Markov Decision Processes

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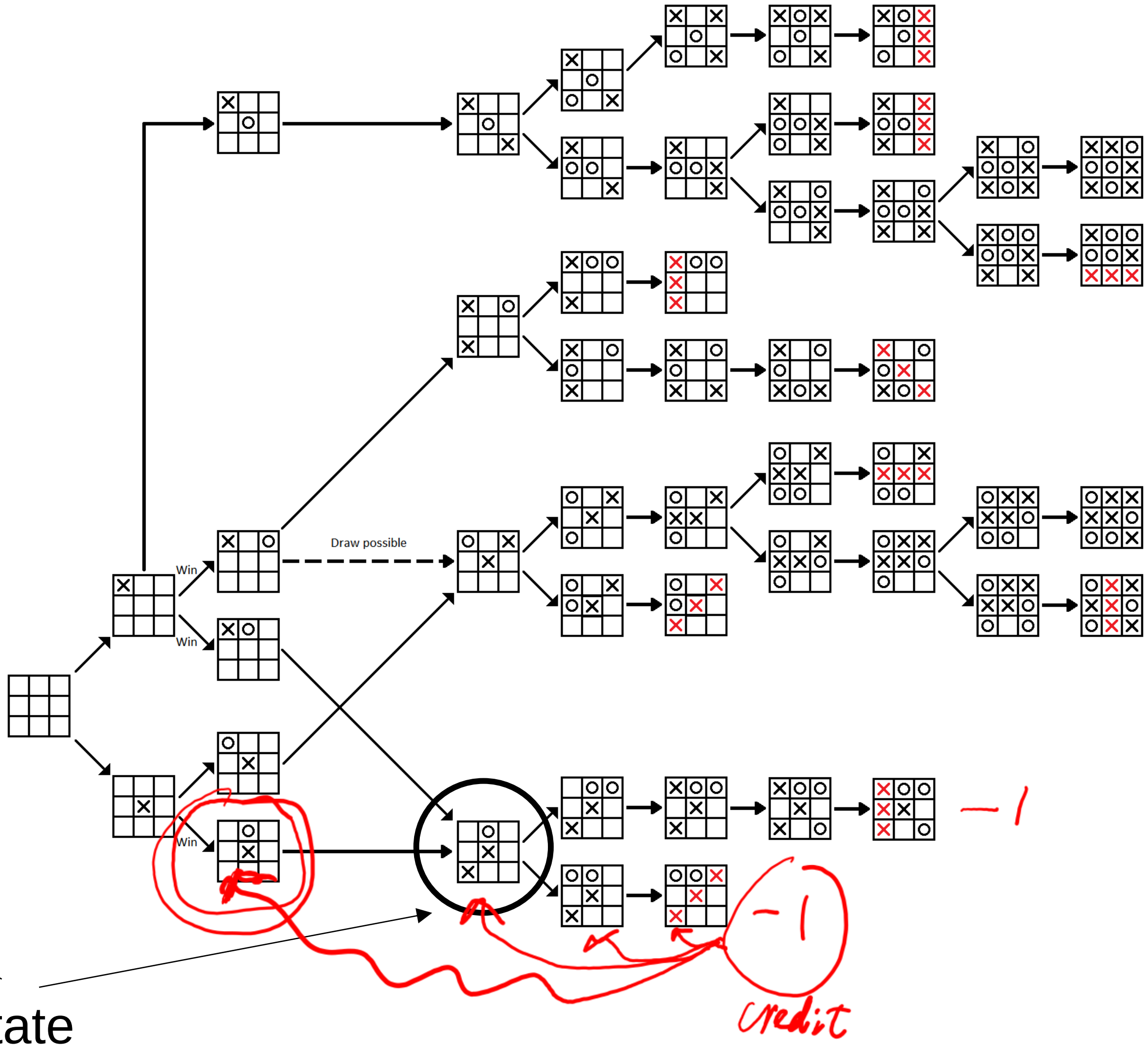
Sequence of Actions

To win the game, the learner has to take a sequence of actions $a_1 \to a_2 \to \cdots \to a_H$.

The effect of a particular action may not be revealed instantaneously.

- Some effect may be revealed instantaneously
- Some may be revealed later

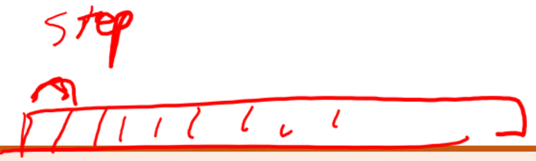
Sequence of Actions



state

(a summary of the current status in a multi-stage game)

Interaction Protocol (Episodic Setting)



For **episode** $t = 1, 2, \dots, T$:

$h \leftarrow 1$

Environment generates initial state $s_{t,1}$

While episode t has not ended:

Learner chooses an action $a_{t,h}$

Learner observes instantaneous reward $r_{t,h}$ with $\mathbb{E}[r_{t,h}] = R(s_{t,h}, a_{t,h})$

Environment generates next state $s_{t,h+1} \sim P(\cdot \mid s_{t,h}, a_{t,h})$

$h \leftarrow h + 1$

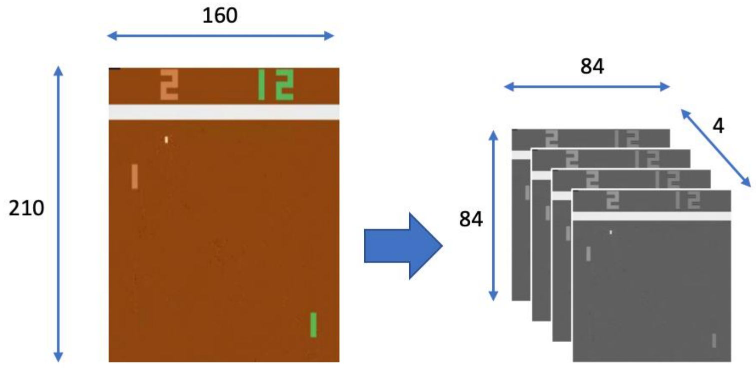
Markov assumption:

$r_{t,h}$ and $s_{t,h+1}$ are conditionally independent of $(s_{t,1}, a_{t,1}, \dots, s_{t,h-1}, a_{t,h-1})$ given $s_{t,h}$

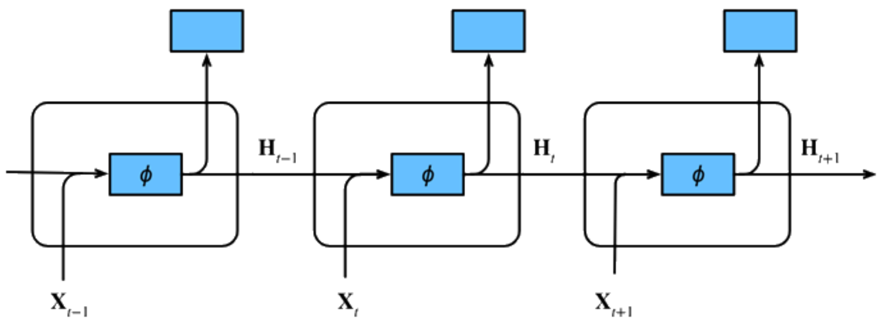
Goal: maximize $\displaystyle\sum_{t=1}^{T} \sum_{h=1}^{\tau_t} R(s_{t,h}, a_{t,h})$

τ_t : length of episode t

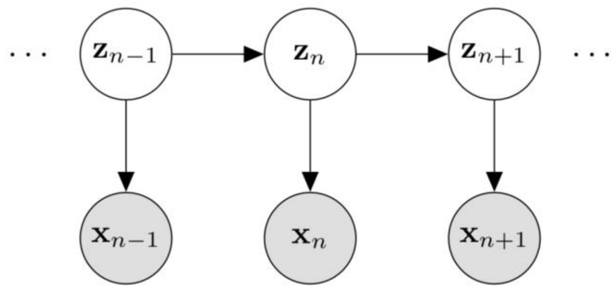
From Observations to States



Stacking recent observations



Recurrent neural network



Hidden Markov model

Regret (Episodic Setting)

$\pi^\star: S \to A$

$$\text{Regret} = \underbrace{\max_{\pi^\star} \mathbb{E}^{\pi^\star}\left[\sum_{t=1}^{T}\sum_{h=1}^{\tilde{\tau}_t} R(\tilde{s}_{t,h}, \pi^\star(\tilde{s}_{t,h}))\right]}_{\text{Benchmark}} - \sum_{t=1}^{T}\sum_{h=1}^{\tau_t} R(s_{t,h}, a_{t,h})$$

CB $\max_{\pi^\star} \sum_{t=1}^{T} R(x_t, \pi^\star(x_t)) - \sum_{t=1}^{T} R(x_t, a_t)$

$$s_{t,1} \xrightarrow{\pi^\star(s_{t,1})} \tilde{s}_{t,2} \xrightarrow{\pi^\star(\tilde{s}_{t,2})} \tilde{s}_{t,3} \xrightarrow{\pi^\star(\tilde{s}_{t,3})} \cdots\cdots \longrightarrow \tilde{s}_{t,\tilde{\tau}_t}$$

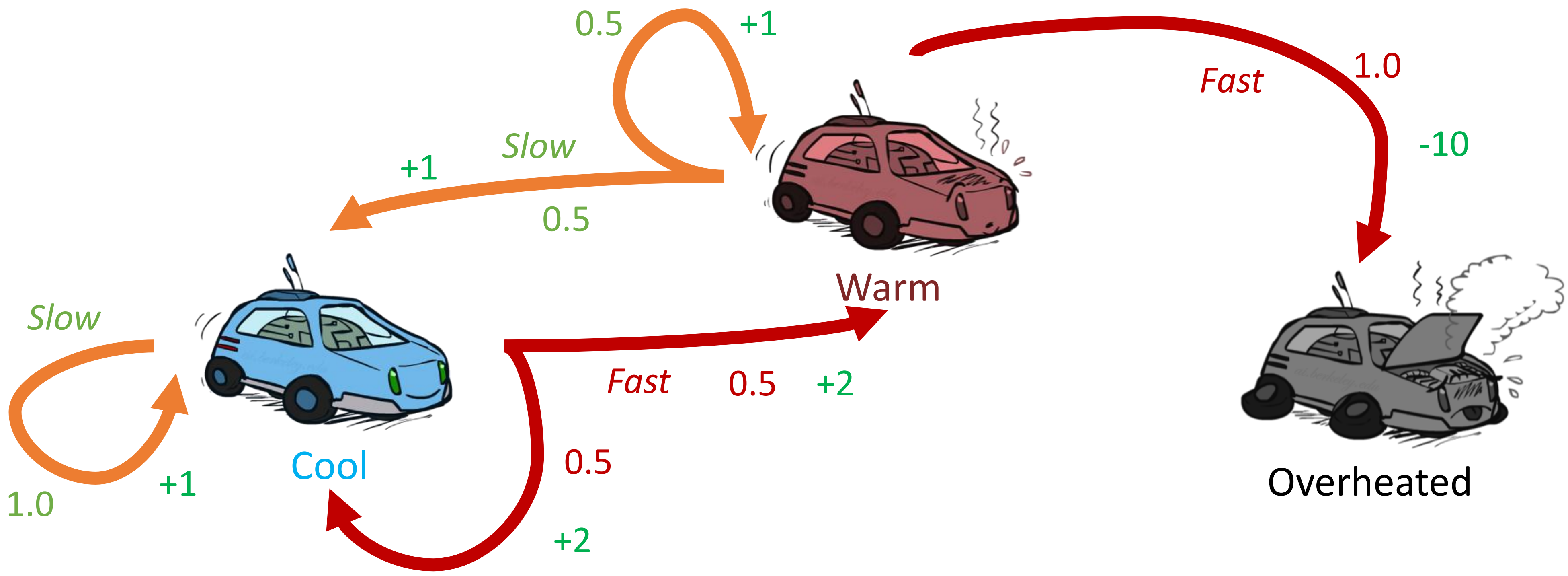
Trajectory generated by $\pi^\star$

$$s_{t,1} \xrightarrow{a_{t,1}} s_{t,2} \xrightarrow{a_{t,2}} s_{t,3} \xrightarrow{a_{t,3}} \cdots \longrightarrow s_{t,\tau_t}$$

Trajectory generated by learner

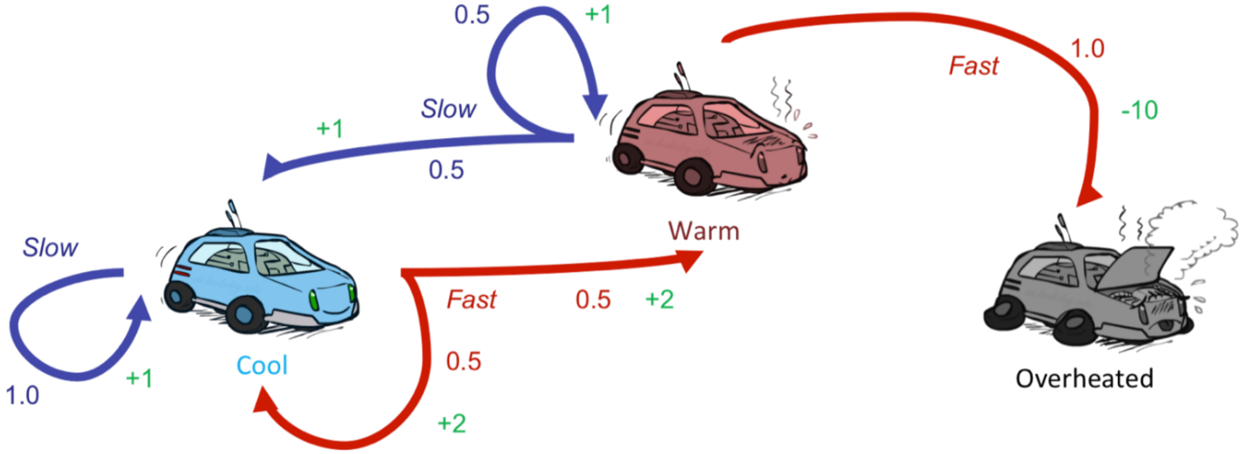
Example: Racing

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward



Example: Racing

s	a	s'	$P(s'\|s,a)$	$R(s,a)$
	Slow		1.0	+1
	Fast		0.5	+2
	Fast		0.5	+2
	Slow		0.5	+1
	Slow		0.5	+1
	Fast		1.0	–10
	(end)		1.0	0



Formulations

- Interaction Protocol
  - Fixed-Horizon
  - Variable-Horizon (Goal-Oriented)
  - Infinite-Horizon
- Performance Metric
  - Total Reward
  - Average Reward
  - Discounted Reward
- Policy
  - Markov policy
  - Stationary policy

Horizon = Length of an episode

Interaction Protocols (1/3): Fixed-Horizon

Horizon length is a fixed number H

$h \leftarrow 1$

Observe initial state $s_1 \sim \rho$

While $h \leq H$:

Choose action a_h

Observe reward r_h with $\mathbb{E}[r_h] = R(s_h, a_h)$

Observe next state $s_{h+1} \sim P(\cdot \mid s_h, a_h)$

Examples: games with a fixed number of time

Interaction Protocols (2/3): Goal-Oriented

The learner interacts with the environment until reaching **terminal states** $\mathcal{T} \subset \mathcal{S}$

$h \leftarrow 1$

Observe initial state $s_1 \sim \rho$

While $s_h \notin \mathcal{T}$**:**

Choose action a_h

Observe reward r_h with $\mathbb{E}[r_h] = R(s_h, a_h)$

Observe next state $s_{h+1} \sim P(\cdot \mid s_h, a_h)$

$h \leftarrow h + 1$

Examples: video games, robotics tasks, personalized recommendations, etc.

Interaction Protocols (3/3): Infinite-Horizon

The learner continuously interacts with the environment

$\cancel{h \leftarrow 1}$

~~Observe initial state~~ $\cancel{s_1 \sim \rho}$

Loop forever:

Choose action a_h

Observe reward r_h with $\mathbb{E}[r_h] = R(s_h, a_h)$

Observe next state $s_{h+1} \sim P(\cdot \mid s_h, a_h)$

$h \leftarrow h + 1$

Examples: network management, inventory management

Formulations

- Interaction Protocol
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Performance Metric

Total Reward (for episodic setting): $\sum_{h=1}^{\tau} r_h$ (τ: the step where the episode ends)

Average Reward (for infinite-horizon setting): $\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} r_h$

Discounted Total Reward (for episodic or infinite-horizon): $\sum_{h=1}^{\tau} \gamma^{h-1} r_h$

τ: the step where the episode ends, or ∞ in the infinite-horizon case
$\gamma \in [0,1)$: discount factor

$\gamma = 0.99$

Interaction Protocols vs. Performance Metrics

"natural" objective

Fixed-Horizon ⟶ Total Reward

Goal-Oriented ⟶ Total Reward — Could be unbounded

Infinite-horizon ⟶ Average Reward — Could have constant change for an infinitesimal change in policy

Discounted Total Reward?

Focusing more on the **recent** reward

There is a potential mismatch between our ultimate goal and what we optimized.

Formulations

- Interaction Protocol
  - Fixed-Horizon
  - Variable-Horizon (Goal-Oriented)
  - Infinite-Horizon
- Performance Metric
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  - Markov policy
  - Stationary policy

Policy for MDPs

$\pi = (\pi_1, \pi_2, \cdots, \pi_H, \cdots)$

Markov Policy

h: step index

$$a_h \sim \pi_h(\cdot \mid s_h) \in \Delta_A \quad \text{(space of dist)}$$
$$a_h = \pi_h(s_h) \in A$$

For **fixed-horizon** setting, there exists an optimal policy in this class ✓

Stationary Policy $\subseteq$ Markov Policy

$$a_h \sim \pi(\cdot \mid s_h)$$
$$a_h = \pi(s_h)$$

For **infinite-horizon/goal-oriented** settings, there exists an optimal policy in this class ✓

Fixed-horizon (Markov Policy) (total reward)

✓ Goal-oriented (Stationary Policy) (Discounted reward)

A **stationary policy** specifies

$\pi(\text{Slow} \mid \text{Cool})$

$\pi(\text{Fast} \mid \text{Cool})$

$\pi(\text{Slow} \mid \text{Warm})$

$\pi(\text{Fast} \mid \text{Warm})$

A **Markov policy** specifies

$\pi_h(\text{Slow} \mid \text{Cool})$

$\pi_h(\text{Fast} \mid \text{Cool})$

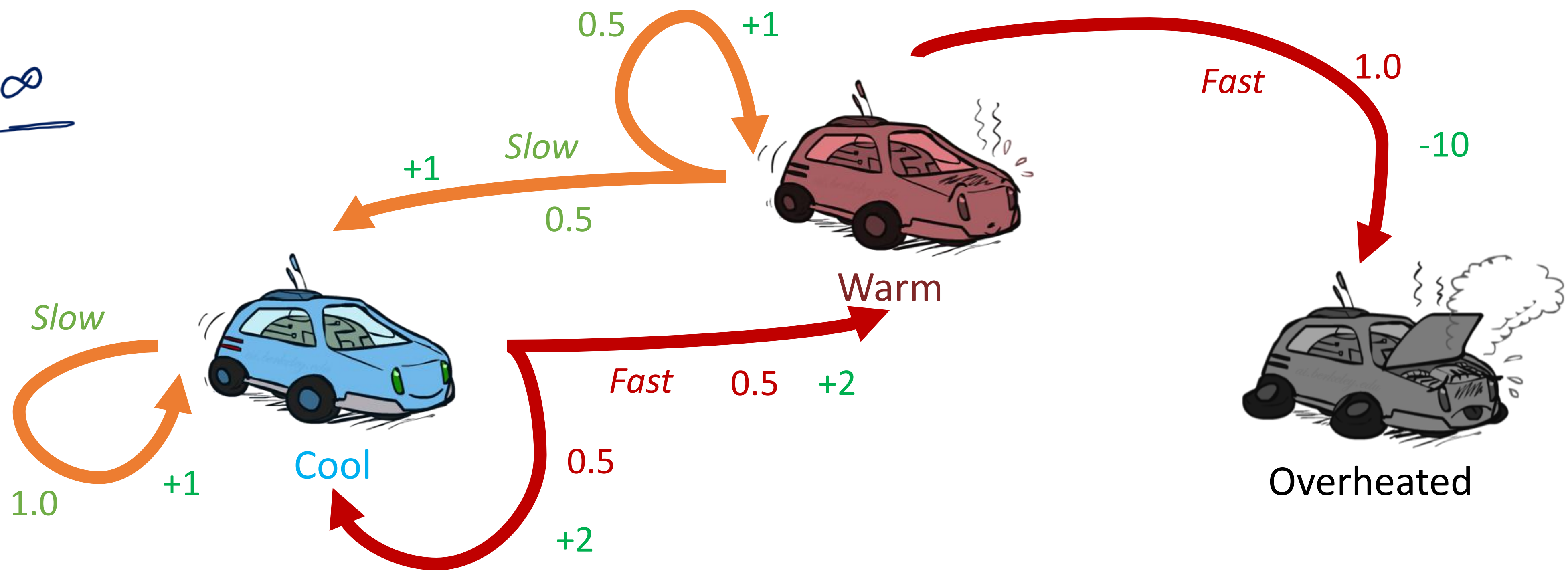
$\pi_h(\text{Slow} \mid \text{Warm})$

$\pi_h(\text{Fast} \mid \text{Warm})$

$\forall h$

$H = 5$

$H = \infty$



Value Iteration

(Fixed-Horizon)

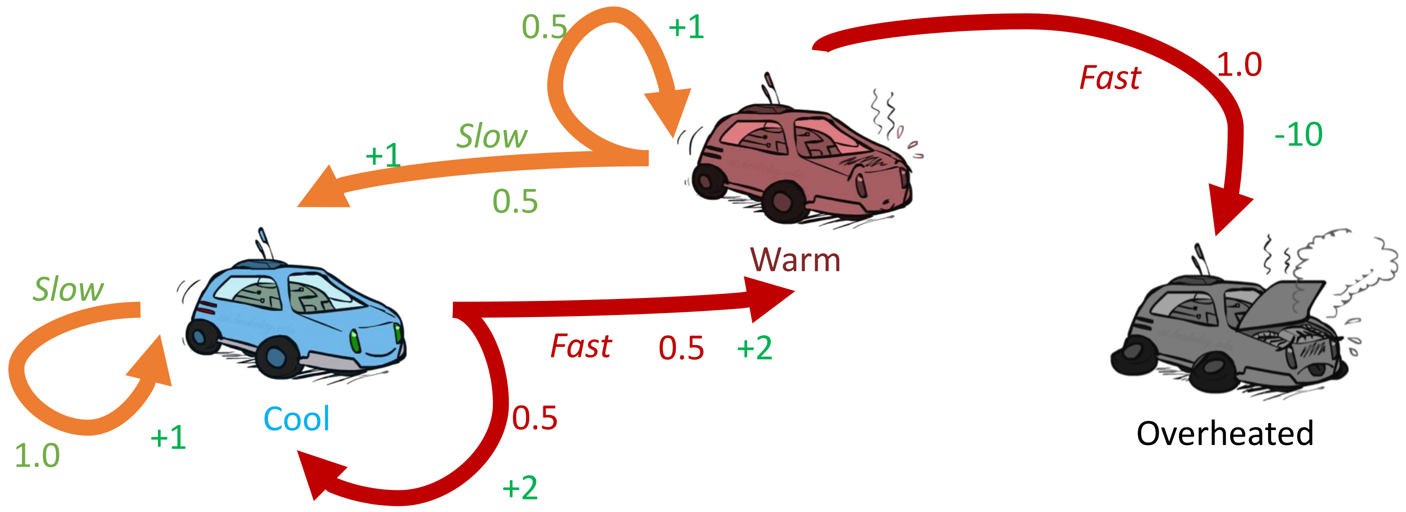
Two Tasks

Policy Evaluation: Calculate the expected total reward of a given policy

What is the expected total reward for the policy $\pi(\text{cool}) = \text{fast}, \pi(\text{warm}) = \text{slow}$?

Policy Optimization: Find the best policy

What is the policy that achieves the highest ~~policy~~ expected total reward?



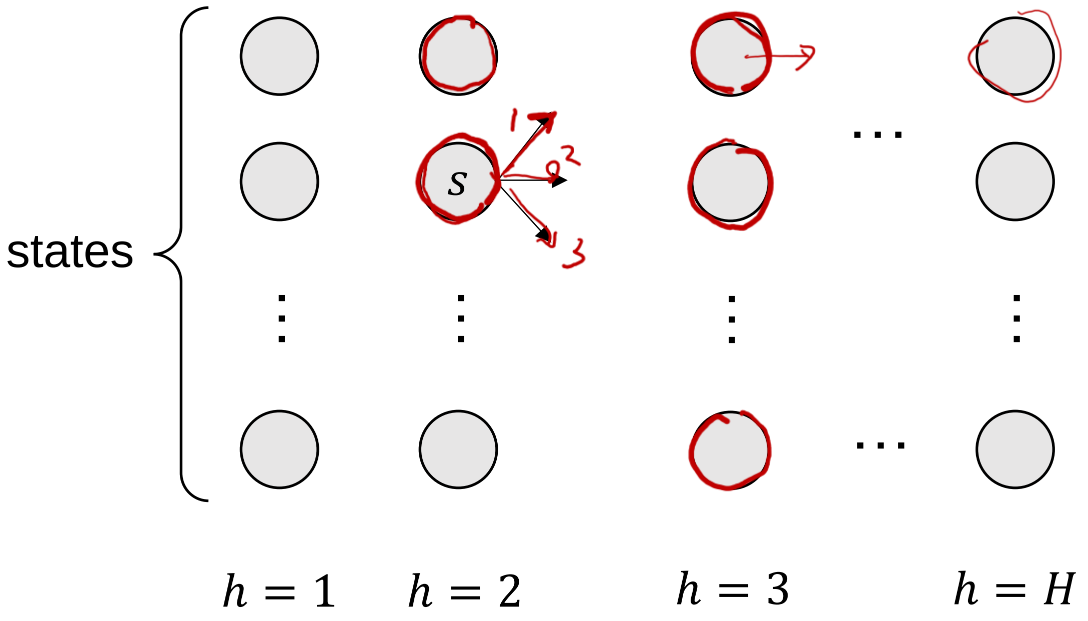
Value Iteration for Policy Evaluation

$\pi = (\pi_1, \cdots, \pi_H)$ $\quad \mathbb{E}^{\pi}\left[\sum_{h=1}^{H} R(s_t, a_t)\right]$

$$Q_h^{\pi}(s,a) = \mathbb{E}^{\pi}\left[\sum_{k=h}^{H} R(s_k, a_k) \;\middle|\; (s_h, a_h) = (s,a)\right]$$

$$V_h^{\pi}(s) = \mathbb{E}^{\pi}\left[\sum_{k=h}^{H} R(s_k, a_k) \;\middle|\; s_h = s\right]$$

$R(s,a)$



State transition: $P(s'|s,a)$

Reward: $R(s,a)$

$\boxed{V_1^{\pi}(s)}$

expected total

$= \sum_s \rho(s) V_1^{\pi}(s)$

Backward induction: $\quad Q_H^{\pi}(s,a) = R(s,a)$

$V_{H+1}^{\pi}(s) = 0 \qquad \forall s$

For $h = H, \dots 1$: $\quad$ for all s, a

$$Q_h^{\pi}(s,a) = R(s,a) + \underbrace{\sum_{s'} P(s'|s,a)\, V_{h+1}^{\pi}(s')}_{\text{Expected total reward of } \pi \text{ from step } h+1}$$

$$V_h^{\pi}(s) = \sum_a \pi_h(a|s)\, Q_h^{\pi}(s,a)$$

Bellman Equation

Q_h^π is called "the state-action value functions of policy π"

V_h^π is called "the state value function of policy π"

Both can be just called "**value functions**"

$$Q_h^\pi(s,a) = R(s,a) + \sum_{s'} P(s'|s,a)\, V_{h+1}^\pi(s')$$

$$V_h^\pi(s) = \sum_a \pi_h(a|\, s) Q_h^\pi(s,a)$$

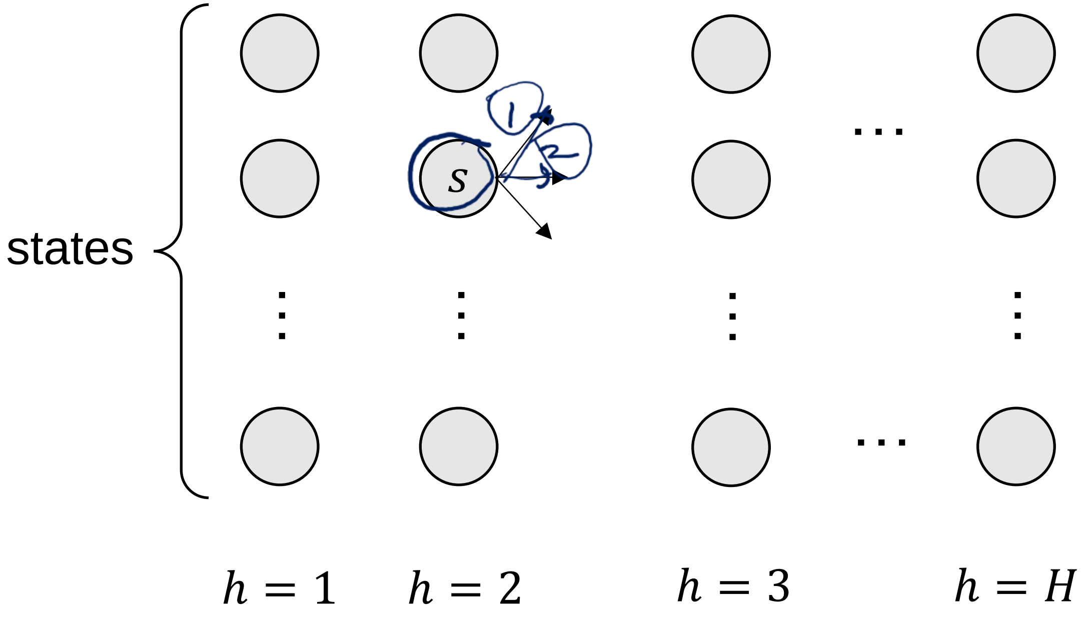
or

$$Q_h^\pi(s,a) = R(s,a) + \sum_{s',a'} P(s'|s,a)\, \pi_{h+1}(a'|s') Q_{h+1}^\pi(s',a')$$

or

$$V_h^\pi(s) = \sum_a \pi_h(a|\, s) \left(R(s,a) + \sum_{s'} P(s'|s,a)\, V_{h+1}^\pi(s') \right)$$

Value Iteration for Policy Optimization



$$Q_h^\star(s,a) = \max_{\pi \in \Pi_M} \mathbb{E}^\pi \left[\sum_{k=h}^{H} R(s_k, a_k) \,\middle|\, (s_h, a_h) = (s, a) \right]$$

$$V_h^\star(s) = \max_{\pi \in \Pi_M} \mathbb{E}^\pi \left[\sum_{k=h}^{H} R(s_k, a_k) \,\middle|\, s_h = s \right]$$

State transition: $P(s'|s,a)$

Reward: $R(s,a)$

Backward induction:

$V_{H+1}^\star(s) = 0 \qquad \forall s$

For $h = H, \ldots 1$: for all s, a

$$Q_h^\star(s,a) = R(s,a) + \underbrace{\sum_{s'} P(s'|s,a)\, V_{h+1}^\star(s')}_{\text{Expected optimal total reward from step } h+1}$$

$$V_h^\star(s) = \max_a Q_h^\star(s,a) \qquad \pi_h^\star(s) = \operatorname{argmax}_a Q_h^\star(s,a)$$

Exercise



s	a	s'	$P(s'\|s,a)$	$R(s,a)$
	Slow		1.0	+1
	Fast		0.5	+2
	Fast		0.5	+2
	Slow		0.5	+1
	Slow		0.5	+1
	Fast		1.0	−10
	(end)		1.0	0

V_2

V_1

V_0

H=3:

$$Q_3^*(s,a) = R(s,a) = \begin{cases} +1 & (\text{cool}, \text{slow}) \\ +2 & (\text{cool}, \text{fast}) \\ +1 & (\text{warm}, \text{slow}) \\ -10 & (\text{warm}, \text{fast}) \end{cases} \quad (s,a)$$

$$V_3^*(s) = \max_a Q_3^*(s,a) = \begin{cases} +2 & \text{cool} \\ +1 & \text{warm} \end{cases}$$

$$Q_2^*(s,a) = R(s,a) + \sum_{s'} P(s'|s,a)\, V_3^*(s') = \{$$

Assume $\gamma = 0.9$ ~~$\pi(\text{cool}) = \text{fast}$, $\pi(\text{warm}) = \text{slow}$~~

Bellman Optimality Equation

$Q_h^\star$: optimal state-action value functions
$V_h^\star$: optimal state value functions
or “**optimal value functions**”

$$Q_h^\star(s,a) = R(s,a) + \sum_{s'} P(s'|s,a)\, V_{h+1}^\star(s')$$

$$V_h^\star(s) = \max_a Q_h^\star(s,a)$$

or

$$Q_h^\star(s,a) = R(s,a) + \sum_{s'} P(s'|s,a) \left(\max_{a'} Q_{h+1}^\star(s',a')\right)$$

or

$$V_h^\star(s) = \max_a \left(R(s,a) + \sum_{s'} P(s'|s,a)\, V_{h+1}^\star(s') \right)$$

$$\pi_h^\star(s) = \underset{a}{\operatorname{argmax}}\ Q_h^\star(s,a)$$

Recall: Regret

$$\text{Regret} = \max_{\pi^\star} \mathbb{E}^{\pi^\star}\left[\sum_{t=1}^{T}\sum_{h=1}^{\tilde{\tau}_t} R(\tilde{s}_{t,h}, \pi^\star(\tilde{s}_{t,h}))\right] - \sum_{t=1}^{T}\sum_{h=1}^{\tau_t} R(s_{t,h}, a_{t,h})$$

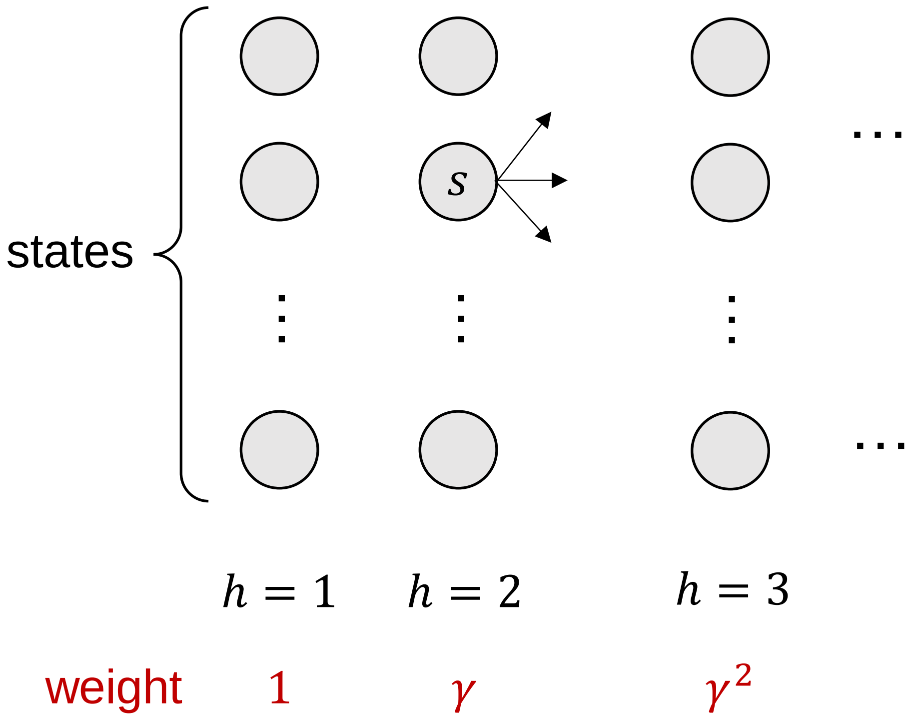
$$\mathbb{E}[\text{Regret}] = \mathbb{E}\left[\sum_{t=1}^{T}\left(V_1^\star(s_{t,1}) - V_1^{\pi_t}(s_{t,1})\right)\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T}\left(V_1^\star(\rho) - V_1^{\pi_t}(\rho)\right)\right]$$

$$V_1^\pi(\rho) \triangleq \mathbb{E}_{s\sim\rho}[V_1^\pi(s)]$$

Value Iteration
(Infinite-Horizon)

Value Iteration for Policy Evaluation



State transition: $P(s'|s,a)$

Reward: $R(s,a)$

$$Q_i^\pi(s,a) = \mathbb{E}^\pi\left[\sum_{h=1}^{i} \gamma^{h-1} R(s_h, a_h) \;\middle|\; (s_0, a_0) = (s,a)\right]$$

$$V_i^\pi(s) = \mathbb{E}^\pi\left[\sum_{h=1}^{i} \gamma^{h-1} R(s_h, a_h) \;\middle|\; s_0 = s\right]$$

$$Q^\pi(s,a) = Q_\infty^\pi(s,a) \qquad V^\pi(s) = V_\infty^\pi(s)$$

$V_0^\pi(s) = 0 \quad \forall s$

For $i = 1, 2, 3, \ldots$: for all s, a

$$Q_i^\pi(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a)\, V_{i-1}^\pi(s')$$

$$V_i^\pi(s) = \sum_a \pi(a|s)\, Q_i^\pi(s,a)$$

Exercise

s	a	s'	$P(s'\|s,a)$	$R(s,a)$
	Slow		1.0	+1
	Fast		0.5	+2
	Fast		0.5	+2
	Slow		0.5	+1
	Slow		0.5	+1
	Fast		1.0	–10
	(end)		1.0	0

			
V_2^π	3.35	2.35	0
V_1^π	2	1	0
V_0^π	0	0	0

Assume $\gamma = 0.9$ $\pi(\text{cool}) = \text{fast},\ \pi(\text{warm}) = \text{slow}$

Bellman Equation

$$Q^{\pi}(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a)\, V^{\pi}(s')$$

$$V^{\pi}(s) = \sum_{a} \pi(a|\, s) Q^{\pi}(s,a)$$

or

$$Q^{\pi}(s,a) = R(s,a) + \gamma \sum_{s',a'} P(s'|s,a)\, \pi(a'|s') Q^{\pi}(s',a')$$

or

$$V^{\pi}(s) = \sum_{a} \pi(a|\, s) \left(R(s,a) + \gamma \sum_{s'} P(s'|s,a)\, V^{\pi}(s') \right)$$

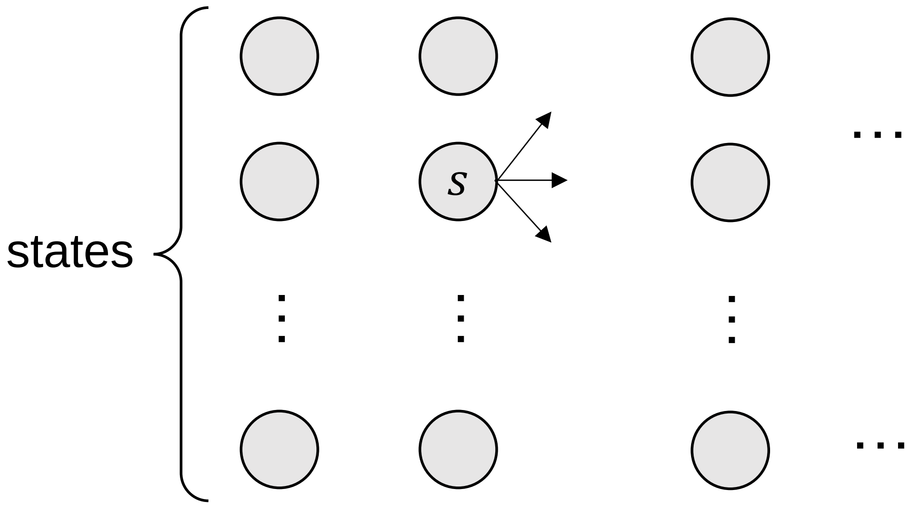
Convergence

Value Iteration ensures

$$\left|Q_i^{\pi}(s,a) - Q^{\pi}(s,a)\right| \leq \gamma^i |Q_0^{\pi}(s,a) - Q^{\pi}(s,a)|$$

$$\left|V_i^{\pi}(s) - V^{\pi}\,(s)\right| \leq \gamma^i |V_0^{\pi}(s) - V^{\pi}(s)|$$

Value Iteration for Policy Optimization



$h = 1$ $\quad$ $h = 2$ $\quad$ $h = 3$

weight $\quad 1 \quad \gamma \quad \gamma^2$

State transition: $P(s'|s,a)$

Reward: $R(s,a)$

$$Q_i^\star(s,a) = \max_\pi \mathbb{E}^\pi\left[\sum_{h=1}^{i} \gamma^{h-1} R(s_h,a_h) \,\middle|\, (s_0,a_0) = (s,a)\right]$$

$$V_i^\star(s) = \max_\pi \mathbb{E}^\pi\left[\sum_{h=1}^{i} \gamma^{h-1} R(s_h,a_h) \,\middle|\, s_0 = s\right]$$

$$Q^\star(s,a) = Q_\infty^\star(s,a) \qquad V^\star(s) = V_\infty^\star(s)$$

$V_0^\star(s) = 0 \quad \forall s$

For $i = 1, 2, 3, \ldots$: $\quad$ for all s, a

$$Q_i^\star(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a)\, V_{i-1}^\star(s')$$

$$V_i^\star(s) = \max_a Q_i^\star(s,a)$$

Bellman Optimality Equation

$$\pi^\star(s) = \underset{a}{\operatorname{argmax}}\ Q^\star(s,a)$$

$$Q^\star(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a)\, V^\star(s')$$

$$V^\star(s) = \max_a Q^\star(s,a)$$

or

$$Q^\star(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q^\star(s',a')$$

or

$$V^\star(s) = \max_a \left(R(s,a) + \gamma \sum_{s'} P(s'|s,a)\, V^\star(s') \right)$$

Convergence

Value Iteration ensures

$$\left|Q_i^\star(s,a) - Q^\star(s,a)\right| \leq \gamma^i |Q_0^\star(s,a) - Q^\star(s,a)|$$

$$\left|V_i^\star(s) - V^\star\,(s)\right| \leq \gamma^i |V_0^\star(s) - V^\star(s)|$$

Question

We know $Q^\star(s,a) = \lim_{i\to\infty} Q_i^\star(s,a)$ recovers the optimal policy by $\pi^\star(s) = \operatorname{argmax}_a Q^\star(s,a)$.

But usually we only have $Q_i^\star(s,a)$ for finite i, or just some $\hat{Q}(s,a)$ that **approximates** $Q^\star(s,a)$

How good is the policy $\hat{\pi}(s) = \operatorname{argmax}_a \hat{Q}(s,a)$?

Policy Iteration

Policy Iteration

Policy Iteration

For $i = 1,\ 2, \ldots$

$$\forall s, \qquad \pi_i(s) \leftarrow \operatorname*{argmax}_a Q^{\pi_i}(s, a)$$

Theorem (monotonic improvement). Policy Iteration ensures

$$\forall s, a, \qquad Q^{\pi_{i+1}}(s, a) \geq Q^{\pi_i}(s, a)$$

(We will prove this later.)

Generalized Policy Iteration

$N = \infty \Rightarrow$ Policy Iteration

$N = 1 \Rightarrow$ Value Iteration for policy optimization

For $i = 1, 2, \ldots$

$$\pi_i(s) = \max_a Q_i(s, a)$$

Policy update

$Q \leftarrow Q_i$

Repeat for N times:

$$Q(s, a) \leftarrow R(s, a) + \gamma \sum_{s', a'} P(s'|s, a)\, \pi_i(a'|s') Q(s', a')$$

Value update

$Q_{i+1} \leftarrow Q$

Notice: in value iteration for PO, there may not exist a policy π such that $Q_i = Q^{\pi}$

In contrast, in policy iteration we have $Q_i = Q^{\pi_{i-1}}$

VI for PO can be viewed as PI **with incomplete policy evaluation**

Summary

- Value Iteration for Policy Optimization (VI for PO)
  - Is essentially a **dynamic programming** algorithm
  - Finds the value functions of the optimal policy
- Value Iteration for Policy Evaluation (VI for PE)
  - Also a **dynamic programming** algorithm
  - Finds the value functions of the given policy
- Policy Iteration (PI)
  - An iterative policy improvement algorithm
  - Each iteration involves a policy evaluation subtask
- VI for PO and PI can be viewed as special cases of Generalized PI

Performance Difference Lemma

Several Unanswered Questions

- For an estimation $\hat{Q}(s,a) \approx Q^\star(s,a)$ with error, how can we bound

$$V^\star(\rho) - V^{\hat{\pi}}(\rho) \qquad\qquad \text{where } \hat{\pi}(s) = \max_a \hat{Q}(s,a)?$$

- How to show that Policy Iteration leads to monotonic policy improvement?
- Also, how are these methods related to the third challenge of online RL: credit assignment?

Performance Difference Lemma

For any two stationary policies π' and π in the discounted setting,

$$\mathbb{E}_{s\sim\rho}\left[V^{\pi'}(s)\right] - \mathbb{E}_{s\sim\rho}[V^{\pi}(s)] = \sum_{s,a} d_\rho^{\pi'}(s)\left(\pi'(a|s) - \pi(a|s)\right)Q^{\pi}(s,a)$$

$$= \sum_{s} d_\rho^{\pi'}(s,a)\left(Q^{\pi}(s,a) - V^{\pi}(s)\right)$$

$$d_\rho^{\pi}(s) \triangleq \mathbb{E}^{\pi}\left[\sum_{h=1}^{\infty}\gamma^{h-1}\mathbb{I}\{s_h = s\} \;\middle|\; s_1 \sim \rho\right]$$

Discounted frequency of visitation to state s

$$d_\rho^{\pi}(s,a) \triangleq \mathbb{E}^{\pi}\left[\sum_{h=1}^{\infty}\gamma^{h-1}\mathbb{I}\{(s_h,a_h) = (s,a)\} \;\middle|\; s_1 \sim \rho\right]$$

Performance Difference Lemma (Fixed-Horizon)

For any two Markov policies π' and π in the fixed-horizon setting,

$$\mathbb{E}_{s_1\sim\rho}\left[V_1^{\pi'}(s_1)\right]-\mathbb{E}_{s_1\sim\rho}[V_1^{\pi}(s_1)]=\sum_{h=1}^{H}\sum_{s,a}d_{\rho,h}^{\pi'}(s)\left(\pi_h'(a|s)-\pi_h(a|s)\right)Q_h^{\pi}(s,a)$$

$$=\sum_{h=1}^{H}\sum_{s,a}d_{\rho,h}^{\pi'}(s,a)\left(Q_h^{\pi}(s,a)-V_h^{\pi}(s)\right)$$

$$d_{\rho,h}^{\pi}(s)\triangleq\mathbb{E}^{\pi}[\mathbb{I}\{s_h=s\}\mid s_1\sim\rho]=\mathbb{P}^{\pi}(s_h=s\mid s_1\sim\rho)$$

$$d_{\rho,h}^{\pi}(s,a)\triangleq\mathbb{E}^{\pi}[\mathbb{I}\{(s_h,a_h)=(s,a)\}\mid s_1\sim\rho]=\mathbb{P}^{\pi}((s_h,a_h)=(s,a)\mid s_1\sim\rho)$$