Discrete Optimization

1. Greedy Algorithms & MATROIDS

1.1. Basics

Def.: An <u>Independent Set System</u> is a pair $(E, \mathcal{F})$, $\mathcal{F} \subseteq \mathcal{P}(E)$ where E is a <u>finite</u> set.

M1) $\emptyset \in \mathcal{F}$

M2) If $X \subseteq Y$, $Y \in \mathcal{F}$ then $X \in \mathcal{F}$.

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$X \in \mathcal{F}$ is called <u>independent</u>

$X \in \mathcal{P}(E) \setminus \mathcal{F}$ is called <u>dependent</u>

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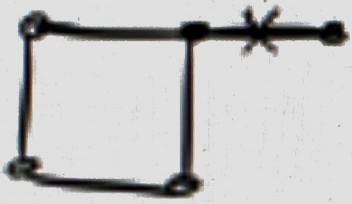
BASIS:

A maximally (inclusion-wise) independent set is called a BASIS.

CIRCLE: Inclusion-wise minimal dependent sets are called CIRCLES.



← not a CIRCLE

Examples

B1) $E = \mathbb{R}^n$ or $E \subseteq_{\text{fin}} \mathbb{R}^n$

$E = K^n$, finite field K

$\mathcal{F} = \{ X \subseteq \mathbb{R}^n$, X endlich :

$\exists\, y\, (\in \mathbb{R}^{|X|}),\ y \neq 0$:

$$\sum_{x \in X} \gamma_x x = 0 \}$$

lin. independance

"representable over the field $K (= \mathbb{R})$."

B2) $E = \{w_i\}_{[n]} \subseteq \mathbb{N}_0$

$[n] := \{1, \dots, n\}$

$B \in \mathbb{N}$

$$\mathcal{F} = \{ X \subseteq E \mid \sum_{w_i \in B} w_i \leq B \}$$

"Knapsack"

B3) E, $|E| = n$

$m \in \mathbb{N}$

$$\mathcal{F} = \{X \subseteq E \mid |X| \leq m\}$$

„uniform Matroid"

B4) $E = E(G)$

$$\mathcal{F} = \{X \subseteq E \mid X \text{ is circlefree}\}$$

„Graphic Matroid"

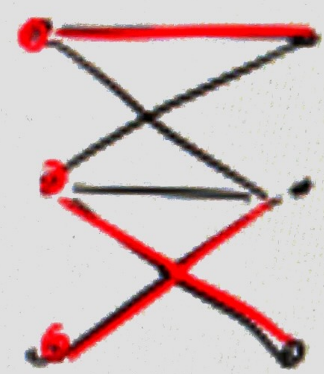
Def.: **Rank & Closure of an ISS**

let $(E, \mathcal{F})$ be an ISS.

For $X \subseteq E$ we call

$$r(X) := \max\{|Y| : Y \subseteq X, Y \in \mathcal{F}\}$$

the RANK of X.

$r\Big($  $\Big) = 3$

$|Y| = 3$

Def.: Closure of $X \subseteq E$

$$\mathrm{cl}(X) := \{ y \in E : r(X \cup \{y\}) = r(X) \}.$$

1.2. Optimization over ISS

Problem 1:

Input $(E, \mathcal{F})$ & $c : E \to \mathbb{R}^+$

Out: $X \in \mathcal{F}$:

$$c(X) = \sum_{e \in X} c(e) \text{ is maximal}$$

in $\mathcal{F}$.

Problem 2:

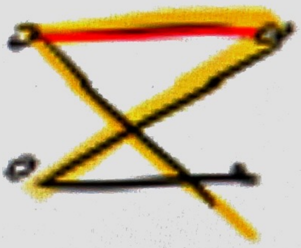
Input: $(E, \mathcal{F})$ & $c: E \to \mathbb{R}$

Out: Basis B with $c(B)$ minimal.

Remark:

There is the term BASIS for an ISS $(E, \mathcal{F})$.

For $X \subseteq E$ a BASIS B is a set $B \subseteq X$ if $B \in \mathcal{F}$ and cannot be independently extended in X.



$K_{2,2}$

B

1.3. Matroid

Def.: <u>Matroid</u>

Let $(E, \mathcal{F})$ be an ISS with

M3) IF: $\forall X, Y \in \mathcal{F}$ & $|X| > |Y|$

THEN: $\exists x \in X \setminus Y$ such that

$$Y \cup \{x\} \in \mathcal{F}.$$

$(E, \mathcal{F})$ fulfills M1 – M3 it is called a <u>Matroid</u>.

<u>Examples:</u> · E set of vectors in K^n