Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
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DATE:......28/10/2024......

MAU22C00 Assignment 1

1) $S \subset T \iff P(S) \subset P(T)$

A) Proof of $P(S) \subset P(T)$ given $S \subset T$

Since every element $s \in S$ must be an element in T as $S \subset T$, and every element of the subsets contained in $P(S)$ must be an element in S, $\forall A \subset P(S)$, $A \subset T$

Then, as every element of T must also exist in an element of $P(T)$ all the elements of $P(S)$ must exist in $P(T)$ at least, if $S \subset T$

B) Proof of $S \subset T$ given $P(S) \subset P(T)$

Since the power set $P(S)$ contains the set S as an element, and $P(S) \subset P(T)$ that means the set S exists as an element of $P(T)$

By the definition of a power set, all the elements of an element x in $P(y)$ must exist in the original set y also. Hence, since $S \subset P(T)$ all the elements in S must exist in T, $\therefore$ $S \subset T$

$\therefore$ $S \subset T \iff P(S) \subset P(T)$

QED

2) (a) R is reflexive if every element is related to itself

$\forall x \in R^*,\ x R x = x/x = 1$

1 is an odd integer so R is reflexive

(b) $\forall x, y \in R^*, x R y \Rightarrow y R x$

$x/y \quad y/x$

$x, y = 2, 1$

$2/1 \quad 1/2$

Neither 2 or $1/2$ are odd integers in this case,
So R is not symmetric

(c) $\forall x, y \in R^*, x R y \wedge y R x \rightarrow x = y$

$x, y = -1, 1 \qquad -1/1 = -1$ so $x R y$ holds

$1/-1 = -1$ so $y R x$ holds

but $x \neq y \ (-1 \neq 1)$
so R is not anti symmetric

2) (d) $\forall x, y, z \in R^*, xRy \wedge yRz \Rightarrow xRz$

Assume:
x/y is an odd integer a
y/z is an odd integer b

$a \times b$ must also be an odd integer

$$(x/y)(y/z) = (a)(b)$$

$$x/z = a \times b$$

$\therefore$ x/z is an odd integer and is transitive

(e) R is not an equivalence relation as while it is reflexive and transitive, it is not symmetric which is required for an equivalence relation

(f) R is not a partial order, as while it is reflexive and transitive, it is not anti symmetric which it must be to be a partial order

3) (a) Injective $\in$ every element in domain maps to one in codomain

This function is not injective as multiple values of the domain share the same codomain

~~[illegible]~~

$x_1, x_2 = -0.5, 0.5$ $y_1, y_2 = \frac{1}{2(-0.5)^4+1}, \frac{1}{2(0.5)^4+1}$

$= 8/9, 8/9$

$y_1 = y_2$ $\therefore$ not injective

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3) (a) $f(x)$ isn't surjective as the range and codomain are ~~equal~~ not equal

The lowest value of $f(x)$ is when $x = -1$ or 1

$$= \frac{1}{2(1)^4+1} = \frac{1}{3}$$

The highest value of $f(x)$ is when $x = 0$

$$= \frac{1}{2(0)^4+1}$$

$$= \frac{1}{1} = 1$$

Range $= [\frac{1}{3}, 1]$

Codomain $= [0, 1]$

Hence $f(x)$ is not surjective

b) To make it surjective, it's codomain could be redefined as $[\frac{1}{3}, 1]$ as then it and the range would match

4) If $m = 0$ and $n = 0$, or, in other words A and B are empty sets, then the Cartesian product $A \times B$ is also the empty set $\emptyset$

Prove the number of elements in $A \times B$ is mn by induction

Base Case:

$m, n = 0$

$|A \times B| = mn = 0$

$\therefore$ when $m, n = 0$ $|A \times B| = mn$

Assume $|A \times B| = mn$, Prove $|A_{+1} \times B_{+1}| = (m+1)(n+1)$

$|A \times B| = mn$

Let a, b = any element which can fit in the set

$|A \cup a \times B \cup b| = (m+1)(n+1)$

$\cancel{|A \times B \cup a \times b| = (m+1)(n+1)}$

$mn + m + n + 1 = mn + m + n + 1$

One to a, one to b + 1 extra for new combination

$\therefore$ $|A_{+1} \times B_{+1}| = (m+1)(n+1)$

From base case

$|0 \emptyset \times \emptyset 0| = 0$

$|1 \emptyset| \times |\emptyset 1| = 1$

$|2| \times |2| = 4$

....

$|m| \times |n| = mn$

5) (a) Prove $(F, \circ)$ is a group

For $(F, \circ)$ to ~~be~~ a group,

- every element must be invertible
- it must have an identity element
- $\circ$ must be a binary operation and be associative

(i) Prove $\circ$ is a binary operation

by definition $\circ$ is a mapping from $F \to F$
so it is a binary operation as it maps from one set to the same set where F is the set of all functions

(ii) Prove $\circ$ is associative

~~For any functions~~

For functions f, g, h $(f \circ g) \circ h = f \circ (g \circ h)$

$$(f \circ g) \circ h = f \circ (g \circ h)$$

$$f(g(x)) \circ h = f \circ (g(h(x)))$$

$$f(g(h(x))) = f(g(h(x)))$$

so $\circ$ is associative

5(a) (iii) Prove $(F, \circ)$ has an identity element

$f_1(z) = e^{i(1)}z$ is the identity element as this will cause no rotation and $1 \in \mathbb{R}$

(iv) Prove every element is invertible

every element is invertible as there exists a value $f_{1/a}$ for every f_a, that when composed with $\circ$ yields f_1, the identity element

So $(F, \circ)$ is a group

(b) Prove $(\mathbb{R}, +)$ is a group

For $(\mathbb{R}, +)$ to be a group,

- $+$ must be a binary operation
- $+$ must be associative
- $(\mathbb{R}, +)$ must have an identity element
- all elements must be invertible

(i) Prove $+$ is a binary operation

$+$ is a binary operation as adding two elements $x, y \in \mathbb{R}$ also yields an element in $\mathbb{R}$

(ii) Prove $+$ is associative

$a, b, c \in \mathbb{R}$

$$a + (b+c) = (a+b) + c$$
$$a + b + c = a + b + c$$

So $+$ is associative

(iii) Show $(R, +)$ has an identity element

The identity element for $(R, +)$ is 0 as $0 \in R$ and $a + 0 = a$

(iv) Show all elements of $(R, +)$ are invertible

For every element $x \in R$, adding the value $-x$ yields the identity element 0, hence every element of $(R, +)$ is invertible

$\therefore (R, +)$ is a group

(c) $\phi : R \to F$

$\forall a \in R, \phi(a) = f_a(z)$

To see if ϕ is isomorphic, it must be bijective, and to be bijective it must be injective and surjective

$a, b \in R$ $\quad \phi(a) = \phi(b)$

$f_a(z) = f_b(z)$

$e^{ia} z = e^{ib} z$ for all $z \in C$

hence, ϕ is injective

For any $f_a \in F$ there is $a \in R$ such that $\phi(a) = f_a(z)$, hence ϕ is surjective and thus bijective and isomorphic