Introduction to Graph Theory

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Ch. 4 Solutions

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May 13, 2019

1 N_1 is certainly planar, and we proved that it is connected. Prove now that it is polygonal by proving that the statement "every edge of N_1 borders on two different faces" is true.

Solution: This is vacuously true because there are no edges in N_1.

2 Omitted. There are many lengthy discussions about fake induction proofs online.

3 Believe it or not, the graph of Figure 104a is planar. Find its number of faces.

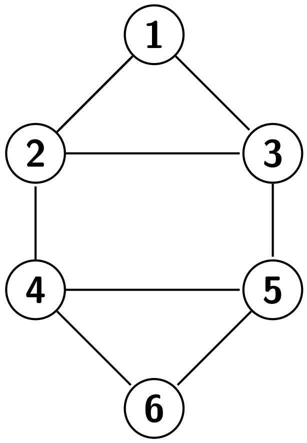
Solution: $v = 9$, $e = 20$, so $v + f - e = 2 \implies f = 13$

4 Imitate the proof of Corollary 12 to construct a proof that the graph in 104b is nonplanar.

Solution: Suppose it is planar. By inspection, the figure is not a supergraph of K_3, so by Theorem 12, we have $e \leq 2v - 4$. But, inspecting the graph, we see that $v = 8$ and $e = 16$ which implies $16 \leq 12$, a contradiction. Therefore, it must not be planar.

5 Find a polygonal graph G having a face bordering the infinite face which, if removed, results in a subgraph H which is not polygonal.

Solution: Removing the face determined by $(2, 3, 5, 4)$ below would leave us with a graph that is not polygonal, since it would be disconnected.



6 Prove this partial converse to Euler's formula: If a graph is planar and $v + f - e = 2$, then the graph is connected.

Solution: Suppose the graph is not connected but is planar. Then we can apply Euler's formula to both pieces and get $v_1 + f_1 - e_1 = 2$ and $v_2 + f_2 - e_2 = 2$. Notice that $v = v_1 + v_2$ and $e = e_1 + e_2$ and $f + 1 = f_1 + f_2$ (because we double-counted the infinite face). Therefore, we get $v + f - e = v_1 + f_1 - e_1 + v_2 + f_2 - e_2 + 1 = 4 \implies v + f - e = 3$. But, this is a contradiction by Euler's formula, so the graph must have been connected.

7 Let 'p' denote the number of components of a graph and prove this generalization of Euler's formula: if a graph is planar, then $v + f - e = 1 + p$.

Solution: We proceed by induction on p. We already know that the case $p = 1$ is true because that is just the familiar Euler's formula. Suppose for a graph with n components we have $v + f - e = 1 + n$. Now, given any graph G with $n+1$ components we can reduce this to a graph H with n components by adding an edge connecting one component to another, which would not change the number of vertices or faces. Therefore, we have $v_H + f_H - e_H = 1+(n+1) \iff v_H + f_H - (e_G + 1) = 1+(n+1) \iff v_G + f_G - e_G = 1 + n$, which completes the inductive step to complete the proof.

Alternative Solution: We could instead treat each component as its own graph and apply Euler's formula to each piece. Suppose we have n components. Then we have $\sum_i v_i + \sum_i f_i + \sum_i e_i = 2$. Summing over all n components we have $V + \sum_i f_i - E = 2n$ but doing it this way we accidentally counted the infinite face n times in total, so we should have $V + \sum_i f_i - E - (n-1) = 2n \implies V + F - E = 1 + n$

8 Use corollary 13 to show the graph in Figure 106 is nonplanar:

Solution: Every vertex has degree 6, so by corollary 13, it cannot be planar.

9 Omitted, see the answer in the book.

10 Prove: Every planar graph with $v \geq 4$ has at least *four* vertices of degree ≤ 5.

Solution: Suppose not. And suppose G is "almost planar" in the sense that if we add a single new edge between any two nodes then the graph will be nonplanar (Note: we can always derive such a graph from planar graphs via connecting successive edges). Now, there can be no vertices with degrees 0 or 1 (if there were, we should connect them to another vertex in the face they are in). And, there are no vertices of degree 2. If there was a vertex of degree 2, then there would be a face of length 4 or more where we could connect it via an edge. So, every vertex has degree ≥ 3. Suppose G has at most 3 vertices of degree < 5. Then $2e \geq 6(v-3)+3\cdot 3 = 6v-9$. But, we know $e \leq 3v-6 \implies 2e \leq 6v-12$. From those two inequalities we derive a contradiction.

11 Find the connectivity c of each graph in Figure 91, 92, 93.

Solution: Respectively: $c = 2, 2, 3, 1, 3, 2$

12 By Exercise 11 of Chapter 2, $2e/v$ is the average of the degrees of a graph. Prove that if a graph has connectivity c then $c \leq 2e/v$.

Solution: Suppose $c > 2e/v$.

$$\begin{aligned} c > 2e/v &\iff cv > 2e = \sum_v \deg(v) && (1)\\ &\iff \sum_v c > \sum_v \deg(v) && (2)\\ &\iff c > \deg(v) && (3) \end{aligned}$$

for some v.

But then if we remove the $\deg(v)$ vertices connected to v, G will be disconnected because v is isolated. That means that c was not the connection because it was not the minimum such value to disconnect the graph G.

13 Use the previous exercise to show that there is no graph with $e = 7$ and $c = 3$, and none with $e = 11$ and $c = 4$.

Solution:

a Since $e = 7$, we must have $v \geq 5$ (using the fact that $v_{K_4} = 6$). Then, $c \leq 2e/v \implies 3 \leq \lfloor 14/5 \rfloor = 2$, a contradiction.

b Since $e = 11$, we must have $v \geq 6$ (using the fact that $v_{K_5} = 10$). Then, $c \leq 2e/v \implies 4 \leq \lfloor 22/6 \rfloor = 3$, a contradiction.