Introduction to Graph Theory

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Ch. 4 Solutions

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1 N_1 is certainly planar, and we proved that it is connected. Prove now that it is polygonal by proving that the statement "every edge of N_1 borders on two different faces" is true.

Solution: This is vacuously true because there are no edges in N_1.

2 Omitted. There are many lengthy discussions about fake induction proofs online.

3 Believe it or not, the graph of Figure 104a is planar. Find its number of faces.

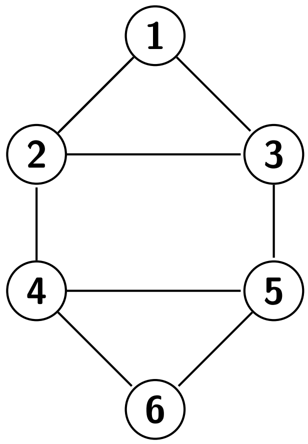
Solution: $v = 9$, $e = 20$, so $v + f - e = 2 \implies f = 13$

4 Imitate the proof of Corollary 12 to construct a proof that the graph in 104b is nonplanar.

Solution: Suppose it is planar. By inspection, the figure is not a supergraph of K_3, so by Theorem 12, we have $e \leq 2v - 4$. But, inspecting the graph, we see that $v = 8$ and $e = 16$ which implies $16 \leq 12$, a contradiction. Therefore, it must not be planar.

5 Find a polygonal graph G having a face bordering the infinite face which, if removed, results in a subgraph H which is not polygonal.

Solution: Removing the face determined by $(2, 3, 5, 4)$ below would leave us with a graph that is not polygonal, since it would be disconnected.



6 Prove this partial converse to Euler's formula: If a graph is planar and $v + f - e = 2$, then the graph is connected.

Solution: Suppose the graph is not connected but is planar. Then we can apply Euler's formula to both pieces and get $v_1 + f_1 - e_1 = 2$ and $v_2 + f_2 - e_2 = 2$. Notice that $v = v_1 + v_2$ and $e = e_1 + e_2$ and $f + 1 = f_1 + f_2$ (because we double-counted the infinite face). Therefore, we get $v + f - e = v_1 + f_1 - e_1 + v_2 + f_2 - e_2 + 1 = 4 \implies v + f - e = 3$. But, this is a contradiction by Euler's formula, so the graph must have been connected.

7 Let 'p' denote the number of components of a graph and prove this generalization of Euler's formula: if a graph is planar, then $v + f - e = 1 + p$.

Solution: We proceed by induction on p. We already know that the case $p = 1$ is true because that is just the familiar Euler's formula. Suppose for a graph with n components we have $v + f - e = 1 + n$. Now, given any graph G with $n+1$ components we can reduce this to a graph H with n components by adding an edge connecting one component to another, which would not change the number of vertices or faces. Therefore, we have $v_H + f_H - e_H = 1+(n+1) \iff v_H + f_H - (e_G+1) = 1+(n+1) \iff v_G + f_G - e_G = 1 + n$, which completes the inductive step to complete the proof.

Alternative Solution: We could instead treat each component as its own graph and apply Euler's formula to each piece. Suppose we have n components. Then we have $\sum_i v_i + \sum_i f_i + \sum_i e_i = 2$. Summing over all n components we have $V + \sum_i f_i - E = 2n$ but doing it this way we accidentally counted the infinite face n times in total, so we should have $V + \sum_i f_i - E - (n-1) = 2n \implies V + F - E = 1 + n$

8 Use corollary 13 to show the graph in Figure 106 is nonplanar:

Solution: Every vertex has degree 6, so by corollary 13, it cannot be planar.

9 Omitted, see the answer in the book.

10 Prove: Every planar graph with $v \geq 4$ has at least *four* vertices of degree ≤ 5.

Solution: Suppose not. And suppose G is "almost planar" in the sense that if we add a single new edge between any two nodes then the graph will be nonplanar (Note: we can always derive such a graph from planar graphs via connecting successive edges). Now, there can be no vertices with degrees 0 or 1 (if there were, we should connect them to another vertex in the face they are in). And, there are no vertices of degree 2. If there was a vertex of degree 2, then there would be a face of length 4 or more where we could connect it via an edge. So, every vertex has degree ≥ 3. Suppose G has at most 3 vertices of degree < 5. Then $2e \geq 6(v-3) + 3 \cdot 3 = 6v - 9$. But, we know $e \leq 3v - 6 \implies 2e \leq 6v - 12$. From those two inequalities we derive a contradiction.

11 Find the connectivity c of each graph in Figure 91, 92, 93.

Solution: Respectively: $c = 2, 2, 3, 1, 3, 2$

12 By Exercise 11 of Chapter 2, $2e/v$ is the average of the degrees of a graph. Prove that if a graph has connectivity c then $c \leq 2e/v$.

Solution: Suppose $c > 2e/v$.

$$c > 2e/v \iff cv > 2e = \sum_v \deg(v) \tag{1}$$

$$\iff \sum_v c > \sum_v \deg(v) \tag{2}$$

$$\iff c > \deg(v) \tag{3}$$

for some v.

But then if we remove the $\deg(v)$ vertices connected to v, G will be disconnected because v is isolated. That means that c was not the connection because it was not the minimum such value to disconnect the graph G.

13 Use the previous exercise to show that there is no graph with $e = 7$ and $c = 3$, and none with $e = 11$ and $c = 4$.

Solution:

a Since $e = 7$, we must have $v \geq 5$ (using the fact that $v_{K_4} = 6$). Then, $c \leq 2e/v \implies 3 \leq \lfloor 14/5 \rfloor = 2$, a contradiction.

b Since $e = 11$, we must have $v \geq 6$ (using the fact that $v_{K_5} = 10$). Then, $c \leq 2e/v \implies 4 \leq \lfloor 22/6 \rfloor = 3$, a contradiction.

14 In a planar graph a bridge necessarily borders on only one face, and an edge bordering on only one face is necessarily a bridge. Thus bridges are the things that prevent planar connected graphs from being polygonal.

Use this fact to prove that if a planar and connected graph G has the property that the boundary of every face is a cyclic graph, then G is polygonal. Then show that the converse statement is false by finding a polygonal graph having a face whose boundary is not a cyclic graph.

Solution: If G is planar and connected, and the boundary of every face is a cyclic graph, then every edge on any face borders two faces by the Jordan Curve Theorem. Therefore there are no bridges, and by the given statement, a planar connected graph without bridges is polygonal.

Counterexample: From problem 4.1, N_1 is polygonal. But, cyclic graphs are only defined when $v \geq 3$, so N_1 is not cyclic.

15 By Theorem 11 we know that ever planar connected graph with $v \geq 3$ has $e \leq 3v - 6$. Prove that if such a graph G has the additional property that every supergraph of G with one more edge is nonplanar, then the boundary of every face of G is C_3, G is polygonal, and G has $e = 3v - 6$.

Solution: If an edge borders only one face, we can add an edge from it to another vertex without creating an edge crossing, therefore the graph is polygonal because every edge will border more than one face. Additionally, if an edge is inside a face bounded by more than 3 vertices, we can connect to at least one of those vertices without creating an edge crossing. Therefore every border must be C_3.

By the previous problem, G must be polygonal.

By Theorem 11, we know $3f = 2e$, so $v + f - e = 2 \implies e - f = (1/3)e = v - 2 \implies e = 3v - 6$.

16 Prove: if G is planar and connected with $v \geq 3$ and the boundary of every face is C_4 then G is polygonal and $e = 2v - 4$.

Solution: In terms of proving G is polygonal, this reduces to the previous problem since we can always turn C_4 graphs into C_3 graphs without edge crossings by connecting two diagonal vertices.

$2f = e$ by theorem 12, using Euler's formula (similar to the previous problem), we get that $e = 2v - 4$.

17 Prove: if the connectivity c of a graph is at least 6, then the graph is nonplanar.

Solution: Suppose G is planar. We know from a previous theorem and using the fact that $c \leq 6$ that $c \leq 2e/v \implies 3v \leq e$. We also know that G is connected because $c \geq 6$, so we can use the fact that $e \leq 3v - 6$. Putting those two inequalities together, we have that $e \leq 3v - 6 \implies e \leq e - 6$, a contradiction. Therefore, G is not planar.

18 Prove: if a nonplanar graph has $v \geq 6$, $c \geq 3$, and a subgraph which is an expansion of K_5, then it also has a subgraph which is an expansion of UG.

Solution: We know an expansion introduces vertices with degree 2, and so if $c \geq 3$ as given, then an expanded vertex must have an additional edge connected to it (else we erase the two adjacent vertices and are left with a disconnected vertex, implying $c \leq 2$).

Additionally, this edge must be connected to another vertex in the K_5 expansion (if it were connected to two vertices in K_5 and one vertex outside K_5 we would remove the outside vertex and have a disconnected graph).

Now, we can then form UG by choosing the expanded vertex to be a utility, and 3 of its adjacent vertices part of the K_5 expansion to be the houses. Of course, since K_5 is complete we can find two more houses and utilities since there are paths between each vertex.

This does not give us a graph isomorphic to UG, but isomorphic to an expansion, because it may be the case that between the original K_5 vertices, there are expanded vertices.