Speech Draft

Page 1: Introduction

Hello everybody, welcome to watch my videos. I am a second year master student who has some interest in math. Last summer in 2023, I am trying to explore the relationship between combinatorial mathematics and the musical space. In particular, I investigate the patterns of hypercubes using its network graph form and explore their musical implications. I appreciate that professor Leah Frederick gave me some helpful guidance for this small project. There are four videos in total. They only require some basic understanding of calculus, algebra and music. Hope you will find it interesting and inspiring.

Page 2: Pitch class space

First, I will give some introduction to the research background and tell you the motivation of why I hope to analyze the patterns of the hypercube graph. Let's first introduce the pitch class space. There are twelve periodic discrete pitches in piano. (audio). Seven white keynotes(audio) and five black keynotes(audio). The pitch class space would be like a clock. The points of this space are not pitches at all but pitch classes. A pitch class contains multiple equivalent pitches. For example, the C represents C pitches in any frequency(audio). The pitches in a pitch class are octave equivalence to each other. In mathematical representation, a pitch X is equivalent to a pitch Y if and only if X minus Y congruences to 0 modulus 12. Or X-Y equal to 12k and k is any integer.

Page 3: Chord and voice leading

The chord is any harmonic set of pitches consisting of multiple notes that are heard as if sounding simultaneously. A triad is a kind of chord that consists of three notes stacked in consecutive thirds. For example, the C major triad consists of C, E and G. Or 0, 4, 7 in numbers(audio). There are three pitches in a triad, so it would be appropriate to relate the triad to the three dimensional spaces, which have x, y, and z axis and each represent a triad. Since we only consider the discrete cases, we could consider multiple lattice points in the three dimensional Euclidean space and each point represents a triad. A cube graph could be seen as a unit of this space. Graphs in general compose of the points, which we called vertices, and lines, which we called edges. We are also interested in tracing a sequence of vertices and edges in a graph, which we call paths or walks. If we impose musical meaning on our vertices, edges and paths, we could transform the graph to a voice-leading space. The vertices represent the chord, the edges represent the chord progression. That is, transformation from one chord to another chord. The paths would be voice-leading paths, which correspond to melodies in songs. So we can see that we actually relate the sound of a song to visual representation of a graph. Therefore, analyzing the patterns, for example, the symmetry property of our graphs, will contribute to more harmonic and smooth songs.

Page 4: Hexatonic triad graph and cube graph

Why study cube graphs? The cube graph is the underlying graph of the hexatonic triad cube graph. For the hexatonic triad cube, it is a part of the three-voice OP equivalence musical space. What is OP equivalence? It is octave equivalence and permutation equivalence. Notice that previously, the pitch class space is octave equivalence. The octave equivalence means every pitch in the triad is octave equivalence. The permutation equivalence means permuting pitches preserve equivalence. So we can view the voice-leading spaces as a composition of chord and chord progression with some kinds of equivalence relations imposed on the chord. Similarly, a 4 dimensional hypercube is a part of the 4-voice OP spaces. So we may believe that a n dimensional hypercube is a part of the OP spaces and that is why I hope to analyze the hypercube graphs.

Page 5: Network graph

The next question is how to analyze the hypercube graph? Notice that in our hypercube graph, we do not consider the spatial information. That is, there is no volume, no area and no length of the graph. We only consider how vertices are connected to each other. Look at the cube graph, if we look at it tilted at 45 degrees, it looks like a neural network in machine learning. Therefore, we can represent the cube graph in a two dimensional network graph. How we create the network graph, we just start from one vertex, define it as the 0_{th} layer, and find the adjacent vertices, define it as the 1_{st} layer, and find the adjacent vertices to the 1_{st} layer to form the second layer. And so on and so forth. For the hexatonic triad graph, we find that the layers are related to the chord quality, or the type of the chord. It shows that our network graph may be a nice way to represent the hypercubes graph and may have some useful patterns.

Page 6-7: Graph of the hypercubes

Let's give a definition of our multi-dimensional cube graph. For 0-cube, the graph is a single vertex. For 1-cube, it is two vertices that are connected by an edge. For 2-cube, we draw another 1-cube and connect the corresponding vertices. It is like a square. Notice that for the graph discussion below, we are only interested in vertices and how these vertices are connected to each other by edges. So the size of the graph does not matter at all. We can also represent our 2-cube like this one. We can relabel the vertices in a way that preserve the vertices' connection or we call adjacency between vertices. We call these two graphs isomorphic to each other. To generate a 3-cube, we draw another 2-cube inside and connect the same labeling vertices. The underlying graph of the 3-cube is the hexatonic triad graph. Similarly, the 4-cube is created by drawing another 3-cube and connecting the same labeling vertices. We call it tesseract. Notice that the 4-cube in our construction is not exactly a 4-dimensional cube. Usually the 4_{th} dimension is time and a 4-dimensional cube cannot be visualized in three dimensional space. By our construction, for the k-cube, we will have 2 to the power of k vertices and each vertex has k adjacent edges. To get the (k+1)-cube, we need to draw another k-cube and connect the vertices with the same labels. In summary, we define the base case cube and define how to draw every successive cube. In this way, we can construct cube with any dimensions. This is an idea of mathematical induction.

Page 6-7: Network graph of the hypercubes

Now we draw the corresponding network graph. We start from one vertex and define it as the 0_{th} layer and find its adjacent vertices to form the 1_{st} layer. And keep finding the adjacent vertices of the 1_{st} layer to form the second layer. So the 2-cube has three layers. Start from one vertex, there are 3 adjacent vertices, the next layer also has 3 vertices, the last layer has one. The 3-cube has four layers. For a 4-cube, every vertex has four edges. So for every vertex in the first layer, there are three edges connected to the vertices in the next layer. Using the symmetry property, for the second layer, it has two edges connected to the previous layer and two edges connected to the next layer. We can generate the k-cube network graph in this way. (You may also view it as a directed graph. From the 0_{th} layer, all the way pointing to the final layer.)

Page 8: Properties of the hypercubes

We can observe some properties of these hypercubes. First, every vertex in the n-cube is equivalent to each other. Every vertex has n edges, which we called n-regular in graph theory. Second, it seems that the n-cube network graph has n+1 layers. Third, for the network graph, there is no edge connected to the vertices in the same layer, while there are edges connected in two consecutive layers. Actually, we can further reduce the network graph into exactly two layers. How can we achieve it? Let me introduce a concept called graph coloring. In graph coloring, we have a constraint that adjacent vertices that are connected by edges should be colored in different colors. We usually will be interested in finding the minimum number of colors needed to color the graph. To color the network graph, we can color a layer as black, the next layer as white, alternatively, we only need two colors to color the graph. If we put all black vertices in one set and all white vertices in another set, we find that there are no edges within each set and there are edge connections between two sets. The graph is called a bipartite graph. We would still focus on network graphs now since it is related to chord quality.

Page 9: Vertices in each layer and Pascal’s Triangle

We notice that in hexatonic triad graphs, each layer has different types of triads. So it would be helpful to analyze the number of vertices in each layer. For 0-cube is 1. One cube is 1,1. 2-cube, 1,2,1. 1,3,3,1. 1,4,6,4,1. I guess you may already see some patterns or find it familiar. Yes, it is Pascal’s triangle, right? We can calculate the next sequence of numbers based on pairwise summation of the previous sequence of numbers. For a 5-cube, it is 1,5,10,10,5,1. Therefore, we can get the number of vertices in each layer for cubes in any dimension. For n-cube, it corresponds to the nth row in the Pascal’s triangle. They are the binomial coefficients. N choose k means the number of ways to select k out of n objects, when order doesn't matter.

Page 10: Musical implication of vertices in each layer of 3-cubes and 4-cubes

The number of vertices in each layer tells us about the chord quality. For the hexatonic triad, the 0th layer contains 1 Augmented triad, the 1st layer contains 3 minor triads. The 2nd layer contains 3 major triads. And the last layer contains 1 augmented triad. Layers are also related to the sum class. To calculate the sum class, we add every pitch in the triad and see its congruence to modulus 12. The sum class will be 9, 10, 11 and 0. For the octatonic seventh chord, it is a part of the 4-voice OP space. Similarly, it corresponds to the 4-cube. The 0th and the last layer is the fully diminished seventh chord, the first layer is the half diminished seventh chord. The middle layer is composed of two French augmented sixth chords and four minor seventh chords. The third layer contains four major seventh chords.

Page 11: Hypercubes and Binomial expansion

Since the number of vertices in each layer are the binomial coefficients. I wonder whether binomial expansion will have something to do with the network graph. I write out the expansion. We find that the number of terms in the expansion is the same as the number of layers in the network graph. Maybe one term will correspond to one layer. The coefficients in each term correspond to the number of vertices in each layer. So it would be appropriate to think what a and b represent and whether the power in a or b have something to do with the graph, for example, the edges. Let's analyze the patterns of edges in the network graph. One vertex will have two kinds of edges, the edges that are connected to the previous layer and the edges that are connected to the next layer. We first focus on the edges that are connected to the next layer. For a 3-cube, it is 3, 2, 1,0. For the 4-cube, it is 4, 3, 2, 1,0. Wow, we seem to observe the patterns. What are these numbers, it seems to correspond to the power of a. For the edges that are connected to the previous layer, the 3-cube is 0, 1, 2, 3. The 4-cube is 0, 1, 2, 3, 4. These numbers are related to the power of b. We observe that the power of a is the number of edges that are connected to the next layer, or we call the right edges. The power of b is the number of edges that are connected to the previous layer, or we call the left edges.

Page 12: A rigorous proof using labeling.

Notice that we just deduce our patterns based on our observations. We observe that the hypercubes can be written as a network graph form. The vertices and edges in each layer correspond to each term in the binomial expansion. But how to rigorously prove it? Below is the intuition of the proof. We could prove it by labeling the vertices. I will show that we can label the vertices using a sequence of binary notations. For the 1-cube, we label it as a and b. For 2-cube, we first label the two vertices as a and b. And then distinguishing the a and b using the same labeling vertices. Similarly for 3-cube. We can find that the vertices that are connected to each other only differ by one number. Vertices' labels composed of all possibilities of sequence of a and b. The connected vertices are in the consecutive layer. Each layer contains a certain number of a and b. That's why the number of vertices in each layer is a combinatorial number. Given a certain number of a and b, how many possible sequences of a and b is the same as how many possible ways to choose the positions of as and left others as bs from the sequence. I use the 3-cube as an example. For the 0th layer, it has one cube. For the 1st layer, it has three squares b. For the second layer, it has three a b squares. For the last layer, it has one b cube. We find that vertices at the extreme side of the network graph tend to have more a or more b, while vertices at the middle part of the network graphs tend to have more even numbers of a and b. To move a vertex to the previous layer, one b changes to a, to move a vertex to the next layer, one a changes to b. Thus, the number of bs is the number of edges connected to the previous layer, which is equal to the power of a, and the number of as is the number of edges connected to the next layer, which is equal to the power of b.

Page 13: Hexatonic triad graph

For the hexatonic triad, it corresponds to the cube of a plus b. The vertices represent chords and the edges indicate single-semitone voice-leading connections. It has one augmented triad in the 0th layer. One augmented triad in the third layer. 3 minor triads in the 1st layer. 3 major triads in the second layer. The cube power of a means there are 3 ways for the base layer augmented triad to transform to minor triads. 2 ways for a minor triad transform to major triads. There is one way for a major triad to transform to an augmented triad. There is one way for a major triad to transform to an augmented triad. For the network graph of the hexatonic triad, we can draw it as a directed graph. The arrows in the graph point in the direction of ascending voice leading.

Page 14-15: Octatonic Seventh-Chord Hypercubes Graph

For the octatonic seventh-chord hypercubes, it is a part of the 4-voice OP space. Similarly, it is corresponds to the 4-cube. The 0th and the last layer is the fully diminished seventh chord, the first layer is the half diminished seventh chord. The middle layer is composed of two French augmented sixth chords and four minor seventh chords. The third layer contains four major seventh chords. Similarly, the power of a is corresponding to the number of ways to transform to the next types of triads. The power of a of a vertex is the number of ways for it to transform to the next types of chords and the power of b is the number of ways for it to transform to the previous type of chords.

Page 16-17: Total number of m-cubes in n-cubes and differentiation

Let's go back to our mathematical analysis. We analyze vertices and their adjacent edges in each layer. Next, we may investigate the total number of vertices and edges. For the total number of vertices, we just sum the number of vertices in each layer. The equation would be like this. Summation of the binomial coefficients is equal to 2 to the power of n. This is a well-known identity. Which can also be shown by defining a and b to be 1 in the binomial expansion of the polynomial a plus b to the power of n. The number of edges is equal to summation of the power of a, that is, the summation of all the right edges. It is the summation of k times each binomial coefficient. There is another way to calculate the total number of edges by the number of vertices. For n-cube, there are 2 to the power of n vertices and each vertex has n edges. There are n times 2 to the power of n edges with double counting since each edge is formed by two vertices. To eliminate double counting, we divide two here. So we got this equation. Actually, there is another way to get this equation, just by differentiating both sides of the vertex equation we got.

Moreover, we find that actually, the number of vertices is just the number of 0-cube. The number of edges in n-cube is just the number of 1-cube in n-cube. So maybe we could also find the number of faces, or the number of 2-cube in n-cube.

Or the number of 3-cube, all the way to the number of (n-1)-cube in n-cube. So how many of them?

The edge equation could be derived from the vertex equation. So we could guess if we keep differentiating the equations, we will get the number of cubes in the following increasing dimension. We can prove that our hypothesis is true. For every vertex in n-cube, there are n choose m ways to choose m edges incident to that vertex. Each of the collections of m edges defines one of the m-dimensional cubes incident to the considered vertex. Doing this for all the vertices of the hypercube, each of the m- dimensional faces of the hypercube is counted 2 to the power of m times since it has that many vertices. So the total number of m-cubes in n-cubes is n choose m times, how many collections, times 2 to the power of n divided by 2 to the power of m. We can store this information in a matrix form. We call it the configuration matrix.(to be continue)

Page 18: Musical implication of the m-cubes in n-cubes

So what is the musical implication of it? The number of vertices means the number of distinct chords. Two chords connected by a line always share two common tones, differing only by a single-semitone displacement in the remaining note. So the number of edges, or the number of one cube, in a hypercubes means the total number of pairs of chords in this hypercubes that has only one different pitch differed by one semitone. The number of m-cubes means the number of m chords that have exactly m different pitches differed by one semitone.

Page 19-20: Shortest voice-leading path

Usually we are interested in the voice leading path where voices are moved between chords in the shortest possible manner to give a smooth connection between chords. Thus, we hope to analyze the shortest paths in the graph. For the 3-cube, the first layer is 0, second layer 1, it has two right edges, so the third layer is 2. To go to the vertices in the second layer, there are 2 shortest paths. The third layer has 3 right edges, so to go to the last layer, we have 2 times 3 equal to 6 shortest paths. We find that we can multiply the number of right edges for one vertex in each layer to get the shortest number of paths. So the k-cube would be 0, 1, 2 factorial, dot dot dot, to k factorial. For the hexatonic triad graph, we can draw a table to show the number of shortest paths from vertices in one layer to vertices in another layer. For the chains of hexatonic triad, we can find that the total number of shortest paths from C augmented triad to E augmented triad is 3 factorial to the power of 4. For the octatonic seventh-chord hypercubes, we can find the shortest path in a similar way. The musical implication of the shortest path is how many total semitones does the chord move.

Page 21-22: Represent graph as adjacency matrix

Graphs are easy to visualize, while the computer is more happy with digits. So how to store these graphs on the computer? We can also represent our hypercube network graph as an adjacency matrix. The rows and columns of the matrix are all the vertices. The element in the matrix will be 1 if the row vertices and column vertices are connected and otherwise, it will be 0. In a musical sense, the columns and vertices will be the chords. The elements in the matrix are 1 if the chords are connected by a single-semitone voice leading. This matrix is a symmetric matrix with the diagonal elements being 0s. The matrix powers have an interesting property. The element of row i and column j in the matrix means the number of walks of length k from vertex i to vertex j. The walks means a sequence of vertices and edges of a graph. So we could find the number of voices leading of length k by taking k power of the adjacency matrix. The adjacency matrix of the hexatonic triad will be like this. Adjacency matrix will make the graph represent in algebraic ways that can be stored in a computer, since the graph and the adjacency matrix have 1-1 correspondence with each other.

Page 23: Extension:

Here is some extension about the music part of today's topic if you hope to know more about fundamental music theory. There are five kinds of common equivalence relation in music. These equivalences can be combined to form multiple musical spaces. For the 3 voice OP space, it contains a chain of hexatonic triads containing triads ranging from all possible sum classes. So there are multiple 3-cubes that connect to each other. We can also study this chord, chord transformation, and voice-leading path in this space.

Page 24: General discussion

Here is a general discussion that you may consider. I told a friend about this. The friend thinks music should be created by people with their own stories and feelings. Generating music in a modeling way will destroy the musical meaning. I think representing music in mathematics has beauty of its own and it will contribute to the harmonic music creation. That’s why I hope to explore something in this project. Besides, creating music using artificial intelligence represents a new form of artistic expression rather than a replacement for traditional music. What do you think about it?

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Page 26: Thank you

Thank you very much for watching this video. Hope you learn something and have fun. I am not really professional in the mathematics and music field, so I would appreciate it if you could point out some mistakes in my video and give some suggestions and advice for my video. Bye and hope you have a great day.