Introduction to Graph Theory
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Ch. 4 Solutions

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1 N_1 is certainly planar, and we proved that it is connected. Prove now that it is polygonal by proving that the statement “every edge of N_1 borders on two different faces” is true.

Solution: This is vacuously true because there are no edges in N_1.

2 Omitted. There are many lengthy discussions about fake induction proofs online.

3 Believe it or not, the graph of Figure 104a is planar. Find its number of faces.

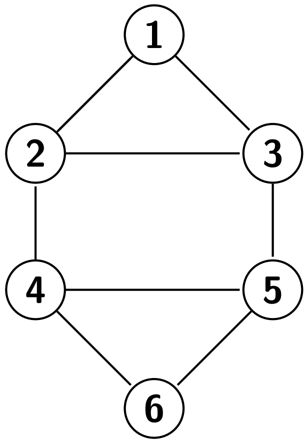
Solution: $v = 9$, $e = 20$, so $v + f - e = 2 \implies f = 13$

4 Imitate the proof of Corollary 12 to construct a proof that the graph in 104b is nonplanar.

Solution: Suppose it is planar. By inspection, the figure is not a supergraph of K_3, so by Theorem 12, we have $e \leq 2v - 4$. But, inspecting the graph, we see that $v = 8$ and $e = 16$ which implies $16 \leq 12$, a contradiction. Therefore, it must not be planar.

5 Find a polygonal graph G having a face bordering the infinite face which, if removed, results in a subgraph H which is not polygonal.

Solution: Removing the face determined by $(2, 3, 5, 4)$ below would leave us with a graph that is not polygonal, since it would be disconnected.



6 Prove this partial converse to Euler's formula: If a graph is planar and $v + f - e = 2$, then the graph is connected.

Solution: We proceed by induction on f. In the case $f = 1$, we have $G = N_1$ which is connected (see problem 4.1). Now, suppose a graph with n faces is planar, has $v + f - e = 2$, and is connected. Given a planar graph G with $n + 1$ faces ....

7 Let 'p' denote the number of components of a graph and prove this generalization of Euler's formula: if a graph is planar, then $v + f - e = 1 + p$.

Solution: We proceed by induction on p. We already know that the case $p = 1$ is true because that is just the familiar Euler's formula. Suppose for a graph with n components we have $v + f - e = 1 + n$. Now, given any graph G with $n+1$ components we can reduce this to a graph H with n components by adding an edge connecting one component to another, which would not change the number of vertices or faces. Therefore, we have $v_H + f_H - e_H = 1+(n+1) \iff v_H + f_H - (e_G+1) = 1+(n+1) \iff v_G + f_G - e_G = 1 + n$, which completes the inductive step to complete the proof.

Alternative Solution: We could instead treat each component as its own graph and apply Euler's formula to each piece. Suppose we have n components. Then we have $\sum_i v_i + \sum_i f_i + \sum_i e_i = 2$. Summing over all n components we have $V + \sum_i f_i - E = 2n$ but doing it this way we accidentally counted the infinite face n times in total, so we should have $V + \sum_i f_i - E - (n-1) = 2n \implies V + F - E = 1 + n$