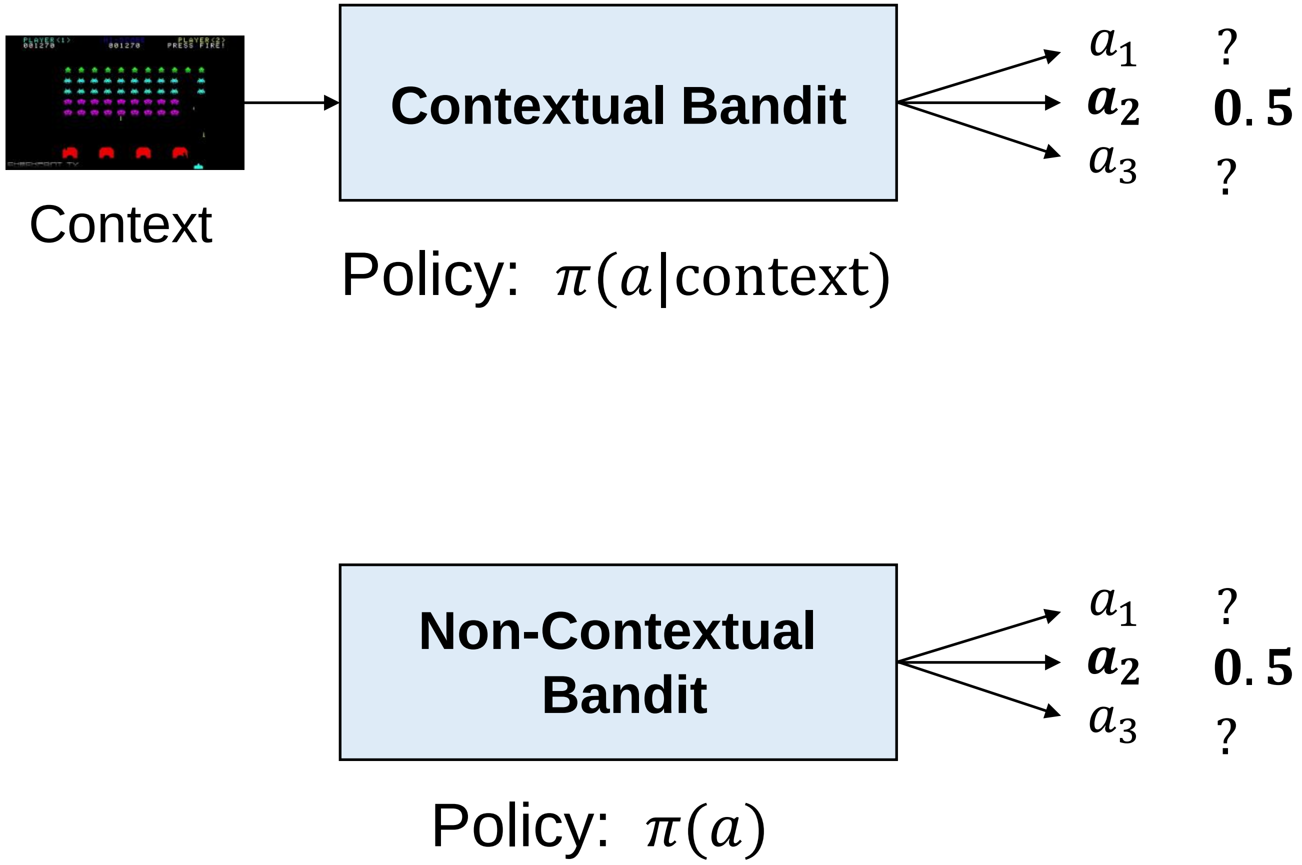
Bandits

Chen-Yu Wei

Contextual Bandits and Non-Contextual Bandits



Multi-Armed Bandits

Multi-Armed Bandits

Given: action set $\mathcal{A} = \{1, \dots, A\}$

For time $t = 1, 2, \dots, T$:

Learner chooses an arm $a_t \in \mathcal{A}$

Learner observes $r_t = R(a_t) + w_t$

Assumption: $R(a)$ is the (hidden) ground-truth reward function

w_t is a zero-mean noise

Goal: maximize the total reward $\sum_{t=1}^{T} R(a_t)$ (or $\sum_{t=1}^{T} r_t$)

How to Evaluate an Algorithm's Performance?

- "My algorithm obtains $0.3T$ total reward within T rounds"
  – Is my algorithm good or bad?
- Benchmarking the problem

$$\text{Regret} := \quad \underbrace{\max_{\pi} \sum_{t=1}^{T} R(\pi)}_{\text{The total reward of the best policy}} - \sum_{t=1}^{T} R(a_t) \quad \underset{\text{In MAB}}{=} \quad \max_{a} T R(a) - \sum_{t=1}^{T} R(a_t)$$

- "My algorithm ensures $\text{Regret} \leq 5T^{\frac{3}{4}}$ "
- $\text{Regret} = o(T) \;\Rightarrow\;$ the algorithm is as good as the optimal policy asymptotically

The Exploration and Exploitation Trade-off in MAB

- To perform as well as the best policy (i.e., best arm) asymptotically, the learner has to pull the best arm most of the time

  ⇒ need to exploit

- To identify the best arm, the learner has to try every arm sufficiently many times

  ⇒ need to explore

A Simple Strategy: Explore-then-Exploit

Explore-then-exploit (Parameter: T_0)

In the first T_0 rounds, sample each arm T_0/A times. **(Explore)**

Compute the **empirical mean** $\hat{R}(a)$ for each arm a

In the remaining $T - T_0$ rounds, draw $\hat{a} = \mathrm{argmax}_a \hat{R}(a)$ **(Exploit)**

What is the *right* amount of exploration (T_0)?

Quantifying the Estimation Error

In the exploration phase, we obtain $N = T_0/A$ i.i.d. samples of each arm.

Key Question:

$$\left| \hat{R}(a) - R(a) \right| \leq ? \;\; f(N)$$

for some decreasing function of N

Empirical mean
of N i.i.d. samples

True mean

Explore-then-Exploit Regret Bound Analysis

Explore-then-Exploit Regret Bound Analysis

Assume $\underbrace{|\hat{R}(a) - R(a)|}$ $\leq f(N) = f\left(\frac{T_0}{A}\right)$

$R(a) \in [0, R_{max}]$

$$\text{Regret} = \sum_{t=1}^{T_0} \left(R(a^*) - R(a_t)\right) + \sum_{t=T_0+1}^{T} \left(R(a^*) - R(a_t)\right)$$

$$\leq T_0 R_{max} + \sum_{t=T_0+1}^{T} \left(\underbrace{R(a^*) - \hat{R}(a^*)}_{\leq f(T_0/A)} + \underbrace{\hat{R}(a_t) - R(a_t)}\right)$$

$a_t = \arg\max \hat{R}(a)$

$$\leq T_0 R_{max} + (T - T_0)\, f(T_0/A)$$

Quantifying the Error: Concentration Inequality

Theorem. Hoeffding's Inequality

Let $X_1, \dots, X_N$ be independent σ**-sub-Gaussian** random variables.

Then with probability at least $1-\delta$,

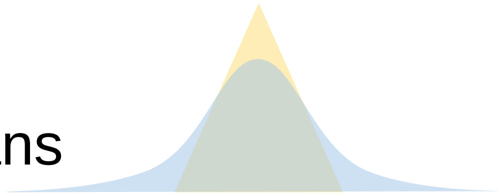
$$\left| \frac{1}{N} \sum_{i=1}^{N} X_i - \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[X_i] \right| \le \sigma \sqrt{\frac{2\log(2/\delta)}{N}} \ .$$

A random variable is called σ-sub-Gaussian if $\mathbb{E}\left[e^{\lambda(X-\mathbb{E}[X])}\right] \le e^{\lambda^2\sigma^2/2} \quad \forall \lambda \in \mathbb{R}$.

Fact 1. $\mathcal{N}(\mu, \sigma)$ is σ-sub-Gaussian.

Fact 2. A random variable $\in [a, b]$ is $(b-a)$-sub-Gaussian.

Intuition: tail probability $\Pr\{|X-\mathbb{E}[X]| \ge z\}$ bounded by that of Gaussians



Regret Bound of Explore-then-Exploit

$a \lesssim b$ means

$a \le \text{const.} \cdot b$

or $a = O(b)$

Theorem. Regret Bound of Explore-then-Exploit

Suppose that $R(a) \in [0,1]$ and w_t is 1-sub-Gaussian.

Then with probability at least $1 - A\delta$, Explore-then-Exploit ensures

$$\text{Regret} \le T_0 + 2(T - T_0)\sqrt{\frac{2A\log(2/\delta)}{T_0}} \; .$$

$$\text{Reg} \lesssim T_0 + 2T\sqrt{\frac{A}{T_0}}$$

$$\lesssim A^{1/3} T^{2/3}$$

By AM-GM: $\dfrac{T_0 + T\sqrt{\frac{A}{T_0}} + T\sqrt{\frac{A}{T_0}}}{3} \ge \sqrt[3]{T_0 \cdot T\sqrt{\frac{A}{T_0}} \cdot T\sqrt{\frac{A}{T_0}}} = A^{1/3} T^{2/3}$

To minimize $T_0 + 2T\sqrt{\frac{A}{T_0}}$, choose $T_0 = T\sqrt{\frac{A}{T_0}} \Rightarrow T_0 = A^{1/3} T^{2/3}$

ϵ-Greedy

Mixing exploration and exploitation in time

ϵ-Greedy (Parameter: ϵ)

In the first A rounds, draw each arm once.

In the remaining rounds $t > A$,

Draw

$$a_t = \begin{cases} \text{uniform}(\mathcal{A}) & \text{with prob. } \epsilon \\ \text{argmax}_a \hat{R}_t(a) & \text{with prob. } 1 - \epsilon \end{cases}$$

where $\hat{R}_t(a) = \frac{\sum_{s=1}^{t-1} \mathbb{I}\{a_s = a\}\, r_s}{\sum_{s=1}^{t-1} \mathbb{I}\{a_s = a\}}$ is the empirical mean of arm a using samples up to time $t - 1$.

Regret Bound of ϵ-Greedy

Theorem. Regret Bound of ϵ-Greedy

With proper choice of ϵ, the expected regret of ϵ-Greedy is bounded by

$$\mathbb{E}[\text{Regret}] \leq \tilde{O}\left(A^{1/3}\, T^{2/3}\right).$$

Can We Do Better?

In explore-then-exploit and ϵ-greedy, every arm receives the same amount of exploration.

… Maybe, for those arms that look worse, the amount of exploration on them can be reduced?

Solution: Refine the amount of exploration for each arm **based on the current mean estimation**.

(Has to do this carefully to avoid **under-exploration**)

Boltzmann Exploration

Boltzmann Exploration (Parameter: λ_t)

In each round, sample a_t according to

$$p_t(a) \propto \exp\big(\lambda_t \, \hat{R}_t(a)\big)$$

where $\hat{R}_t(a)$ is the empirical mean of arm a using samples up to time $t-1$.

Cesa-Bianchi, Gentile, Lugosi, Neu. **Boltzmann Exploration Done Right**, 2017.
Bian and Jun. **Maillard Sampling: Boltzmann Exploration Done Optimally**. 2021.

Another adaptive exploration $p_t(a) = \frac{1}{\gamma - \lambda_t \hat{R}_t(a)}$ will work! (later in the course)

Another Idea: “Optimism in the Face of Uncertainty”

In words:

Act according to the **best plausible world**.

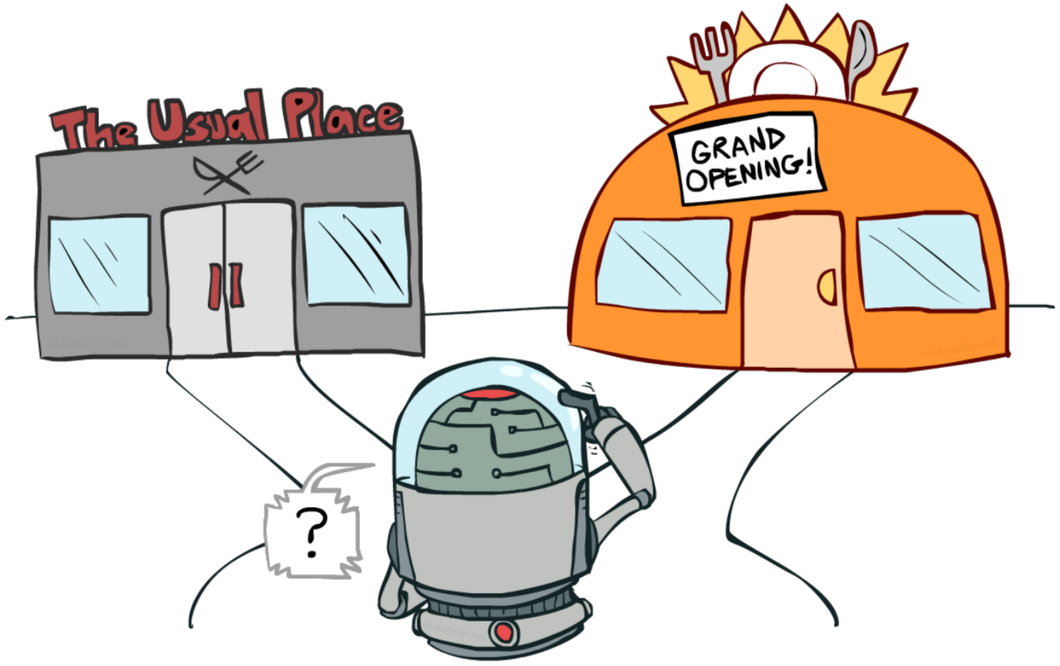


Image source: UC Berkeley AI course slide, lecture 11.

Another Idea: “Optimism in the Face of Uncertainty”

In words:

Act according to the **best plausible world**.

At time t, suppose that arm a has been drawn for $N_t(a)$ times, with empirical mean $\hat{R}_t(a)$.

What can we say about the true mean $R(a)$?

$$\left| R(a) - \hat{R}_t(a) \right| \leq \sqrt{\frac{2\log(2/\delta)}{N_t(a)}} \quad \text{w.p.} \geq 1 - \delta$$

What’s the most optimistic mean estimation for arm a?

$$\hat{R}_t(a) + \sqrt{\frac{2\log(2/\delta)}{N_t(a)}}$$

UCB

UCB (Parameter: δ)

In the first A rounds, draw each arm once.

For the remaining rounds: in round t, draw

$$a_t = \mathrm{argmax}_a \ \ \hat{R}_t(a) + \sqrt{\frac{2\log(2/\delta)}{N_t(a)}}$$

where $\hat{R}_t(a)$ is the empirical mean of arm a using samples up to time $t-1$.
$N_t(a)$ is the number of samples of arm a up to time $t-1$.

P Auer, N Cesa-Bianchi, P Fischer. **Finite-time analysis of the multiarmed bandit problem**, 2002.

Regret Bound of UCB

Theorem. Regret Bound of UCB

With probability at least $1 - AT\delta$,

$$\text{Regret} \le O\left(\sqrt{AT\log(1/\delta)}\right) = \tilde{O}(\sqrt{AT}) \;.$$

UCB Regret Bound Analysis

Define $\tilde{R}_t(a) = \hat{R}_t(a) + \sqrt{\frac{2\log(T/\delta)}{N_t(a)}}$

Assume $|R(a) - \hat{R}_t(a)| \leq \sqrt{\frac{2\log T/\delta}{N_t(a)}}$ $\forall t, a$

$$\text{Regret} \leq A + \sum_{t=A+1}^{T} \left(R(a^*) - R(a_t) \right)$$

$$\leq A + \sum_{t=A+1}^{T} \left(\underbrace{R(a^*) - \tilde{R}_t(a^*)}_{\leq 0} + \underbrace{\tilde{R}_t(a_t) - R(a_t)}_{\underbrace{\tilde{R}_t(a_t) - \hat{R}_t(a_t)}_{= \sqrt{\frac{2\log(T/\delta)}{N_t(a_t)}}} + \underbrace{\hat{R}_t(a_t) - R(a_t)}_{\leq \sqrt{\frac{2\log(T/\delta)}{N_t(a_t)}}}} \right)$$

$$\lesssim A + \sum_{t=A+1}^{T} 2\sqrt{\frac{2\log(T/\delta)}{N_t(a_t)}}$$

$$= A + \sum_{a} \sum_{t=A+1}^{T} \mathbb{I}\{a_t = a\} \cdot 2\sqrt{\frac{2\log(T/\delta)}{N_t(a)}} = A + \sum_{a} \underbrace{\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N_t(a)}} \right)}_{\lesssim \sqrt{N_t(a)}} \cdot 2\sqrt{2\log T/\delta}$$

$$\lesssim A + \sum_{a} \sqrt{N_t(a) \log 1/\delta}$$

$$\leq A + \sqrt{A \sum_{a} N_t(a) \log 1/\delta} \leq A + \sqrt{AT \log 1/\delta}$$

Comparison

	Regret Bound	Exploration
Explore-then-Exploit ϵ-Greedy	$A^{1/3}\, T^{2/3}$	Non-adaptive
Boltzmann Exploration	---	Adaptive
UCB Thompson Sampling	$\sqrt{AT}$	Adaptive

Bayesian Setting for MAB

Assumptions:

- At the beginning, the environment draws a parameter θ from some prior distribution $\theta \sim P_{\text{prior}}$
- In every round, the reward vector $\boldsymbol{r_t} = (r_t(1), \ldots, r_t(A))$ is generated from $\boldsymbol{r_t} \sim P_\theta$

E.g., Gaussian Case

- At the beginning, $\theta(a) \sim \mathcal{N}(0,1)$ for all $a \in \{1, \ldots, A\}$.
- In every round, the reward of arm a is generated by $r_t(a) \sim \mathcal{N}(\theta(a), 1)$.

For the learner, P_{prior} is known; θ is unknown; P_θ is known for any θ.

Thompson Sampling

William Thompson. **On the likelihood that one unknown probability exceeds another in view of the evidence of two samples**, 1933.

In words:

Randomly pick an arm according to the probability you **believe** it is the optimal arm.

At time t, after seeing $\mathcal{H}_t = (a_1, r_1(a_1), a_2, r_2(a_2), \ldots, a_{t-1}, r_{t-1}(a_{t-1}))$, the learner has a **posterior distribution** for θ:

$$P(\theta|\mathcal{H}_t) = \frac{P(\mathcal{H}_t, \theta)}{P(\mathcal{H}_t)} = \frac{P_\theta(\mathcal{H}_t) P_{\text{prior}}(\theta)}{P(\mathcal{H}_t)} \propto P_\theta(\mathcal{H}_t) P_{\text{prior}}(\theta)$$

In math:

Sample a_t according to $p_t(a) = \int_\theta \; P(\theta|\mathcal{H}_t)\, \mathbb{I}\{a^\star(\theta) = a\} = \mathbb{E}_{\theta \sim P(\cdot\,|\,\mathcal{H}_t)}[\mathbb{I}\{a^\star(\theta) = a\}]$

Implementation: Sample $\theta_t \sim P(\cdot\,|\,\mathcal{H}_t)$, and choose $a_t = a^\star(\theta_t)$.

Thompson Sampling in the Gaussian Case

More on Thompson Sampling

Besides Gaussian, the **Beta** prior is a prior distribution commonly used with Bernoulli reward.

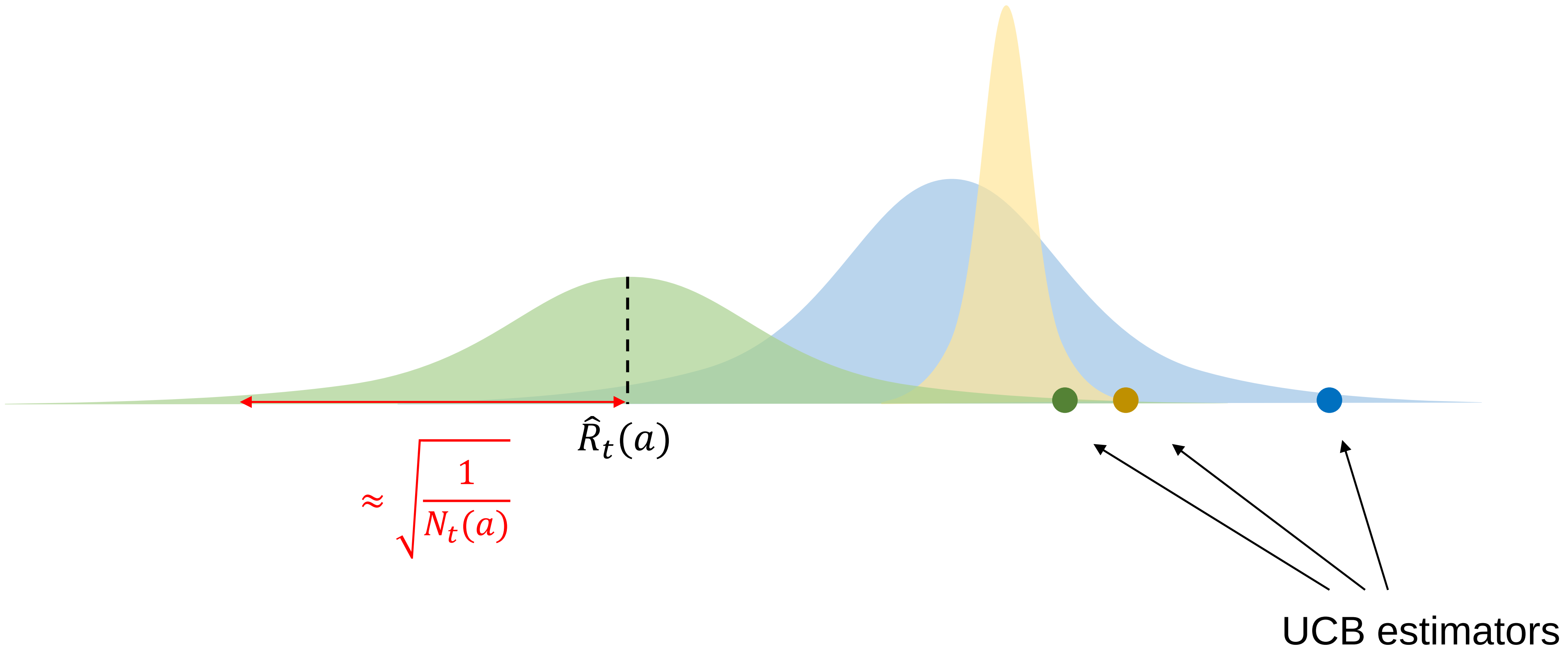
Regret bound analysis of Thompson sampling

Shipra Agrawal, Navin Goyal. Near-optimal Regret Bounds for Thompson Sampling. 2017.

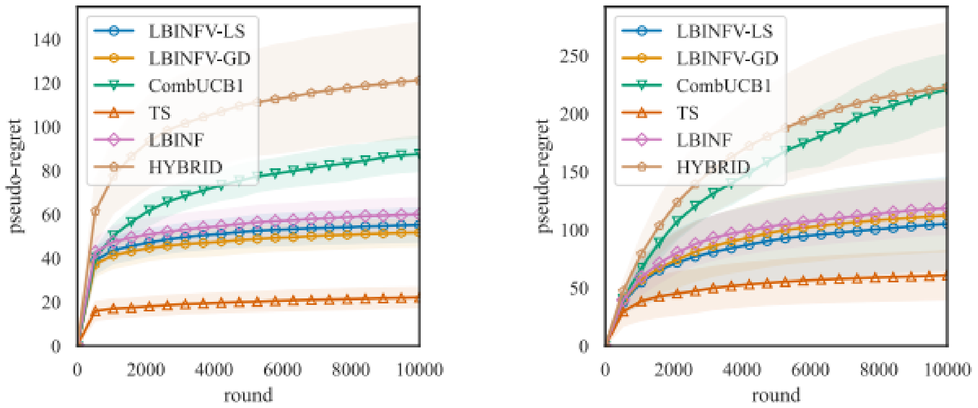
Daniel Russo and Ben Van Roy. An Information-Theoretic Analysis of Thompson Sampling. 2016.

Tor Lattimore and Ssaba Szepesvári. Bandit Algorithms Chapter 36. 2020.

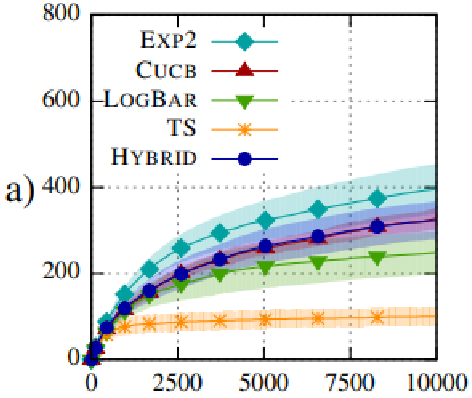
TS, UCB, and BE



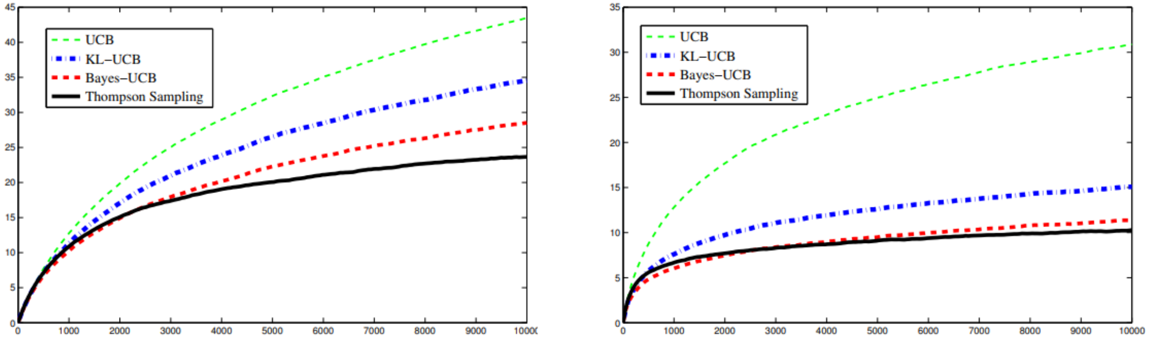
Superior Empirical Performance of TS



Tsuchiya, Ito, Honda. Further Adaptive Best-of-Both-Worlds Algorithm for Combinatorial Semi-Bandits. 2023



Zimmert, Luo, Wei. Beating Stochastic and Adversarial Semi-bandits Optimally and Simultaneously. 2019.



Kaufmann, Korda Munos. Thompson Sampling: An Asymptotically Optimal Finite Time Analysis. 2012.