

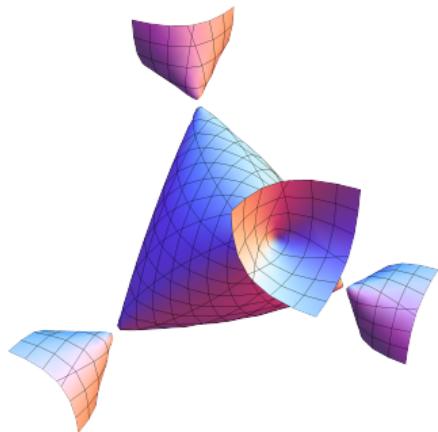
# COUNTING POINTS ON CHARACTER VARIETIES

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$\textcolor{blue}{X}$  is built from groups like  $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$ , etc.

$\textcolor{orange}{Y}$  is built from Lie algebras like  $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$ , etc.

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## Main Theorem

$|\textcolor{blue}{X}(\mathbb{F}_q)|$  and  $|\textcolor{orange}{Y}(\mathbb{F}_q)|$  are polynomials in  $q$

# CHARACTER VARIETIES

Fix: A ‘big’ conjugacy class  $\mathcal{C}$  in  $G$

The multiplicative character variety is

$$\textcolor{blue}{X} := \left\{ (a_1, b_1, \dots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \mid \prod_{i=1}^g [a_i, b_i] c = 1 \right\} / G$$

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Fix: A ‘big’ orbit  $\mathcal{O}$  in  $\mathfrak{g}$

The additive character variety is

$$\textcolor{brown}{Y} := \left\{ (x_1, y_1, \dots, x_g, y_g, z) \in \mathfrak{g}^{2g} \times \mathcal{O} \mid \sum_{i=1}^g [x_i, y_i] + z = 0 \right\} / G$$

## A $\mathbf{GL}_2$ EXAMPLE

If  $G = \mathbf{GL}_2$  and  $g = 1$  then

$$\textcolor{blue}{X} = \left\{ (a, b, c) \in \mathbf{GL}_2 \times \mathbf{GL}_2 \times \mathcal{C} \mid [a, b]c = 1 \right\} \Big/ \mathbf{GL}_2$$

$$\textcolor{red}{Y} = \left\{ (x, y, z) \in \mathfrak{gl}_2 \times \mathfrak{gl}_2 \times \mathcal{O} \mid [x, y] + z = 0 \right\} \Big/ \mathbf{GL}_2$$

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One computes by hand

$$|\textcolor{blue}{X}(\mathbb{F}_q)| = q^4 - q^3 - q + 1, \quad |\textcolor{red}{Y}(\mathbb{F}_q)| = q^4 + q^3$$

# CHARACTER VARIETIES IN NATURE

$$\pi_1 \left( \text{Diagram} \right) = \left\langle a_1, b_1, \dots, a_g, b_g, c \mid \prod_{i=1}^g [a_i, b_i] c = 1 \right\rangle$$

# CHARACTER VARIETIES IN NATURE

$$\pi_1\left(\text{Diagram of a surface with genus } g \text{ and punctures}\right) = \left\langle a_1, b_1, \dots, a_g, b_g, c \mid \prod_{i=1}^g [a_i, b_i] c = 1 \right\rangle$$

**Fact:** There is a bijection

$$\left\{ f: \pi_1\left(\text{Diagram of a surface with genus } g \text{ and punctures}\right) \rightarrow G \mid f(c) \in \mathcal{C} \right\} / G$$



$$\left\{ (a_1, b_1, \dots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \mid \prod_{i=1}^g [a_i, b_i] c = 1 \right\} / G$$

# BREAK

# PURITY

$$H_{\text{pure}}^*(\textcolor{orange}{Y}) \longleftrightarrow H^*(\textcolor{orange}{Y})$$

$$H_{\text{pure}}^*(\textcolor{blue}{X}) \longleftrightarrow H^*(\textcolor{blue}{X})$$

# PURITY

$$H_{\text{pure}}^*(Y) \longleftrightarrow H^*(Y)$$

$$H_{\text{pure}}^*(X) \longleftrightarrow H^*(X)$$

Theorem (Hausel–Letellier–Rodriguez–Villegas)

If  $G = \text{GL}_n$  then  $H_{\text{pure}}^*(Y) = H^*(Y)$

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**Theorem (Hausel–Letellier–Rodriguez–Villegas)**

If  $G = \text{GL}_n$  then  $H_{\text{pure}}^*(Y) = H^*(Y)$

**Conjecture (Hausel–Letellier–Rodriguez–Villegas)**

If  $G = \text{GL}_n$  then  $H^*(Y) \simeq H_{\text{pure}}^*(X)$

$$H_{\text{pure}}^*(Y) \xrightarrow{\sim} H^*(Y) \dashrightarrow H_{\text{pure}}^*(X) \xrightarrow{\sim} H^*(X)$$

# COUNTING POINTS

We access cohomology by counting points over finite fields

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Weil conjectures

$$|\textcolor{blue}{X}(\mathbb{F}_q)| \rightsquigarrow H^*(\textcolor{blue}{X})$$

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**GL<sub>2</sub>-example:**  $|\textcolor{blue}{X}(\mathbb{F}_q)| = q^4 - q^3 - q + 1,$      $|\textcolor{orange}{Y}(\mathbb{F}_q)| = q^4 + q^3$

# PATTERNS AND OBSERVATIONS

## Main Theorem

$|\text{X}(\mathbb{F}_q)|$  and  $|\text{Y}(\mathbb{F}_q)|$  are polynomials in  $q$  with explicit formulas

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E.g.  $|\text{X}(\mathbb{F}_q)| = q^4 - q^3 - q + 1$

$$\rightsquigarrow 1, -1, 0, -1, 1$$

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$|\textcolor{orange}{Y}(\mathbb{F}_q)| \in \mathbb{N}[q]$  when  
 $\text{rank } G \leq 4$  and  $g \leq 10$

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