04_1osc_superHopf

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1 Oscillations in Two Coupled Variables

1.1 Mathematical Model

Single Variable, first-order differential equation

The equation:

$$\frac{dEx}{dt} = h_{ex} - Ex + c_1 * tanh(Ex) - c_2 * tanh(In)$$
$$\frac{dIn}{dt} = h_{in} - In + c_3 * tanh(Ex) - c_4 * tanh(In)$$

where Ex and In are variables that changes with time t, h_{ex} and c_i are model parameters, $tanh$ is the tangens hyperbolicus.

The model dynamics can be characterised by scans of parameter h_ex.

Note that for $c_2 = c_3 = 0$, the two variables are uncoupled and independent. We want to study the interaction between the two variables as a function of one of the coupling parameters, c_2.

1.2 Import Functions

[18]:
```
from scipy.integrate import odeint
from scipy.signal import find_peaks

from numpy import zeros, tanh, linspace, ndarray, var
from numpy import asarray, array, around, flip

from numpy.random import uniform, seed

from matplotlib.pyplot import subplots
```

1.3 Model and Functions

[21]:
```
def sigmoid(u):

    return tanh(u)
```

```
def N_oscillators(y, t, h_ex, h_in, pars):

    tau_ex, tau_in, c_1, c_2, c_3, c_4 = pars

    dydt = (
        (h_ex - y[0] + c_1*sigmoid(y[0]) - c_2*sigmoid(y[1]))*tau_ex,
        (h_in - y[1] + c_3*sigmoid(y[0]) - c_4*sigmoid(y[1]))*tau_in
       )

    return dydt
```

[23]:
```
def run_bif_diagram(time_stop, sr, par_set, h_in, y_ini, pars):

    time = linspace(start=0, stop=time_stop, num=time_stop*sr)

    results_max_f      = dict()
    results_max_inds_f = dict()
    results_min_f      = dict()
    results_min_inds_f = dict()

    rows = time.size

    num = 0
    # Simulation "forward"
    for par in par_set:

        h_ex = par

        y_f = odeint(func=N_oscillators, y0=y_ini, t=time,
                  args=(h_ex, h_in, pars),
                  hmax=0.1)

        for num, series in enumerate(y_f[rows//2:,:-1:2].T):

            if var(series) < 0.00005:

                if num not in results_max_f:

                    results_max_f[num]      = [series[-1]]
                    results_max_inds_f[num] = [0]
                    results_min_f[num]      = [series[-1]]
                    results_min_inds_f[num] = [0]

                else:
                    results_max_f[num].append(series[-1])
                    results_max_inds_f[num].append(0)
```

```python
                results_min_f[num].append(series[-1])
                results_min_inds_f[num].append(0)


        else:
            y_f_max_inds = find_peaks(series, distance=100)
            y_f_maxs     = series[y_f_max_inds[0]]

            y_f_min_inds = find_peaks(-series, distance=100)
            y_f_mins     = series[y_f_min_inds[0]]

            if num not in results_max_f:

                results_max_f[num]      = [y_f_maxs]
                results_max_inds_f[num] = [y_f_max_inds]
                results_min_f[num]      = [y_f_mins]
                results_min_inds_f[num] = [y_f_min_inds]

            else:

                results_max_f[num].append(y_f_maxs)
                results_max_inds_f[num].append(y_f_max_inds)
                results_min_f[num].append(y_f_mins)
                results_min_inds_f[num].append(y_f_min_inds)

    if par != par_set[-1]:

        y_ini = y_f[-1, :]


results_max_b      = dict()
results_max_inds_b = dict()
results_min_b      = dict()
results_min_inds_b = dict()


# Simulation "backward"
for par in flip(par_set):

    h_ex = par

    y_b = odeint(func=N_oscillators, y0=y_ini, t=time,
             args=(h_ex, h_in, pars),
             hmax=0.1)

    for num, series in enumerate(y_b[rows//2:,:-1:2].T):
```

```
            if var(series) < 0.00005:

                if num not in results_max_b:

                    results_max_b[num]      = [series[-1]]
                    results_max_inds_b[num] = [0]
                    results_min_b[num]      = [series[-1]]
                    results_min_inds_b[num] = [0]

                else:
                    results_max_b[num].append(series[-1])
                    results_max_inds_b[num].append(0)
                    results_min_b[num].append(series[-1])
                    results_min_inds_b[num].append(0)


            else:
                y_b_max_inds = find_peaks(series, distance=100)
                y_b_maxs     = series[y_b_max_inds[0]]

                y_b_min_inds = find_peaks(-series, distance=100)
                y_b_mins     = series[y_b_min_inds[0]]

                if num not in results_max_b:

                    results_max_b[num]      = [y_b_maxs]
                    results_max_inds_b[num] = [y_b_max_inds]
                    results_min_b[num]      = [y_b_mins]
                    results_min_inds_b[num] = [y_b_min_inds]

                else:

                    results_max_b[num].append(y_b_maxs)
                    results_max_inds_b[num].append(y_b_max_inds)
                    results_min_b[num].append(y_b_mins)
                    results_min_inds_b[num].append(y_b_min_inds)

        # y_ini = y_b[-1, :]

    return results_min_f, results_max_f, results_min_b, results_max_b


def plot_bifdiagram(results_min_f, results_max_f,
                    results_min_b, results_max_b,
                    par_set):

    N = len(results_min_f)
```

```
fig, ax = subplots()

for xe, ye in zip(par_set, results_max_f[0]):

    if not isinstance(ye, ndarray):
        ax.scatter(xe, ye, c='r', s=5)
    else:
        ax.scatter([xe] * len(ye), ye, c='m', s=50, marker='x')

for xe, ye in zip(par_set, results_min_f[0]):

    if not isinstance(ye, ndarray):
        ax.scatter(xe, ye, c='r', s=5)
    else:
        ax.scatter([xe] * len(ye), ye, c='m', s=50, marker='x')

for xe, ye in zip(flip(par_set), results_max_b[0]):

    if not isinstance(ye, ndarray):
        ax.scatter(xe, ye, c='b', s=5)
    else:
        ax.scatter([xe] * len(ye), ye, c='b', s=20, marker='P')

for xe, ye in zip(flip(par_set), results_min_b[0]):

    if not isinstance(ye, ndarray):
        ax.scatter(xe, ye, c='b', s=5)
    else:
        ax.scatter([xe] * len(ye), ye, c='b', s=20, marker='P')

ax.set_xticks(linspace(par_min, par_max, 5));
ax.set_xticklabels(around(linspace(par_min, par_max, 5), 2), fontsize=16);
ax.set_xlabel('Parameter', fontsize=16)

ax.set_ylabel('Ex', fontsize=14)

y_min, y_max = ax.get_ylim()

ax.set_yticks(linspace(y_min, y_max, 3));
ax.set_yticklabels(around(linspace(y_min, y_max, 3),2), fontsize=14);

ax.set_title('Bifurcation Diagram')


fig.tight_layout()
```

```
    return fig, ax
```

[25]:
```
def plot_series(time, data, time_begin, time_end, sr):

    N = data.shape[1]//2

    name_vars = ('Ex', 'In')

    no_vars = 2*N

    fig, ax = subplots(ncols=2, figsize=(6, 4))

    # for ind in arange(no_vars):

    ax[0].plot(time[time_begin*sr:time_end*sr], data[time_begin*sr:time_end*sr,␣
 ↪0], linewidth=2, c='b', label='Excitatory')
    ax[0].plot(time[time_begin*sr:time_end*sr], data[time_begin*sr:time_end*sr,␣
 ↪1], linewidth=2, c='r', label='Inhibitory')
    ax[0].set_xticks(linspace(0, time_end-time_begin, 5));
    ax[0].set_xticklabels(linspace(0, time_end-time_begin, 5));
    ax[0].set_xlabel('Time', fontsize=12);
    ax[0].set_ylabel('Variables', fontsize=12)
    y_min, y_max = ax[0].get_ylim()
    ax[0].set_yticks(linspace(y_min, y_max, 3));
    ax[0].set_yticklabels(around(linspace(y_min, y_max, 3),1), fontsize=14);
    ax[0].legend()
    ax[0].set_title('Time Series')
    ax[1].set_title('State Space')

    ax[1].plot(data[time_begin*sr:time_end*sr, 1], data[time_begin*sr:
 ↪time_end*sr, 0], linewidth=2, c='m')
    ax[1].set_xlabel('Inhibitory', fontsize=12);
    ax[1].set_ylabel('Excitatory', fontsize=12)

    fig.tight_layout()

    return fig, ax


def plot_state_space(time, data, time_begin, time_end, sr):

    N = data.shape[1]//2

    name_vars = ('Ex', 'In')

    no_vars = 2*N
```

```
    fig, ax = subplots(figsize=(6, 4))

    ax.plot(data[time_begin*sr:time_end*sr, 1], data[time_begin*sr:time_end*sr,␣
 ↪0], linewidth=2, c='b')
    # ax.set_xticks(linspace(0, time_end-time_begin, 5));
    # ax.set_xticklabels(linspace(0, time_end-time_begin, 5));
    ax.set_xlabel('In', fontsize=12);
    ax.set_ylabel('Ex', fontsize=12)
    y_min, y_max = ax.get_ylim()
    x_min, x_max = ax.get_xlim()
    ax.set_yticks(linspace(y_min, y_max, 3));
    ax.set_yticklabels(around(linspace(y_min, y_max, 3),1), fontsize=14);
    ax.set_xticks(linspace(x_min, x_max, 3));
    ax.set_xticklabels(around(linspace(x_min, x_max, 3),1), fontsize=14);

    fig.tight_layout()

    return fig, ax
```

1.4 Time Series

[28]:

```
# Excitatory input parameter
h_ex_0    = -2.5
h_in_0    = -4

# Supercritical Hopf parameters
pars = (1, 1, 4, 6, 6, 0)
# Bistability parameters
# pars = (1, 1, 4, 1, 6, 0)

# Initial conditions
seed(1108)
y_ini = uniform(size=2)
# y_ini = y[-1, :]

# Time array
time_stop = 10
sr        = 1000
time      = linspace(start=0, stop=time_stop, num=time_stop*sr)

# Simulation
y = odeint(func=N_oscillators, y0=y_ini, t=time,
          args=(h_ex_0, h_in_0, pars),
          hmax=0.1)

# Show final values of all variables
print('End of run:', list(around(y[-1,:],3)))
```

```
print('')
```

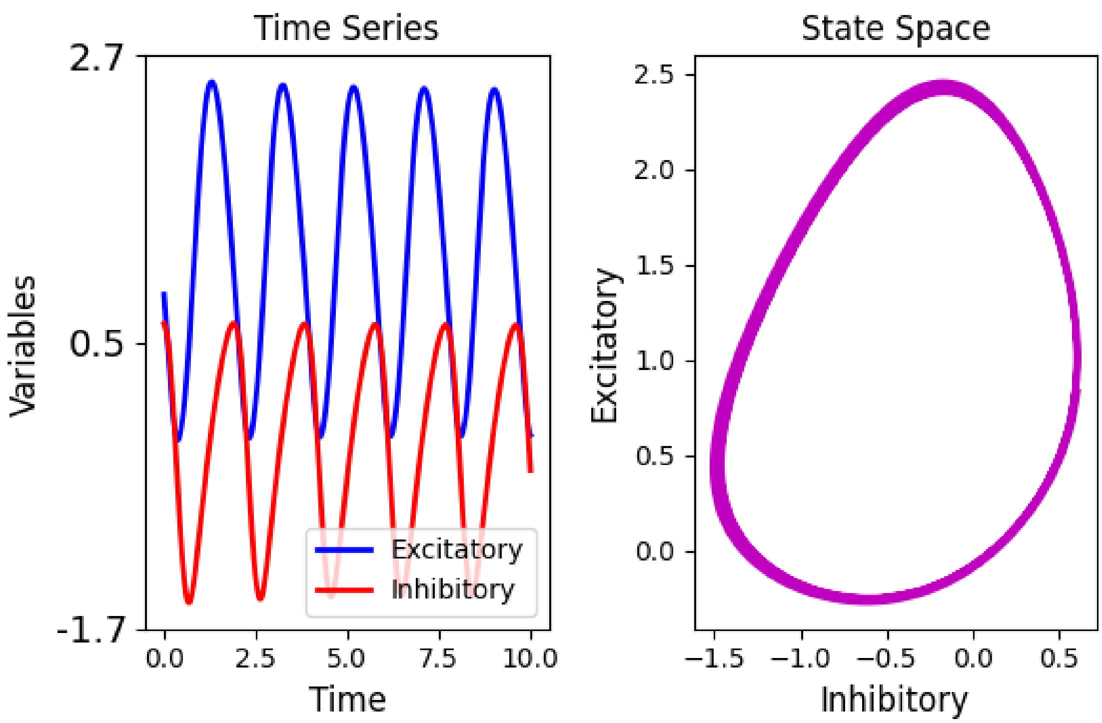
```
End of run: [np.float64(-0.237), np.float64(-0.502)]
```

[30]:
```
time_begin, time_end = 0, time_stop

fig, ax = plot_series(time, y, time_begin, time_end, sr)

title_chars = 'Figs/SuperHopf_Timeseries_h_ex' + str(h_ex_0) + '.png'
# fig.savefig(title_chars, format='png')
print(title_chars)
```

```
Figs/SuperHopf_Timeseries_h_ex-2.5.png
```



1.5 Bifurcation Diagram

[33]:
```
# Initial conditions
y_ini = y[-1, :]

# Bifurcation parameter range
steps = 50
par_min, par_max = -5, 5
```

```
par_set = linspace(par_min, par_max, steps)

# Stop time
time_stop = 100

results_min_f, results_max_f, results_min_b, results_max_b =␣
↪run_bif_diagram(time_stop, sr, par_set, h_in_0, y_ini, pars)

print('Scan complete!')
print('')
```

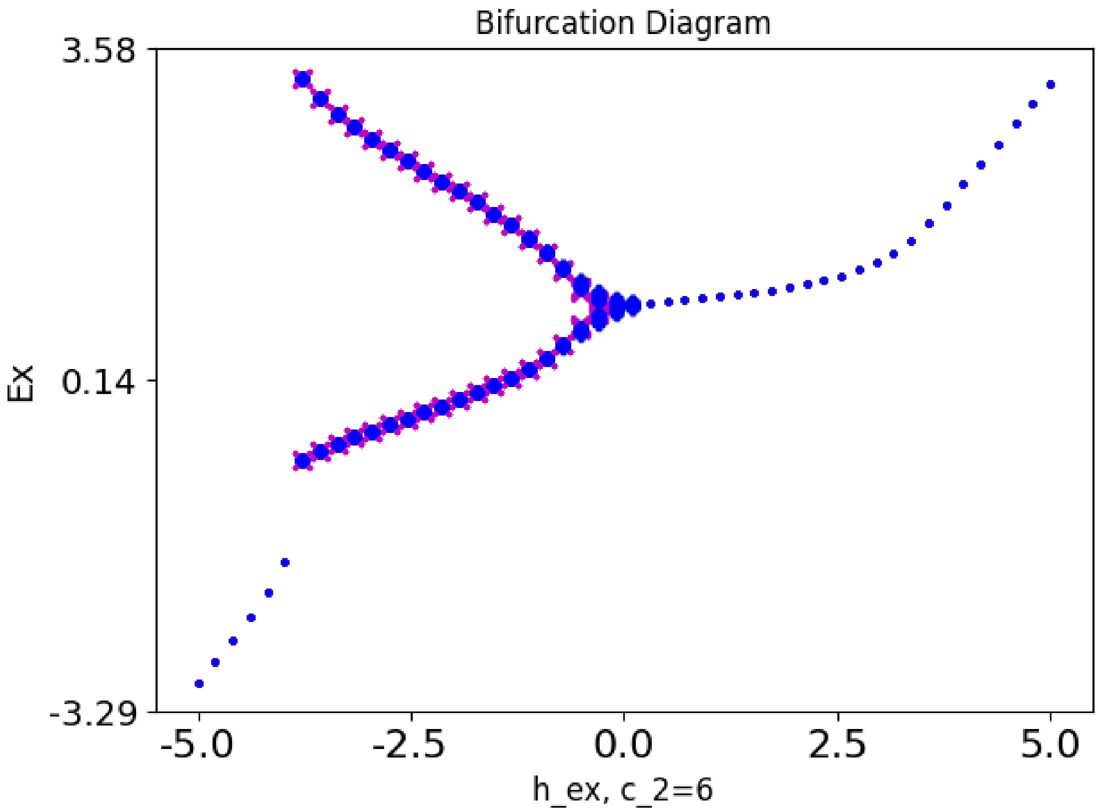
```
Scan complete!
```

[34]:
```
# Plot
fig, ax = plot_bifdiagram(results_min_f, results_max_f,
                          results_min_b, results_max_b, par_set)


ax.set_xlabel('h_ex, c_2=' + str(pars[3]), fontsize=12)

title_chars = '2Var_Bifs' + '_h_ex, c_2=' + str(pars[3]) + '.png'
# fig.savefig(title_chars, format='png')
title_chars
```

[34]:
```
'2Var_Bifs_h_ex, c_2=6.png'
```



The result for $c_2 = 6$ involves a SNIC bifurcation (saddle-node on invariant cycle) and a supercritial Hopf bifurcation (real-part of complex pair of eigenvalues of the Jacobian passing through zero). SNIC oscillations are large and slow, often non-sinusoidal. Hopf oscillations are sinusoidal and have a characteristic increase in amplitude as the parameter moves away from the bifurcation point.

1.6 New Time Series

[39]:
```
# Excitatory input parameter
h_ex_0    = -4.05 # Excitable at -4.05, small limit cycle at -0.6
h_in_0    = -4

# Supercritical Hopf parameters
pars = (1, 1, 4, 6, 6, 0)
# Bistability parameters
# pars = (1, 1, 4, 1, 6, 0)

# Initial conditions
seed(1108)
# y_ini = uniform(size=2)
y_ini = [1, -0]
# y_ini = y[-1, :]
```

```python
# Time array
time_stop = 100
sr        = 1000
time      = linspace(start=0, stop=time_stop, num=time_stop*sr)

# Simulation
y = odeint(func=N_oscillators, y0=y_ini, t=time,
          args=(h_ex_0, h_in_0, pars),
          hmax=0.1)

# Show final values of all variables
print('End of run:', list(around(y[-1,:],3)))
print('')
```

```
End of run: [np.float64(-1.861), np.float64(-9.717)]
```

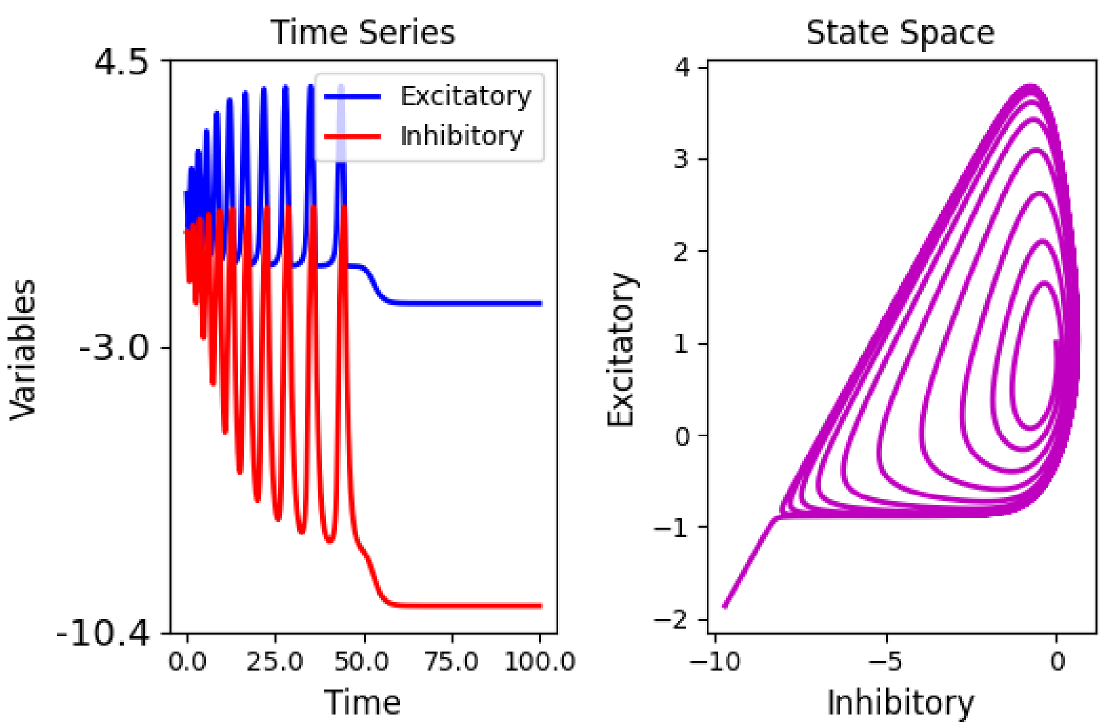
[41]:

```python
time_begin, time_end = 0, time_stop

fig, ax = plot_series(time, y, time_begin, time_end, sr)

title_chars = 'Figs/SuperHopf_Timeseries_h_ex' + str(h_ex_0) + '.png'
# fig.savefig(title_chars, format='png')
print(title_chars)
```

```
Figs/SuperHopf_Timeseries_h_ex-4.05.png
```



2 Conclusion

- Coupling between two variables can either result in a bistability of steady states or in periodic oscillations.
- Amplitude and frequency of oscillations depend on parameter settings.
- Interpretation: In invasive EEG, mesoscopic oscillations may results from local self-organised neural activity.

3 Try it Yourself

Re-run the above code with these parameter settings:

$pars = (1.2, 0.1, 4, 6, 6, 0)$

Here, the time constants have been made unequal. The excitatory population is now operating at a faster rate compared to the slow population. This leads to the disappearance of the supercritical Hopf bifurcation.

4 Notes on the Reading

4.1 Equilibrium by E. Itzikevich

This tutorial introduces the formal treatment of the concept of **State** in dynamical systems. It uses linear perturbation theory and shows how states and state transitions can be derived from the eigenvalues of the Jabian matrix of a system of coupled ordinary differential equations.

A similar treatment can be done on a linearised version of a matrix around a limit cycle. This is the Floquet theory. We can thus expect, oscillatory solutions (as the ones above) to also become unstable and leading to even more complex solutions.

4.2 The Dynamic Brain: From Spiking Neurons to Neural Masses and Cortical Fields, G Deco et al, and K Friston

This review summarises the reasoning why mesoscopic (large-scale) recordings like the sEEG can be modelled with a comparatively small number of collective variables. This helpt to bypass the need to consider every neuron in the brain or a specific brain area in order to describe the clinical (or experimental) findings. It shows how the mean field approximation is used to derive simple ODE models (page 7, Neural Modes and Masses) which allow a reasonable good description, specifically of pathological findings in the EEG.

[ ]:
```

```