1 Derivation of Strassen algorithm

1.1 Gauss's Hint

An example to reduce the time of multiplication

$$(A + Bi)(C + Di) = (A \cdot C - B \cdot D) + (A \cdot D + B \cdot C)i \tag{1}$$

The equation (1) we used four multiplications

If we make some changes

$$(A \cdot C - B \cdot D) + [(A + B)(D + C) - A \cdot C - B \cdot D]i \tag{2}$$

After making some changes, we used three multiplications

$$\begin{cases} A \cdot C \\ B \cdot D \\ A \cdot D \\ B \cdot C \end{cases} \rightarrow \begin{cases} A \cdot C \\ B \cdot D \\ (A + B) \cdot (D + C) \end{cases} \tag{3}$$

1.2 Strassen Algorithm

For three Matrices A B and C

$$C = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} = A \times B = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \times \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

If we use ordinary Matrix arithmetic

$$\begin{cases} c_{00} & = a_{00} \times b_{00} + a_{01} \times b_{10} \\ c_{01} & = a_{00} \times b_{01} + a_{01} \times b_{11} \\ c_{10} & = a_{10} \times b_{00} + a_{11} \times b_{10} \\ c_{11} & = a_{10} \times b_{01} + a_{11} \times b_{11} \end{cases} \tag{4}$$

We used eight multiplications

$$
\begin{cases}
a_{00} \times b_{00} \\
a_{01} \times b_{10} \\
a_{00} \times b_{01} \\
a_{01} \times b_{11} \\
a_{10} \times b_{00} \\
a_{11} \times b_{10} \\
a_{10} \times b_{01} \\
a_{11} \times b_{11}
\end{cases}
\tag{5}
$$

Based on Gauss's Hint, we can reduce the time of multiplications

1.2.1 Step 1: Convert C_{00}

$$
\begin{aligned}
C_{00} &= a_{00} \times b_{00} + a_{01} \times b_{10} \\
&= (a_{00} + a_{11})(b_{00} + b_{11}) + (a_{01} - a_{11})(b_{10} + b_{11}) \\
&\quad - a_{00}b_{11} - a_{11}b_{00} - a_{11}b_{11} - a_{01}b_{11} + a_{11}b_{10} + a_{11}b_{11} \\
&= (a_{00} + a_{11})(b_{00} + b_{11}) + (a_{01} - a_{11})(b_{10} + b_{11}) \\
&\quad - a_{00}b_{11} - a_{11}b_{00} - a_{01}b_{11} + a_{11}b_{10}
\end{aligned}
$$

Further extraction of common factors:

$$
\begin{aligned}
C_{00} = {} & (a_{00} + a_{11})(b_{00} + b_{11}) + (a_{01} - a_{11})(b_{10} + b_{11}) \\
& - b_{11}(a_{01} + a_{00}) + a_{11}(-b_{00} + b_{10})
\end{aligned}
$$

1.2.2 Step 2: Convert C_{11}

$$
\begin{aligned}
C_{11} &= a_{10} \times b_{01} + a_{11} \times b_{11} \\
&= (a_{10} - a_{00})(b_{01} + b_{00}) + (a_{11} + a_{00})(b_{11} + b_{00}) \\
&\quad - a_{10}b_{00} + a_{00}b_{01} + a_{00}b_{00} - a_{11}b_{00} - a_{00}b_{11} - a_{00}b_{00} \\
&= (a_{10} - a_{11})(b_{01} + b_{00}) + (a_{11} + a_{00})(b_{11} + b_{00}) \\
&\quad - a_{10}b_{00} + a_{00}b_{01} - a_{11}b_{00} - a_{00}b_{11}
\end{aligned}
$$

Further extraction of common factors:

$$
\begin{aligned}
C_{11} = {} & (a_{10} - a_{11})(b_{01} + b_{00}) + (a_{11} + a_{00})(b_{11} + b_{00}) \\
& - b_{00}(a_{11} + a_{10}) + a_{00}(b_{01} - b_{11})
\end{aligned}
$$

1.2.3 Step 3: Extraction of common factors in C_{00} and C_{11}

$$\begin{cases} M_1 = (a_{10} + a_{11})(b_{01} + b_{00}) \\ M_2 = b_{00}(a_{11} + a_{10}) \\ M_3 = a_{00}(b_{01} - b_{11}) \\ M_4 = a_{11}(b_{10} - b_{00}) \\ M_5 = (a_{01} + a_{00})b_{11} \\ M_6 = (a_{10} - a_{11})(b_{01} + b_{00}) \\ M_7 = (a_{01} - a_{11})(b_{10} + b_{11}) \end{cases}$$

1.2.4 Step 4: Convert C_{01} and C_{10}

$$C_{01} = M_3 + M_5$$
$$C_{10} = M_2 + M_4$$

1.2.5 Step 5: Express C by using M_i

$$C = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} = \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{pmatrix}$$