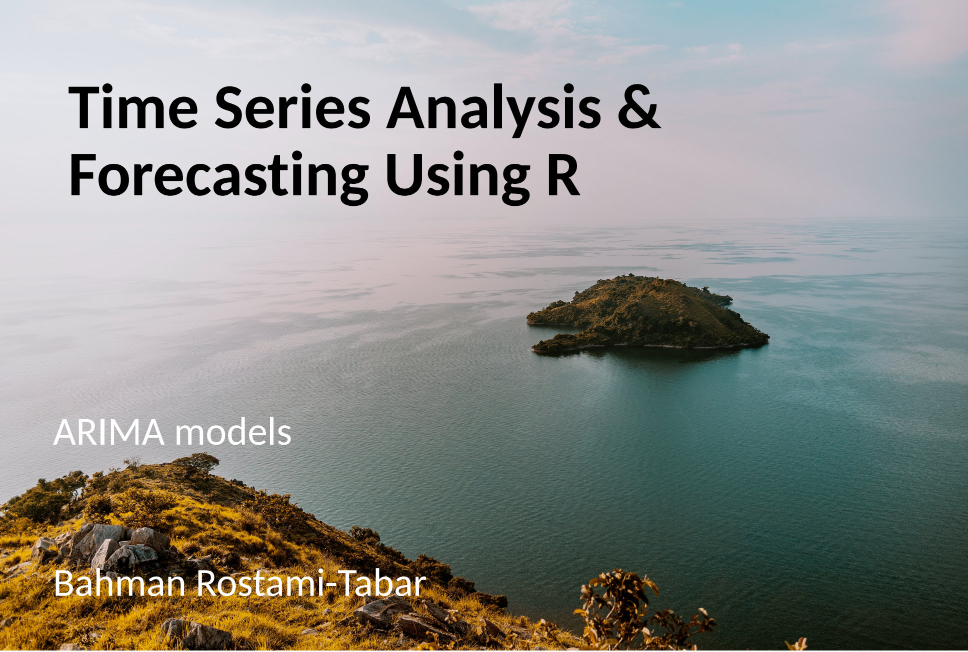
Time Series Analysis & Forecasting Using R



ARIMA models

Bahman Rostami-Tabar

Outline

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Learning objectives

- Describe model building strategy for ARIMA models
- Explain criteria for best model selection
- Produce forecast using ARIMA models

Outline

Exponential smoothing vs ARIMA models

- Exponential smoothing models were based on a description of trend and seasonality in the data,
- ARIMA models aim to describe the autocorrelations in the data.
- Exponential smoothing and ARIMA models are the two most widely-used approaches to time series forecasting

ARIMA models

Autoregressive Integrated Moving Average models

AR: autoregressive (lagged observations as inputs)
I: integrated (differencing to make series stationary)
MA: moving average (lagged errors as inputs)

ARIMA model

- Combine ARMA model with **differencing**.
- $(1 - B)^d y_t$ follows an ARMA model.

Autoregressive Moving Average(ARMA) models:

$$
\begin{aligned}
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\
+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.
\end{aligned}
$$

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\end{aligned}
$$

ARIMA(p, d, q) model

AR: p = number of preceding/lagged y values

I: d = number of times series have to be "differenced"

MA: q = number of preceding/lagged values for the error term .

What does ARIMA account for?

- Previous observations
- Rate of change in the previous observations
- Error term in the previous observations
- Perform weel for short term horizons

Stationarity

ARIMA models are stationary

Definition

If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \ldots, y_{t+s})$ does not depend on t.

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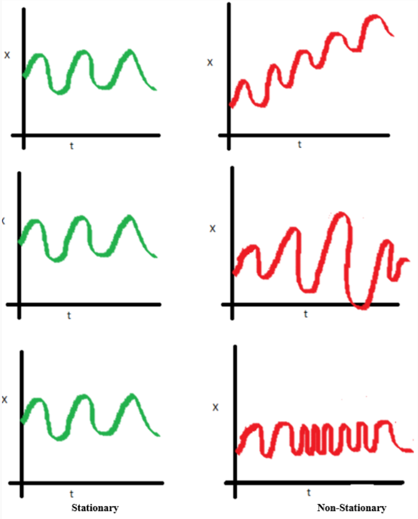
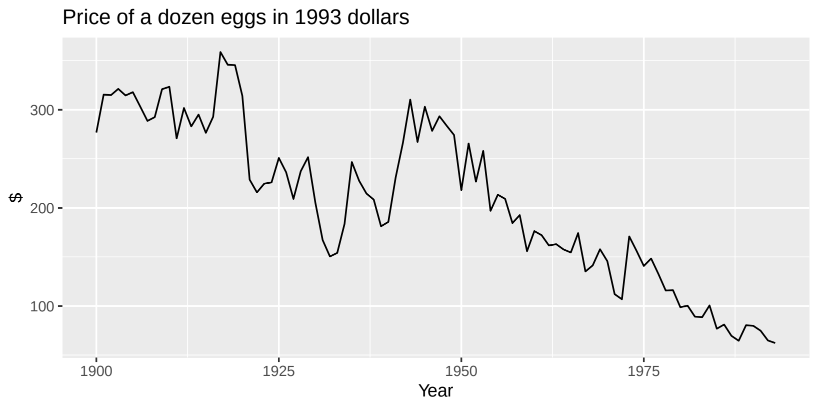
A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

Stationarity vs. Non-Stationarity

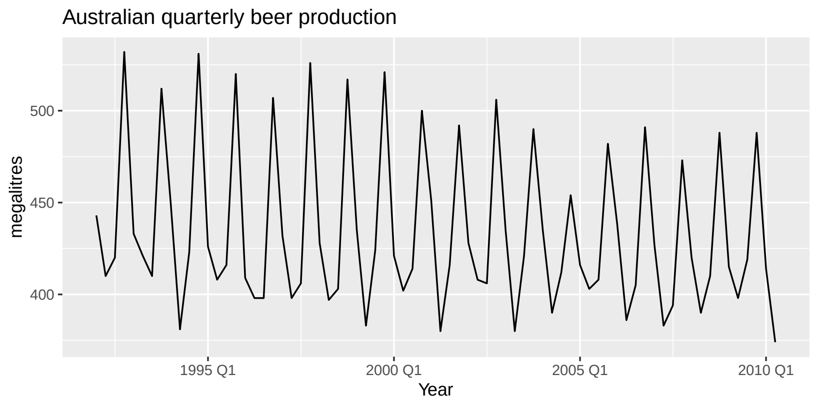


Stationary?



Price of a dozen eggs in 1993 dollars
300
200
100
$
1900
1925
1950
1975
Year

Stationary?


Australian quarterly beer production
megalitres
500
450
400
1995 Q1
2000 Q1
2005 Q1
2010 Q1
Year

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Transformations help to **stabilize the variance**.

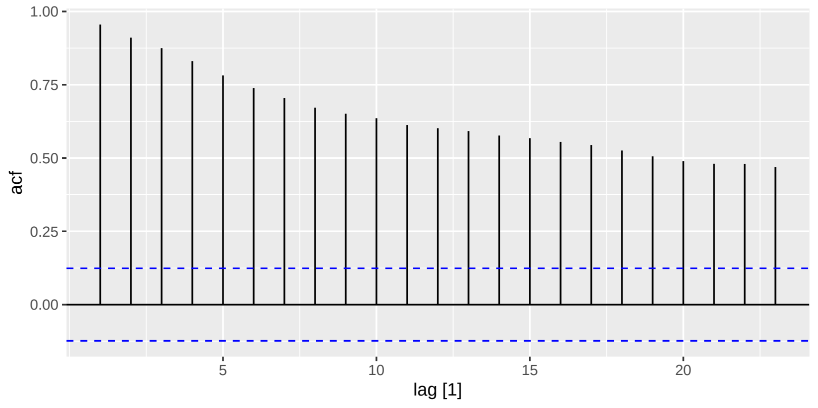
For ARIMA modelling, we also need to **stabilize the mean**.

Is your data stationarity?

Identifying non-stationary series

- Time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of r_1 is often large and positive.

Example: Google stock price



Unit root tests

- One way to determine more objectively whether data is non stationary to use a unit root test
- These are statistical hypothesis tests of stationarity that are designed for determining whether differencing is required

Statistical tests to determine the required order of differencing.

- Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary

KPSS test

```r
google_2018 %>%
  features(Close, unitroot_kpss)
```

```
## # A tibble: 1 x 3
##   Symbol kpss_stat kpss_pvalue
##   <chr>      <dbl>       <dbl>
## 1 GOOG       0.573      0.0252
```

Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series.
- Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time.
- In practice, it is almost never necessary to go beyond second-order differences.

Number of differencing

```r
google_2018 %>%
  features(Close, unitroot_ndiffs)
```

```
## # A tibble: 1 x 2
##   Symbol ndiffs
##   <chr>   <int>
## 1 GOOG        1
```

```r
#seasonal differencing
#features(Close, unitroot_nsdiffs)
```

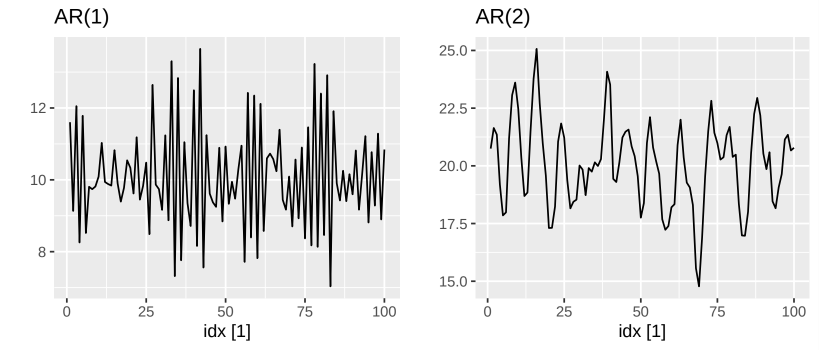
Outline

Autoregressive models

Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. We use **lagged values** of y_t as predictors.



Autoregressive models

- In an autoregression model, we forecast the variable of interest using a linear combination of past values of the variable.
- Where c is a constant and e_t i.i.d. (white noise) random variable with zero mean and known variance, σ^2.
- Changing the parameters $\phi_1, \phi_2, \ldots, \phi_p$ results in different time series patterns.

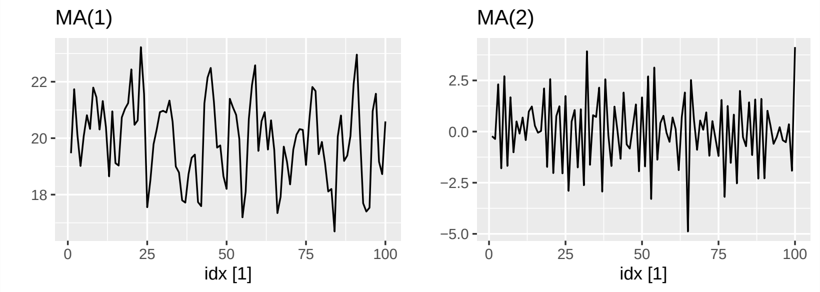
Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where ε_t is white noise.

We use **past *errors*** as predictors. *Don't confuse this with moving average smoothing!*



Moving Average (MA) models

- We forecast the variable of interest using a linear combination of **past *errors***
- *c* is a constant and e_t i.i.d. (white noise) random variable with zero mean and known variance, σ^2.
- Changing the parameters $\theta_1, \theta_2, \ldots, \theta_q$ results in different time series patterns.

ARMA(p,q) models

Autoregressive Moving Average(ARMA) models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

Outline

Select order of p,d,q

- Once you have a stationary time series, the next step is to select the appropriate ARIMA model. - Number of differencing determine *d*
- This means finding the most appropriate values for *p* and *q* in the $ ARIMA(p, d, q) model.
- To do so, you need to examine the Autocorrelation and Partial Autocorrelation of the stationary time series.

Partial autocorrelations

Partial autocorrelations measure relationship between y_t and y_{t-k}, when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

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$$\begin{aligned}\alpha_k &= k\text{th partial autocorrelation coefficient}\\ &= \text{equal to the estimate of } \phi_k \text{ in regression:}\end{aligned}$$

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_k y_{t-k}.$$

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- Varying number of terms on RHS gives α_k for different values of k.
- There are more efficient ways of calculating α_k.
- $\alpha_1 = \rho_1$
- same critical values of $\pm 1.96/\sqrt{T}$ as for ACF.

ACF and PACF interpretation

AR(1)

$$\rho_k = \phi_1^k \qquad \text{for } k = 1, 2, \dots;$$

$$\alpha_1 = \phi_1 \qquad \alpha_k = 0 \qquad \text{for } k = 2, 3, \dots.$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

ACF and PACF interpretation

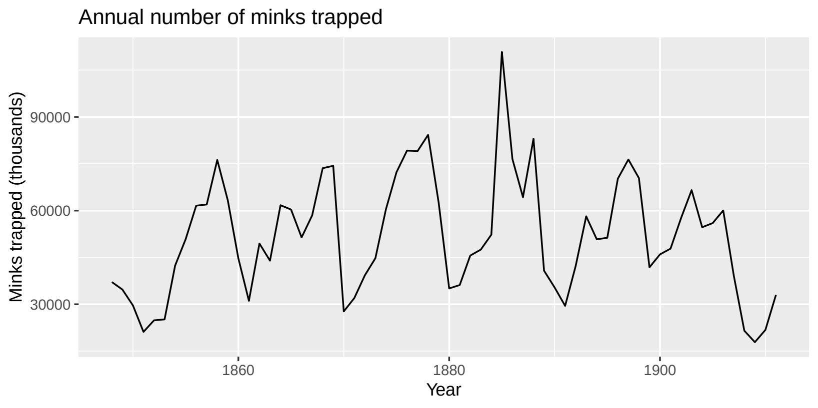
MA(1)

$$\rho_1 = \theta_1 \qquad \rho_k = 0 \qquad \text{for } k = 2, 3, \ldots;$$
$$\alpha_k = -(-\theta_1)^k$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF

Example: Mink trapping



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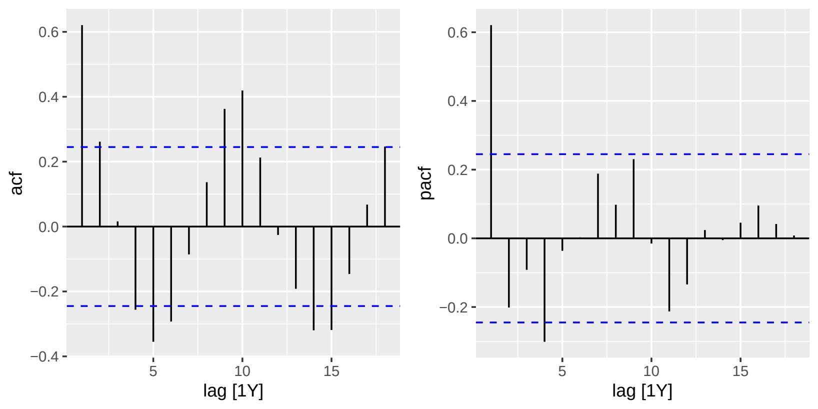

Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$.

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- MLE is very similar to least squares estimation obtained by minimizing

$$\sum_{t-1}^{T} e_t^2$$

Information criteria

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2\log(L) + 2(p + q + k + 1),$$

where L is the likelihood of the data,

$k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p+q+k+1)(p+q+k+2)}{T-p-q-k-2}.$$

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Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p+q+k-1).$$

Good models are obtained by minimizing either the AIC, AICc or BIC. Our preference is to use the AICc.

Outline

How does ARIMA() work?

A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: p, q, d

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d and D via KPSS test and seasonal strength measure.
- Select p, q by minimising AICc.
- Use stepwise search to traverse model space.

How does ARIMA() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \tfrac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

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where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

Step1: Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$
ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

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ARIMA$(0, d, 1)$

Step 2: Consider variations of current model:

- vary one of p, q, from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

Repeat Step 2 until no lower AICc can be found.

Modelling procedure with `ARIMA`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
3. If the data are non-stationary: take first differences of the data until the data are stationary.
4. Examine the ACF/PACF: Is an AR(p) or MA(q) model appropriate?
5. Try your chosen model(s), and use the AICc to search for a better model.
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

Automatic modelling procedure with `ARIMA`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
3. Use `ARIMA` to automatically select a model.
4. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
5. Once the residuals look like white noise, calculate forecasts.

Outline

Point forecasts

1. Rearrange ARIMA equation so y_t is on LHS.
2. Rewrite equation by replacing t by $T + h$.
3. On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \dots$.

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \ldots.$$

Outline

Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$ Non-seasonal part of the model	$\uparrow$ Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)$_{12}$ will show:

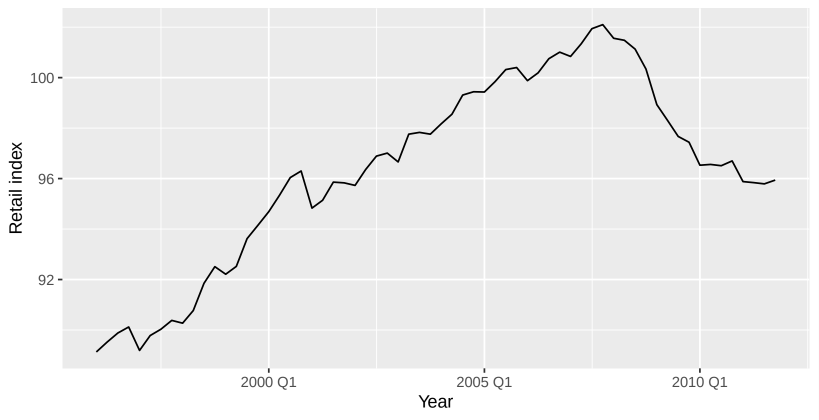
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ...

ARIMA(0,0,0)(1,0,0)$_{12}$ will show:

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

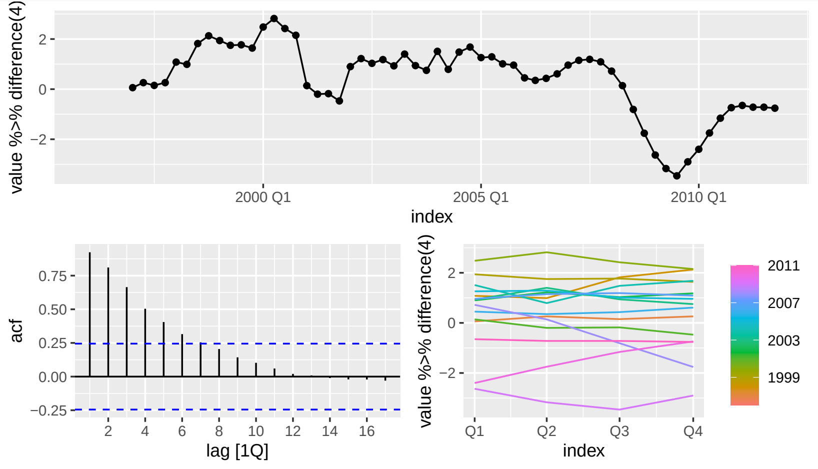
European quarterly retail trade

```
eu_retail %>% autoplot(value) +
  xlab("Year") + ylab("Retail index")
```



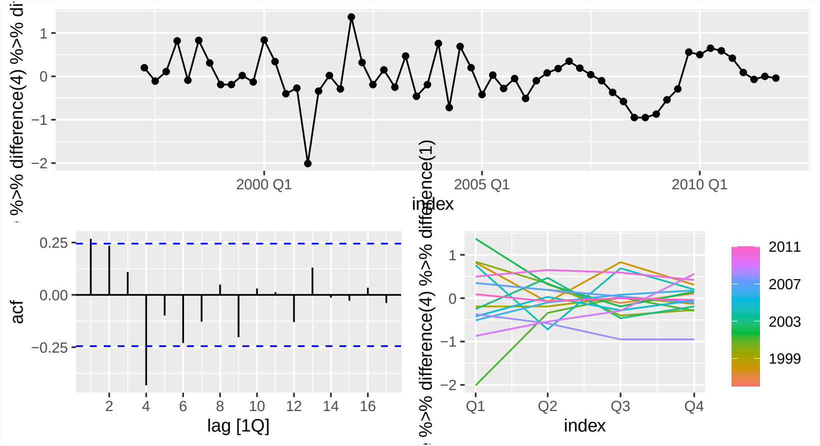
European quarterly retail trade

```
eu_retail %>% gg_tsdisplay(
  value %>% difference(4))
```



European quarterly retail trade

```
eu_retail %>% gg_tsdisplay(
  value %>% difference(4) %>% difference(1))
```



European quarterly retail trade

- $d = 1$ and $D = 1$ seems necessary.
- Significant spike at lag 1 in ACF suggests non-seasonal MA(1) component.
- Significant spike at lag 4 in ACF suggests seasonal MA(1) component.
- Initial candidate model: ARIMA$(0,1,1)(0,1,1)_4$.
- We could also have started with ARIMA$(1,1,0)(1,1,0)_4$.

European quarterly retail trade

```r
fit <- eu_retail %>%
  model(arima = ARIMA(value ~ pdq(0,1,1) + PDQ(0,1,1)),
        auto_arima = ARIMA(value))
fit %>% report()
```

```
## # A tibble: 2 x 8
##   .model     sigma2 log_lik   AIC  AICc   BIC ar_ro~1 ma_r
##   <chr>       <dbl>   <dbl> <dbl> <dbl> <dbl> <list>  <lis
## 1 arima       0.188   -34.6  75.3  75.7  81.5 <cpl>   <cpl
## 2 auto_ari~   0.156   -28.6  67.3  68.4  77.6 <cpl>   <cpl
## # ... with abbreviated variable names 1: ar_roots,
## #   2: ma_roots
```

```r
fit %>% select(arima) |> gg_tsresiduals()
```

European quarterly retail trade

```r
augment(fit) %>%
  features(.resid, ljung_box, lag = 8, dof = 2)
```

```
## # A tibble: 2 x 3
##   .model     lb_stat lb_pvalue
##   <chr>        <dbl>     <dbl>
## 1 arima       10.7      0.0997
## 2 auto_arima   0.511    0.998
```