Modus Ponens}

- Can KB entail α?
- Can the algorithm derive α from KB?

$$\frac{P,\quad P \Rightarrow \zeta}{\zeta}$$

$$\frac{P,\quad \neg P \lor \zeta}{\zeta}$$

Forward Inference with Modus Ponens is **sound** but **not complete**

A Sound and Complete Algorithm?

Fact 1. If KB only consists of **Horn clauses,**

then Forward Inference with **Modus Ponens** is sound and complete.

Fact 2. In general, Forward Inference with **Resolution** is sound and complete.

Horn Clauses + Modus Ponens is Complete

Horn clause: sentence that have the following forms

$$X_1 \wedge X_2 \wedge \cdots \wedge X_{k-1} \Rightarrow X_k \quad \text{or} \quad X_1 \wedge X_2 \wedge \cdots \wedge X_k \Rightarrow \text{False}$$

$$\equiv \qquad\qquad\qquad\qquad \equiv$$

$$\neg X_1 \vee \neg X_2 \vee \cdots \vee \neg X_{k-1} \vee X_k \qquad\qquad \neg X_1 \vee \neg X_2 \vee \cdots \vee \neg X_k$$

Disjunction with only one positive symbol
(Definite clause)

Disjunction with no positive symbol
(Goal clause)

Horn Clauses + Modus Ponens is Complete

KB

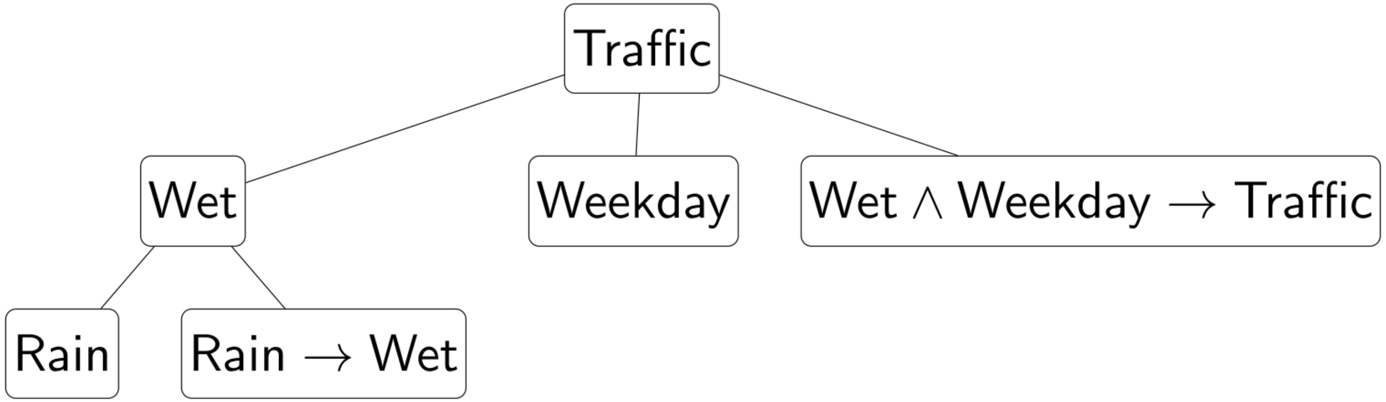
Rain

Weekday

Rain $\rightarrow$ Wet

Wet $\wedge$ Weekday $\rightarrow$ Traffic

Traffic $\wedge$ Careless $\rightarrow$ Accident



Intuition: The inference procedure of horn clauses is *direct*, in the sense that there is no branching.

Horn clause: Rain ∧ Snow → Dark ∧ Traffic

$R \wedge S \rightarrow D$

$R \wedge S \rightarrow T$

Non- horn clause: Wet → $\underbrace{\text{Rain} \vee \text{Snow}}$

Has to branch into the cases ¬Rain, ¬Snow etc.

A pseudocode for Forward Inference with Modus Ponens (this algorithm is also called **Forward Chaining**). This pseudocode assumes that all sentences are definite clauses (but it's easy to extend it to handle goal clauses as well).

The time complexity is linear in the "**size of KB**", i.e., the sum of the lengths of all sentences in KB.

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  inputs: KB, the knowledge base, a set of propositional definite clauses
          q, the query, a proposition symbol
  count ← a table, where count[c] is the number of symbols in c's premise
  inferred ← a table, where inferred[s] is initially false for all symbols
  agenda ← a queue of symbols, initially symbols known to be true in KB

  while agenda is not empty do
     p ← POP(agenda)
     if p = q then return true
     if inferred[p] = false then
        inferred[p] ← true
        for each clause c in KB where p is in c.PREMISE do
           decrement count[c]
           if count[c] = 0 then add c.CONCLUSION to agenda
  return false
```

Figure 7.15 The forward-chaining algorithm for propositional logic. The *agenda* keeps track of symbols known to be true but not yet "processed." The *count* table keeps track of how many premises of each implication are as yet unknown. Whenever a new symbol p from the agenda is processed, the count is reduced by one for each implication in whose premise p appears (easily identified in constant time with appropriate indexing.) If a count reaches zero, all the premises of the implication are known, so its conclusion can be added to the agenda. Finally, we need to keep track of which symbols have been processed; a symbol that is already in the set of inferred symbols need not be added to the agenda again. This avoids redundant work and prevents loops caused by implications such as $P \Rightarrow Q$ and $Q \Rightarrow P$.

General Case: Resolution is Complete

Resolution

$$\frac{\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_k \vee p, \quad \neg p \vee \beta_1 \vee \beta_2 \vee \cdots \vee \beta_m}{\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_k \vee \beta_1 \vee \beta_2 \vee \cdots \vee \beta_m}$$

Example

$$\frac{\text{Rain} \vee \text{Shine}, \quad \neg\text{Rain} \vee \text{Wet}}{\text{Shine} \vee \text{Wet}}$$

Converting Sentences to CNF Before Applying Resolution

Conjunctive Normal Form (CNF)

Example: $(A \lor B \lor \neg C) \land (\neg B \lor D)$

Converting Sentences to CNF: Example

Initial formula:

$$(\text{Summer} \rightarrow \text{Snow}) \rightarrow \text{Bizzare}$$

Remove implication ($\rightarrow$):

$$\neg(\neg\text{Summer} \vee \text{Snow}) \vee \text{Bizzare}$$

Push negation ($\neg$) inwards (de Morgan):

$$(\neg\neg\text{Summer} \wedge \neg\text{Snow}) \vee \text{Bizzare}$$

Remove double negation:

$$(\text{Summer} \wedge \neg\text{Snow}) \vee \text{Bizzare}$$

Distribute $\vee$ over $\wedge$:

$$(\text{Summer} \vee \text{Bizzare}) \wedge (\neg\text{Snow} \vee \text{Bizzare})$$

Converting Sentences to CNF: General Rules

Conversion rules:

- Eliminate $\leftrightarrow$: $\frac{f \leftrightarrow g}{(f \rightarrow g) \wedge (g \rightarrow f)}$
- Eliminate $\rightarrow$: $\frac{f \rightarrow g}{\neg f \vee g}$
- Move $\neg$ inwards: $\frac{\neg(f \wedge g)}{\neg f \vee \neg g}$
- Move $\neg$ inwards: $\frac{\neg(f \vee g)}{\neg f \wedge \neg g}$
- Eliminate double negation: $\frac{\neg\neg f}{f}$
- Distribute $\vee$ over $\wedge$: $\frac{f \vee (g \wedge h)}{(f \vee g) \wedge (f \vee h)}$

Resolution-Based Inference Algorithm

Note that $KB \vDash \alpha$ is equivalent to $\mathcal{M}(KB \land \neg \alpha)$ = empty set

KB' $\leftarrow$ KB U $\{\neg \alpha\}$

Convert all sentences in KB' to CNF

Repeatedly apply Resolution Rule until

1) False is derived $\rightarrow$ return $KB \vDash \alpha$
2) No new sentence can be derived $\rightarrow$ return $KB \nvDash \alpha$

Resolution-Based Inference Algorithm

```
function PL-RESOLUTION(KB, α) returns true or false
    inputs: KB, the knowledge base, a sentence in propositional logic
            α, the query, a sentence in propositional logic

    clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
    new ← { }
    loop do
        for each Ci, Cj in clauses do
            resolvents ← PL-RESOLVE(Ci, Cj)
            if resolvents contains the empty clause then return true
            new ← new ∪ resolvents
        if new ⊆ clauses then return false
        clauses ← clauses ∪ new
```

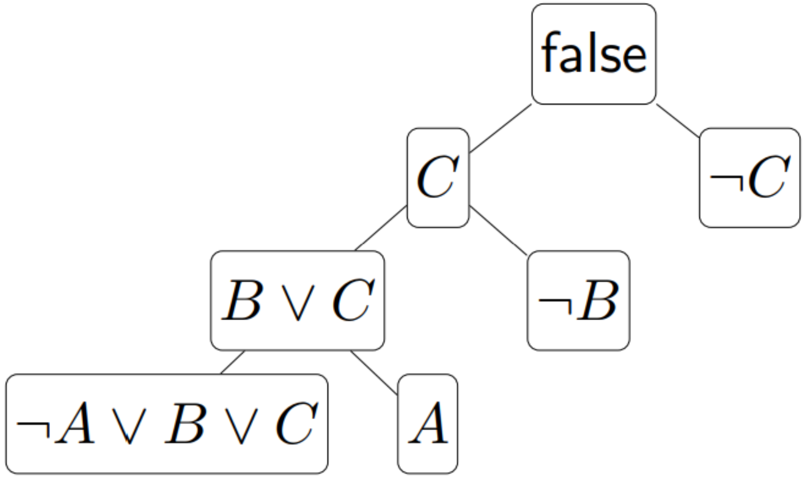
Resolution-Based Inference Algorithm

$$\text{KB}' = \{A \to (B \vee C), A, \neg B, \neg C\}$$

Convert to CNF:

$$\text{KB}' = \{\neg A \vee B \vee C, A, \neg B, \neg C\}$$

Repeatedly apply **resolution** rule:



Conclusion: KB **entails** f

Time Complexity

- Modus Ponens

$$\frac{\alpha_1, \alpha_2, \ldots, \alpha_k, \quad (\alpha_1 \land \alpha_2 \land \cdots \land \alpha_k) \Rightarrow \beta}{\beta}$$

Each rule application adds sentence with **one** propositional symbol
→ **linear time**

- Resolution

$$\frac{\alpha_1 \lor \alpha_2 \lor \cdots \lor \alpha_k \lor p, \quad \neg p \lor \beta_1 \lor \beta_2 \lor \cdots \lor \beta_m}{\alpha_1 \lor \alpha_2 \lor \cdots \lor \alpha_k \lor \beta_1 \lor \beta_2 \lor \cdots \lor \beta_m}$$

Each rule application adds sentence with **many** propositional symbol
→ **exponential time**

Recap

	Modus Ponens	Resolution
Sound?	Yes	Yes
Complete?	No	Yes
Complete for horn clauses?	Yes	Yes
Time complexity	linear	exponential