1 Introduction

The motivation of this note is the following fact, used in proving the supporting hyperplane theorem in Bazarra's textbook[1]:

Theorem 1 (Bazarra)

Let V be a finite dimensional normed space, C be a nonempty convex set in V, and $\bar{x} \in \mathrm{Bd}(C)$. Then $\exists y_k \notin \mathrm{Cl}(C), k = 1, 2, \ldots$ s.t. $y_k \to \bar{x}$.

The theorem would be trivial if we only require $y_k \notin C, k = 1, 2, \ldots$. By the definition of $\mathrm{Bd}(C)$, within any open ball $B(\bar{x}; \epsilon)$[2] there is a point not in C. By reducing ϵ, we get a desired sequence. The theorem will not hold if C is a general set. For example, let $C = B(a; r) - \{a\}$ for some $a \in V$ and $r > 0$, then $a \in \mathrm{Bd}(C)$. Clearly, the theorem does not hold at a. In this example, C is not convex. Thus, we can expect that convexity plays a crucial role in the proof of the theorem.

We assume that the background space is a normed space V throughout this note. Some conclusions further require $\dim V < \infty$. Our goal is to demonstrate the following theorem.

Theorem 2 (Interior of Closure)

$\dim V < \infty$, C is a convex set in V. Then

$$\mathrm{Int}(C) = \mathrm{Int}(\mathrm{Cl}(C))$$

Then Thm. 1 follows immediately.

Proof (Proof of Thm. 1)

$\bar{x} \in \mathrm{Bd}(C)$, then $\bar{x} \notin \mathrm{Int}(C) = \mathrm{Int}(\mathrm{Cl}(C))$. Thus, $\exists y_n \in B(\bar{x}; 1/n)$ s.t. $y \notin \mathrm{Cl}(C)$. Then $\{y_n\}$ is a desired sequence.

In the following, we first prove Thm. 2 when $\mathrm{Int}(C) \neq \emptyset$. Then we consider the case of $\mathrm{Int}(C) = \emptyset$.

2 The Case of $\mathrm{Int}(C) \neq \emptyset$

For convenience, we introduce the following notations:

Definition 3 Line Segments

Given $x, y \in V$,

$$\begin{aligned}
(x, y) &= \{tx + (1-t)y | t \in (0, 1)\} \\
[x, y] &= \{tx + (1-t)y | t \in [0, 1]\} \\
[x, y) &= \{tx + (1-t)y | t \in (0, 1]\} \\
(x, y] &= \{tx + (1-t)y | t \in [0, 1)\}
\end{aligned}$$

The proof of Thm. 2 when $\mathrm{Int}(C) \neq$ is based on the following theorem.

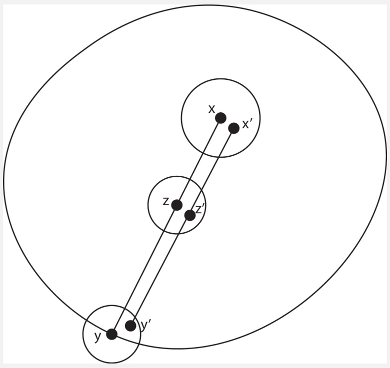
Theorem 4 General Line Segment Property

C convex, $x \in \mathrm{Int}(C)$ and $y \in \mathrm{Cl}(C)$. Then $(x, y) \subseteq \mathrm{Int}(C)$, namely

$$tx + (1-t)y \in \mathrm{Int}(C) \quad \forall t \in (0, 1).$$

[1] In the textbook, the background space is $\mathbb{R}^n$. We discuss the theorem in a normed space.
[2] An open ball $B(x; r)$ is defined as $B(x; r) = \{y \in V | \|y - x\| < r\}$.

Proof



Let $z = \lambda x + (1-\lambda)y$. $x \in \mathrm{Int}(C)$, then $\exists B(x;h) \subseteq C$. To show $z \in \mathrm{Int}(C)$, we need to find some $B(z;r) \subseteq C$. For any $z' \in B(z;r)$, we wish it to be a line combination of two points in C, so that $z' \in C$. More specifically, we wish

$$z' = \lambda x' + (1-\lambda)y'$$

where $x', y' \in C$, $x' \in B(x;h)$, and y' is near y. To ensure $x' \in B(x;h)$, we compute

$$\|x' - x\| = \|\frac{z' - (1-\lambda)y'}{\lambda} - \frac{z - (1-\lambda)y}{\lambda}\|$$
$$= \frac{1}{\lambda}\|(z' - z) - (1-\lambda)(y' - y)\| \leq \frac{1}{r}\|z' - z\| + \frac{1-\lambda}{\lambda}\|y' - y\| < \frac{r}{\lambda} + \frac{1-\lambda}{\lambda}\epsilon$$

Then we can choose

$$\frac{r}{\lambda} = \frac{h}{2}$$

Then $\|x' - x\| < h$ because ϵ is arbitrary.
Revert the reasoning gives a proof. $x \in \mathrm{Int}(C)$, then $\exists B(x;h) \subseteq C$. $y \in \mathrm{Cl}(C)$, then $\exists y' \in B(y;\epsilon) \cap C$ where $\epsilon = h\lambda/(2(1-\lambda))$. For any given $\lambda \in (0,1)$, let $r = \lambda h/2$. For any $z' \in B(z;r)$, let

$$x' = \frac{z' - (1-\lambda)y'}{\lambda}$$

Then

$$\|x' - x\| = \|\frac{z' - (1-\lambda)y'}{\lambda} - \frac{z - (1-\lambda)y}{\lambda}\|$$
$$= \frac{1}{\lambda}\|(z' - z) - (1-\lambda)(y' - y)\| \leq \frac{1}{r}\|z' - z\| + \frac{1-\lambda}{\lambda}\|y' - y\| < \frac{r}{\lambda} + \frac{1-\lambda}{\lambda}\epsilon = h$$

which implies $x' \in B(x;h) \subseteq C$. Thus $z' = \lambda x' + (1-\lambda)y' \in C$ since C is convex.

Now we can prove the theorem.

Theorem 5 The Case of $\mathrm{Int}(C) \neq \emptyset$

$\dim V < \infty$, C is convex nonempty set in V. Then $\mathrm{Int}(C) = \mathrm{Int}(\mathrm{Cl}(C))$.

Proof

Since $C \subseteq \mathrm{Cl}(C)$, we have $\mathrm{Int}(C) \subseteq \mathrm{Int}(\mathrm{Cl}(C))$. Conversely, let $z \in \mathrm{Int}(\mathrm{Cl}(C))$. We show $z \in \mathrm{Int}(C)$. Since $z \in \mathrm{Int}(\mathrm{Cl}(C))$, there is some $B(z;r) \subseteq \mathrm{Cl}(C)$. Our strategy is to find some $x \in \mathrm{Int}(C)$ and $y \in \mathrm{Cl}(C)$ s.t. $z = \lambda x + (1-\lambda)y$, then $z \in \mathrm{Int}(C)$ due to Thm. 4. Since $\mathrm{Int}(C) \neq \emptyset$, we can find some $x \in \mathrm{Int}(C)$. Then we can solve for y and choose λ s.t $y \in B(z; r) \subseteq \mathrm{Cl}(C)$. Namely

$$\begin{aligned} y &= \frac{z - \lambda x}{1 - \lambda} \\ \Rightarrow \|y - z\| &= \frac{\lambda}{1-\lambda}\|z - x\| \\ \Rightarrow \|y - z\| < r &\to \lambda < \frac{r}{r + \|z - x\|} \end{aligned}$$

3 The Case of $\mathrm{Int}(C) = \emptyset$

When $\mathrm{Int}(C) = \emptyset$, we hope to show $\mathrm{Int}(\mathrm{Cl}(C)) = \emptyset$. This requires knowledge on affine sets. Especially, we need the following. The details can be found in my notes on optimization on Github.

Theorem 6 Nonempty interior of simplex

$\dim V = n$, $x_0, \ldots, x_n$ aff idp, $H = \mathrm{Conv}(\{x_0, \ldots, x_n\})$. Then $\mathrm{Int}(H) \neq \emptyset$.

Theorem 7 Empty interior

$\dim V = n$, C convex. Then $\mathrm{Int}(C) = \emptyset \Leftrightarrow \dim \mathrm{Aff}(C) < n$.

Theorem 8 Affine hull of closure

$\dim V < \infty$, $S \subseteq V$. Then $\mathrm{Aff}(\mathrm{Cl}(S)) = \mathrm{Aff}(S)$.

With these results, we can prove the theorem.

Theorem 9

$\dim V < \infty$, C convex. Then $\mathrm{Int}(C) = \emptyset \Rightarrow \mathrm{Int}(\mathrm{Cl}(C)) = \emptyset$.

Proof

By Thm. 7 and Thm. 8, $\mathrm{Int}(C) = \emptyset \Rightarrow \dim \mathrm{Aff}(C) < n \Rightarrow \dim \mathrm{Aff}(\mathrm{Cl}(C)) < n \Rightarrow \mathrm{Int}(\mathrm{Cl}(C)) = \emptyset$.