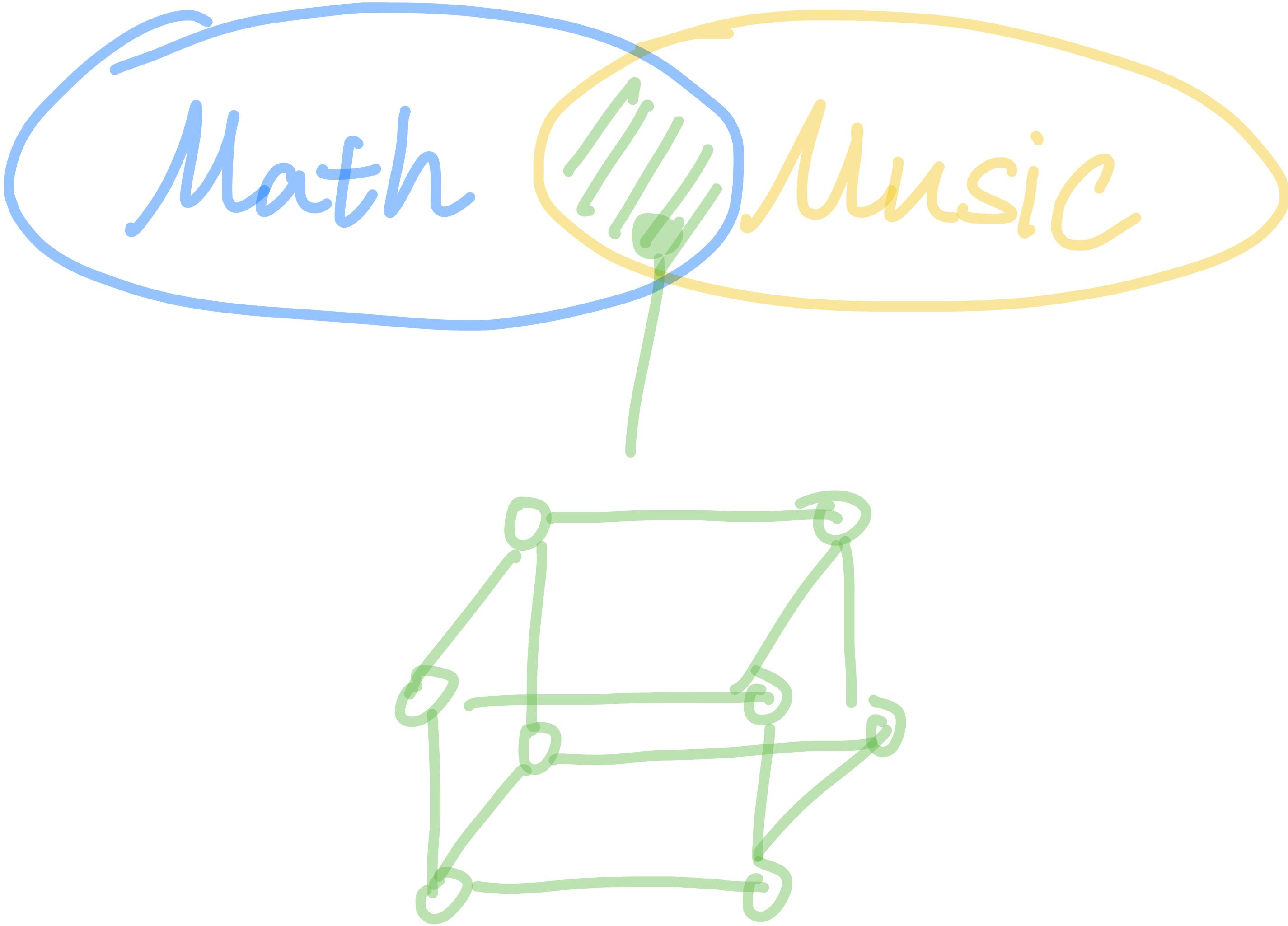
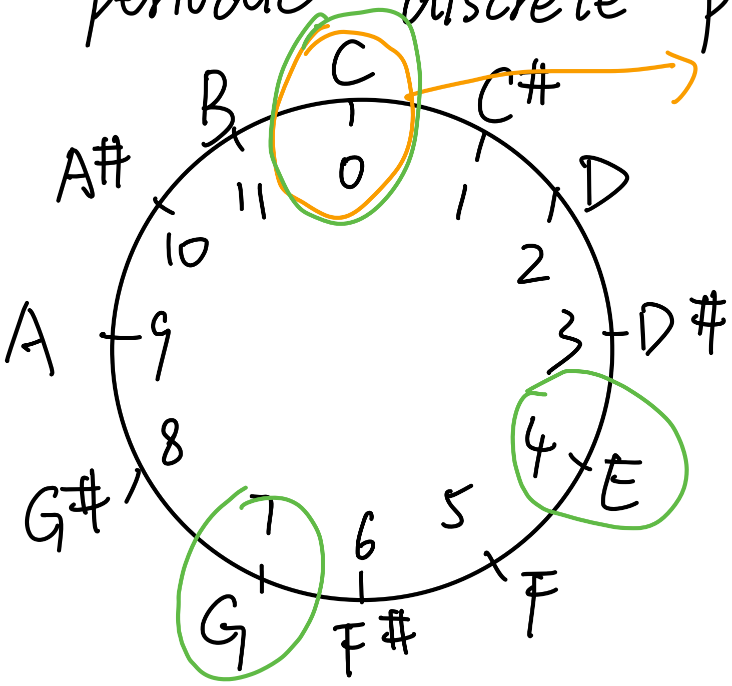

Math
Music

Pitch class space

Piano 12 periodic discrete pitches



pitch class

$(C_0 \sim C_1 \sim C_2 \sim \cdots)$

Octave equivalence.

$x \sim y$

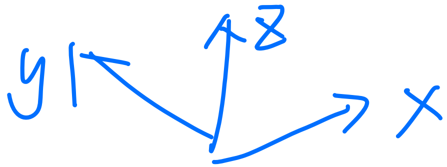
$\Leftrightarrow x - y \equiv 0 \pmod{12}$ /

$x - y = 12k, \ k \in \mathbb{Z}$.

Chord $\geq$ triad

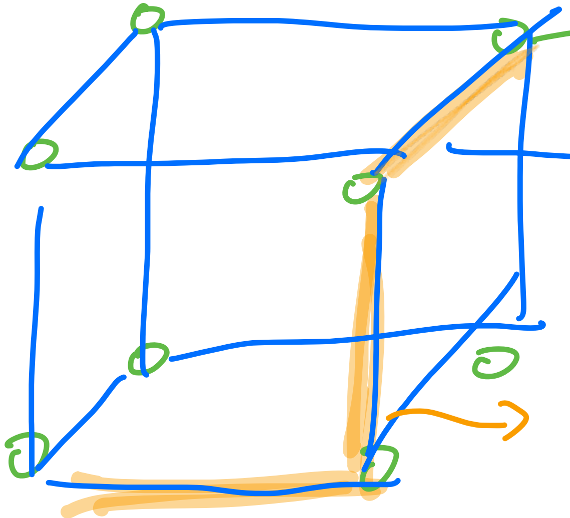
n pitches.

C major triad (C, E, G) (0, 4, 7).



Chord and voice-leading

Graph. Voice-leading space



vertices ⟷ chord. ↙ OP.

edges. ⟷ chord progression.

paths/walks ⟷ voice-leading path.

Melody.

Patterns, Symmetry ⟷ Harmonic. Smooth

Melody

Visual ⟷ sound.

hexatonic triad graph.

Cube $\subseteq$ 3-voice OP equivalence musical space

(3 pitches)

① octave

$(x, y, z) \sim (x+12, y-12, z)$

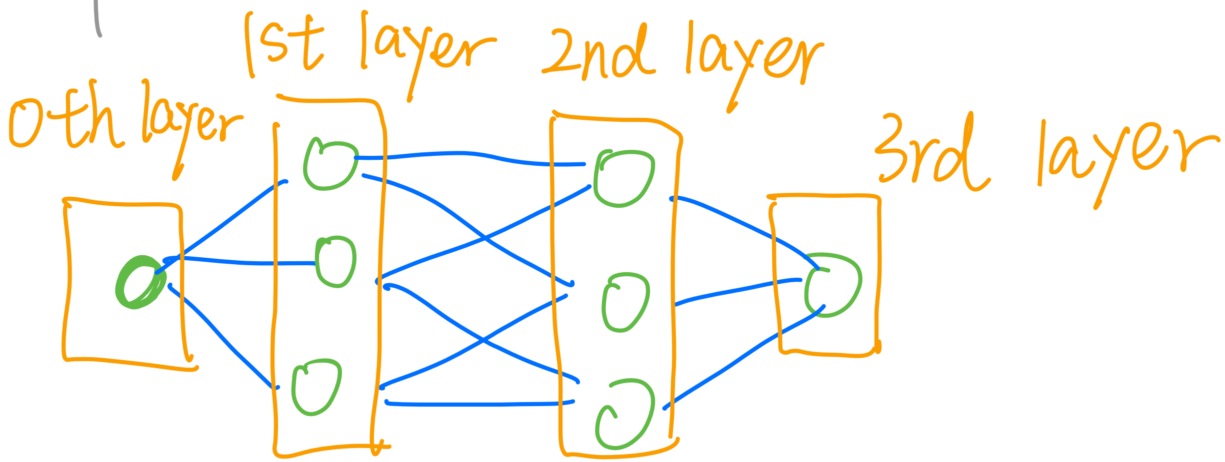
② permutation

$(x, y, z) \sim (x, z, y)$.

Hypercube ⟷ network graphs

No spatial information

Networks



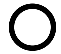
layers → Types of chords / chord quality.

We are only interested in vertices and positions of edges.

There is no spatial information (area, volume) in these graphs

n-cube | Graph | Network Graph.

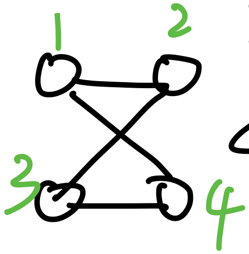
0

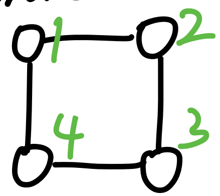


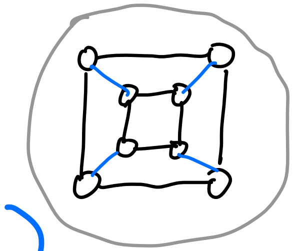
1

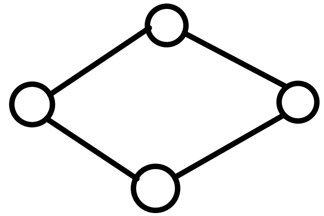
isomorphic

2



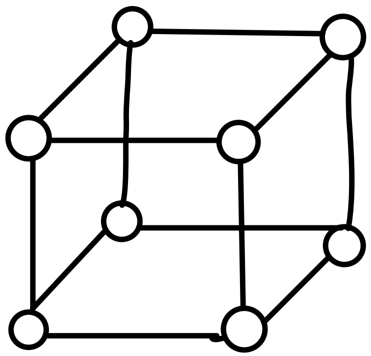


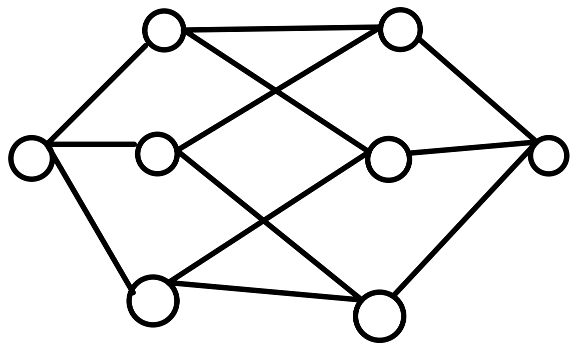




(Relabel and preserve adjacent vertices)

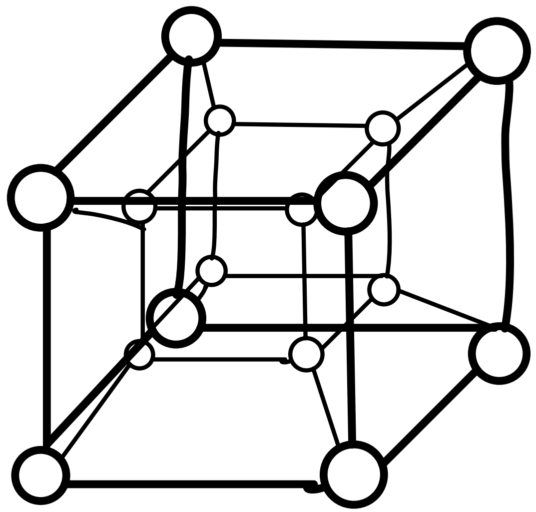
3

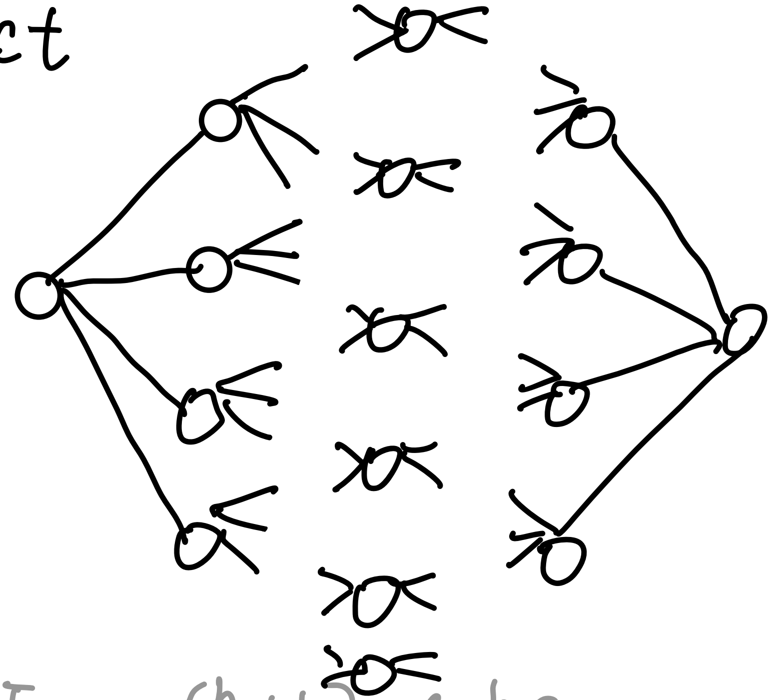




4.

Tesseract

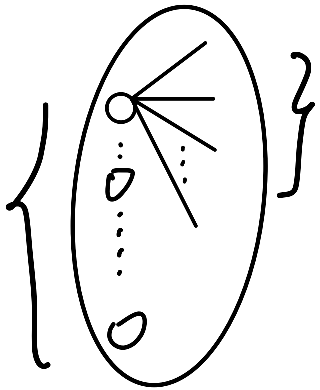




⋮

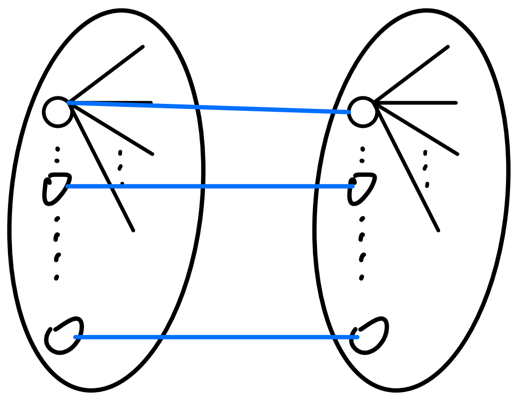
K

2^k vertices



k adjacent edges

K+1



For (k+1)-cube, double k-cube and connect the corresponding vertices.

Mathematic Induction.

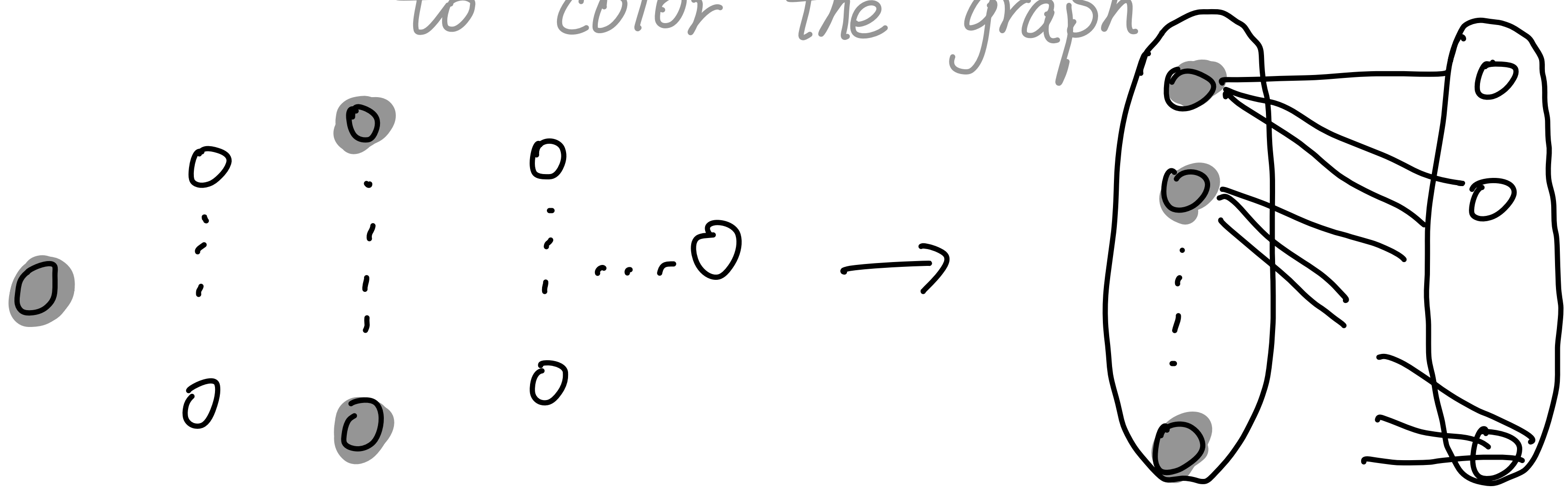
property : ① n- regular (vertices equivalent)

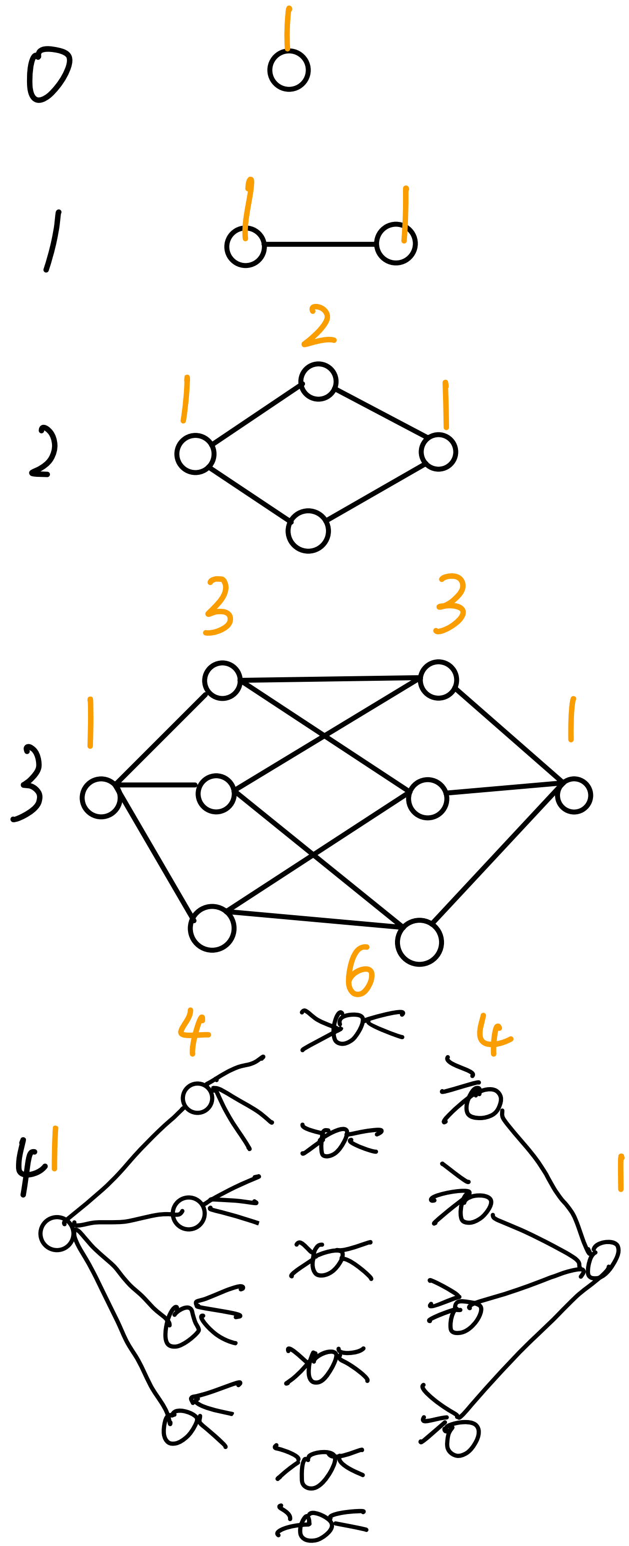
② Bipartite. ③ n+1 Layers

Graph coloring :

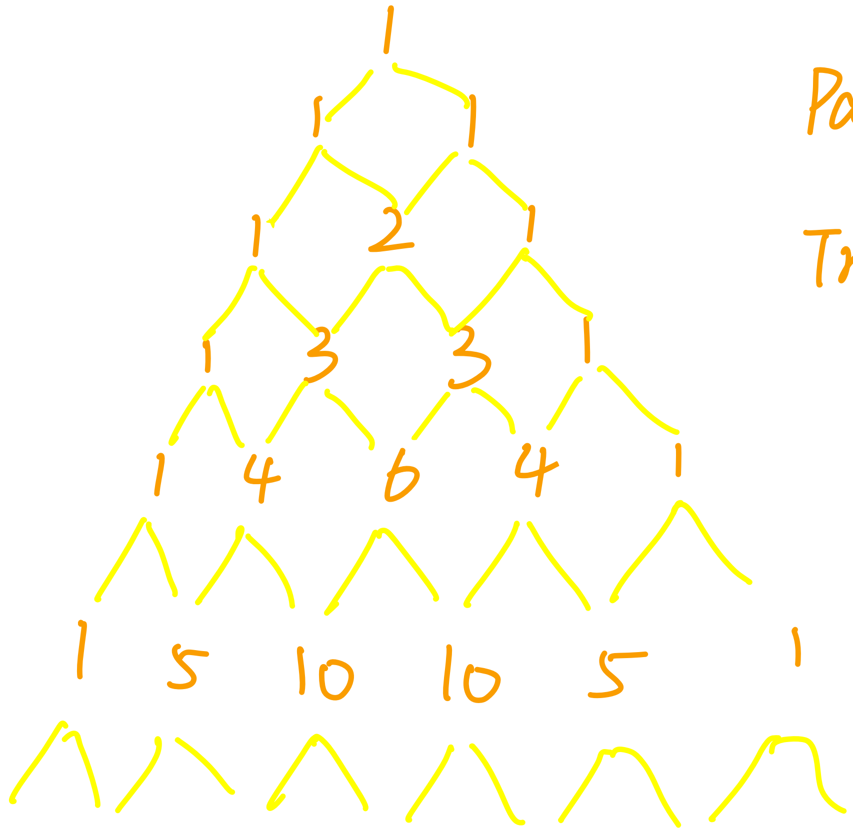
constraint: Adjacent vertices should be colored in different colors

Interested: The minimum number of colors needed to color the graph





# of vertices in each layer

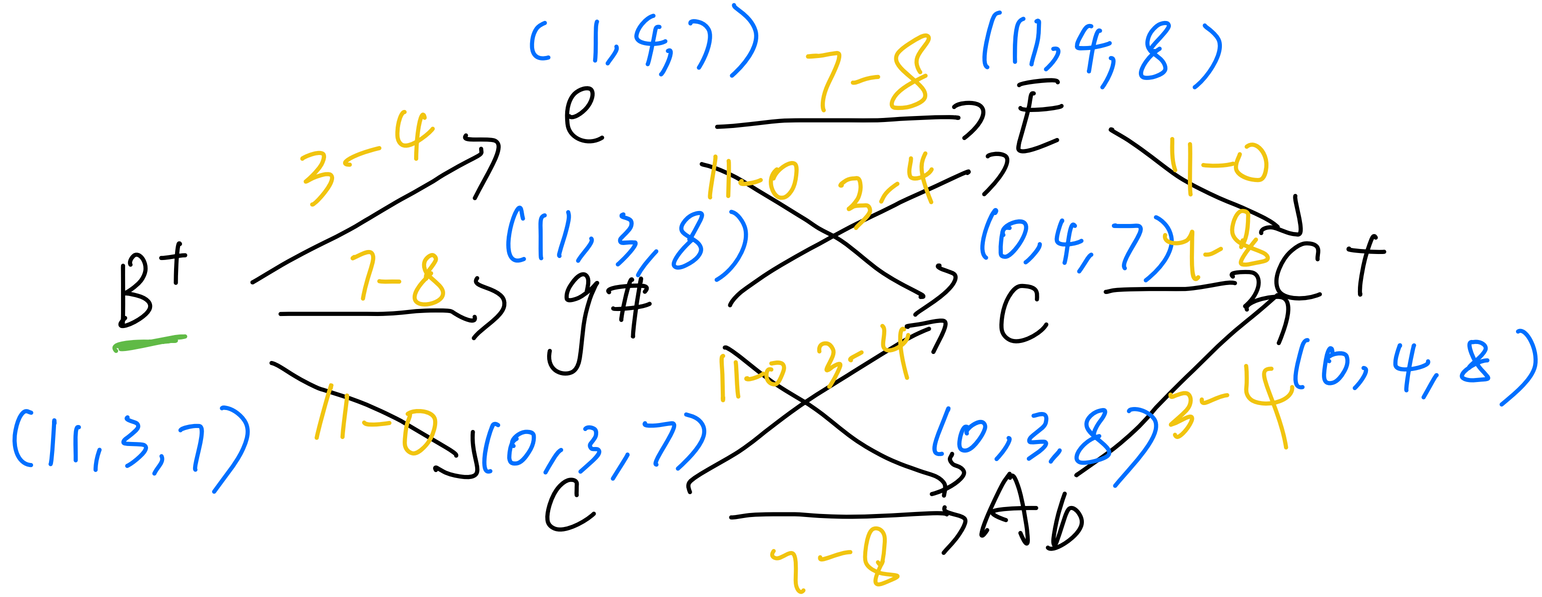


Pascal's Triangle!

Combinations:

nth-row : $\binom{n}{0}\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}$

Hexatonic triad graph



# of chords	1	3	3	1
chord quality	Augmented	Minor	Major.	Augmented
sum class	9	10	11	0.

$11+3+7 \equiv 9 \pmod{12}$

Binomial expansion

a^p: $\rightarrow$ # of right edges

b^q: $\rightarrow$ # of left edges.

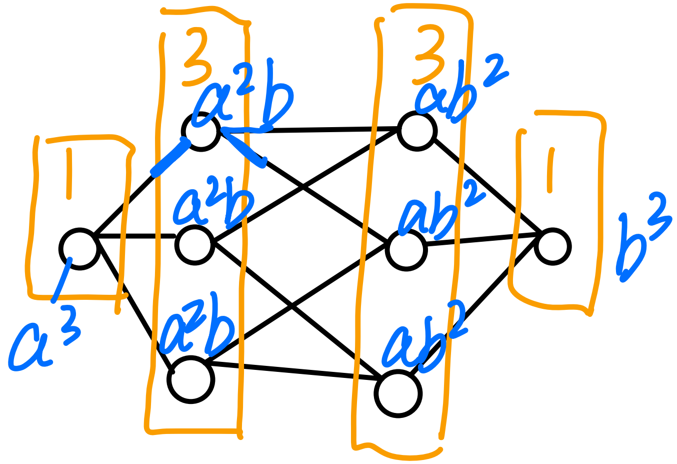
$$(a+b)^0 = 1$$

$$a+b = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)(a+b)$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$



Terms $\leftrightarrow$ layers

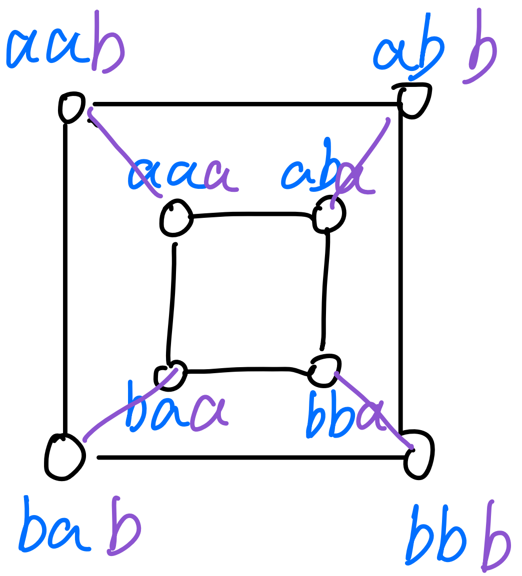
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

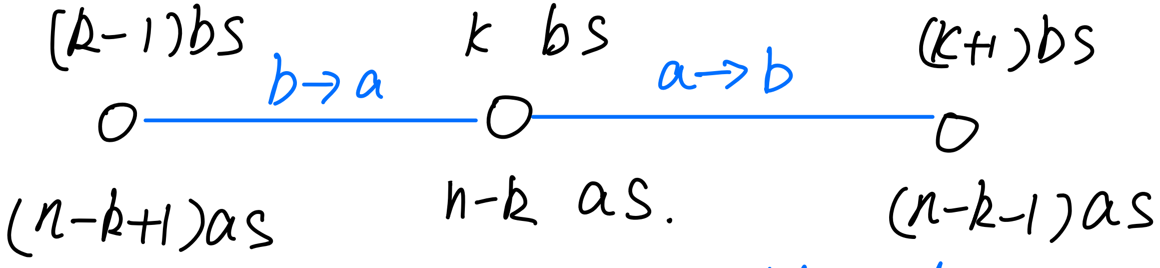
$$\vdots$$

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}a^0b^n$$

Rigorous proof

V: $x_1 x_2 \cdots x_n$. $x_i = a$ or b.





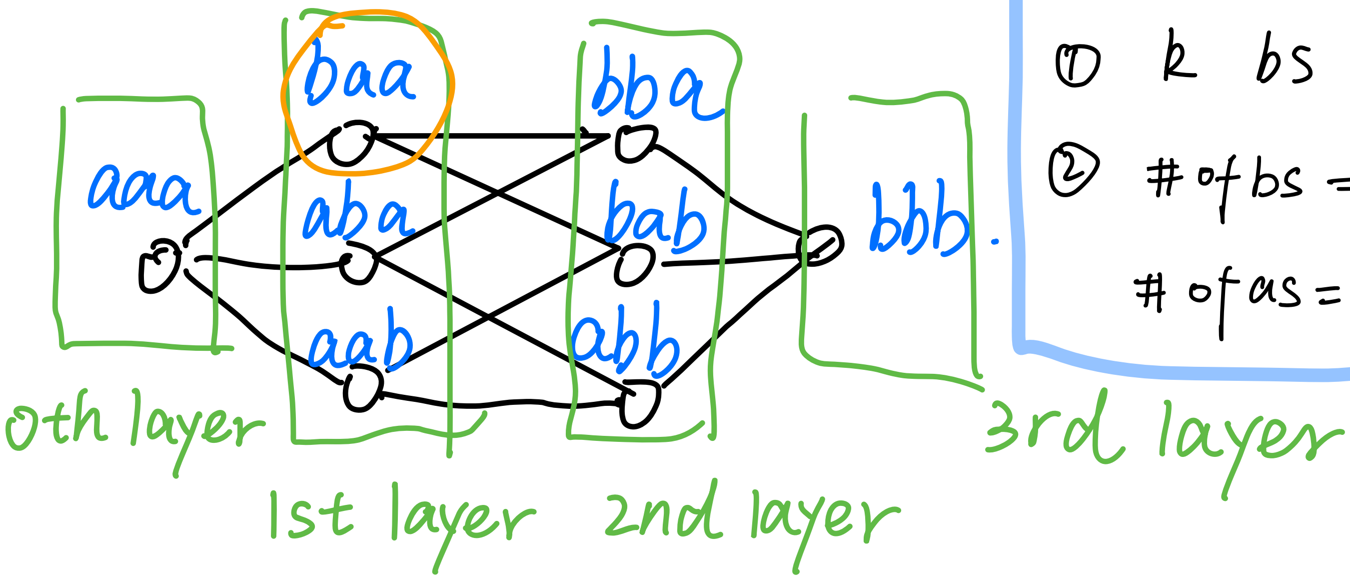
k left edges   n-k right edges

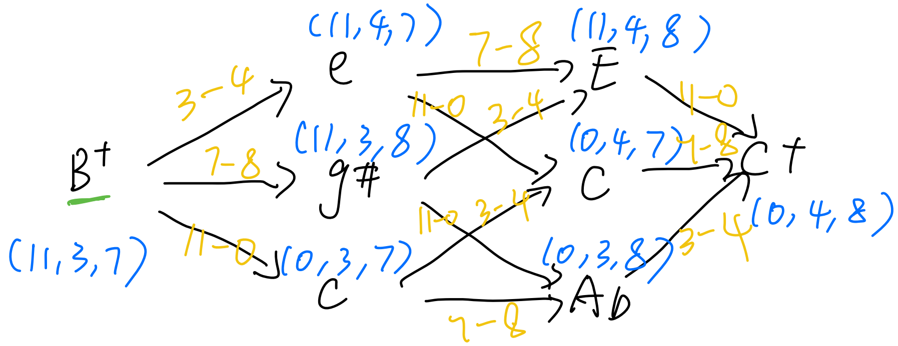
Kth layer: $(a+b)^k = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

① k bs, (n-k) as.

② # of bs = # of left edges = k

# of as = # of right edges = n-k.





3 augmented → minor

2 minor → major

1 major → augmented

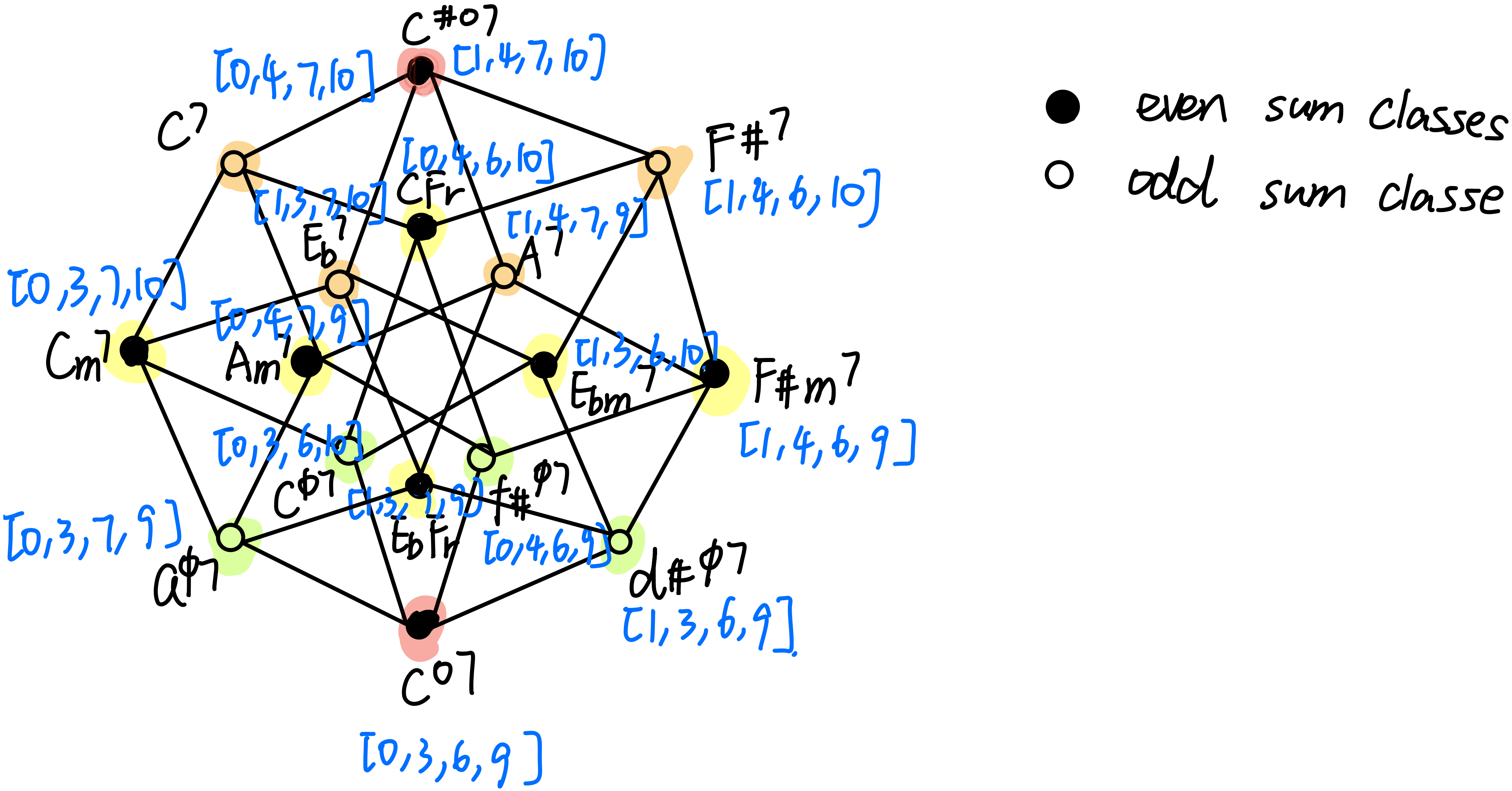
$$-(a+b)^3 = 1a^3 + 3a^2b + 3a^1b^2 + 1b^3$$

1 augment

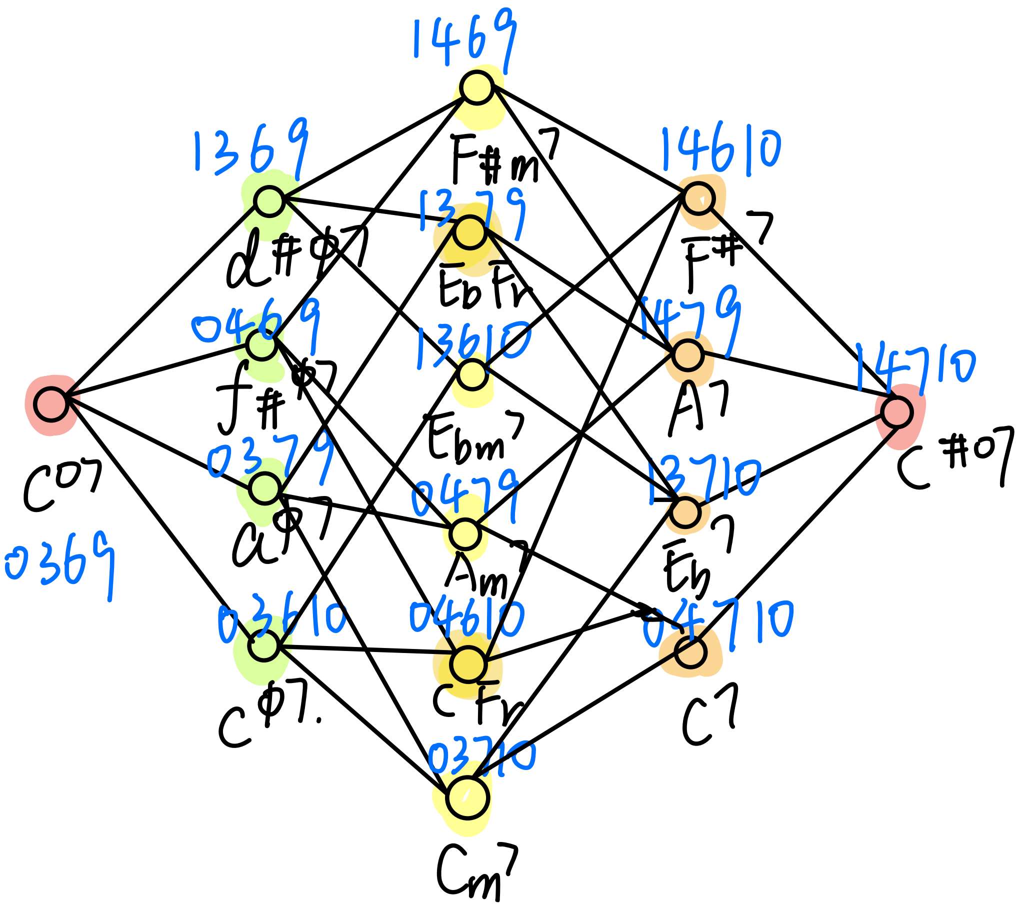
3 minor

3 major

1 augmented



Octatonic seventh-chord hypercubes.



Fully diminished seventh chord

Half diminished seventh

Minor seventh chord

French augmented sixth chord

Major seventh. chord

$$(a+b)^4 = 1a^4 + 4a^3b + (4a^2b^2 + 2a^2b^2) + 4ab^3 + 1b^4$$

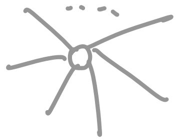
n-cube	# of vertices	# of edges
0	$1 = 1$	0
1	$1+1 = 2$	$1+0 = 1$
2	$1+2+1 = 4$	$1\cdot 2 + 2\cdot 1 = 4$
3	$1+3+3+1 = 8$	$1\cdot 3 + 3\cdot 2 + 3\cdot 1 = 12$
4	$1+4+6+4+1 = 16$	$1\cdot 4 + 4\cdot 3 + 6\cdot 2 + 4\cdot 1 + 1\cdot 0 = 32$
$\vdots$	$\vdots$	$\vdots$
n	$\sum_{k=0}^{n} \binom{n}{k} = 2^n$	$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$

differentiate

# of faces   # of cubes. $\cdots$ # of m-cubes

differentiate $\cdots$

$\binom{n}{2}2^{n-2}$ $\quad\to\quad$ $\binom{n}{3}2^{n-3}$ $\quad\to\quad$ $\binom{n}{m}2^{n-m}$

$\binom{n}{m}$ ways to choose  m edges.

one of m-cube

# of vertices

$$\frac{2^n\binom{n}{m}}{2^m}$$

Each m-cube has 2^m vertices. Each is counted 2^m times.

Musical implication

① # of vertices = # of distinct chords.

② # of edges (1-cube) = # of pairs of chords that has only one different pitch differed by one semitone

③ # of m-cube = # of set of m chords that has exactly m different pitches differed by one semitone

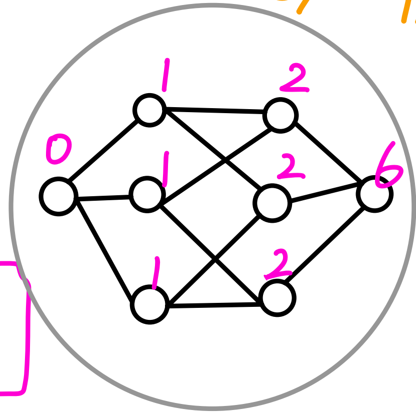
Cube # shortest paths from the first vertex to each vertices.

Multiply # of right edges for one vertex in each layer

1D   0   1

2D   0   1   2

3D $\boxed{0 \quad 1 \quad 2 \quad 6}$



4D   0   1   2   6   24

5D   0   1   2   6   24   120

6D $0 \underbrace{\quad}_{\times 1} 1 \underbrace{\quad}_{\times 2} 2 \underbrace{\quad}_{\times 3} 6 \underbrace{\quad}_{\times 4} 24 \underbrace{\quad}_{\times 5} 120 \underbrace{\quad}_{\times 6} 720$.

⋮

KD   0   1   2!   3!   . . . . . .   k!

Total # of shortest path from X to Y.

Y \ X	B^+	e/g#/c	E/C/Ab	C^+
B^+	0	1	2	3! = 6
e/g#/c	1	0	1	2
E/C/Ab	2	1	0	1
C^+	3! = 6	2	1	0

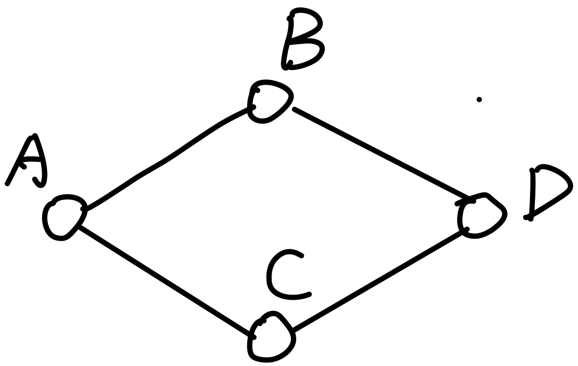
example:

Total # of paths from B^+ to C^+ : 3! = 6.

How many total semitones does the chord move

= The difference of layer.

Adjacency Matrix

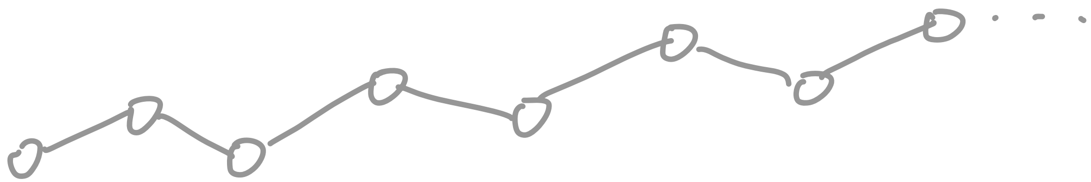


$$M = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 0 \\ B & 1 & 0 & 0 & 1 \\ C & 1 & 0 & 0 & 1 \\ D & 0 & 1 & 1 & 0 \end{array}$$

$$M_{ij} = \begin{cases} 1 & i, j \text{ is adjacent} \\ 0 & \text{o.w.} \end{cases}$$

Matrix powers.

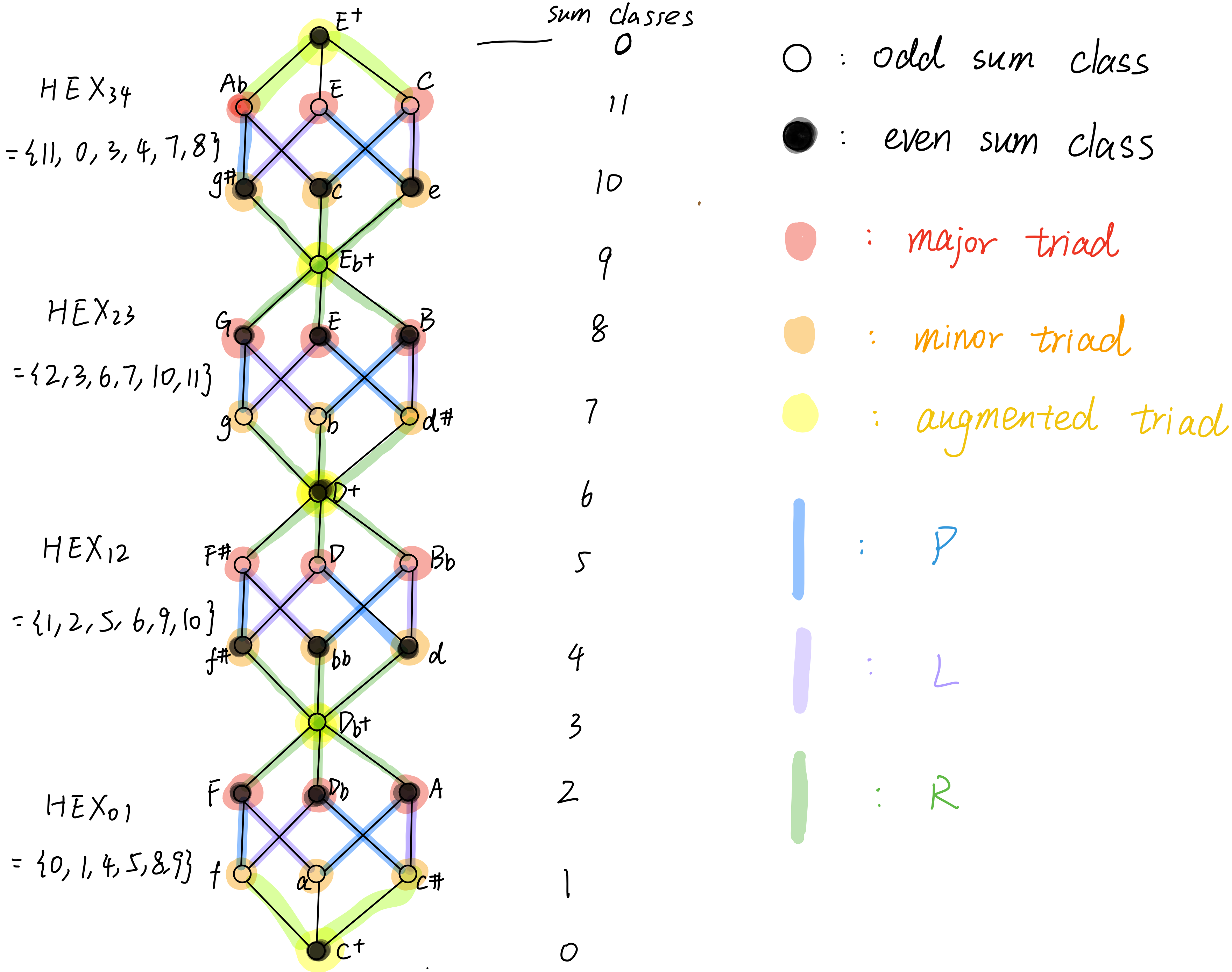
$$M^k = \underbrace{M \cdots M}_{k}.$$



M^k_{ij} : # of walks of length k from vertex i to vertex j.

Adjacency Matrix

	B^+	e	$g\#$	c	E	C	A^b	C^+
B^+	0	1	1	1	0	0	0	0
e	1	0	0	0	1	1	1	0
$g\#$	1	0	0	0	1	1	1	0
c	1	0	0	0	1	1	1	0
E	0	1	1	1	0	0	0	1
C	0	1	1	1	0	0	0	1
A^b	0	1	1	1	0	0	0	1
C^+	0	0	0	0	1	1	1	0



sum classes
E^+
0
HEX_{34}
$=\{11, 0, 3, 4, 7, 8\}$
Ab
E
C
11
g#
c
e
10
Eb^+
9
HEX_{23}
$=\{2, 3, 6, 7, 10, 11\}$
G
E
B
8
g
b
d#
7
D^+
6
HEX_{12}
$=\{1, 2, 5, 6, 9, 10\}$
F#
D
Bb
5
f#
bb
d
4
Db^+
3
HEX_{01}
$=\{0, 1, 4, 5, 8, 9\}$
F
Db
A
2
f
a
c#
1
C^+
0
: odd sum class
: even sum class
: major triad
: minor triad
: augmented triad
: P
: L
: R

General Discussion

Music

Math

Human

Model?

Feeling?

Page 22: References:

Adjacency Matrix. Retrieved from: https://en.wikipedia.org/wiki/Adjacency_matrix

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L.gargano, M., F.Malerba., Lewinter, M.. Hypercubes and Pascal's Triangle: A Tale of Two Proofs. *Mathematics Magazine*. p.216.

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Thanks you