Hypercubes and Pascal's Triangle: A Tale of Two Proofs

MICHAEL L. GARGANO
JOSEPH F. MALERBA
Pace University
NY, NY 10038

MARTY LEWINTER
Purchase College
Purchase, NY 10577

The entries of the nth row of Pascal's triangle consists of the combinatorial numbers

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-2}, \binom{n}{n-1}, \binom{n}{n}, \quad \text{where } \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

These numbers are called the binomial coefficients, because they satisfy the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1}$$

Upon setting $x = 1$, we obtain

$$2^n = \sum_{k=0}^{n} \binom{n}{k}. \tag{2}$$

Differentiating both sides of (1) with respect to x, we have

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}. \tag{3}$$

Setting $x = 1$, we finally obtain the well-known identity [**4**, p. 11],

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}. \tag{4}$$

This last identity can also be proven without calculus. For a typical short proof, see Rosen [**9**, Section 4.3, Exercise 51] or Buckley and Lewinter [**3**, Section 1.4, Exercise 9].

We shall prove identity (4) using graph theory. In contrast to the previously mentioned proofs, which suggest that (4) is an algebraic accident, our approach here will count a combinatorial object in two different ways, thereby yielding insight into *why* the identity is true. The hypercube, Q_n, is an important graph, with applications in computer science [**1**]–[**3**], [**5**]–[**8**]. Its vertex set is given by $V(Q_n) = \{(x_1, x_2, \ldots, x_n) \mid x_i = 0 \text{ or } 1;\ i = 1, 2, \ldots, n\}$, i.e., each vertex is labeled by a binary n-dimensional vector. It follows that $|V(Q_n)| = 2^n$. Vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are adjacent if and only if $\sum_{i=1}^{n} |x_i - y_i| = 1$, from which it follows that Q_n is n-regular. Since the degree sum is $n2^n$, we find that Q_n has $n2^{n-1}$

edges, that is, $|E(Q_n)| = n2^{n-1}$. The distance between vertices x and y is given by $\sum_{i=1}^{n} |x_i - y_i|$, that is, the number of place disagreements in their binary vectors.

Calling the vertex $(0, 0, \ldots, 0)$ the *origin*, define the ith *distance set* D_i, as the set of vertices whose distance from the origin is i. Then for each $i = 0, 1, 2, \ldots, n$, we have $D_i = \{(x_1, x_2, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = i\}$, that is, D_i consists of those vertices with exactly i 1s in their binary n-vectors. Moreover, we have $|D_i| = \binom{n}{i}$. The fact that the D_is partition $V(Q_n)$ demonstrates Equation (2) rather nicely.

Now observe that the induced subgraph on any D_i contains no edges, since all of the binary vectors of the vertices in D_i contain the same number of 1s. (If two vertices are adjacent, the number of 1s in their binary vectors must differ by exactly one.) Furthermore, if $|i - j| \geq 2$, then if $x \in D_i$ and $y \in D_j$, it follows that x and y are nonadjacent, that is, $xy \notin E(Q_n)$. Then all edges are of the form uv, where $u \in D_i$ and $v \in D_{i+1}$, for $i = 0, 1, 2, \ldots, n-1$. Since each vertex in D_{i+1} has $i+1$ 1s in its binary vector, it is adjacent to exactly $i+1$ vertices in D_i. (These vertices are obtained by replacing one 1 by 0 in the binary vector of the chosen vertex in D_{i+1}.) This implies that the number of edges with endpoints in both D_i and D_{i+1} is $(i+1)|D_{i+1}| = (i+1)\binom{n}{i+1}$. It follows that the total number of edges in Q_n is given by $\sum_{i=0}^{n-1} (i+1)\binom{n}{i+1}$. Finally, since $|E(Q_n)| = n2^{n-1}$, we are done with the proof of (4).

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A Derivation of Taylor's Formula with Integral Remainder

DIMITRI KOUNTOUROGIANNIS
69 Dover street #2
Somerville, MA 02144
dimitrik@alum.mit.edu

PAUL LOYA
Binghamton University
Binghamton, NY 13902-6000
paul@math.binghamton.edu

Taylor's formula with integral remainder is usually derived using integration by parts [**4, 5**], or sometimes by differentiating with respect to a parameter [**1, 2**]. According