Speech Draft

Page 1: Introduction

Hello everybody, welcome to watch my videos. I am a second year master student who have some interest in math. In this summer, I am trying to explore the relationship between combinatorial mathematics and the musical space. In particular, the relationship between hypercubes and the voice-leading space, to which I will give definitions later. I appreciate that professor Leah Frederick gave me some helpful guidance for this small project. This video only requires some basic understanding of calculus, algebra and music. Hope you will find it interesting and inspiring.

Page 2: Pitch class space

First, I will give some introduction to the research background and tell you motivation of why I hope to analyze the patterns of the hypercube graph. Let's first introduce the pitch class space. There are twelve periodic discrete pitches in piano. Seven white keynotes and five black keynotes. The pitch class space would be like a clock. The points of this space are not pitches at all but pitch classes. A pitch class contain multiple equivalent pitches. For example, the C represent C pitches in any frequencies. The pitches in a pitch class are octave equivalence to each other. In mathematical representation, a pitch X is equivalent to a pitch Y if and only if X minus Y congruence to 0 modulus 12. Or X-Y equal to 12k and k is any integers.

Page 3: Chord and voice leading

The chord is any harmonic set of pitches consisting of multiple notes that are heard as if sounding simultaneously. A triad is a kind of chord that consists of three notes stacked in consecutive thirds. For example, the C major triad is consisting of C, E and G. Or 0, 4, 7 in numbers. There are three pitches in a triad, so it would be appropriate to relate the triad to the three dimensional spaces. Since we only consider the discrete cases, instead of drawing a cube, we draw a cube graph. Graphs in general compose of the points, which we called vertices, and lines, which we called edges. We also interested in tracing a sequence of vertices and edges in graph, which we call paths or walks. If we impose the meaning of our vertices, edges and paths, we could transform the graph as a voice-leading space graph. The vertices represent the chord, the edges represent the chord progression. That is, transformation from one chord to another chord. The path would be a voice-leading path, which corresponds to melodies in songs. So we can see that we actually relate the sound of a songs to visual representation of a graph. Therefore, analyzing the patterns, for example, the symmetry property of our graphs, will contribute to a more harmonic and smooth songs.

Page 4: Hexatonic triad graph and cube graph

The cube graph is the underlying graph of the hexatonic triad cube graph. For the hexatonic triad cube, it is a part of the three-voice OP equivalence musical space. What is OP equivalence? It is octave equivalence and permutation equivalence. Notice that previously, the pitch class space is octave equivalence. The octave equivalence means every pitch in the triad is octave equivalence. The premutation equivalence means we can permuting pitches preserve equivalence. So we can view the voice-leading spaces as a composition of chord and chord progression with some kinds of equivalence relations imposed to the chord. Similarly, a 4 dimensional hypercube is a part of the 4-voice OP spaces. So we may believe that a n dimensional hypercube is a part of the OP spaces and that is why I hope to analyze the hypercube graphs.

Page 5: Network graph

The next question is how to analyze the hypercube graph? Notice that in our hypercube graph, we do not consider the spatial information. That is, there is no volume, no area and no length of the graph. We only consider how vertices are connected to each other. Therefore, we find that can represent the cube graph in a two dimensional network graph. How we create the network graph, we just start from one vertices, define it as the 0^{th} layer, and find the adjacent vertices, define it as the 1^{st} layer, and find the adjacent vertices to the 1^{st} layer to form the second layer. And so on and so forth. For the hexatonic triad graph, we find that the layers are related to the chord quality, or the type of the chord. It shows that our network graph maybe a nice way to represent the hypercubes graph and may have some useful patterns.

*Voice leading is the linear progression of individual melodic lines and their interaction with one another to create harmonies.

Page 6-7: Graph of the hypercubes

Let's give a definition of our multi-dimensional cube graph. For 0-cube, the graph is a single vertex. For 1-cube, it is two vertices that are connected by an edge. For 2-cube, we draw another 1-cube and connect the corresponding vertices. It is like a square. Notice that for the graph discussion below, we are only interested in vertices and how these vertices are connected to each other by edges. So the size of graph does not matter at all. we can also represent our 2-cube like this one. We can relabel the vertices in a way that preserve the vertices' connection or we call adjacency between vertices. We call these two graphs are isomorphic to each other. To generate a 3-cube, we draw another 2-cube inside and connect the same labelling vertices. The underlying graph of 3-cube is the hexatonic triad graph. Similarly, the 4-cube is created by drawing another 3-cube and connect the same labelling vertices. We call it tesseract. Notice that 4-cube in our construction is not exactly a 4-dimensional cube. Usually the 4^{th} dimension is time and a 4-dimensional cube cannot be visualized in three dimensional space. By our construction, for the k-cube, we will have 2 to the power of k vertices and each vertex have k adjacent edges. To get the (k+1)-cube, we need to draw another k-cube and connect the vertices with the same labels. In summary, we define base case cube and define how to draw every successive cube. In these way, we can construct cube with any dimensions. This is an idea of mathematical induction.

Page 6-7: Network graph of the hypercubes

Now we draw the corresponding network graph. We start from one vertex and define it as the 0^{th} layer and find its adjacent vertices to form the 1^{st} layer. And keep finding the adjacent vertices of the 1^{st} layer to form the second layer. So 2-cube has three layers. Start from one vertex, there are 3 adjacent vertices, the next layer also has 3 vertices, the last layer has one. 3-cube has four layers. For 4-cube, every vertex has four edges. So for every vertex in this layer, there are three edges connected to the vertices in the next layer. Using the symmetry property, for the second layer, it has two edges connected to the previous layer and two edges connected to the next layer. We can generate the k-cube network graph in this way. You may also view it as a directed graph. From the 0^{th} layer, all the way pointing to the final layer.

Page 8: Properties of the hypercubes

We can observe some properties of these hypercubes. First, every vertices in the n-cube is equivalent to each other. Every vertex has n edges, which we called n-regular in graph theory. Second, it seems that n-cube network graph has n+1 layers. Third, for the network graph, there is no edge connected the vertices in the same layer, while there are edges connected in two consecutive layers. Actually, we can further reduce the network graph into exactly two layers. How can we achieve it? Let me introduce a concept called graph coloring. In graph coloring, we have a constraint that adjacent vertices that are connected by edges should be colored in different colors. We usually will be interested in finding the minimum number of colors needed to color the graph. To color the network graph, we can color the o layer as black, the next layer as white, alternatively, we only need two color to color the graph. If we put all black vertices in one set and all white vertices in another set, we find that there are no edges within each set and there are edges connection between two sets. The graph is called bipartite graph.

Page 9: Vertices in each layer and Pascal's Triangle

We notice that in hexatonic triad graph, each layer has different types of triads. So it would be helpful to analyze the number of vertices in each layer. For 0-cube is 1. One cube is 1,1. 2-cube, 1,2,1. 1,3,3,1. 1,4,6,4,1. I guess you may already see some patterns or find it familiar. Yes, it is Pascal's triangle, right? We can calculate the next sequence of numbers based on pairwise summation of previous sequence of numbers. For 5-cube, it is 1,5,10,10,5,1. Therefore, we can get the number of vertices in each layer for cubes in any dimensions. For n-cube, it corresponds to the nth row in the Pascal's triangle. They are the binomial coefficients. N choose k means the number of ways to select k out of n objects, when order does not matter.

Page 10: Musical implication of vertices in each layer of 3-cubes and 4-cubes

The number of vertices in each layer tells us about the chord quality. For the hexatonic triad, the 0^{th} layer contains 1 Augmented triad, the 1^{st} layer contains 3 minor triads. The 2^{nd} layer contains 3 major triads. And the last layer contains 1 augmented triad. Layers are also related to the sum class. To calculate the sum class, we add every pitches in the triad and see what it congruence to modulus 12. The sum class will be 9, 10, 11 and 0. For the octatonic seventh chord, it is a part of the 4-voice OP space. Similarly, it is corresponds to the 4-cube. The 0th and the last layer is the fully diminished seventh chord, the first layer is the half diminished seventh chord. The middle layer compose of two French augmented sixth chord and four minor seventh chord. The third layer contains four major seventh chord.

Page 11: Hypercubes and Binomial expansion

Since the number of vertices in each layer are the binomial coefficients. I wonder whether binomial expansion will have something to do with the network graph. I write out the expansion. We find that the number of terms in the expansion is the same as the number of layers in network graph. Maybe one term will correspond to one layer. The coefficients in each term corresponds to the number of vertices in each layer. So it would be appropriate to think what a and b represent and whether the power in a or b have something to do with the graph, for example, the edges. Let’s analyze about the patterns of edges in the network graph. One vertex will have two kinds of edges, the edges that are connected to the previous layer and the edges that are connected to the next layer. We first focus on the edges that are connected to the next layer. For 3-cube, it is 3,2,and 1,0. For the 4-cube, it is 4, 3, 2, 1,0. Wow, we seems to observe the patterns. What is these numbers, it seems to corresponds to the power of a. For the edges that are connected to the previous layer. 3-cube is 0, 1, 2, 3. 4-cube is 0, 1, 2, 3, 4. These numbers are related to the power of b. We observe that the power of a is the number of edges that are connected to the next layer, or we call the right edges. The power of b is the number of edges that are connected to the previous layer, or we call the left edges.

Page 12: A rigorous proof using labelling.

Notice that we just deduce our patterns based on our observations. We observe that the hypercubes can be written as a network graph form. The vertices and edges in each layer corresponds to each term in the binomial expansion. But how to rigorously prove it? Below is the intuition of the proof. We could prove it by labelling the vertices. I will show that we can label the vertices using a sequence of binary notations. For the 1-cube, we label it as a and b. For 2-cube, we first label the two vertices as a and b. And then distinguishing the a and b using the same labelling vertices. Similarly for 3-cube. We can find that the vertices that are connected to each other only differ by one number. And the vertices is composed of all possibility of sequence of a and b. So that the connected vertices is in the consecutive layer. Each layer contains a certain number of a and b. That's why the number of vertices in each layer is combinatorial number. Given a certain number of a and b, how many possible sequence of a and b is the same as how many possible way to choose those number of as and left others as bs from the sequence. I use 3-cube as an example. For the 0^{th} layer, it has one a cube. For the 1^{st} layer, it has three a square b. For the second layer, it has three a b square. For the last layer, it has one b cube. We find that vertices at the extreme side of the network graph tends to have more a or more b, while vertices at the middle part of the network graphs tends to have a more even numbers of a and b.

Page 13: Hexatonic triad graph

For the hexatonic triad, it corresponds to the cube of a plus b. The vertices represent chords and the edges indicate single-semitone voice-leading connections. It has one augmented triad in the 0^{th} layer. one augmented triad in the third layer. 3 minor triads in the 1^{st} layer. 3 major triads in the second layer. the cube power of a means there are there are 3 ways for the based layer augmented triad to transform to minor triads. 2 ways for a minor triad transform to major triads. There are one way for a major triad transform to an augmented triads. There are one way for a major triad transform to augmented triad. For the network graph of hexatonic triad, we can draw it as a directed graph. The arrows in the graph point in the direction of ascending voice leading.

Page 14-15: Octatonic Seventh-Chord Hypercubes Graph

For the octatonic seventh-chord hypercubes, it is a part of the 4-voice OP space. Similarly, it is corresponds to the 4-cube. The 0th and the last layer is the fully diminished seventh chord, the first layer is the half diminished seventh chord. The middle layer compose of two French augmented sixth chord and four minor seventh chord. The third layer contains four major seventh chord. Similarly, the power of a corresponding to the number of ways to transform to the next types of triads. The power of a of a vertex is the number of ways for it to transform to the next types of chords and the power of b is the number of ways for it to transform to the previous type of chords.

Page 16-17: Total number of m-cubes in n-cubes and differentiation

Let's go back to our mathematical analysis. We analyze vertices and its adjacent edges in each layer. Next, we may investigate the total number of vertices and edges. For the total number of vertices, we just sum the number of vertices in each layer. The equation would be like this. Summation of the binomial coefficients is equal to 2 to the power of n. This is a well-known identity. Which can also be shown by defining a and b to be 1 in the binomial expansion of the polynomial a plus b to the power of n. The number of edges is equal to summation of the power of a, that is, the summation of all the right edges. It is the summation of k times each binomial coefficient. There is another way to calculate the total number of edges by the number of vertices. For n-cube, there are 2 to the power of n vertices and each vertex has n edges. There are n times 2 to the power of n edges with double counting since each edge is formed by two vertices. To eliminate double counting, we divide two here. So we got this equation. Actually, there is to get this equation, just by differentiated both sides of the vertices equation we got.

Moreover, we find that actually, the number of vertices is just the number of 0-cube. The number of edges in n-cube is just the number of 1-cube in n-cube. So maybe we could also find the number of faces, or the number of 2-cube in n-cube. Or the number of 3-cube, all the way to the number of (n-1)-cube in n-cube. So how many of them?

We know that the edges equation is derived from vertices equation. So we could guess if we keep differentiate the equations, we will get the number of cubes in the following increasing dimension. We can prove that our hypothesis is true. For every vertices in n-cube, there are n choose m ways to choose m edges incident to that vertex. Each of the collections of m edges defines one of the m-dimensional cubes incident to the considered vertex. Doing this for all the vertices of the hypercube, each of the m-dimensional faces of the hypercube is counted 2 to the power of m times since it has that many vertices. So the total number of m-cubes in n-cubes is n choose m times, how many collections, times 2 to the power of n divided by 2 to the power of m. We can store this information to a matrix form. We call it configuration matrix. Interested people can search about it.

Page 18: Musical implication of the m-cubes in n-cubes

So what is the musical implication of it? The number of vertices means the number of distinct chords. Two chords connected by a line always share two common tones, differing only by a single-semitone displacement in the remaining note. So the number of edges, or the number of one cube, in a hypercubes means the total number of pairs of chords in this hypercubes that has only one different pitch differed by one semitone. The number of m-cubes means the number of m chords that has exactly m different pitches differed by one semitone.

Page 19-20: Shortest voice-leading path

Usually we are interested in the voice leading path that voices are moved between chords in the shortest possible manner to give a smooth connection between chords. Thus, we hope to analyze the shortest paths in the graph. For the 3-cube, first layer is 0, second layer 1, it has two right edges, so the third layer is 2. To go to the vertices in the second layer, there is 2 shortest paths. The third layer has 3 right edges, so to go to the last layer, we have 2 times 3 equal to 6 shortest paths. We find that we can multiply the number of right edges for one vertex in each layer to get the shortest number of paths. So the k-cube would be 0, 1, 2 factorial, dot dot dot, to k factorial. For the hexatonic triad graph, we can draw a table to show the number of shortest path from vertices in one layer to vertices in another layer. For the chains of hexatonic triad, we can find that the total number of shortest path from C augmented triad to E augmented triad is 3 factorial to the power of 4. For the octatonic seventh-chord hypercubes, we can find the shortest path in a similar way. The musical implication of the shortest path is how many total semitones does the chord move.

Page 21-22: Represent graph as adjacency matrix

Graph is easy to visualize, while the computer is more happy with digits. So how to store these graphs in computer? We can also represent our hypercube network graph as an adjacency matrix. The rows and columns of the matrix is all the vertices. The element in the matrix will be 1 if the row vertices and column vertices are connected and otherwise, it will be 0. In musical sense, the columns and vertices will be the chords. The elements in the matrix is 1 if the chords are connected by a single-semitone voice leading. This matrix is a symmetric matrix with the diagonal elements being 0s. The matrix powers have an interesting property. The element of row i and column j in the matrix means the number of walks of length k from vertex i to vertex j. The walks means a sequence of vertices and edges of a graph. So we could find the number of voice leading of length k by taking k power of the adjacency matrix. The adjacency matrix of the hexatonic triad will be like this. Adjacency matrix will make the graph represent in an algebra ways that can be stored in computer, since the graph and the adjacency matrix have 1-1 correspondence with each other.

Page 23: Extension:

Here is some extension about the music part of today's topic if you hope to know more about the fundamental music theory. There are five kinds of common equivalence relation in music. These equivalence can be combined to form multiple musical spaces. For the 3 voice OP space, we have a chain of hexatonic triads containing triads ranging from all possible sum classes. So there are multiple 3-cubes that connect to each other. We can also study this chord, chord transformation, and voice-leading path in this space.

Page 24: General discussion

Here is general discussion that you may consider. I think represent music in mathematics has beauty of its own and it will contribute to the harmonic music creation. That’s why I hope to explore something in this project. However, some people think music should be created by people with their own stories and feeling. Generating music in a modelling way will destroy the musical meaning. What do you think about it?

Page 25: References:

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Page 26: Thank you

Thank you very much for watching this video. Hope you learn something and have fun. I am not really professional in the mathematics and music field, so I would be appreciated it if you could point out some mistakes in my video and give some suggestions and advice for my video. Bye and hope you have a great day.