Algorithm 1 Algorithm to determine optimal parallelization strategy for an operator DAG

1: **procedure** FindOptStrategy($G = (V, E)$, $nprocs$)
2: **Input(s):** $G = (V, E)$: Operator DAG.
$(\forall v \in V)[v.dim]$: Dimension of the iteration domain of the operation of v.
$nprocs$: Number of processors.
3: **Output(s):** $\mathcal{S}$: Optimal parallelization starategy for G

4: ▷ Initialize vertex properties
5: **for all** $v \in V$ **do**
6: $v.n \leftarrow \{u \mid (u, v) \in E \vee (v, u) \in E\}$ ▷ Neighbors
7: $v.un \leftarrow v.n$ ▷ Unprocessed neighbors
8: $v.cfg \leftarrow \{(c_1, c_2, \ldots, c_{v.dim}) \mid (\forall i \in [1, v.dim])[c_i \in \mathbb{Z}^+] \wedge \prod_{i=1}^{v.dim} c_i \leq nprocs\}$
9: ▷ Set of all possible configurations
10: **end for**

11: ▷ Populate $v.tbl$ with all possible combinations of neighboring configurations. Each entry in $v.tbl$ is a sub-strategy for the nodes $\{v\} \cup v.n$
12: **for all** $v \in V$ **do**
13: $v.tbl \leftarrow \prod_{u \in \{v\} \cup v.n} \{(u, cfg) \mid cfg \in u.cfg\}$
14: **end for**

15: $U \leftarrow V$ ▷ Set of unprocessed vertices
16: **while** $U \neq \emptyset$ **do**
17: ▷ Choose a vertex with least no. of unprocessed neighbors from U
18: $v_{min} \leftarrow \arg\min_{v \in U} |v.un|$

19: ▷ Partition $v_{min}.tbl$ such that in each set, all the neighbor configurations are the same. Refer Alg. 2 for the function IsValid
20: $\mathcal{P} \leftarrow \{S_1, S_2, \ldots \mid \cup_{S_i} = v_{min}.tbl \wedge i \neq j \implies S_i \cap S_j = \emptyset \wedge$
$(\forall s_x \in S_i)(\forall s_y \in S_i)[\text{IsValid}(s_x, s_y, v_{min}.n) = \top] \wedge$
$i \neq j \implies (\forall s_x \in S_i)(\forall s_y \in S_j)[\text{IsValid}(s_x, s_y, v_{min}.n) = \bot]\}$
21: ▷ Iterate over each set $S \in \mathcal{P}$ and update $v_{min}.tbl$ so that it only contains the sub-strategy in S that has the least cost.
22: **for all** $S \in \mathcal{P}$ **do**
23: $best \leftarrow \arg\min_{s \in S} \text{Cost}(G, s)$
24: $rem \leftarrow S \setminus \{best\}$
25: $v_{min}.tbl \leftarrow v_{min}.tbl \setminus rem$
26: **end for**

27: **for all** $v \in v_{min}.n$ **do**
28: ▷ Restrict the tables of the neighbors based on the best sub-strategies found for v_{min} in Line 23
29: $common = (v_{min}.n \cap v.n) \cup \{v_{min}, v\}$
30: $v.tbl \leftarrow \{t_1 \in v.tbl \mid (\exists t_2 \in v_{min}.tbl)[\text{IsValid}(t_1, t_2, common) = \top])\}$

31: $v.un \leftarrow v.un \setminus \{v_{min}\}$
32: **end for**

33: $U \leftarrow U \setminus \{v_{min}\}$
34: **end while**

35: ▷ Collect all valid strategies for G from $v.tbl$, and pick the one that minimizes the cost.
36: $\mathcal{S}' \leftarrow$ ValidStrategies(G) ▷ Refer Alg. 3
37: $\mathcal{S} \leftarrow \arg\min_{S \in \mathcal{S}'} \text{Cost}(G, S)$
38: **return** $\mathcal{S}$
39: **end procedure**

Algorithm 2 Returns $\top$ if the two sub-strategies S_1 and S_2 have same configurations for the vertices in V.

```
procedure IsValid(S_1, S_2, V)
    N_1 ← {(n, c) ∈ S_1 | n ∈ V}
    N_2 ← {(n, c) ∈ S_2 | n ∈ V}

    if N_1 ≠ N_2 then
        return ⊥
    end if

    return ⊤
end procedure
```

Algorithm 3 Returns a set of valid strategies

```
procedure ValidStrategies(G = (V, E))
    𝒮 ← ∅
    for all v ∈ V do
        if 𝒮 = ∅ then
            𝒮 ← v.tbl
            Continue
        end if

        𝒮' ← ∅
        for all S ∈ 𝒮 do
            for all T ∈ v.tbl do
                U_1 ← {u | (u, c) ∈ S}
                U_2 ← {u | (u, c) ∈ v.tbl}

                if IsValid(S, T, U_1 ∩ U_2) = ⊤ then
                    𝒮' ← 𝒮' ∪ {S ∪ T}
                end if
            end for
        end for
        𝒮 ← 𝒮'
    end for

    return 𝒮
end procedure
```

1 Proof of optimality

Notation Given a strategy $\mathcal{S}$ of a graph $G=(V,E)$ and $V' \subseteq V$, $\mathcal{S}_{|V'}$ denotes the sub-strategy restricted to the subgraph of G induced by V', and $\mathcal{S}(v)$ denotes the configuration for the vertex v in $\mathcal{S}$. Let $\widehat{\mathcal{S}}$ be an optimal strategy for an operator DAG $G=(V,E)$, and $\widehat{T} = \textsc{Cost}(G, \widehat{\mathcal{S}})$ be the optimal parallelization cost for $\widehat{\mathcal{S}}$.

Alg. 3 takes a DAG G as input, whose vertices have an associated property *tbl* that contains a set of sub-strategies, and returns a set of all possible valid strategies for G. Line 13 of Alg. 1 assigns all possible sub-strategies to each vertex in G. It can be seen that just before entering the loop at line 16 in Alg. 1, we have $\widehat{\mathcal{S}} \in \textsc{ValidStrategies}(G)$.

We will show that the invariant $(\exists \mathcal{S} \in \textsc{ValidStrategies}(G))[\textsc{Cost}(G,\mathcal{S}) \leq \widehat{T}]$ is maintained everywhere after line 14 in Alg. 1.

Theorem 1. *Given an operation DAG $G=(V,E)$ and an optimal parallelization strategy $\widehat{\mathcal{S}}$ for G, the invariant $(\exists \mathcal{S} \in \textsc{ValidStrategies}(G))[\textsc{Cost}(G,\mathcal{S}) \leq \widehat{T}]$ is maintained in any iteration of the loop L in lines 16–34 of Alg. 1.*

Proof. As a contradiction, consider that there exists an iteration l of loop L where the invariant is violated for the first time. Lines 25 and 30 are the only places in loop L where some of the valid sub-strategies are removed from $v.tbl$. We will consider both the cases below.

Let ϕ_l be the set of valid strategies returned by $\textsc{ValidStrategies}(G)$ at the beginning of an iteration l, and ϕ_{l+1} be the set of valid strategies returned at the end of iteration l.

Case I Consider line 25 of Alg. 1. Let $V' = V \setminus \{v_{min}\}$, and $E' = E \cap (V' \times V')$. Note that line 25 only removes some of the configurations of the vertex v_{min}, i.e., $\phi_{l|V'}$ does not change after execution of line 25.

Let B denote the set of all *best* sub-strategies picked in line 25 in iteration l. For some $\mathcal{S} \in B$, let $B(\mathcal{S}_{|v.n})$ denote the configuration c for v s.t. $\{(v,c)\} \cup \mathcal{S}_{|v.n} = \mathcal{S}$. It can be seen that $B(\mathcal{S}_{|v.n})$ is unique from the construction of $\mathcal{P}$ in line 20.

We have,

$$\begin{aligned}
\widehat{T} = \quad & \min_{\mathcal{S}\in\phi_l}\{\sum_{v\in V} t_V(v,\mathcal{S}(v)) + \sum_{(u,v)\in E} t_E((u,v),\mathcal{S}(u),\mathcal{S}(v))\} \\
= \quad & \min_{\mathcal{S}\in\phi_l}\{t_V(v_{min},\mathcal{S}(v_{min})) + \sum_{(u,v_{min})\in E} t_E((u,v_{min}),\mathcal{S}(u),\mathcal{S}(v_{min})) + \\
& \sum_{(v_{min},v)\in E} t_E((v_{min},v),\mathcal{S}(v_{min}),\mathcal{S}(v)) + \sum_{v\in V'} t_V(v,\mathcal{S}(v)) + \sum_{(u,v)\in E'} t_E((u,v),\mathcal{S}(u),\mathcal{S}(v))\} \\
\geq \quad & \min_{\mathcal{S}\in\phi_l}\{t_V(v_{min},B(\mathcal{S}_{|v_{min}.n})) + \sum_{(u,v_{min})\in E} t_E((u,v_{min}),\mathcal{S}(u),B(\mathcal{S}_{|v_{min}.n})) + \\
& \sum_{(v_{min},v)\in E} t_E((v_{min},v),B(\mathcal{S}_{|v_{min}.n}),\mathcal{S}(v)) + \sum_{v\in V'} t_V(v,\mathcal{S}(v)) + \sum_{(u,v)\in E'} t_E((u,v),\mathcal{S}(u),\mathcal{S}(v))\} \qquad (1)
\end{aligned}$$

contradicting our assumption, as there exists a strategy $\mathcal{S} \in \phi_l$ and $\{(v_{min}, B(\mathcal{S}_{|v_{min}.n})) \cup \mathcal{S}_{|V'}\} \in \phi_{l+1}$ whose cost is equal to $\widehat{T}$.

Eq. 1 above follows from the fact that

$$best = \arg\min_{\mathcal{S}_{|V''}} \sum_{v\in\{v_{min}\}\cup v_{min}.n} t_V(v,\mathcal{S}_{|V''}(v)) + \sum_{(u,v)\in E\setminus E'} t_E((u,v),\mathcal{S}_{|V''}(u),\mathcal{S}_{|V''}(v))$$

Case II Consider line 30 in Alg. 1. As the computation in line 30 removes a sub-strategy S from *tbl* only if there doesn't exist any valid strategy that comprises of S, no optimal strategy is eliminated in this step.

□