ESE 680 Autonomous Racing
Lab 5: Scan_matching - PLICP method - Localization
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1 Theoretical Questions (20)

1. $M_i = \begin{pmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \end{pmatrix}$

1. Show that $B_i := M_i^T M_i$ is symmetric.

1. $M_i = \begin{pmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \end{pmatrix}$

1. $B_i = M_i^T M_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p_{i0} & p_{i1} \\ -p_{i1} & p_{i0} \end{pmatrix} \begin{pmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \\ p_{i0} & p_{i1} & p_{i0}^2+p_{i1}^2 & 0 \\ -p_{i1} & p_{i0} & 0 & p_{i0}^2+p_{i1}^2 \end{pmatrix}$$

$\therefore B_i = M_i^T M_i$ is symmetric.

2. Demonstrate that B_i is positive semi-definite.

As from http://theanalysisofdata.com/probability/C_4.html , Proposition C.4.1. We know: A symmetric matrix is psd (positive semi definite) if and only if all eigenvalues are non-negative. Eigenvalues for B_i are calculated using Matlab:

```
>> positive_semi_definite_matrix

e =

                  0
                  0
 pi0^2 + pi1^2 + 1
 pi0^2 + pi1^2 + 1
```

All eigenvalues are non-negative. Thus, B_i is a positive semi-definite matrix.

2. The following is the optimization problem

$$x^* = \text{argmin}_{x \in \mathbb{R}^4} \sum_{i=1}^{n} \|M_i x - \pi_i\|_2^2$$
$$\text{s.t.} \quad x_3^2 + x_4^2 = 1$$

a. Find the matrices M, W and g which give you the formulation

$$x^* = \text{argmin}_{x \in \mathbb{R}^4} x^T M x + g^T x$$
$$\text{s.t. } x^T W x = 1$$

2. $\sum_{i=1}^{n} \|M_i x_i - \pi_i\|_2^2$

$= \sum_{i=1}^{n} (M_i x - \pi_i)^T (M_i x - \pi_i)$

$= \sum [(M_i x)^T - \pi_i^T] (M_i x - \pi_i)$

$= \sum (x^T M_i^T - \pi_i^T)(M_i x - \pi_i)$ $\quad = x^T M_i^T \pi_i$

$= \sum x^T M_i^T M_i x - x^T M_i^T \pi_i - \pi_i^T M_i x - \pi_i^T \pi_i$

$= \sum x^T \underbrace{M_i^T M_i}_{M} x - 2x^T M_i^T \pi_i - \pi_i^T \pi_i$

$\because -\pi_i^T \pi_i$ is a constant, $\therefore$ can be ignored

$= \sum x^T M x - \underbrace{2\pi_i^T M_i}_{g^T} x$

$\cancel{M = \sum M_i^T \pi_i M}$

$\therefore \cancel{M = M_i^T M_i}$, $\cancel{g = M_i^T \pi_i}$ $\quad \cancel{M = \sum_i M_i^T \pi_i M_i}$, $g = \sum_i -2\pi_i^T M_i$

$M = \sum_i M_i^T M_i$

when $W = \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{bmatrix}$, $\quad x_3^2 + x_4^2 = 1$

$\Downarrow$

$x^T W x = 1$

Assuming a non-zero column vector z

$z^T W z = [z_1 \; z_2 \; z_3 \; z_4] \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$

$= z_3^2 + z_4^2 \geq 0$

$\therefore$ W is positive semi-definite matrix.

As from the calculation shown in the screenshot, we can tell that:

$$M = \sum_{i=1}^{n} M_i^T M_i \, , \, g = \sum_{i=1}^{n} -2\pi^T M_i \, , \, W = \begin{bmatrix} 0_{2*2} & 0_{2*2} \\ 0_{2*2} & I_{2*2} \end{bmatrix}$$

Which are the answers for question a.

B. Show that M and W are positive semi definite matrix.
The prove of W is shown in the above screenshot using the psd definition.
The prove of M is shown in the below screenshot using the prove in question 1.

From Q1, we know $M_i^T M_i$ is psd.

$\therefore M = \sum_{i=1} M_i^T M_i$, $z^T M z = z^T \cdot \sum_{i=1}^{n} M_i^T M_i z$

$= z^T (M_1^T M_1 + M_2^T M_2 + \cdots + M_n^T M_n) z$

$= z^T M_1^T M_1 z + z^T M_2^T M_2 z + \cdots z^T M_n^T M_n z$.

$\because$ every items on the right side are non-negative as from the results in Q1.

$\therefore$ the sum of them are non-negative.

$\therefore M \geq 0$

$\therefore$ M is psd.