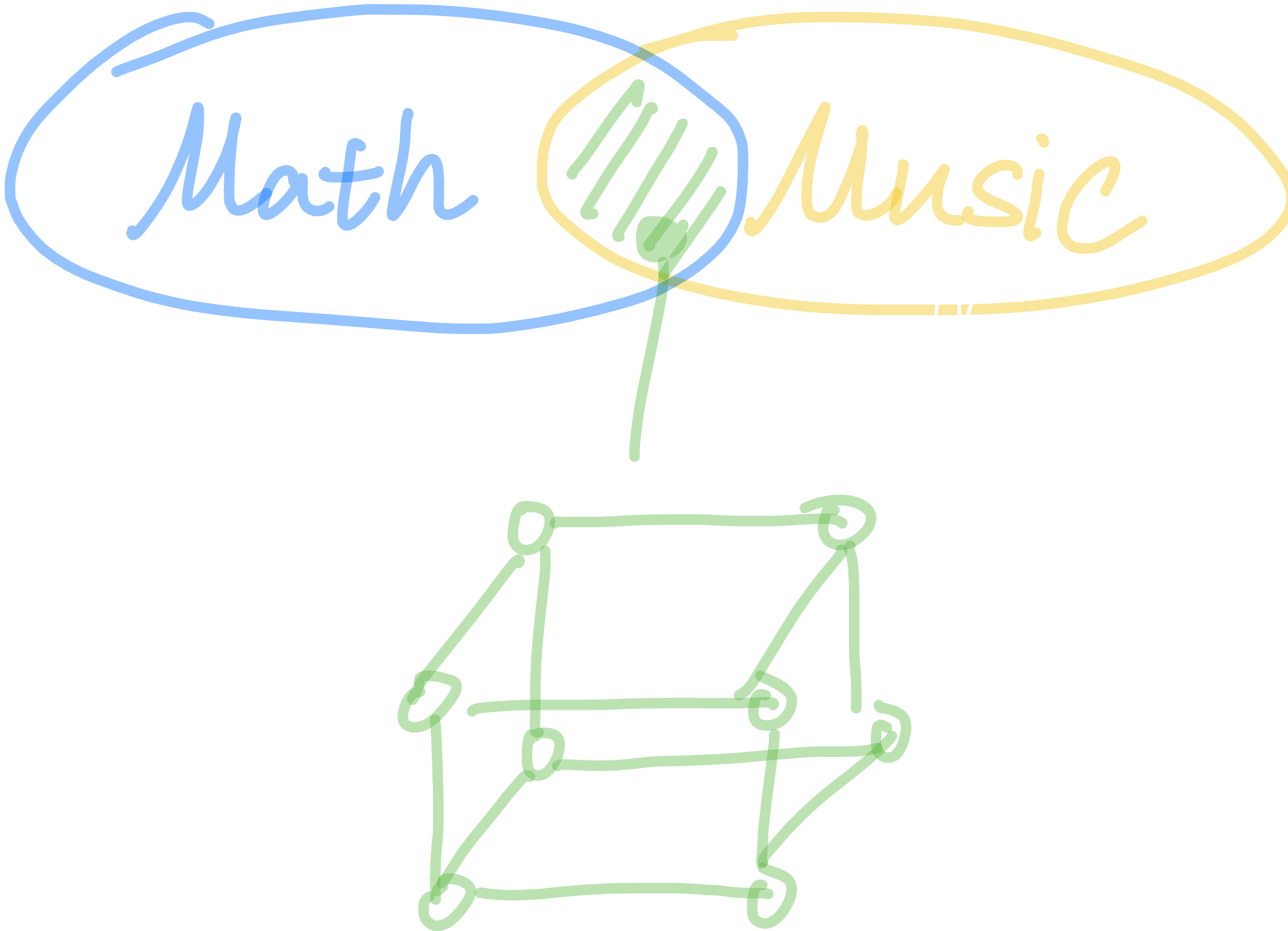
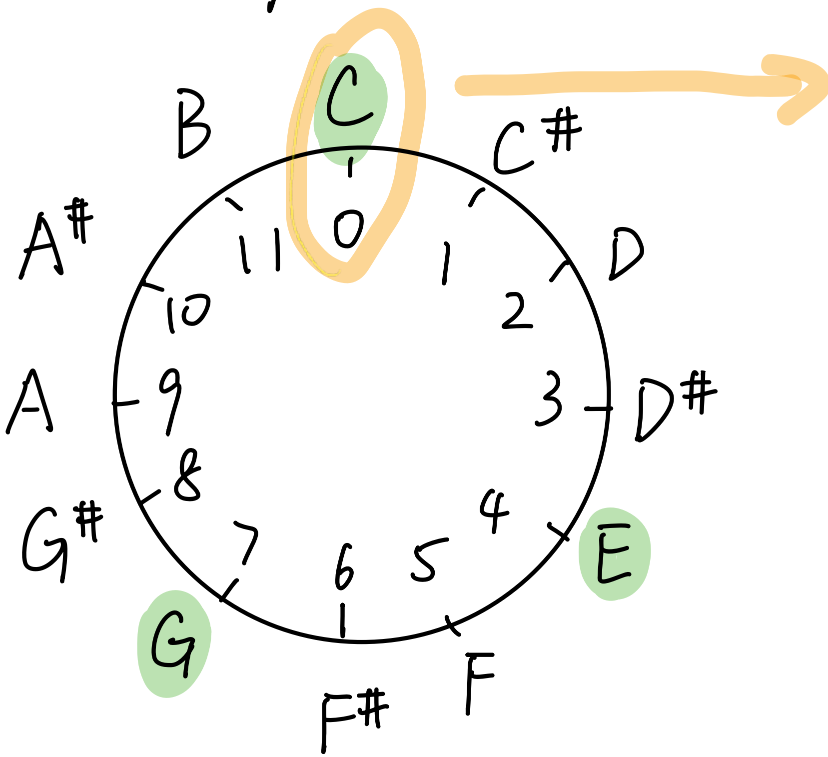

Math
Music

Pitch class space

Piano: 12 periodic discrete pitches



pitch class

octave equivalence

$(C_0 \sim C_1 \sim C_2 \sim \cdots)$

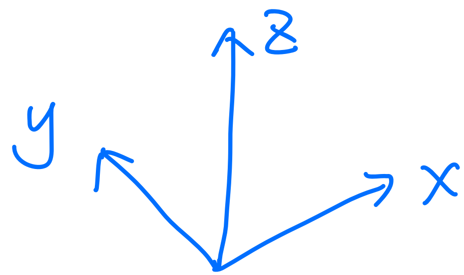
Math Notation:

$X \sim Y \iff X - Y \equiv 0 \pmod{12}$

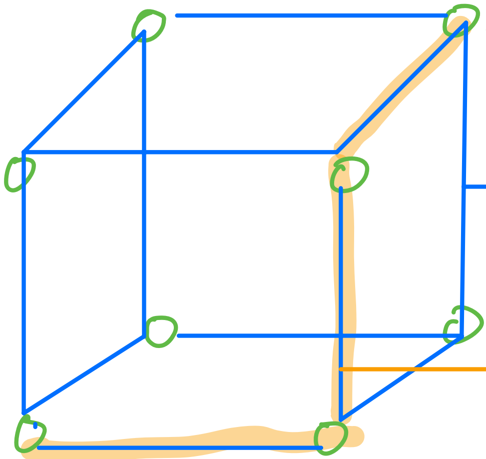
or $X - Y \equiv 12k,\ k \in \mathbb{Z}$

triad $\subseteq$ chord

C major triad (C, E, G) / (0, 4, 7)



Chord and voice-leading



Graph		Voice-leading space
vertices	⟷	chord ↙ OP
edges	⟷	chord progression
paths/walks	⟷	voice-leading path/melody
Patterns, symmetry	⟷	Harmonic, Smooth Melody
Visual	⟷	Sound

Underlying graph of the hexatonic triad graph

(Discrete) 3-voice OP equivalence musical space

① Octave equivalence

$(x, y, z) \sim (x+12, y-12, z)$

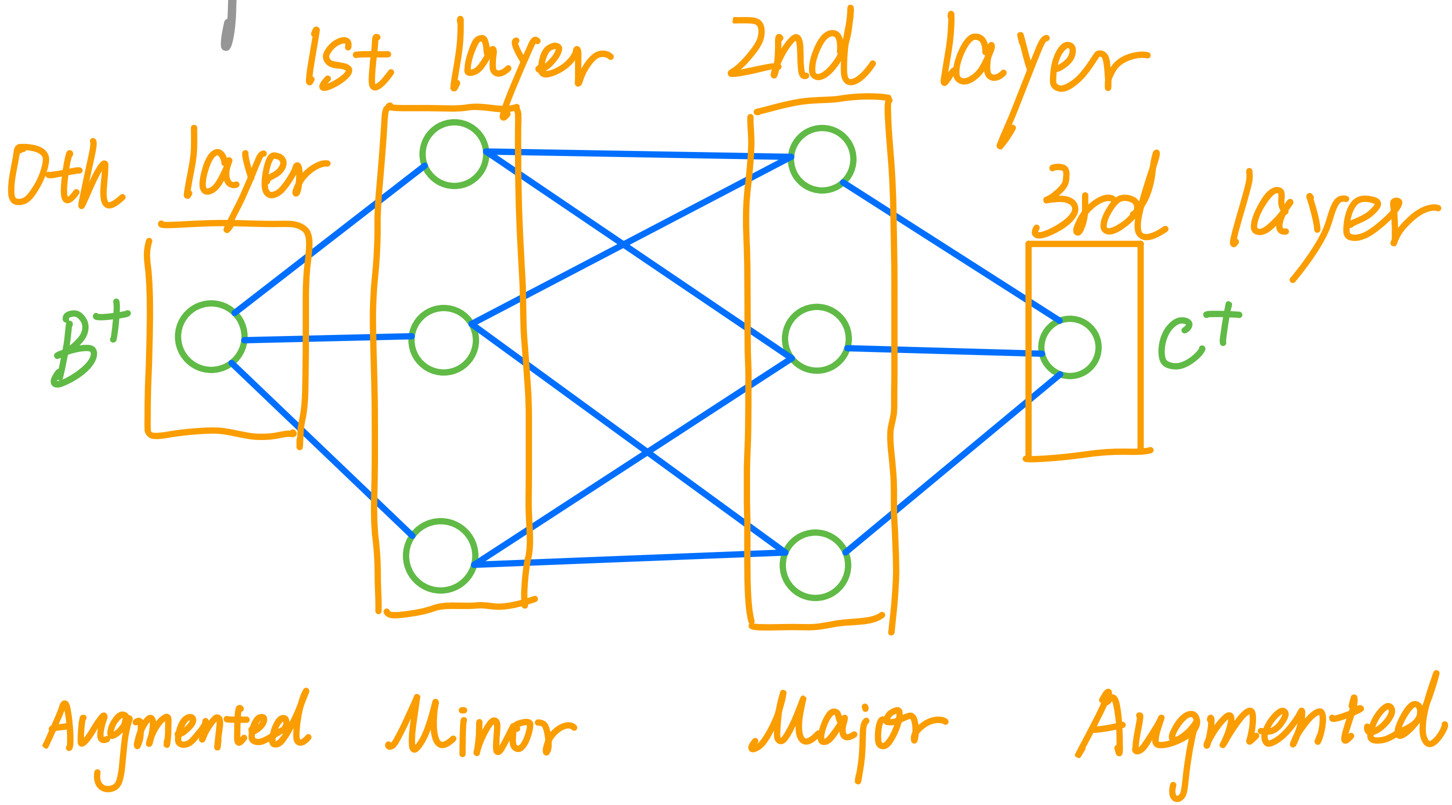
② Permutation equivalenee

$(x, y, z) \sim (x, z, y)$

Hypercube ⟷ Network Graphs

No spatial information

Networks



layers → Types of chords / chord quality

We are only interested in vertices and positions of edges.

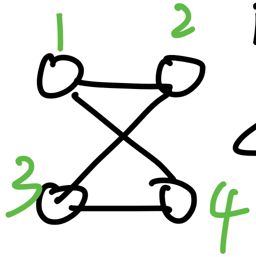
There is no spatial information (area, volume) in these graphs

n-cube | Graph | Network Graph.

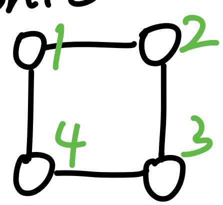
0

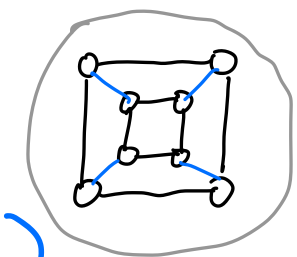
1

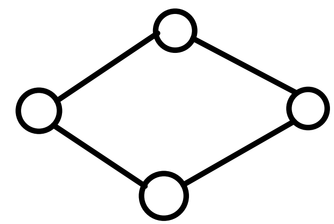
2



isomorphic

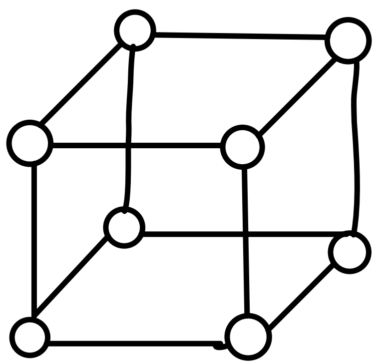


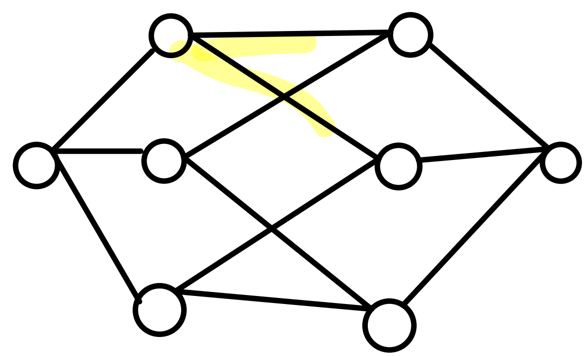




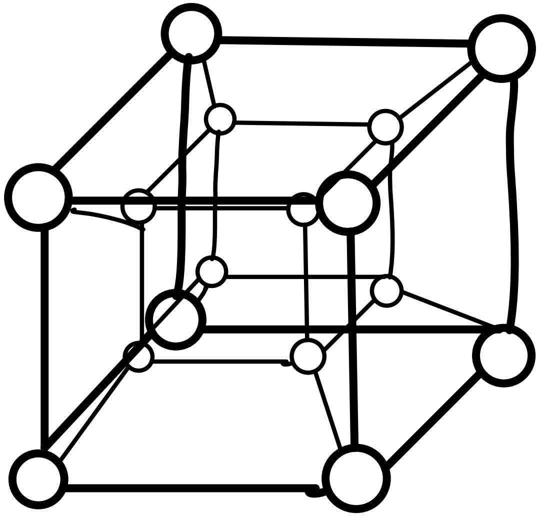
(Relabel and preserve adjacent vertices)

3

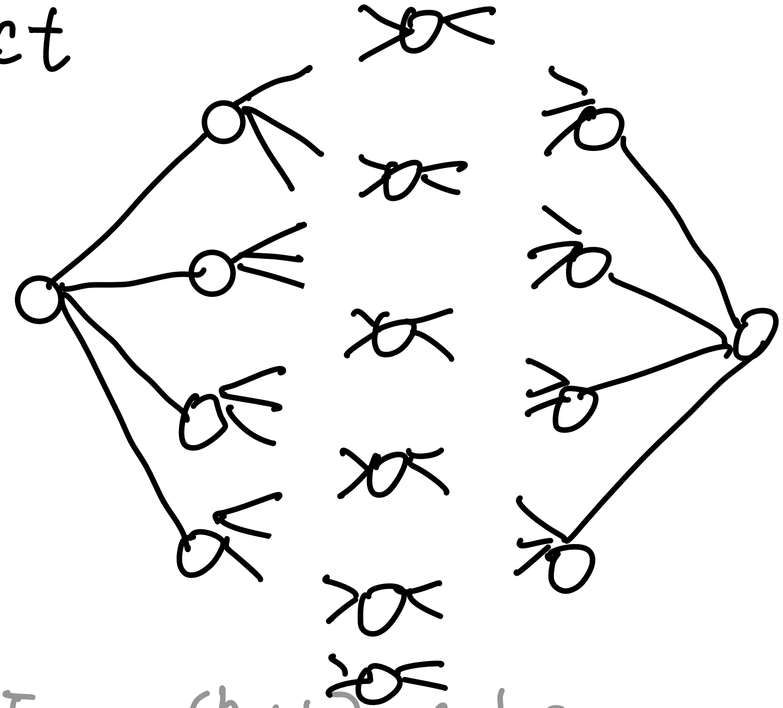




4.

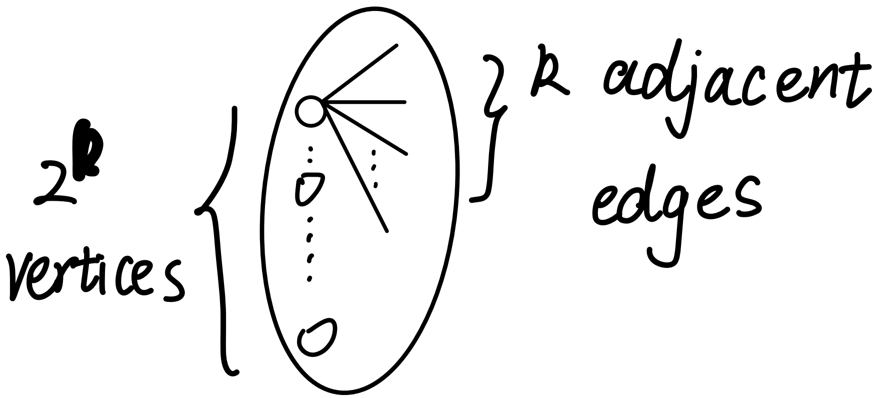


Tesseract

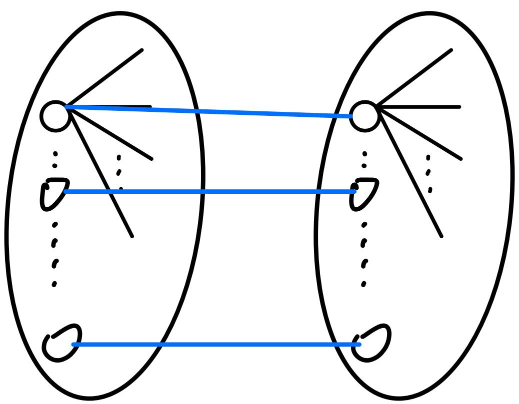


⋮

k



$k+1$



For $(k+1)$-cube, double k-cube and connect the corresponding vertices.

Mathematic Induction.

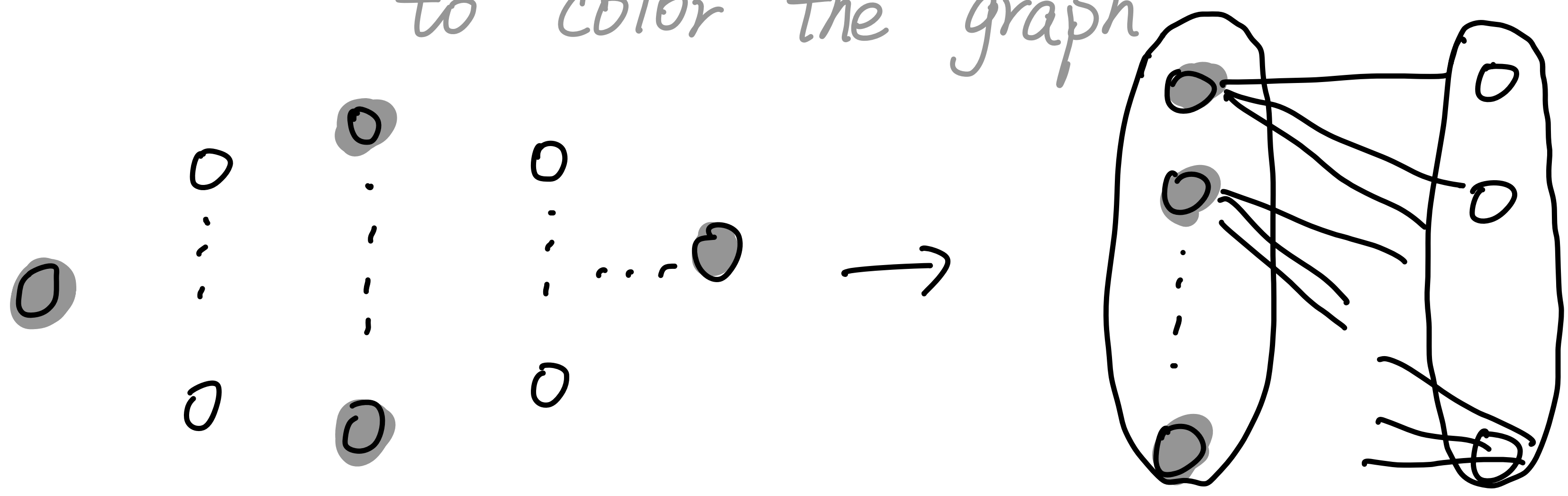
Property: ① n-regular (vertices equivalent)

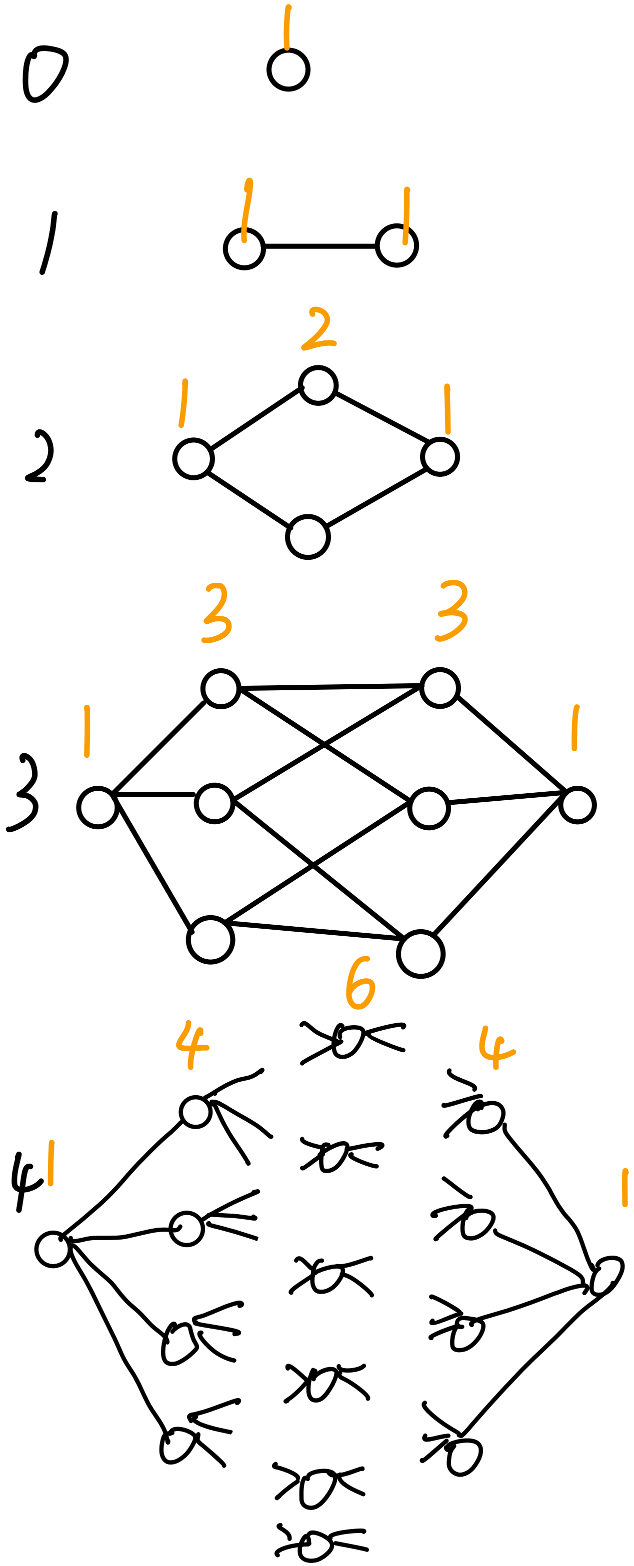
② Bipartite. ③ n+1 Layers

Graph coloring:

Constraint: Adjacent vertices should be colored in different colors

Interested: The minimum number of colors needed to color the graph





# of vertices in each layer

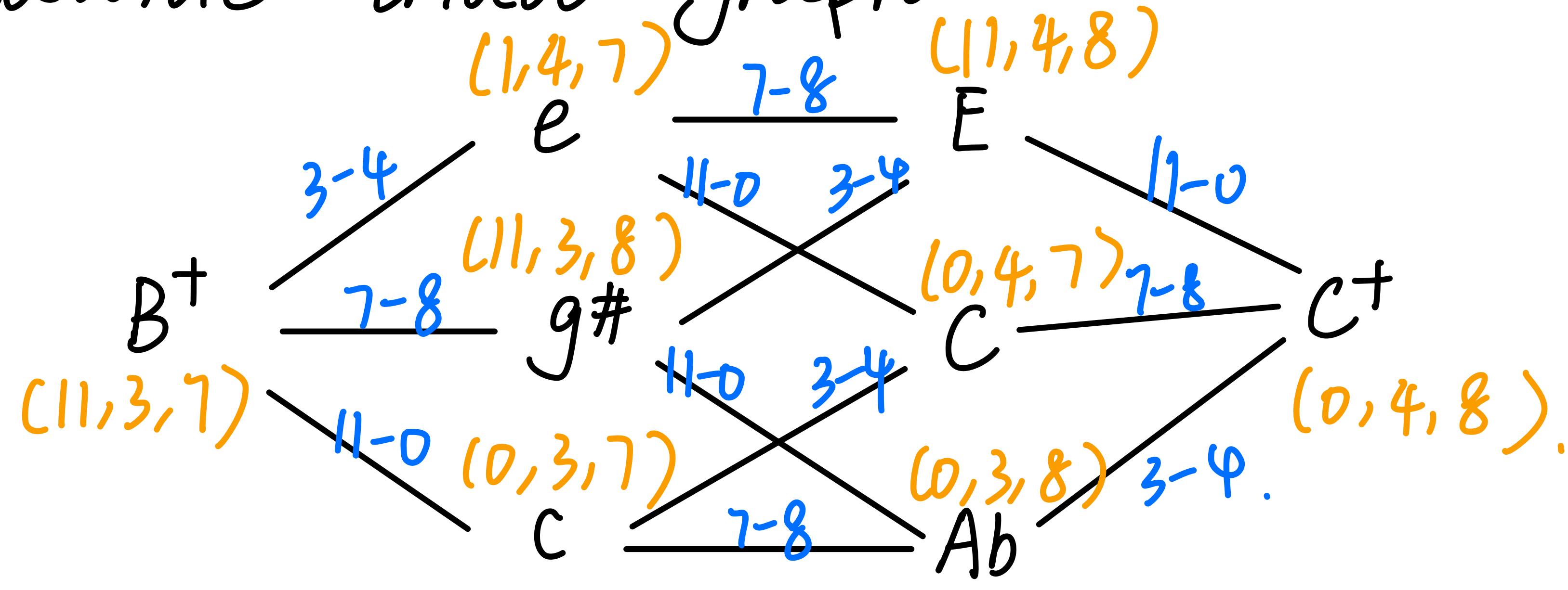
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's Triangle!

combinations:

nth-row : $\binom{n}{0}\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}$

Hexatonic triad graph



# of chord	1	3	3	1
Chord Quality	Augmented	Major	Minor	Augmented
Sum class	9	10	11	0

(e.g. $(11+3+7 \equiv 9 \pmod{12})$)

Binomial expansion

a^p: → # of right edges

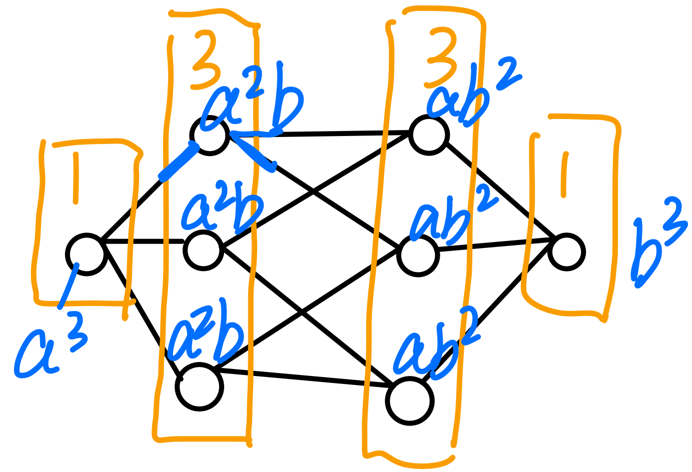
b^q: → # of left edges.

$$(a+b)^0 = 1$$

$$a+b = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)(a+b)$$



Terms ⟷ layers

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

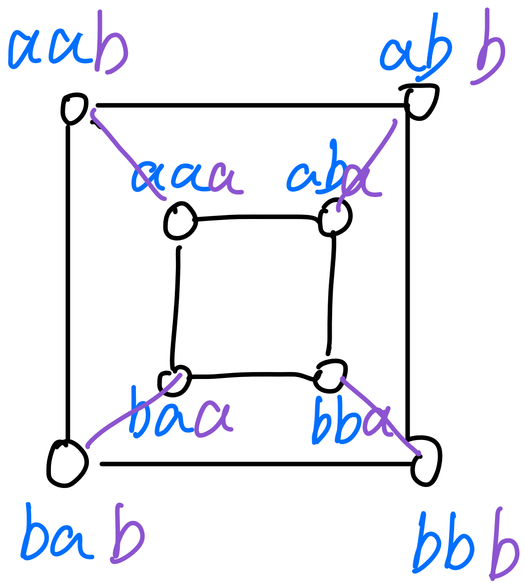
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

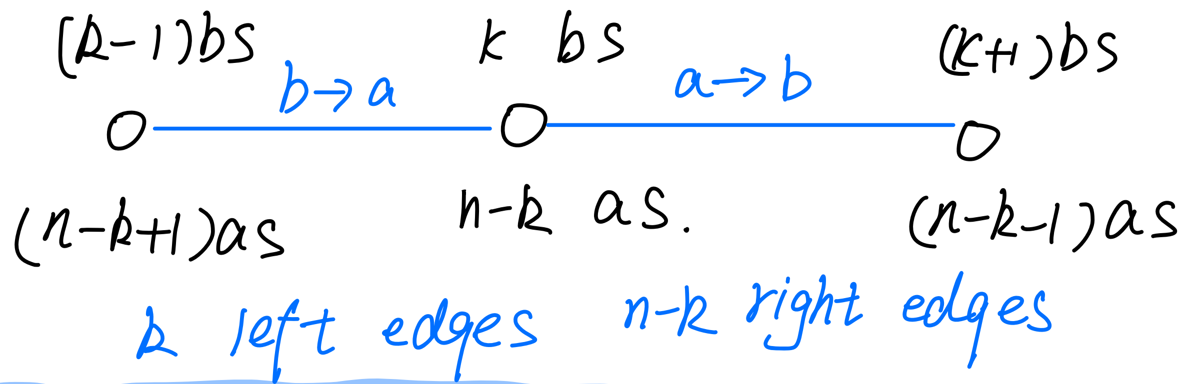
$\vdots$

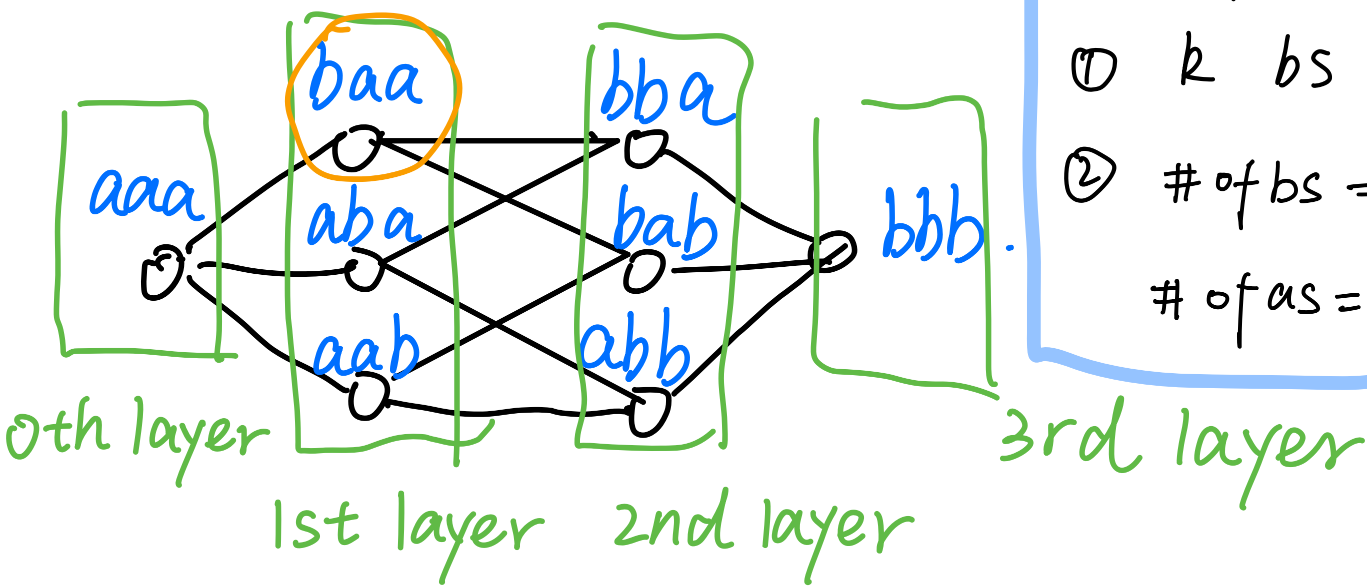
$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} a^0 b^n$$

Rigorous proof

V: $x_1 x_2 \cdots x_n$. $x_i = a$ or b.





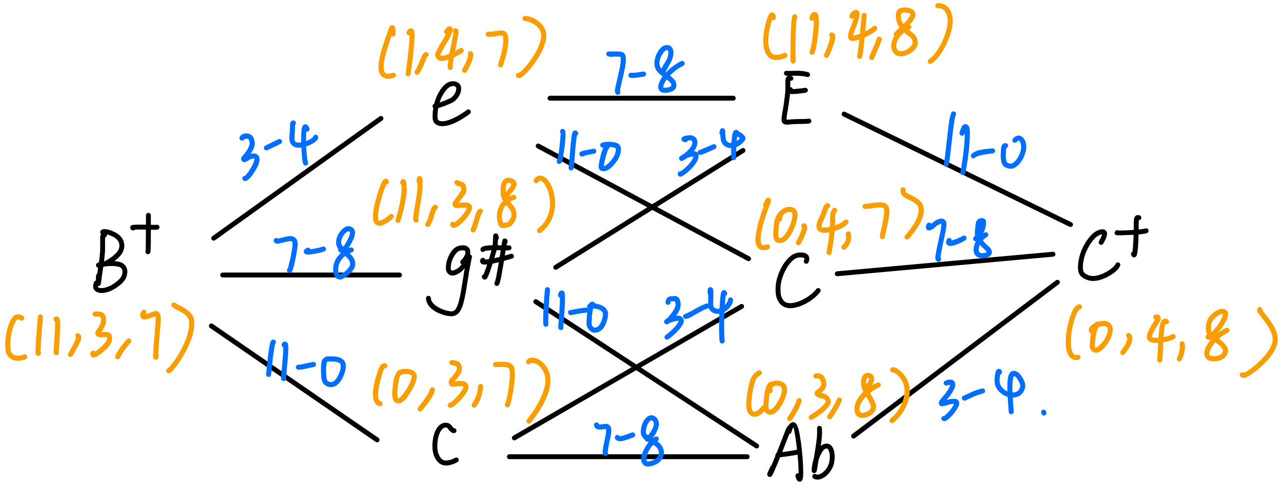


kth layer: $(a+b)^k = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

① k bs, $(n-k)$ as.

② # of bs = # of left edges = k

# of as = # of right edges = $n-k$.



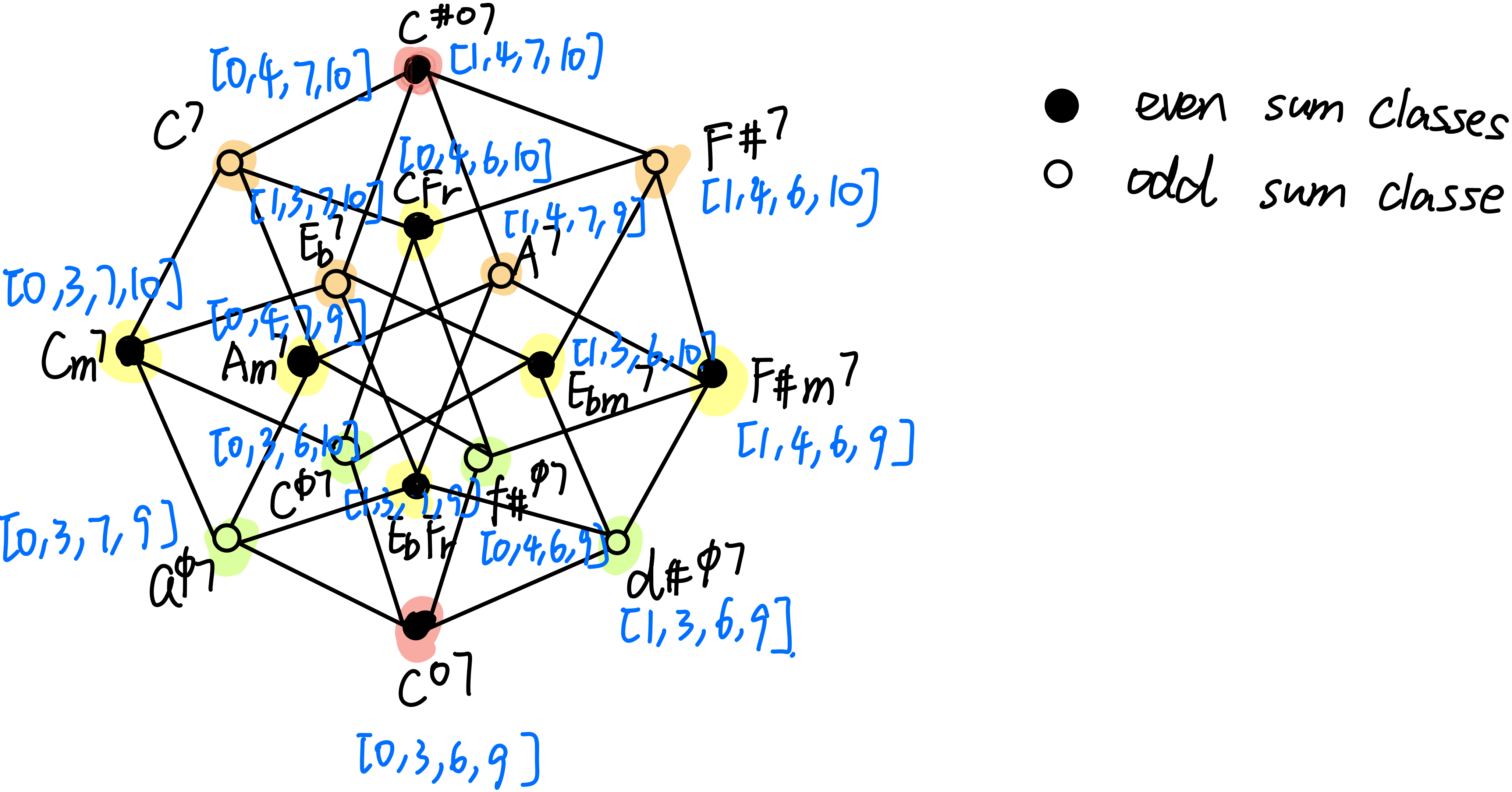
2 minor → major

3 augmented → minor

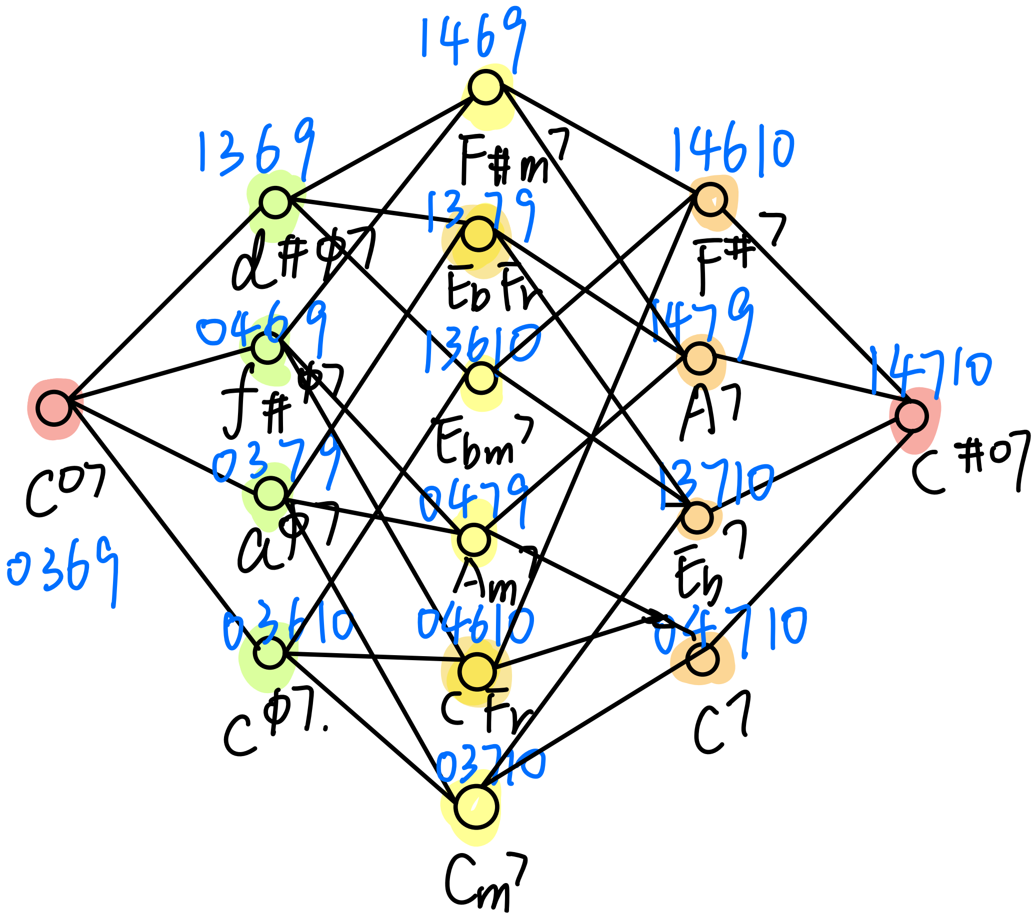
1 major → augmented

$$(a+b)^3 = 1a^3 + 3a^2b + 3a^1b^2 + 1b^3$$

1 augmented 3 minor 3 major 1 augmented



Octatonic seventh-chord hypercubes.



Fully diminished seventh chord

Half diminished seventh

Minor seventh chord

French augmented sixth chord

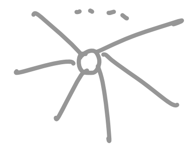
Major seventh. chord

$$(a+b)^4 = 1a^4 + 4a^3b + (4a^2b^2 + 2a^2b^2) + 4ab^3 + 1b^4$$

n-cube	# of vertices	# of e ges
0	$1 = 1$	0
1	$1+1 = 2$	$1+0 = 1$
2	$1+2+1 = 4$	$1\cdot 2 + 2\cdot 1 = 4$
3	$1+3+3+1 = 8$	$1\cdot 3 + 3\cdot 2 + 3\cdot 1 = 12$
4	$1+4+6+4+1 = 16$	$1\cdot 4 + 4\cdot 3 + 6\cdot 2 + 4\cdot 1 + 1\cdot 0 = 32$
$\vdots$	$\vdots$	$\vdots$
		differentiate
n	$\sum_{k=0}^{n} \binom{n}{k} = 2^n$	$\sum_{k=0}^{n} k\binom{n}{k} = n\cdot 2^{n-1}$

# of faces | **# of cubes. ...** | **# of m-cubes**

$$\binom{n}{2}2^{n-2} \xrightarrow{\text{differentiate}} \binom{n}{3}2^{n-3} \xrightarrow{\cdots} \binom{n}{m}2^{n-m}$$

$\binom{n}{m}$ ways to choose  m edges.

$\downarrow$

one of m-cube

\# of vertices

$$\frac{2^n\binom{n}{m}}{2^m}$$

Each m-cube has 2^m vertices. Each is counted 2^m times.

Musical implication

① # of vertices = # of distinct chords.

② # of edges (1-cube) = # of pairs of chords that has only one different pitch differed by one semitone

③ # of m-cube = # of set of m chords that has exactly m different pitches differed by one semitone

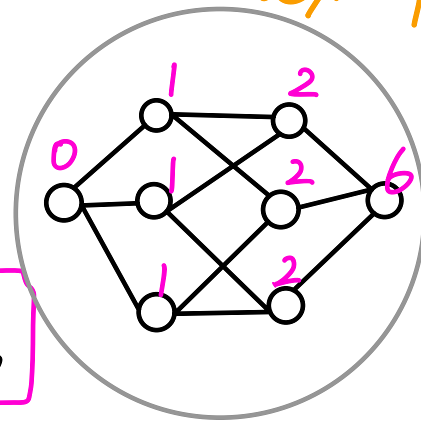
Cube # shortest paths from the first vertex to each vertices.

Multiply # of right edges for one vertex in each layer

1D 0 1

2D 0 1 2

3D $\boxed{0 \quad 1 \quad 2 \quad 6}$



4D 0 1 2 6 24

5D 0 1 2 6 24 120

6D $0 \underbrace{\quad}_{\times 1} 1 \underbrace{\quad}_{\times 2} 2 \underbrace{\quad}_{\times 3} 6 \underbrace{\quad}_{\times 4} 24 \underbrace{\quad}_{\times 5} 120 \underbrace{\quad}_{\times 6} 720$.

⋮

KD 0 1 2! 3! ...... k!

Total # of shortest path from X to Y.

Y \ X	B^+	e/g#/c	E/C/Ab	C^+
B^+	0	1	2	3! = 6
e/g#/c	1	0	1	2
E/C/Ab	2	1	0	1
C^+	3! = 6	2	1	0

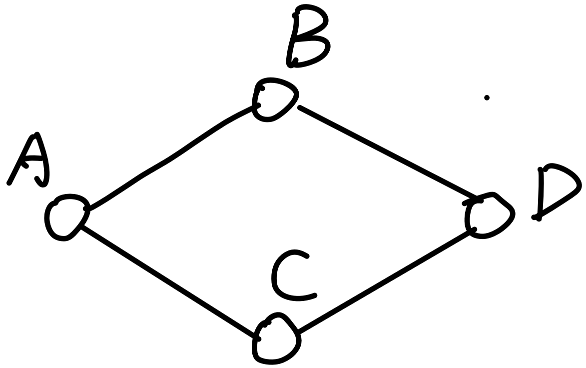
example:

Total # of paths from B^+ to C^+ : 3! = 6.

How many total semitones does the chord move

= The difference of layer.

Adjacency Matrix

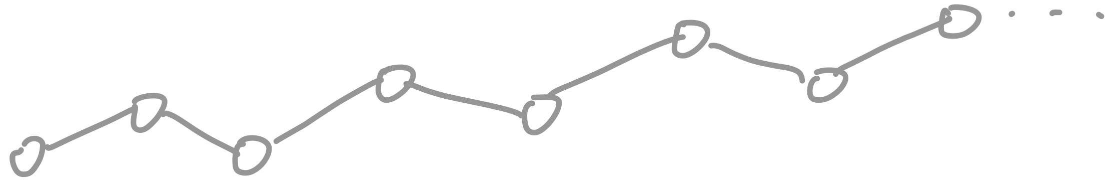


$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

$$M_{ij} = \begin{cases} 1 & i,j \text{ is adjacent} \\ 0 & \text{o.w.} \end{cases}$$

Matrix powers.

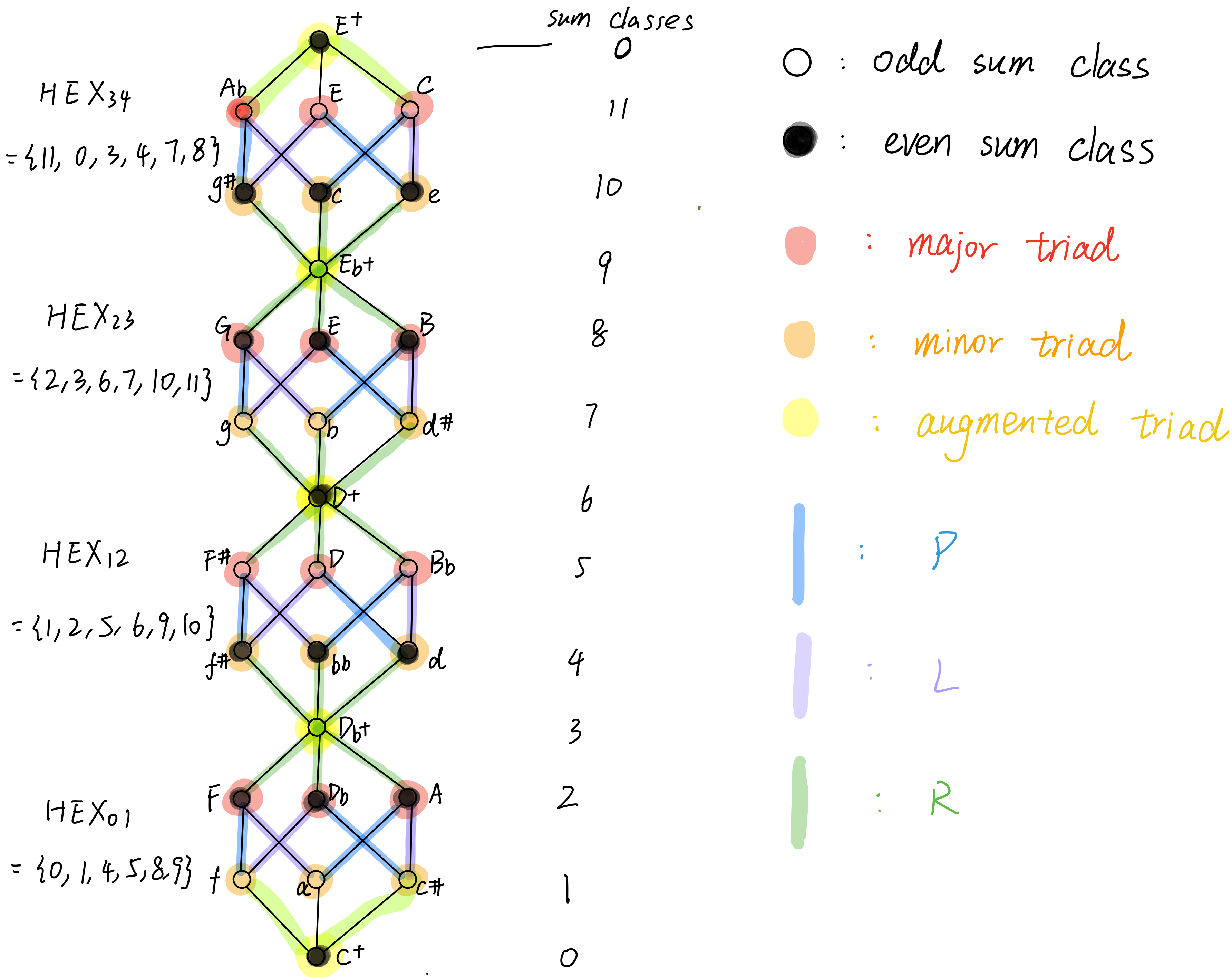
$$M^k = \underbrace{M \cdots M}_{k}.$$



M^k_{ij} : # of walks of length k from vertex i to vertex j.

Adjacency Matrix

	B^+	e	g#	c	E	C	A^b	C^+
B^+	0	1	1	1	0	0	0	0
e	1	0	0	0	1	1	1	0
g#	1	0	0	0	1	1	1	0
c	1	0	0	0	1	1	1	0
E	0	1	1	1	0	0	0	1
C	0	1	1	1	0	0	0	1
A^b	0	1	1	1	0	0	0	1
C^+	0	0	0	0	1	1	1	0



sum classes
E+
0
Ab
E
C
11
HEX34
={11, 0, 3, 4, 7, 8}
g#
c
e
10
Eb+
9
HEX23
={2,3,6,7,10,11}
G
E
B
8
g
b
d#
7
D+
6
HEX12
={1,2,5,6,9,10}
F#
D
Bb
5
f#
bb
d
4
Db+
3
HEX01
={0,1,4,5,8,9}
F
Db
A
2
f
a
c#
1
C+
0
: odd sum class
: even sum class
: major triad
: minor triad
: augmented triad
: P
: L
: R

General Discussion

Music

?

?

Math

?

Human

Model?

Feeling?

Page 22: References:

Adjacency Matrix. Retrieved from: https://en.wikipedia.org/wiki/Adjacency_matrix

Hypercube. Retrieved from: https://en.wikipedia.org/wiki/Hypercube

L.gargano, M., F.Malerba., Lewinter, M.. Hypercubes and Pascal's Triangle: A Tale of Two Proofs. *Mathematics Magazine*. p.216.

Hook, J. (2023). Exploring Musical Spaces. *Oxford University Press*.

Thanks you