Introduction to Graph Theory

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Ch. 3 Solutions

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1 Prove the following statements:

- a If 3 edges were added to the graph in Figure 63a, then at least 2 of the new edges will be adjacent.
- b Every graph with $v = 5$ and $e = 3$ has at least two adjacent edges.
- c If v is an odd number, then every graph with v vertices and $\frac{1}{2}(v+1)$ edges has at least two adjacent edges.

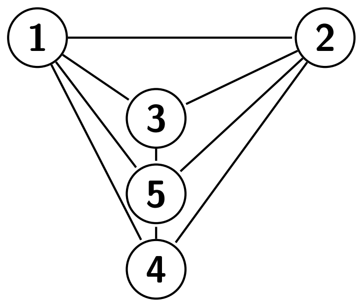
Solution:

- a This is a straightforward application of the pigeonhole principle. If we add one edge, we connect two vertices. If we add another vertex it will either touch an existing vertex (and we are done) or not. If not, then we now have 4 vertices connected, none of which have adjacent edges, and one isolated vertex. For the final edge, no matter how we attach it, it must be adjacent to at least one other edge.
- b We proved this in item a because we did not utilize the fact that Figure 63a was C_5.
- c This follows by the same pigeonhole argument as the previous two items. If we attach $\frac{1}{2}(v-1)$ edges, we either have adjacent edges already, or we don't. If we don't, then we have connected a single isolated vertex and a final edge to add. Therefore, any connection will create two adjacent edges.

2 Prove that in Figure 58, graph a) is planar, and graphs b) and c) are non-planar.

Solution:

- a This graph is isomorphic to the following planar graph:



b This is a supergraph of UG.

c K_7 is a supergraph of K_5.

3 Find a graph G such that *every* expansion of G is also a supergraph of G.

4 - 5 Omitted, answers in book.

6 Draw a nonplanar graph whose complement is nonplanar.

7 Prove the "Petersen graph" is nonplanar.

8 Omitted for size.

9 Prove: if H is an expansion of G then $v_G + e_H = v_H + e_G$.

10 Prove: if H and J are expansions of GG then $v_H + e_J = V_J + e_H$.

11 Find all integers v for which $\overline{C_v}$ is nonplanar. Prove that your answer is correct.

12 Use the pigeonhole principle to prove that there are at least two red maples in the United States having the same number of leaves.

Solution This appears to be false as written (at least without additional assumptions). Counterexample: It is conceivable that there are only two trees, one with 0 leaves and one with 1 leaf.

13 Prove the following statements:

a Except for UG itself, no expansion of UG is also a supergraph of UG.

b Except for K_5 itself, no expansion of K_5 is also a supergraph of K_5.

14 Let S be the set of all expansions of supergraphs of UG or K_5, and let T be the set of all supergraphs of expansions of UG or K_5. Prove that T is *not* a subset of S and therefore $S \neq T$, by finding a supergraph of an expansion of K_5 that is not also an expansion of a supergraph of K_5.

15 Isomorphism is "transitive", that is, if $G \cong H$ and $H \cong J$, then $G \cong J$. This enables us to prove the following theorem.

Theorem: If H is planar and $G \cong H$, then G is planar too.

Use this fact to devise new proofs that the pairs of graphs in Figures 46, 51, 54, and 55 are not isomorphic.

16 Prove that the graphs in Figure 89 are planar.

17 Prove that the graphs in Figure 90 are nonplanar.

18 - 20 In Figures 91 - 93, decide whether each graph is planar or nonplanar and then prove that your choice is correct.