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Arithmetic geometry of character varieties

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Abstract

We study character varieties associated to punctured orientable surfaces and connected split reductive groups. To study these varieties, we count points over finite fields and find the number of points is a polynomial, called the counting polynomial.

We compute features of the counting polynomial such as its degree, its leading coefficient and its value at 1, yielding topological information such as the dimension, number of components and Euler characteristic of character varieties, respectively. We prove the counting polynomial is palindromic which suggests a curious Poincaré duality for character varieties. We also implement our formula for the counting polynomial using the Chevie system in the Julia programming language.

There are two main ideas appearing in this thesis: representation-theoretic data called G-types elucidates the point-count of character varieties, and choosing conjugacy classes 'generically' simplifies the point-count.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

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Publications included in this thesis

No publications included in this thesis.

Submitted manuscripts included in this thesis

1. [KNWG24] M. Kamgarpour, G. Nam, **B. Whitbread**, and S. Giannini, *Counting points on character varieties* (2024). Preprint, arXiv:2409.04735.

Other publications during candidature

No other publications during candidature.

Contributions by others to the thesis

All new results were obtained in collaboration with Masoud Kamgarpour, GyeongHyeon Nam, and Stefano Giannini, who, along with Ole Warnaar, have given feedback on versions of this thesis.

Statement of parts of the thesis submitted to qualify for the award of another degree

No works submitted towards another degree have been included in this thesis.

Research involving human or animal subjects

No animal or human subjects were involved in this research.

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Contents

Chapter 1

Introduction

1.1 Overview

This thesis lies at the intersection between two branches of mathematics: algebra and geometry. Specifically, it lies at the intersection between representation theory, algebraic geometry and arithmetic geometry. Algebraic and arithmetic geometry study spaces defined by polynomial equations, while representation theory is the study of algebraic objects through linearisation techniques. In this thesis, we study the aforementioned spaces using the toolboxes of representation theory, algebraic geometry and arithmetic geometry.

A basic object linking the two worlds of algebra and geometry is an (affine) algebraic group. Briefly, this is an affine algebraic variety G over a field k such that G is an abstract group and the associated multiplication and inversion maps are morphisms of varieties. When the field k equals $\mathbb{C}$ or $\mathbb{R}$, we recover many important examples of complex and real Lie groups including $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{SO}_3(\mathbb{R})$, and so on. Although there are many important common themes, the theory of algebraic groups and the theory of Lie groups are distinct because, for instance, some Lie groups cannot be realised as an algebraic group over $\mathbb{R}$ or $\mathbb{C}$.

Character varieties are built from algebraic groups. Roughly speaking, they are spaces whose points are homomorphisms from the fundamental group of an orientable surface to a connected algebraic group G. These varieties are related to numerous topics in mathematics and physics, including the Langlands program, the Yang–Mills equations, gauge theory, Higgs bundles, the Hitchin system, Calabi–Yau manifolds, non-abelian Hodge theory, Hitchin's equations, the $P = W$ conjecture, and mirror symmetry [AB83, Hit87, Sim91, Sim92, BD96, HT03, DP12, Hau13, BPG-PNT14, BP16, BZN18, HMMS22, Hos23, MS24].

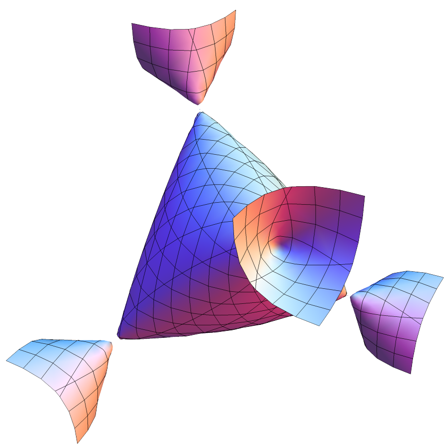


Figure 1.1: An SL_2-character variety [CFLO16].

Character varieties are generally not well-understood. Significant progress has been made when G is the general linear group GL_n. This is thanks to the seminal work of Hausel, Letellier and Rodriguez-Villegas [HRV08, HLRV11] and subsequent work [Let15, Sch16, Mel18, Mel19, Mel20, Bal23]. In light of the Langlands program, which seeks to connect algebraic geometry, representation theory and number theory, we study character varieties built from a wide class of algebraic groups.

We investigate character varieties by studying their cohomology, obtaining useful topological invariants such as dimension, Euler characteristic, and the number of irreducible components. We access these invariants through techniques of arithmetic geometry. Specifically, our work relies on the Weil conjectures, a jewel of 20th century mathematics. Their statements are complicated, but they teach us an important philosophy: cohomological information can be obtained by counting points over finite fields.

A formula first revealed by Frobenius links the number of points on the character variety over finite fields to the complex representation theory of the underlying finite group.[1] This provides a clear strategy to analyse character varieties: use the representation theory of finite groups to evaluate Frobenius' formula and extract cohomological information from the resulting expression. This is the strategy employed in this thesis.

The finite groups appearing in this thesis are algebraic groups over $\mathbb{F}_q$, the finite field with order a prime power $q = p^r$. In this setting, we recover many important families of finite groups, such as $\mathrm{SL}_n(\mathbb{F}_q)$, $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{SO}_n(\mathbb{F}_q)$. Such groups are called finite groups of Lie type, due to their connections to Lie groups, and are closely related to the classification of finite simple groups [Gal76, §12]. Since these groups are finite, their complex representation theory is well-behaved in the sense that it is sufficient to understand the finite list of so-called irreducible representations.

Figure 1.2: Frobenius created representation theory, prompted by work of Dedekind [Fro68].

While analysing finite groups and their representations individually can yield insights, we need to understand the situation uniformly. Such an understanding has already been achieved, up to some conditions on G and p. Specifically, the representation theory of finite groups of Lie type is well-understood uniformly when G is connected and reductive (the latter meaning G contains no non-trivial closed connected normal 'unipotent' subgroups) with connected centre and p is not too small. This is primarily due to the eponymous characters of Deligne and Lusztig [DL76, Lus84].

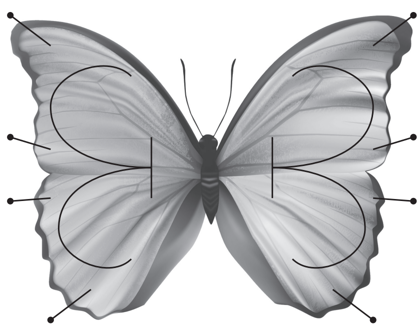


Figure 1.3: Grothendieck's visualisation of connected reductive groups [Mil17].

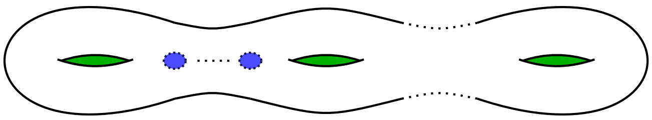
[1] The connection to Frobenius' work is as follows. Given three conjugacy classes C_1, C_2, C_3 of a finite group, Frobenius used representation theory to determine the number of pairs $(x,y) \in C_1 \times C_2$ with $xy \in C_3$ [Fro68, Band III, p. 1].

The novelty of this thesis is our type-independent approach; i.e., we do not make assumptions about the type of (the underlying root system of) G and our proofs are not case-by-case in the type of G. When in such generality, we find fundamental objects arising in the Langlands program (such as Langlands dual groups, pseudo-Levi subgroups and endoscopy groups) play key roles in our analysis of character varieties; such phenomena does not arise when $G = \mathrm{GL}_n$. Our findings appear in the preprint [KNWG24].

The remainder of this introduction is as follows. In §1.2, we define the three closely related spaces we study: the representation variety, the character variety, and the character stack. We explain how we study these spaces in §1.3. Namely, we state two ideas fundamental to this thesis (polynomial and rational count spaces) and we explain how one uses these to extract cohomological information. Equipped with this knowledge, we detail what is already known about character varieties in §1.4.

1.2 Character varieties

Suppose G is a connected reductive group over a finite field $\mathbb{F}_q$ and let Σ be an orientable surface with genus $g \geq 0$ and $n \geq 1$ punctures, depicted as follows.



This surface has the fundamental group

$$\pi_1(\Sigma) \simeq \left\langle a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_n \;\middle|\; [a_1, b_1] \cdots [a_g, b_g] y_1 \cdots y_n = 1 \right\rangle$$

and therefore a group homomorphism $\pi_1(\Sigma) \to G$ is determined by the images of the generators, subject to the relation of the fundamental group. Thus, we have a bijection

$$\mathrm{Hom}(\pi_1(\Sigma), G) \simeq \left\{ (A_1, B_1, \ldots, A_g, B_g, Y_1, \ldots, Y_n) \in G^{2g+n} \;\middle|\; [A_1, B_1] \cdots [A_g, B_g] Y_1 \cdots Y_n = 1 \right\}.$$

Reductive groups carry the structure of a variety, so this space of homomorphisms does too. Choosing conjugacy classes $\mathcal{C} = (C_1, \ldots, C_n)$ in G, we define

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G) := \left\{ (A_1, B_1, \ldots, A_g, B_g, Y_1, \ldots, Y_n) \in \mathrm{Hom}(\pi_1(\Sigma), G) \;\middle|\; Y_i \in C_i \right\}.$$

We call $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)$ the **representation variety** associated to G, Σ and $\mathcal{C}$. Recall two representations $\pi_1(\Sigma) \to G$ are equivalent if they are conjugate by an element of G. Under the identification $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G) \subseteq G^{2g+n}$, the representation variety admits an action of G by simultaneous conjugation in each entry. Thus, it is natural to consider the collection of orbits $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G$. However, this does not necessarily inherit the algebro-geometric structure of G.

There are two ways of endowing the collection of orbits with an algebro-geometric structure:

(i) We consider the geometric-invariant-theory (GIT) quotient, denoted $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/\!\!/G$. Historically, this was the first solution to the orbit-space problem, due to Mumford [Mum65]. Over algebraically closed fields, the points of the GIT quotient are in bijection with the closed orbits.

(ii) We consider the stack quotient, denoted $[\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G]$. Stacks are a higher algebraic object defined in the wake of Grothendieck, Mumford, Deligne and Artin, who resolved the orbit-space problem by keeping track of additional data which is forgotten by the GIT quotient. In a sense, the stack quotient is the 'correct' quotient, but its construction requires some work.

The GIT quotient $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G) /\!\!/ G$ is called the **character variety** and has a close relationship to the problems from other areas stated earlier. In general, when the action of G is not free, it is not clear how to relate the point-counts of the character variety and representation variety. On the other hand, the stack quotient $[\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G]$ is called the **character stack**, and it has essentially the same point-count as that of the representation variety. Since the centre Z of G acts trivially on the representation variety, the G-action on the representation variety is usually not well-behaved. However, by definition of the GIT quotient, we only need the G/Z-action to be well-behaved.

1.3 Counting points

Our strategy for analysing character varieties is to count their points over finite fields. We say a variety X defined over $\mathbb{F}_q$ is **polynomial count** with **counting polynomial** $\|X\| \in \mathbb{Q}[t]$ if

$$|X(\mathbb{F}_{q^n})| = \|X\|(q^n) \text{ for all } n \geq 1.$$

More generally, we say X is **potentially polynomial count** if it becomes polynomial count after passing to a finite extension of $\mathbb{F}_q$. We also allow ourselves to exclude finitely many primes, in which case we say X is polynomial count **away from** those primes.

Fine cohomological information is encoded in counting polynomials (see [LRV23, §2.2] for details). From these polynomials, which we can extract topological information. For example,

(i) The dimension of X is the degree of $\|X\|$,

(ii) The Euler characteristic of X is given by $\|X\|(1)$, and

(iii) The number of irreducible components of X is the leading coefficient of $\|X\|$.[2]

Counting polynomials encode cohomological information, so it is worthwhile asking if they are palindromic, meaning their coefficients are the same when read backwards and forwards. If X is smooth, projective and polynomial count then Poincaré duality implies $\|X\|$ is palindromic. The character varieties in this project are affine but we find they have palindromic counting polynomials too. This suggests several properties concerning the mixed Hodge structure of character varieties: they obey curious Poincaré duality, curious hard Lefschetz and the $P = W$ conjecture, cf. [HRV08, HLRV11, Hau13].

The story above can be extended to a larger class of algebraic objects. We will need this extended story in order to analyse the character stack. We say an algebraic stack $\mathfrak{X}$ of finite type over $\mathbb{F}_q$ is

[2]The leading coefficient is actually the number of irreducible components of maximum dimension, because the counting polynomial captures information about the compactly supported cohomology of X. Often we deal with equidimensional spaces, so this technicality can be ignored.

rational count with **counting function** $\|\mathfrak{X}\| \in \mathbb{Q}(t)$ if

$$|\mathfrak{X}(\mathbb{F}_{q^n})| = \|\mathfrak{X}\|(q^n) \text{ for all } n \geq 1,$$

and **potentially rational count** stacks are defined analogously. Clearly, if $\mathfrak{X}$ is polynomial count then it is rational count, and the converse is true if $\mathfrak{X}$ is a variety defined over $\mathbb{F}_q$ [LRV23, Lemma 2.8].

In general, counting points over finite fields is not an easy problem. However, in our setting, there is a formula due to Frobenius telling us how to point-count on the representation variety [HLRV11, Proposition 3.1.4].[3] **Frobenius' formula** says

$$|\operatorname{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G(\mathbb{F}_q))| = |G(\mathbb{F}_q)| \sum_{\chi \in \operatorname{Irr}(G(\mathbb{F}_q))} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-2} \prod_{i=1}^{n} \frac{\chi(C_i(\mathbb{F}_q))}{\chi(1)} |C_i(\mathbb{F}_q)|,$$

where $\operatorname{Irr}(G(\mathbb{F}_q))$ is the set of irreducible complex characters of the finite group $G(\mathbb{F}_q)$.

We see evaluating Frobenius' formula is a problem in the world of the representation theory of finite reductive groups. Note we do not need to impose reductivity on the algebraic group G to make use of Frobenius' formula.[4] However, assuming G is reductive gives us access to powerful techniques of Deligne–Lusztig theory.

1.4 Literature

Our work is inspired by the ground-breaking work of Hausel, Letellier and Rodriguez-Villegas. In [HRV08, HLRV11], they studied the character variety when $G = \mathrm{GL}_n$ and $\mathcal{C}$ consists of semisimple conjugacy classes chosen in a 'generic' sense. The primary benefit of the generic assumption is G/Z acts freely on the representation variety, so the character stack and character variety coincide [HLRV11, Proposition 2.1.4]. The authors count points on character varieties over finite fields, conclude the character variety is polynomial count and analyse the counting polynomial. A significant feature of this work is the use of symmetric functions which arise naturally due to the combinatorial description of $\operatorname{Irr}(\mathrm{GL}_n(\mathbb{F}_q))$ originally given in [Gre55].

In [Cam17], the author studies the character variety when $G = \mathrm{Sp}_{2n}$ and Σ is an orientable surface of genus $g \geq 0$ with one puncture. At the puncture, the conjugacy class is semisimple, regular and 'generic' in a sense similar to that of [HRV08, HLRV11]. In this setting, the generic assumption does not imply G/Z acts freely on the representation variety. Thus, the main difficulties are twofold: understand the G/Z-action on the representation variety and understand the representation theory of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. The author finds the stabilisers of this action are finite [Cam17, Proposition 3.1.6] and begins to draw on Deligne–Lusztig theory [Cam17, §2.2]. This allows the author to count points on character varieties over finite fields, conclude the character variety is polynomial count and analyse the counting polynomial.

In [BK23], the authors consider a general connected reductive group with connected center and an orientable surface of genus $g \geq 0$ with no punctures. Rather than study the character variety

[3] We assume $C_i \subseteq G$ is a single $G(\mathbb{F}_q)$-conjugacy class.

[4] Understanding character varieties when the algebraic group is not reductive is closely-related to Higman's conjecture [Hig60, p. 29] which asks whether the number of conjugacy classes of the group of unipotent matrices in $\mathrm{GL}_n(\mathbb{F}_q)$ is a polynomial in q. There is evidence to suggest Higman's conjecture fails for $n \geq 59$ [PS15, Conjecture 1.6].

directly, the authors study the character stack. This is because, without the presence of punctures, the G/Z-action on the representation variety is difficult to understand and control, but one can still study the representation variety and character stack. The main ingredient in the authors' work is a deep theorem in Deligne–Lusztig theory called Lusztig's Jordan decomposition which describes the irreducible representations of $G(\mathbb{F}_q)$. The authors count points on character stacks over finite fields, conclude the character stack is potentially polynomial count and analyse the counting polynomial.

In [KNP23], the authors add punctures to the setting of [BK23]; they choose both semisimple regular and unipotent regular conjugacy classes. They study the character variety directly, since the mixture of semisimple regular and unipotent regular conjugacy classes means G/Z acts freely on the representation variety [KNP23, Lemma 3]. In view of Frobenius' formula, the authors must deal with character values at semisimple regular and unipotent regular elements. However, in light of a theorem of Green, Lusztig and Lehrer, many of these character values are zero [KNP23, Theorem 14]. This simplifies calculations, reducing them to a problem involving Weyl groups acting on tori [KNP23, §4] and the evaluation of certain character sums [KNP23, §5]. The authors then count point on character varieties over finite fields, conclude the character variety is polynomial count and analyse the counting polynomial.

In this thesis, we study the same character varieties as [KNP23], but only semisimple regular conjugacy classes are chosen. The lack of a unipotent regular conjugacy classes means we are forced to address several problems: the G/Z-action on the representation variety may not be free, Frobenius' formula involves many non-zero character values, and we must explicitly evaluate the character sums seen in [KNP23]. To navigate these problems, we define a reductive generalisation of the 'generic' condition seen in [HRV08, HLRV11, Cam17] and inspired by [Boa14]. Therefore, one can view this thesis as a step towards a reductive generalisation of [HRV08, HLRV11].[5]

The contents of this thesis are as follows. We begin by fixing terminology, presenting our new results, and stating some unexplored directions warranting further exploration in §2. In §3, we recall necessary and deep results from the representation theory of finite reductive groups. In §4, we use this theory to define the notion of a G-type which are the lens through which we view Frobenius' formula, reducing the evaluation of Frobenius' formula to the evaluation of certain character sums. In §5, we define and develop a key idea which is a generic choice of conjugacy classes simplifies the point-count of character varieties. We perform some technical analysis of the aforementioned character sums in §6, allowing us to prove our main theorems in §7.

The appendix of this thesis contains worked examples of counting points on character varieties using our new formulas in §A, and we quickly and automatically compute counting polynomials using the Chevie system in §B.

[5]This thesis is not a generalisation of [Cam17] since Sp_{2n} has disconnected centre.

Chapter 2

Main results

2.1 Results

Let G be a connected reductive group over $\mathbb{F}_q$ with connected centre Z and maximal split torus $T \subseteq G$. Let Σ be an orientable surface with genus $g \geq 0$ and $n \geq 1$ punctures and fix conjugacy classes $\mathcal{C} = (C_1, \dots, C_n)$ in G. Denote the representation variety by $\mathbf{R} := \mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)$ and recall G acts on $\mathbf{R}$ by simultaneous conjugation with Z acting trivially. Thus, we form the character variety

$$\mathbf{X} := \mathbf{R} /\!\!/ (G/Z) = \mathbf{R} /\!\!/ G$$

and the character stack

$$\mathfrak{X} := [\mathbf{R}/(G/Z)].$$

In this thesis, we make two assumptions on the conjugacy classes $\mathcal{C} = (C_1, \dots, C_n)$:

Assumption 1. *Assume*

(i) Each C_i is the conjugacy class of a strongly regular element $S_i \in T(\mathbb{F}_q)$, and

(ii) The product $S_1 \cdots S_n$ lies in $[G,G]$; i.e., $C_1 \cdots C_n \subseteq [G,G]$.

Strongly regular is meant in the sense of [Ste65] (i.e., $C_G(S_i) = T$) and such elements form a dense open set in G [Ste65, 2.15]. The first assumption implies the centraliser of each C_i is connected, as promised in §1.3, and the second assumption is necessary for $\mathbf{R}$ to be non-empty.

We are ready to state our first main theorem:

Theorem 2. *Away from finitely many primes, the character stack $\mathfrak{X}$ is potentially rational count with counting function given in Theorem 57. Furthermore, if $g \geq 1$ then $\mathfrak{X}$ is potentially polynomial count.*

We exclude finitely many primes so that the structure and representation theory of $G(\mathbb{F}_q)$ is well-behaved. The primes we exclude depend only on the root datum of G:

Assumption 3. *Assume the prime $p > 0$ is very good for G.*[1]

[1] The definition and role of very good primes is explained in [Let05, §2.5].

Working in very good characteristic is necessary in many places throughout this thesis. The following table summarises the very good primes for groups with irreducible root systems:

Type of G	Very good primes
A_n	$p \nmid n+1$
$B_n,\ C_n,\ D_n$	$p > 2$
$G_2,\ F_4,\ E_6,\ E_7$	$p > 3$
E_8	$p > 5$

Table 2.1: The very good primes for various G [Let05].

We also make a light assumption on g and n:

Assumption 4. *Assume $2g-2+n \geq 1$, in addition to our requirement that $g \geq 0$ and $n \geq 1$.*

This assumption excludes the cases $(g,n) = (0,1)$ and $(0,2)$ which can be studied by hand since, in these cases, we have $\pi_1(\Sigma) \simeq 1$ and $\pi_1(\Sigma) \simeq \mathbb{Z}$, respectively. In the next theorem only, we will also exclude the cases $(g,n) = (0,3)$ and $(1,1)$ by requiring that $2g-2+n \geq 2$. (This is because we rely on estimates concerning the ranks of root systems. For details, see §7.2.)

From the counting function, we extract the following information:

Theorem 5. *If $2g-2+n \geq 2$ then the character stack is non-empty of dimension*

$$\dim(\mathfrak{X}) = (2g-2+n)\dim(G) + 2\dim(Z) - n\,\mathrm{rank}(G)$$

with number of components equal to

$$|\pi_0(\mathfrak{X})| = |\pi_0(Z(\check{G}))|$$

where $Z(\check{G})$ is the centre of the Langlands dual group $\check{G}$.

So far, we have only analysed the character stack, and we now turn our attention to the character variety. To analyse the character variety, we choose conjugacy classes generically:

Definition 6. *We say a tuple $\mathcal{C} = (C_1, \ldots, C_n)$ of semisimple conjugacy classes of G is generic if*

$$\prod_{i=1}^{n} X_i \notin [L,L]$$

for all proper Levi subgroups L of G and for all $X_i \in C_i \cap L$.

This notion of choosing conjugacy classes generically is a generalisation of the one seen in [HLRV11]. In this paper, the authors consider $G = \mathrm{GL}_n$ and a generic choice of semisimple conjugacy classes implies G/Z acts freely on $\mathbf{R}$ [HLRV11, Proposition 2.1.4]. Thus, in their setting, it is straightforward to show $\mathfrak{X}$ and $\mathbf{X}$ are isomorphic.

Our situation is slightly more subtle. In particular, we have our next main theorem:

Theorem 7. *Suppose $\mathcal{C}$ is generic. Then* *(i)* *G/Z acts on* $\mathbf{R}$ *with finite étale stabilisers,*

(ii) $\mathbf{R}$ *is smooth and equidimensional,*

(iii) $\mathfrak{X}$ *is a smooth Deligne-Mumford stack,*

(iv) $\mathbf{X}$ *is a coarse moduli space for* $\mathfrak{X}$*, and*

(v) $\mathfrak{X}$ *and* $\mathbf{X}$ *have the same number of points over finite fields.*

Combining Theorems 2 and 7 yields the following:

Theorem 8. *If* $\mathcal{C}$ *is generic then* $\mathfrak{X}$ *and* $\mathbf{X}$ *are potentially polynomial count (away from finitely many primes). Moreover, they have equal counting polynomials (with an expression given in Theorem 62) and this counting polynomial is independent of* $\mathcal{C}$*.*

We compute the character variety's dimension and number of components in all cases:

Theorem 9. *If* $\mathcal{C}$ *is generic then the character variety is non-empty of dimension*

$$\dim(\mathbf{X}) = (2g-2+n)\dim(G) + 2\dim(Z) - n\cdot\operatorname{rank}(G)$$

with number of components equal to

$$|\pi_0(\mathbf{X})| = |\pi_0(Z(\check{G}))|$$

where $Z(\check{G})$ *is the centre of the Langlands dual group* $\check{G}$*.*

We also compute the character variety's Euler characteristic in all cases:

Theorem 10. *Suppose* $\mathcal{C}$ *is generic.*

(i) *If either* $g > 1$*, or* $g > 0$ *and* $\dim(Z) > 0$*, then* $\chi(\mathbf{X}) = 0$*,*

(ii) *If* $g = 1$ *and* $\dim(Z) = 0$ *then* $\chi(\mathbf{X})$ *may be non-zero, with a formula given in Theorem 65, and*

(iii) *If* $g = 0$ *and* $n \geq 3$ *then* $\chi(\mathbf{X})$ *may be non-zero, with a formula given in Theorem 66.*

When $G = \mathrm{GL}_n$ and $g = 0$, one can calculate $\chi(\mathbf{X})$ using [HLRV11, Theorem 1.2.3] or [Mel20, Theorem 7.10] but this is "complicated due to the presence of high-order poles" [HLRV11, Remark 5.3.4]. Our formula does not have this issue; it only involves differentiating a smooth function.

Lastly, we prove the character variety obeys a specialisation of curious Poincaré duality:

Theorem 11. *If* $\mathcal{C}$ *is generic then* $\|\mathbf{X}\|$ *is a palindromic polynomial; i.e.,*

$$\|\mathbf{X}\|(q) = q^{\dim(\mathbf{X})}\|\mathbf{X}\|(1/q).$$

This is the first time this specialisation of curious Poincaré duality has been demonstrated for character varieties associated to general reductive groups; in [KNP23], the counting polynomials were not always palindromic.

2.2 Further directions

We conclude by discussing research directions warranting further attention:

(i) Associated to $\mathbf{X}$ is the **mixed Poincaré polynomial** $H_c(\mathbf{X};q,t)$ which provides important information about the Frobenius' action on the (compactly supported) cohomology of $\mathbf{X}$ [LRV23, §2.2]. When $\mathbf{X}$ is polynomial count, the counting polynomial is recovered by setting $t=-1$ in the mixed Poincaré polynomial [LRV23, Theorem 2.9]. Therefore, proving $\mathbf{X}$ is polynomial count and obtaining an explicit expression for $\|\mathbf{X}\|$ is a step towards a formula for $H_c(\mathbf{X};q,t)$. For instance, when $G=\mathrm{GL}_n$, there is a conjectural formula for $H_c(\mathbf{X};q,t)$ when $\mathcal{C}$ is a generic collection of (not necessarily regular) semisimple conjugacy classes [HLRV11, Conjecture 1.2.1]. Moreover, there is a known formula for $H_c(\mathbf{X};q,t)$ when $G=\mathrm{GL}_2$ and $n=1$ with conjugacy class representative $\left(\begin{smallmatrix}-1 & \\ & -1\end{smallmatrix}\right)$ [HRV08, Theorem 1.1.3].[2]

(ii) There is an additive analogue to our situation. Let $\mathfrak{g}=\mathrm{Lie}(G)$ with Lie bracket $[\cdot,\cdot]$ and recall G acts on $\mathfrak{g}$ by the adjoint action $g\cdot x:=\mathrm{Ad}_g(x)$. Let $\mathfrak{t}=\mathrm{Lie}(T)$, fix adjoint orbits $O_1,\ldots,O_n$ of regular elements $x_1,\ldots,x_n\in\mathfrak{t}$ and define the **additive representation variety**

$$\mathbf{A}:=\left\{(a_1,b_1,\ldots,a_g,b_g,y_1,\ldots,y_n)\in\mathfrak{g}^{2g+n}\;\middle|\;[a_1,b_1]+\cdots+[a_g,b_g]+y_1+\cdots+y_n=0,\ y_i\in O_i\right\}.$$

This inherits the adjoint action of G on $\mathfrak{g}$, so we form the **additive character variety** $\mathbf{Y}:=\mathbf{A}/\!\!/G$. It has recently been shown $\mathbf{Y}$ is polynomial count when G is a connected split reductive group over $\mathbb{F}_q$ with connected centre [Gia24]. Moreover, the additive character variety is conjectured to have deep links to the multiplicative character variety when $G=\mathrm{GL}_n$ [HLRV11, Remark 1.3.2]. Precisely, at the level of polynomials, it is conjectured $\|\mathbf{Y}\|$ is equal to the 'pure part' of $H_c(\mathbf{X};q,t)$ [HLRV11, §1.2.1], providing constraints for what $H_c(\mathbf{X};q,t)$ can be.

(iii) In this thesis, we assume T is split and we require an understanding of principal series representations. We appeal to an exclusion theorem (Proposition 25) and a well-known Hecke algebra (Proposition 26). In particular, $\mathrm{End}_G R_T^G\theta$ is isomorphic to a Hecke algebra. To remove the split assumption, we must instead understand the aforementioned endomorphism algebra when T is non-split. To do so, one appeals to the general **exclusion theorem of Deligne–Lusztig characters** [GM20, Theorem 2.3.2] and **cyclotomic Hecke algebras** [GM20, §A.6].

(iv) We expect our results hold when G has **disconnected centre**. In this thesis, we assume G has connected centre to ensure semisimple centralisers in the dual group are connected reductive groups, and to use a simplified version of Lusztig's Jordan decomposition. One reason for expecting generalisation is our results closely mirror those of [Cam17], where an Sp_{2n}-character variety is studied. Our work does not address this character variety since the centre of Sp_{2n} is disconnected. An instance of Lusztig's Jordan decomposition without the assumption of a connected centre is [GM20, Theorem 4.8.24].

[2] In [HRV08, HLRV11], the authors consider the mixed Hodge polynomial, rather than the mixed Poincaré polynomial, which is an invariant associated to the character variety over $\mathbb{C}$, rather than the character variety over $\mathbb{F}_q$.

Chapter 3

Recollections on finite reductive groups

Frobenius' formula involves irreducible characters of finite reductive groups. Therefore, we dedicate this chapter to recalling the relevant theory. The primary references are [DL76, Car93, DM20, GM20].

Throughout this chapter, we work with a connected split reductive group G over $k = \mathbb{F}_q$ with connected centre $Z(G) = Z$. Fix a maximal split torus $T \subseteq G$ and let (G,T) have root datum $(X, \Phi, \check{X}, \check{\Phi})$ with Weyl group W. We also fix a Borel subgroup $B \subseteq G$ so that Φ has positive roots Φ^+ and simple roots Δ. Then G has Langlands dual $\check{G}$, which is a connected split reductive group over k with maximal split torus $\check{T} := \mathrm{Spec}(k[X])$ and with $(\check{G}, \check{T})$ having root datum $(\check{X}, \check{\Phi}, X, \Phi)$.

We start by defining several families of groups which are crucial to this thesis (namely, pseudo-Levi subgroups, Levi subgroups and endoscopy groups) in §3.1. Afterwards, we explain a deep result in representation theory which parameterises the irreducible characters of finite reductive groups in §3.2. We briefly review polynomials describing the cardinality of connected split reductive groups and the degrees of irreducible characters in §3.3. In §3.4, we closely examine an important family irreducible characters, called principal series characters, which play a key role in our point-count of the character variety. Lastly, in §3.5, we gather necessary facts about Alvis–Curtis duality, an important duality of characters which will help us to analyse counting polynomials of character varieties.

3.1 Pseudo-Levi subgroups and endoscopy groups

In this section, we review several families of groups associated to G which are crucial in this thesis. To this end, let $\Psi \subseteq \Phi$ be a root subsystem and denote by $G(\Psi)$ the connected split reductive group over k with root datum $(X, \Psi, \check{X}, \check{\Psi})$; this always contains T but is not always a subgroup of G.

In Definition 6, we referred to the so-called Levi subgroups of G, defined as follows:

Definition 12. *A subsystem $\Psi \subseteq \Phi$ is a **Levi subsystem** if it is of the form $\Phi \cap E$ for some vector subspace $E \subseteq \mathrm{span}_{\mathbb{R}}(\Phi)$.*[1] *The groups $G(\Psi)$ are called **Levi subgroups** of G containing T.*

We have a convenient description of the Levi subsystems of Φ:

[1] Such a subsystem will automatically be closed; $\Psi \subseteq \Phi$ is closed if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Psi$.

Proposition 13 (Proposition 24 of [Bou02])**.** *A subsystem $\Psi \subseteq \Phi$ is a Levi subsystem if and only if it is of the form $w \cdot \langle S \rangle$ for some $w \in W$ and $S \subseteq \Delta$. Here, $\langle S \rangle$ means the closed subsystem* $\mathrm{span}_{\mathbb{Z}}(S) \cap \Phi$.

It is important for us to consider a larger class of closely-related subsystems and subgroups:

Definition 14. *A subsystem $\Psi \subseteq \Phi$ is a **pseudo-Levi subsystem** if it arises as the root system of a connected centraliser subgroup of semisimple element in G. The groups $G(\Psi)$ are called **pseudo-Levi subgroups** of G containing T.*

Pseudo-Levi subsystems are characterised by Deriziotis' Criterion [Hum95, §2.15]:

Theorem 15. *Suppose Φ is irreducible with simple roots Δ and highest root θ. Let H be a connected reductive subgroup of G containing T with root system $\Psi \subseteq \Phi$. Then Ψ is a pseudo-Levi subsystem if and only if $\Psi = w \cdot \langle S \rangle$ for some $w \in W$ and some proper subset $S \subset \Delta \sqcup \{-\theta\}$.*[2]

This means the collection of pseudo-Levi subgroups of G containing T is independent of the ground field and we know exactly what they are. It also means we can take Deriziotis' Criterion as a definition of pseudo-Levi subsystems.

There is also an analogue of Deriziotis' Criterion for groups with reducible root systems which we briefly state now. Suppose $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$ with each Φ_i irreducible and simple roots Δ_i. Let θ^i be the highest root of Φ_i and write $\tilde{\Delta}_i := \Delta_i \sqcup \{-\theta^i\}$. Then Deriziotis' Criterion is the same as before, but we we say Ψ is a pseudo-Levi subsystem if and only if $\Psi = w \cdot \langle S \rangle$ for some $w \in W$ and some proper subset $S \subseteq \tilde{\Delta}_1 \sqcup \cdots \sqcup \tilde{\Delta}_r$.

We distinguish another important family of pseudo-Levi subgroups which will be key later:

Definition 16. *A subsystem of Φ is **isolated in** Φ if it is not contained in a proper Levi subsystem of Φ, and a subgroup of G containing T is **isolated in** G if its root system is an isolated subsystem in Φ.*

We have a description of the isolated pseudo-Levi subgroups à la Deriziotis' Criterion:

Proposition 17. *Suppose L is a pseudo-Levi subgroup of G containing T with root system of the form $w \cdot \langle S \rangle$ for some $w \in W$ and $S \subset \Delta \sqcup \{-\theta\}$. Then L is isolated in G if and only if $|S| = |\Delta|$.*

Proof. If $|S| < |\Delta|$ then the Levi subgroup of G containing T with root system $w \cdot \langle S \rangle$ would be a proper Levi subgroup of G containing T and L, so L could not be isolated. Conversely, if $|S| = |\Delta|$ and L were not isolated then there would be some proper Levi subgroup of G containing L, but $|S| = |\Delta|$ contradicts properness of the Levi subgroup. □

Using Proposition 17, we summarise the isolated pseudo-Levi subsystems in Table 3.1.

Rather than work with the pseudo-Levi subsystems of $\check{\Phi}$ and pseudo-Levi subgroups of $\check{G}$, we often work with endoscopy subsystems of Φ and endoscopy groups of G, defined as follows:

Definition 18. *A subsystem $\Psi \subseteq \Phi$ is an **endoscopy system** if $\check{\Psi}$ is a pseudo-Levi subsystem of $\check{\Phi}$. A connected split reductive group K (not necessarily a subgroup of G) is an **endoscopy group** of G if the dual group $\check{K}$ is a pseudo-Levi subgroup of $\check{G}$ containing $\check{T}$.*

[2] Deriziotis' formulation assumes G is simple but this is not necessary, cf. [MS03, Proposition 30, Remark 31, Proposition 32]. It also assumes G is simply connected so that centralisers of semisimple elements of G are connected.

Type of Φ	Isolated pseudo-Levi subsystems of Φ
$A_n,\ n \geq 1$	A_n only
$B_n,\ n \geq 2$	$B_n,\ A_1 \times A_1 \times B_{n-2},\ A_3 \times B_{n-3},$ $A_1 \times D_{n-1},\ D_n,\ B_{n-r} \times D_r\ (n-2 \geq r \geq 4)$
$C_n,\ n \geq 2$	$C_n,\ A_1 \times C_{n-1},$ $C_{n-r} \times C_r\ (n-2 \geq r \geq 2)$
$D_n,\ n \geq 4$	$D_n,\ A_1 \times A_1 \times D_{n-2},$ $D_{n-r} \times D_r\ (n-4 \geq r \geq 4)$
G_2	$G_2,\ A_2,\ A_1 \times A_1$
F_4	$F_4,\ A_1 \times C_3,\ A_2 \times A_2,\ A_1 \times A_3,\ B_4$
E_6	$E_6,\ A_1 \times A_5,\ A_2 \times A_2 \times A_2$
E_7	$E_7,\ A_7,\ A_1 \times D_6,\ A_2 \times A_5,\ A_1 \times A_3 \times A_3$
E_8	$E_8,\ A_1 \times E_7, A_2 \times E_6,\ A_3 \times D_5,\ A_4 \times A_4,$ $A_1 \times A_2 \times A_5,\ A_1 \times A_7,\ A_8,\ D_8$

Table 3.1: The isolated pseudo-Levi subsystems of irreducible root systems.

Endoscopy groups of G need not lie in G. For instance, consider $G = \mathrm{SO}_{13}$. Then $\check{G} = \mathrm{Sp}_{12}$ contains a pseudo-Levi subgroup $\mathrm{Sp}_6 \times \mathrm{Sp}_6$, so $\mathrm{SO}_7 \times \mathrm{SO}_7$ is an endoscopy group of SO_{13}. However, it cannot be a subgroup of SO_{13} because $B_3 \times B_3$ does not arise in the Borel–de Siebenthal algorithm (which determines all possible closed subsystems of Φ) applied to B_7 [Kan01, §12].

We define isolated endoscopy groups analogously:

Definition 19. *An endocopy subsystem $\Psi \subseteq \Phi$ is **isolated in** Φ if $\check{\Psi}$ is not contained in a proper Levi subsystem of $\check{\Phi}$. Then an endoscopy group of G containing T is **isolated with respect to** G if its root system is an endoscopy subsystem isolated in Φ.*[3]

Lastly, we collect important properties of isolated endoscopy groups and Levi subgroups:

Proposition 20. *If L is an isolated endoscopy group of G containing T then $Z(L)^\circ = Z(G)^\circ$. Moreover, the same is true if L is an isolated pseudo-Levi subgroup of G containing T.*

Proof. Since L is isolated with respect to G, we have $\mathrm{rank}(L) = \mathrm{rank}(G)$. Then

$$\dim(T) - \dim(Z(L)) = \mathrm{rank}(L) = \mathrm{rank}(G) = \dim(T) - \dim(Z(G)),$$

so $\dim(Z(L)) = \dim(Z(G))$ and we must have $\dim(Z(L)^\circ) = \dim(Z(G)^\circ)$, cf. [Gec13, Proposition 1.3.13, Corollary 1.3.14] and therefore $Z(L)^\circ = Z(G)^\circ$ since both lie in T. □

Proposition 21. *If L is an isolated endoscopy group of G containing T then $T \cap [L,L] = T \cap [G,G]$.*

Proof. If G is semisimple then L is semisimple too and $T \cap [L,L]$ and $T \cap [G,G]$ both equal T. If G is not semisimple then observe three facts: $[G,G]$ and $[L,L]$ are semisimple, $[L,L]$ is isolated with respect to $[G,G]$, and $T \cap [G,G]$ is a maximal split torus of $[G,G]$. Then the semisimple case implies

$$T \cap [L,L] = T \cap ([G,G] \cap [[L,L],[L,L]]) = T \cap ([G,G] \cap [[G,G],[G,G]]) = T \cap [G,G].$$ □

[3] We use phrases 'endoscopy group of G' and 'isolated with respect to G' because the endoscopy may not lie in G.

Proposition 22. *If L is a Levi subgroup of G containing T with* $\operatorname{rank}[L,L] = \operatorname{rank}[G,G]$ *then $L = G$.*

Proof. Observe L and G have the same number of simple roots since $[L,L]$ and $[G,G]$ do. Since the root system of L must be generated by a subset of the simple roots of G, we must have $L = G$. □

We give two important examples to keep in mind when navigating pseudo-Levi subsystems, Levi subsystems and isolated pseudo-Levi subsystems.

Firstly, consider $G = \mathrm{SO}_5$. We have $\Phi = B_2 = \langle \alpha, \beta \rangle$ with highest root $\theta = 2\alpha + \beta$. Computing $w \cdot \langle S \rangle$ for all $w \in W$ and $S \subset \Delta \sqcup \{-\theta\}$ yields seven pseudo-Levi subsystems:

(i) B_2,

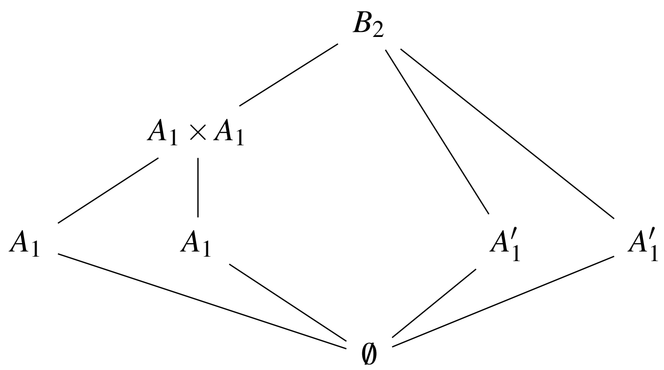
(ii) $A_1 \times A_1 \simeq \langle \beta, \theta \rangle$,

(iii) $A_1 \simeq \langle \beta \rangle \simeq \langle \theta \rangle$,

(iv) $A_1' \simeq \langle \alpha \rangle \simeq \langle \alpha + \beta \rangle$, and

(v) $\emptyset$.

Clearly, the Levi subsystems are B_2, $\langle \alpha \rangle$ and $\langle \beta \rangle$ and the isolated pseudo-Levi subsystems are B_2 and $A_1 \times A_1$. We visualise the Hasse diagram of pseudo-Levi subsystems ordered by inclusion below:



Secondly, consider $G = G_2$. We have $\Phi = G_2 = \langle \alpha, \beta \rangle$ with highest root is $\theta = 3\alpha + 2\beta$. Computing $w \cdot \langle S \rangle$ for all $w \in W$ and $S \subset \Delta \sqcup \{-\theta\}$ yields twelve pseudo-Levi subsystems:

(i) G_2,

(ii) $A_2 \simeq \langle \beta, 3\alpha + \beta \rangle$,

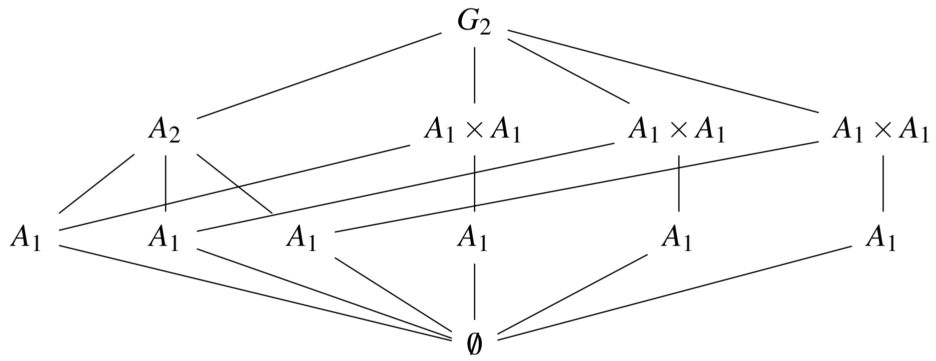
(iii) $A_1 \times A_1' \simeq \langle \alpha, 3\alpha + 2\beta \rangle$
$\simeq \langle \beta, 2\alpha + \beta \rangle \simeq \langle \alpha + \beta, 3\alpha + \beta \rangle$,

(iv) $A_1 \simeq \langle \beta \rangle \simeq \langle 3\alpha + \beta \rangle \simeq \langle 3\alpha + 2\beta \rangle$,

(v) $A_1' \simeq \langle \alpha \rangle \simeq \langle \alpha + \beta \rangle \simeq \langle 2\alpha + \beta \rangle$,

(vi) $\emptyset$.

The Levi subsystems are G_2, $\langle \alpha \rangle$ and $\langle \beta \rangle$ and the isolated pseudo-Levi subsystems are G_2, A_2 and all copies of $A_1 \times A_1$. We visualise the Hasse diagram of pseudo-Levi subsystems ordered by inclusion below:



3.2 Lusztig's Jordan decomposition

One of the deepest results in the representation theory of finite reductive groups is Lusztig's Jordan decomposition of the set of complex irreducible characters $\mathrm{Irr}(G(\mathbb{F}_q))$, which we will call the set of $G(\mathbb{F}_q)$-characters from now on. In essense, it says we can parameterise $G(\mathbb{F}_q)$-characters using two pieces of data: a conjugacy class of a semisimple element $x \in \check{G}$, and a so-called unipotent character of the semisimple centraliser subgroup $\check{G}_x$.

In general, centraliser subgroups $\check{G}_x$ of semisimple elements are not connected reductive groups, complicating the definition of unipotent characters (e.g., take $\check{G} = \mathrm{PGL}_2$ and $x = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$). However, under our assumptions, the aforementioned centraliser subgroups are connected. This is guarenteed if the derived subgroup of $\check{G}$ is simply connected [Car93, Theorem 3.5.4, Theorem 3.5.6], which happens if G has a smooth connected centre [DL76, Proposition 5.23]. We assume $\mathrm{char}(\mathbb{F}_q)$ is very good for G, ensuring smoothness of the connected centre [DL76, p. 131].

We say a $G(\mathbb{F}_q)$-character is **unipotent** if it appears as a summand in the Deligne–Lusztig character $R_{T'}^G 1$ for some maximal torus $T' \subseteq G$. Deligne–Lusztig characters are defined using a deep blend of algebraic geometry, number theory and representation theory which we will not discuss, but these characters are explained in [DL76, Car93, DM20, GM20]. The set of unipotent characters is denoted $\mathrm{Uch}(G(\mathbb{F}_q))$. A unipotent character is called **principal** if it appears as a summand in $R_T^G 1 = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} 1$.[4] Such characters are key to our point-count of the character variety.

Remarkably, Lusztig completely classified unipotent characters. Moreover, they admit uniform parameterisations and their degrees are known [GM20, Theorem 4.5.8]. This is due to Lusztig's theory of symbols in classical type [Lus77] and case-by-case analysis in exceptional type [Lus84].

We now state Lusztig's Jordan decomposition:

Theorem 23 (Theorem 4.23 of [Lus84]). *If G has connected centre then there is a bijection*

$$\mathrm{Irr}(G(\mathbb{F}_q)) \longleftrightarrow \bigsqcup_{\substack{[x]\subseteq \check{G}(\mathbb{F}_q) \\ [x] \text{ semisimple} \\ \text{conjugacy class}}} \mathrm{Uch}(\check{G}_x(\mathbb{F}_q)),$$

such that if $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))$ is paired with $\rho \in \mathrm{Uch}(\check{G}_x(\mathbb{F}_q))$ then $\chi(1)$ and $\rho(1)$ are related by

$$\frac{|G(\mathbb{F}_q)|}{\chi(1)} = q^{|\Phi(\check{G})^+|-|\Phi(\check{G}_x)^+|}\frac{|\check{G}_x(\mathbb{F}_q)|}{\rho(1)}.$$

[4]The fact that Deligne–Lusztig induction reduces to plain induction is proven in [Car93, Proposition 7.2.4].

In other words, there is a bijection from the set of $G(\mathbb{F}_q)$-characters to the set of pairs $([x],\rho)$ where $[x]$ is a $\check{G}(\mathbb{F}_q)$-conjugacy class of a semisimple $x\in\check{G}$ and ρ is a unipotent character of $\check{G}_x(\mathbb{F}_q)$. Moreover, this bijection allows us to keep track of dimensions of $G(\mathbb{F}_q)$-characters.

It will be important later in Chapter 4 to pin down an exact bijection; there are potentially multiple bijections satisfying the degree formula. Precisely, we use the unique bijection guarenteed by [GM20, Theorem 4.7.1]. We will use this bijection to define a map later in §4.1.

3.3 Order polynomials and degree polynomials

It is well-known that the orders of connected split reductive groups over $\mathbb{F}_q$ and the degrees of their irreducible characters are polynomials in q. That is, we have $|G(\mathbb{F}_q)| = \|G\|(q)$ for some polynomial $\|G\|$ and, given $\chi\in\mathrm{Irr}(G(\mathbb{F}_q))$, we have $\chi(1)=\|\chi\|(q)$ for some polynomial $\|\chi\|$ [GM20]. We call $\|G\|$ the **order polynomial** of G and $\|\chi\|$ the **degree polynomial** of χ.[5]

To count points on the character variety, we need a description of $|G(\mathbb{F}_q)|$:

Proposition 24 (Theorem 1.6.7 of [GM20])**.** *Let B be a Borel subgroup of G containing T, let U be the unipotent radical of B and write $P_W(q) := \sum_{w\in W} q^{\mathrm{length}(w)}$ for the Poincaré polynomial of W. Then*

$$|G(\mathbb{F}_q)| = |U(\mathbb{F}_q)|\cdot|T(\mathbb{F}_q)|\cdot|(G/B)(\mathbb{F}_q)| = q^{|\Phi^+|}(q-1)^{\dim(T)}P_W(q).$$

We give three examples of order polynomials:

(i) If $G=\mathrm{GL}_3$ then $\Phi\simeq A_2$, $\dim(T)=3$ and $W\simeq S_3$. Thus,

$$\|\,\mathrm{GL}_3\,\|(q) = |\,\mathrm{GL}_3(\mathbb{F}_q)| = q^3(q-1)^3(q^3+2q^2+2q+1).$$

(ii) If $G=\mathrm{SO}_5$ then $\Phi\simeq B_2$, $\dim(T)=2$ and $W\simeq D_8$. Thus,

$$\|\,\mathrm{SO}_5\,\|(q) = |\,\mathrm{SO}_5(\mathbb{F}_q)| = q^4(q-1)^2(q^4+2q^3+2q^2+2q+1).$$

(iii) If G is the semisimple group of adjoint type G_2 then $\Phi\simeq G_2$, $\dim(T)=2$ and $W\simeq D_{12}$. Thus,

$$\|G\|(q) = |G(\mathbb{F}_q)| = q^6(q-1)^2(q^6+2q^5+2q^4+2q^3+2q^2+2q+1).$$

3.4 Principal series characters

In view of Frobenius' formula, we must evaluate $G(\mathbb{F}_q)$-characters at strongly regular elements of T in order to count points on the character variety. A deep theorem due to Deligne–Lusztig describes these character values using the so-called principal series characters of $G(\mathbb{F}_q)$ [DL76, Corollary 7.6]. To this end, we review these characters and their relevant properties now.

A **principal series character** is a $G(\mathbb{F}_q)$-character appearing as summand in

$$R_T^G\,\theta = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\,\tilde{\theta}$$

[5] We do not need a formula for $\|\chi\|(q)$, but one is given in [GM20, Definition 2.3.25].

for some $\theta \in T(\mathbb{F}_q)^\vee$, where $T(\mathbb{F}_q)^\vee := \mathrm{Hom}(T(\mathbb{F}_q), \mathbb{C}^\times)$ denotes the Pontryagin dual of the finite abelian group $T(\mathbb{F}_q)$ and $\tilde{\theta}\colon B(\mathbb{F}_q) \to \mathbb{C}^\times$ is the usual inflation of θ from $T(\mathbb{F}_q)$ to $B(\mathbb{F}_q)$.

A key observation is principal series characters obey a special dichotomy.[6] Before we state it, recall W acts on $T(\mathbb{F}_q)^\vee$ in the following manner. The Weyl group W acts on $T(\mathbb{F}_q)$ by $w \cdot S := \dot{w} S \dot{w}^{-1}$, where $\dot{w}$ is any lift of w (i.e., $w = \dot{w}T \in W = N_G(T)/T$). This action is well-defined since $\dot{w}$ normalises T. Then W acts on $T(\mathbb{F}_q)^\vee$ by $(w \cdot \theta)(S) := \theta(w \cdot S)$.

We now state the dichotomy of principal series characters:

Proposition 25 (Corollary 6.3 of [DL76])**.** *Given $\theta, \theta' \in T(\mathbb{F}_q)^\vee$, exactly one of the following is true:*

(i) $R_T^G\,\theta$ and $R_T^G\,\theta'$ share no irreducible summands (up to isomorphism), or

(ii) θ and θ' are related by the action of W on $T(\mathbb{F}_q)^\vee$ in which case $R_T^G\,\theta \simeq R_T^G\,\theta'$.

A deeper understanding of principal series characters is afforded by a certain **Hecke algebra** denoted $\mathcal{H}(G,\theta)$. This is the unital associative $\mathbb{C}$-algebra of functions $f\colon G(\mathbb{F}_q) \to \mathbb{C}$ satisfying $f(bgb') = \tilde{\theta}(b) f(g) \tilde{\theta}(b')$ for all $g \in G(\mathbb{F}_q)$ and $b, b' \in B(\mathbb{F}_q)$, with convolution product

$$(ff')(g) := \sum_{xy=g} f(x) f'(y).$$

The utility of Hecke algebras is as follows. Irreducible finite-dimensional complex $\mathcal{H}(G,\theta)$-characters are in bijection with irreducible constituents of $R_T^G\,\theta$. Moreover, the multiplicity of an irreducible consistituent of $R_T^G\,\theta$ is recorded by the dimension of the associated $\mathcal{H}(G,\theta)$-character. Both of these claims follow from the Double Centraliser Theorem [EGH$^+$11, Theorem 5.18.1] in light of the isomorphism $\mathcal{H}(G,\theta) \simeq \mathrm{End}_G R_T^G\,\theta$. Another proof is given in [CR81, Theorem 11.25].

Furthermore, it is well-known (e.g., via Tits' deformation theorem [GP00, Theorem 7.4.6], originally proven in [CIK71, Theorem 1.11]) that $\mathcal{H}(G,\theta)$ is isomorphic to the group algebra $\mathbb{C}[W_\theta]$, and this isomorphism preserves isomorphism classes of irreducible representations. Here, W_θ is the stabiliser subgroup of θ under the action of W on $T(\mathbb{F}_q)^\vee$, and is a Coxeter group because G has connected centre [DL76, 5.13].

We summarise the above in the following proposition:

Proposition 26. *If G has connected centre then there are canonical bijections*

$$\left\{ \begin{array}{c} \textit{Irreducible} \\ \textit{constituents of } R_T^G\,\theta \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Irreducible} \\ \textit{representations of } \mathcal{H}(G,\theta) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Irreducible} \\ \textit{representations of } W_\theta \end{array} \right\}$$

such that if $\chi \in R_T^G\,\theta$ has image $\zeta \in \mathrm{Irr}(\mathcal{H}(G,\theta))$ and $\phi \in \mathrm{Irr}(W_\theta)$, then

$$\langle \chi, R_T^G\,\theta \rangle = \dim(\zeta) = \dim(\phi).$$

[6]This dichotomy is a special case of the exclusion theorem [GM20, Theorem 2.3.2].

3.5 Alvis–Curtis duality of characters

Alvis–Curtis duality was originally defined in [Alv79, Cur80] as a generalisation of the relationship between the trivial and Steinberg representations of $G(\mathbb{F}_q)$. It has been noted as early as [HRV08, Hau13] that this duality is responsible for the palindromicity of the counting polynomials of GL_n-character varieties. As we shall see, this is also the case for the character varieties in this thesis. For our purposes, Alvis–Curtis duality is useful because it yields an expression for $\|\chi\|(1/q)$, originally proven in [Alv82, Corollary 3.6]. That is, it allows us to invert q in the polynomial describing the character degree $\chi(1)$.

We recall the necessary properties of Alvis–Curtis duality now:

Proposition 27. *There is an involution D_G on the space of complex-valued $G(\mathbb{F}_q)$-class functions, defined explicitly in* [DM20, §7.2] *and* [GM20, §3.4]*, with the following properties:*

(i) If $\chi \in R_T^G\,\theta$ then $D_G(\chi) \in R_T^G\,\theta$,

(ii) If $\chi \in R_T^G\,\theta$ is matched with $\phi \in \mathrm{Irr}(W_\theta)$ according to Proposition 26 then $D_G(\chi)$ is matched with $\phi \otimes \mathrm{sgn}$, where $\mathrm{sgn} \in \mathrm{Irr}(W_\theta)$ is the sign character of W_θ, and

(iii) If $\chi \in R_T^G\,\theta$ then $\|D_G(\chi)\|(q) = q^{|\Phi(G)^+|}\|\chi\|(1/q)$,

Proof. The first part is a weaker version of [DM20, Corollary 7.2.9], the second part is [DM20, Proposition 7.2.13], and the third part is [GM20, Proposition 3.4.21]. □

For example, if $G = \mathrm{GL}_3$ and θ is trivial then $W_\theta \simeq S_3$ and

$$R_T^G\,1 = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\,1 = \mathrm{triv} \oplus \chi_{q^2+q}^{\oplus 2} \oplus \mathrm{St}.$$

Here, χ_{q^2+q} is the unipotent character of degree q^2+q and St is the Steinberg character. According to Proposition 26, the trivial $\mathrm{GL}_3(\mathbb{F}_q)$-representation is matched with the trivial S_3-character, the Steinberg representation is matched with the sign character of S_3, and χ_{q^2+q} is matched with the two-dimensional character $\phi_{2D} \in \mathrm{Irr}(S_3)$. We know $\mathrm{sgn} \otimes \mathrm{sgn} = \mathrm{triv}$ so $D_G(\mathrm{St}) = \mathrm{triv}$ and $D_G(\mathrm{triv}) = \mathrm{St}$, and $\phi_{2D} \otimes \mathrm{sgn} = \phi_{2D}$ so $D_G(\chi_{q^2+q}) = \chi_{q^2+q}$. We give a visualisation below:

$$\begin{tikzcd} S_3\text{-characters:} & \mathrm{triv} \arrow[r, bend right, "-\otimes\mathrm{sgn}"'] & \mathrm{sgn} \arrow[l, bend right, "-\otimes\mathrm{sgn}"'] & \phi_{2D} \arrow[loop right, "-\otimes\mathrm{sgn}"] \end{tikzcd}$$

$$\begin{tikzcd} \text{unipotent } \mathrm{GL}_3(\mathbb{F}_q)\text{-characters:} & \mathrm{triv} \arrow[r, bend right, "D_G"'] & \mathrm{St} \arrow[l, bend right, "D_G"'] & \chi_{q^2+q} \arrow[loop right, "D_G"] \end{tikzcd}$$

Chapter 4

The type of a $G(\mathbb{F}_q)$-character

In this chapter, we introduce the type of a $G(\mathbb{F}_q)$-character. This is data independent of the ground field which remembers enough information about the character to evaluate expressions appearing in Frobenius' formula. The idea of a type follows naturally from Lusztig's Jordan decomposition of $\mathrm{Irr}(G(\mathbb{F}_q))$ (cf. Theorem 23). Our types are closely connected to those used in [HLRV11, Cam17] to count points on GL_n- and Sp_{2n}-character varieties; these connections are explained in §4.3 and §4.4. But first, we define types in §4.1 and explain their main benefit in §4.2.

4.1 G-types and the type map

Consider the collection of pairs (L,ρ) where L is an endoscopy group of G containing T and ρ is a principal unipotent character of $L(\mathbb{F}_q)$. Since W acts on the root system of G, it also acts on the collection of pairs (L,ρ) by $w\cdot(L,\rho) := (L',\rho')$, where $L' = \dot{w}L\dot{w}^{-1}$ and $\rho'(\ell) := \rho(\dot{w}\ell\dot{w}^{-1})$.[1]

We are ready to define G-types:

Definition 28. *A **G-type** is the W-orbit of a pair (L,ρ), denoted $\tau = [(L,\rho)] =: [L,\rho]$.*

The set of G-types is denoted $\mathcal{T}(G)$. This set is independent of q and depends only on the root datum of G by Proposition 15 and [GM20, Theorem 4.5.8]. We have defined G-types for two reasons, which we explain now.

The first reason is Lusztig's Jordan decomposition implies there is a **type map**

$$\mathscr{T}\colon \{\text{Principal series characters of } G(\mathbb{F}_q)\} \to \mathcal{T}(G), \quad \mathscr{T}(\chi) = [L,\rho],$$

where L and ρ are determined using Lusztig's Jordan decomposition. Explicitly, $\mathscr{T}$ is defined as follows. A principal series character $\chi \in R_T^G\,\theta$ is well-defined up to W-conjugacy by Proposition 25. Under the identification $\theta \in T(\mathbb{F}_q)^\vee \simeq \check{T}(\mathbb{F}_q)$, the dual of the centraliser subgroup $\check{G}_\theta$, denoted G_θ, is an endoscopy group of G containing T. Moreover, a choice of irreducible summand in $R_T^G\,\theta$ is the same as a choice of principal unipotent character of $\check{G}_\theta(\mathbb{F}_q)$ (cf. Propositition 26) which is the same as a choice of principal unipotent character of $G_\theta(\mathbb{F}_q)$ [GM20, Remark 2.6.5].

[1]The fact that ρ' is unipotent follows from considering $\pi\colon L \to \dot{w}L\dot{w}^{-1},\ \ell \mapsto \dot{w}\ell\dot{w}^{-1}$ in [GM20, Proposition 2.3.15].

The second reason is Lusztig's Jordan decomposition implies if χ is a principal series $G(\mathbb{F}_q)$-character with G-type $\tau = [L,\rho]$ then

$$\frac{|G(\mathbb{F}_q)|}{\chi(1)} = q^{|\Phi(G)^+|-|\Phi(L)^+|}\frac{|L(\mathbb{F}_q)|}{\rho(1)}.$$

This expression appears in Frobenius' formula and we explain how this helps us count points in §4.2.

We remark that one can broaden the definition of G-types to include all endoscopy groups of G (not just those containing T), allowing one to define an extended type map defined on all $G(\mathbb{F}_q)$-characters, not just the principal series characters, using Lusztig's Jordan decomposition. For instance, if χ is a cuspidal character of $\mathrm{GL}_2(\mathbb{F}_q)$ then this extended type map sends χ to $[T_{\text{non-split}}, \mathrm{triv}]$ where $T_{\text{non-split}}$ is a maximal non-split torus in GL_2. This is essentially the approach taken in [HLRV11, §4.1] to count points on GL_n-character varieties; see §4.3 for details.

4.2 Frobenius' formula with types

Recall from §2.1 the representation variety

$$\mathbf{R} := \left\{ (A_1,B_1,\dots,A_g,B_g,Y_1,\dots,Y_n) \in G^{2g}\times\prod_{i=1}^n C_i \;\middle|\; [A_1,B_1]\cdots[A_g,B_g]Y_1\cdots Y_n = 1 \right\}$$

and recall from §1.3 that we can count points on $\mathbf{R}$ using Frobenius' formula

$$\frac{|\mathbf{R}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \sum_{\chi\in\mathrm{Irr}(G(\mathbb{F}_q))}\left(\frac{|G(\mathbb{F}_q)|}{\chi(1)}\right)^{2g-2}\prod_{i=1}^n\frac{\chi(S_i)}{\chi(1)}|C_i(\mathbb{F}_q)|.$$

In this section, we rewrite Frobenius' formula using the fibres of the type map, which will elucidate several aspects of the point-count of the representation variety. To do so, we must first recall a deep result of Deligne–Lusztig telling us how to evaluate characters at strongly regular elements:

Proposition 29 (Corollary 7.6 of [DL76]). *If $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))$ and $S \in T$ is strongly regular then*

$$\chi(S) = \sum_{\theta\in T(\mathbb{F}_q)^\vee}\langle \chi, R_T^G\,\theta\rangle\,\theta(S).$$

In particular, $\chi(S) = 0$ unless χ is a principal series character.

This means **only principal series characters contribute to the point-count of the representation variety**. In light of this fact, we are ready to reformulate Frobenius' formula using types. To this end, fix a G-type $\tau = [L,\rho]$ and define

$$\|\tau\|(q) := q^{|\Phi(G)^+|-|\Phi(L)^+|}\frac{\|L\|(q)}{\|\rho\|(q)}$$

and the character sum

$$S_\tau(q) := \sum_{\chi\in\mathscr{T}^{-1}(\tau)}\prod_{i=1}^n\chi(S_i),$$

where $\|L\|(q) := |L(\mathbb{F}_q)|$ and $\|\rho\|(q) := \rho(1)$ (cf. §3.3).

Proposition 30. *Let* $\mathbf{R}$ *be the representation variety under Assumption 1; i.e., each* C_i *is the conjugacy class of a strongly regular* $S_i \in T$ *and* $C_1 \cdots C_n \subseteq [G,G]$. *Then* ***Frobenius' formula with types*** *is*

$$\frac{|\mathbf{R}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \frac{1}{|T(\mathbb{F}_q)|^n} \sum_{\tau \in \mathfrak{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q).$$

Proof. In view of Proposition 29, expand Frobenius' formula using the fibres $\mathscr{T}^{-1}(\tau)$ to obtain

$$\frac{|\mathbf{R}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \sum_{\tau \in \mathfrak{T}(G)} \sum_{\chi \in \mathscr{T}^{-1}(\tau)} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-2} \prod_{i=1}^{n} \frac{\chi(S_i)}{\chi(1)} |C_i(\mathbb{F}_q)|.$$

Apply the formula $\frac{|G(\mathbb{F}_q)|}{\chi(1)} = q^{|\Phi(G)^+|-|\Phi(L)^+|} \frac{|L(\mathbb{F}_q)|}{\rho(1)}$ given in §4.1 to obtain

$$\frac{|\mathbf{R}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \left(\prod_{i=1}^{n} \frac{|C_i(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} \right) \sum_{\tau \in \mathfrak{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q).$$

Each $S_i \in T$ is strongly regular, meaning $C_G(S_i) = T$, so the proof is concluded by applying the orbit-stabiliser theorem to the conjugaction action of $G(\mathbb{F}_q)$ on itself. □

It is important to note $\|\tau\|(q)$ is a polynomial since the root system of L is contained in the root system of G and $\|\rho\|(q)$ always divides $\|L\|(q)$ [GM20, Remark 2.3.27]. Therefore **polynomiality of** $|\mathbf{R}(\mathbb{F}_q)|$ **is reduced to the polynomiality of** $S_\tau(q)$; the latter is the focus of Chapter 6.

4.3 Green-types

In this section, we explain why G-types generalise the types seen in [HLRV11], which we call **Green-types** due to their original formulation by Green [Gre55]. To this end, we recall some definitions and fix some notation. By a partition, we mean a decreasing list of non-negative integers

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

with only finitely many non-zero λ_i. We also denote a partition by $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \ldots$ where $m_i > 0$ is the number of times i appears in λ. We keep track of two important statistics: the **length** of λ is the smallest non-negative integer $\ell = \ell(\lambda)$ such that $\lambda_\ell > 0$, and the **weight** of λ is

$$|\lambda| := \sum_{i \geq 0} \lambda_i = \sum_{i \geq 0} i m_i.$$

Lastly, denote by $\mathcal{P}^+$ the set of partitions with positive weight. With the above in mind, Green-types are defined using the following total order:

Definition 31. *Define a total order* $\geq$ *on* $\mathbb{Z}^+ \times \mathcal{P}^+$ *by the following three conditions:*

(i) If $d > d'$ *then* $(d, \lambda) > (d', \lambda')$,

(ii) If $d = d'$ *and* $|\lambda| > |\lambda'|$ *then* $(d, \lambda) > (d', \lambda')$, *and*

(iii) If $d = d'$, $|\lambda| = |\lambda'|$ *and* $\lambda > \lambda'$ *(according to the lexicographic order) then* $(d, \lambda) > (d', \lambda')$.

Definition 32. *A **Green-type** is a finite chain in $(\mathbb{Z}^+ \times \mathcal{P}^+, \geq)$ denoted $\omega = (d_1, \lambda_1) \dots (d_s, \lambda_s)$.*

For Green-types, there is one important statistic

$$|\omega| := \sum_{i=1}^{s} d_i |\lambda_i|$$

called the **weight** of ω. The Green-types of weight n are key to counting points on GL_n-character varieties. For example, the four types of weight 2 are

$$(1,1^2), \quad (1,2^1), \quad (1,1^1)(1,1^1), \quad (2,1^1)$$

and the eight types of weight 3 are

$$(1,1^3), \quad (1,2^1 1^1), \quad (1,3^1), \quad (3,1^1),$$

$$(1,1^2)(1,1^1), \quad (1,2^1)(1,1^1), \quad (1,1^1)(1,1^1)(1,1^1), \quad (2,1^1)(1,1^1).$$ [2]

The relationship between Green-types and G-types is clear in light of two well-known facts:

Proposition 33. *(i) The set $\mathrm{Uch}(\mathrm{GL}_n(\mathbb{F}_q))$ is in bijection with the set of partitions of n, and*

(ii) Centralisers of semisimple elements of $\mathrm{GL}_n(\mathbb{F}_q)$ are of the form

$$\mathrm{GL}_{n_1}(\mathbb{F}_{q^{d_1}}) \times \cdots \times \mathrm{GL}_{n_s}(\mathbb{F}_{q^{d_s}}).$$

The former is explained in [GM20, §4.3] and the latter is explained in [DF18, §3.4.1].

Given a character $\chi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))$, its Green-type is determined as follows. Use Lusztig's Jordan decomposition to associate to χ a pair $([s], \rho)$, where $[s]$ is a semisimple conjugacy class in $\mathrm{GL}_n(\mathbb{F}_q)$. Then its centraliser defines two lists of positive integers $n_1, \dots, n_s$ and $d_1, \dots, d_s$, and the unipotent character ρ defines a list of partitions $\lambda_1, \dots, \lambda_s$ with each λ_i a partition of n_i. Then $\omega = (d_1, \lambda_1) \dots (d_s, \lambda_s)$ is a Green-type of weight n.

Note the above process, and hence Green-types, are defined for any $\mathrm{GL}_n(\mathbb{F}_q)$-character. However, in this thesis, we have only defined G-types for principal series $G(\mathbb{F}_q)$-characters. Therefore there are some Green-types with no analogous GL_n-type. For instance, a cuspidal $\mathrm{GL}_2(\mathbb{F}_q)$-character has Green-type $(2,1^1)$ but we do not define its GL_n-type. We resolved this at the end of §4.1.

We end this section by giving a table to translate between Green-types and GL_n-types:

Green-type	GL_n-type
$(1,1^1) \dots (1,1^1)$	$[T, \mathrm{triv}]$
$(1,1^{n_1}) \dots (1,1^{n_r})$	$[\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}, \mathrm{triv}]$
$(1,\lambda_1) \dots (1,\lambda_r)$	$[\mathrm{GL}_{\lvert\lambda_1\rvert} \times \cdots \times \mathrm{GL}_{\lvert\lambda_r\rvert}, \lambda_1 \otimes \cdots \otimes \lambda_r]$

Table 4.1: A translation between Green-types and GL_n-types. The tensor product of partitions $\lambda_1 \otimes \cdots \otimes \lambda_r$ denotes the unipotent character of $\prod_i \mathrm{GL}_{n_i}$ labeled by the λ_i.

[2] The number of types of weight n is described in the OEIS entry A003606.

4.4 Cambò-types

In this section, we explain the relationship between G-types and the types seen in [Cam17], which we call **Cambò-types**. The G-types of this thesis do not generalise Cambò-types because the center of Sp_{2n} is disconnected. Nevertheless, we still detail these types and explain their relationship to G-types. To this end, we recall some definitions and fix some notation.

Fix $G = \mathrm{Sp}_{2n}$ and an odd prime power q. A character $\theta \in T(\mathbb{F}_q)^\vee$ is identified with an element of $(\mathbb{F}_q^\times)^n \simeq C_{q-1}^n$. Under this identification, we fix a collection of W-orbit representatives in $T(\mathbb{F}_q)^\vee$:

Proposition 34 (Proposition 2.4.14 of [Cam17]). *The set*

$$\left\{ \left(\underbrace{k_1,\dots,k_1}_{\lambda_1}, \underbrace{k_2,\dots,k_2}_{\lambda_2}, \dots, \underbrace{k_\ell,\dots,k_\ell}_{\lambda_\ell}, \underbrace{0,\dots,0}_{\alpha_1}, \underbrace{\frac{q-1}{2},\dots,\frac{q-1}{2}}_{\alpha_\varepsilon} \right) \; : \; \begin{array}{c} 1 \le k_i \le \frac{q-3}{2}, \\ k_i > k_j \text{ if } \lambda_i = \lambda_j, \\ |\lambda| + \alpha_1 + \alpha_\varepsilon = n \end{array} \right\}$$

is a complete collection of W-orbit representatives; i.e., every orbit has exactly one representative.

For instance, if $n = 2$, then choosing a W-orbit amounts to choosing a pair of the form

$$\underbrace{\underbrace{\left(\frac{q-1}{2},\frac{q-1}{2}\right)}_{\alpha_1=0,\ \alpha_\varepsilon=2}, \underbrace{\left(0,\frac{q-1}{2}\right)}_{\alpha_1=\alpha_\varepsilon=1}, \underbrace{(0,0)}_{\alpha_1=2,\ \alpha_\varepsilon=0},}_{|\lambda|=0} \quad \underbrace{\underbrace{\left(k_1,\frac{q-1}{2}\right)}_{\alpha_1=0,\ \alpha_\varepsilon=1}, \underbrace{(k_1,0)}_{\alpha_1=1,\ \alpha_\varepsilon=0},}_{|\lambda|=1} \quad \underbrace{\underbrace{(k_1,k_1)}_{\lambda=2^1}, \underbrace{(k_2,k_2)}_{\lambda=1^2}}_{|\lambda|=2}$$

where $1 \le k_1, k_2 \le \frac{q-3}{2}$ and $k_1 > k_2$.

Definition 35. *The **Cambò-type** of $\chi \in R_T^G\theta$ is the quadruple $\tau = (\lambda, \alpha_1, \alpha_\varepsilon, \beta)$ where $(\lambda, \alpha_1, \alpha_\varepsilon)$ is the W-orbit representative of θ in Proposition 34 and $\beta \in \mathrm{Irr}(W_\theta)$ corresponds to χ under the bijection of Proposition 26.*

In both settings, a type keeps track of a principal series character $R_T^G\theta$ and one of its irreducible summands. As an example, the sixteen Cambò-types of Sp_4 are below:

W-orbit rep. θ	W_θ	λ	α_1	α_ε	$\beta \in \mathrm{Irr}(W_\theta)$
$(\frac{q-1}{2}, \frac{q-1}{2})$	D_8	0	0	2	$\chi_1, \chi_2, \chi_3, \chi_4, \chi_{2D}$
$(0, \frac{q-1}{2})$	1	0	1	1	triv
$(0,0)$	D_8	0	2	0	$\chi_1, \chi_2, \chi_3, \chi_4, \chi_{2D}$
$(k_1, \frac{q-1}{2})$	1	1^1	0	1	triv
$(k_1,0)$	1	1^1	1	0	triv
(k_1,k_1)	S_2	2^1	0	0	triv, sgn
(k_1,k_2)	1	1^2	0	0	triv

Table 4.2: Cambò-types for Sp_4. The zero partition is denoted 0. The four 1-dimensional characters of D_8 are denoted $\chi_1, \dots, \chi_4$ and the 2-dimensional irreducible character of D_8 is denoted χ_{2D}.

Chapter 5

Generic conjugacy classes

In this chapter, we develop the idea of choosing semisimple conjugacy classes generically. We generalise the ideas of [HLRV11] and are inspired by [Boa14]. In the latter, a similar idea for complex reductive groups was used to conclude stability and irreducibility of certain representations and allowed the author to study the so-called irregular Deligne–Simpson problem. In our setting, choosing conjugacy classes generically has four key advantages:

(i) The G/Z-action on the representation variety has finite étale stabilisers,

(ii) The character sums $S_\tau(q)$ defined in §4.2 greatly simplify,[1]

(iii) The representation variety is smooth and equidimensional,

(iv) The character stack and the character variety have the same point-count,

We address the first advantage in §5.3, the second advantage in §6.4, and the third and fourth advantages in §7.3. But first, we give the definition and examples of generic conjugacy classes in §5.1 and then prove conjugacy classes can be chosen generically in the first place in §5.2.

5.1 Definition and examples

Recall from §2.1 the definition of generic conjugacy classes:

Definition 36. *We say a tuple $\mathcal{C} = (C_1, \dots, C_n)$ of semisimple conjugacy classes of G is generic if*

$$\prod_{i=1}^{n} x_i \notin [L, L]$$

for all proper Levi subgroups L of G (not necessarily those containing T) and for all $x_i \in C_i \cap L$.

Before seeing examples, we prove only the Levi subgroups of G containing T need to be checked:

[1] We obtain an expression for $S_\tau(q)$ without generic conjugacy classes, but its computation is less clear.

Proposition 37. *The collection of semisimple conjugacy classes $\mathcal{C} = (C_1, \dots, C_n)$ is generic if and only if $\prod_{i=1}^n x_i \notin [L,L]$ for all proper Levi subgroups L of G containing T and for all $x_i \in C_i \cap L$.*

Proof. The forward implication is clear. Conversely, suppose L is an arbitrary Levi subgroup of G and choose $g \in G$ so that gLg^{-1} contains T. Then the result follows by observing the equivalences

$$\prod_{i=1}^n x_i \notin [L,L] \iff g\left(\prod_{i=1}^n x_i\right)g^{-1} \notin g[L,L]g^{-1} \iff \prod_{i=1}^n gx_ig^{-1} \notin [gLg^{-1}, gLg^{-1}]$$

and

$$x_i \in C_i \cap L \iff gx_ig^{-1} \in g(C_i \cap L)g^{-1} = C_i \cap gLg^{-1}. \qquad \square$$

Next, we explain that ...

Consider the semisimple element $S = \mathrm{diag}(a,b,c)$ in $G = \mathrm{GL}_3$ representing the conjugacy class C, where $a,b,c \in \mathbb{F}_q^\times$. In this section, we give necessary and sufficient conditions for C to be generic and for C to be strongly regular, allowing us to produce four examples:

(i) A strongly regular and generic conjugacy class,

(ii) A strongly regular but not generic conjugacy class,

(iii) A generic but not strongly regular conjugacy class, and

(iv) A conjugacy class which is neither strongly regular nor generic.

Since $S \in T$, we only need to consider the proper Levis of G containing T, which are

$$L_1 := \left\{\begin{pmatrix} \mathrm{GL}_2 & \\ & \mathrm{GL}_1 \end{pmatrix}\right\}, \quad L_2 := \left\{\begin{pmatrix} \mathrm{GL}_1 & \\ & \mathrm{GL}_2 \end{pmatrix}\right\}, \quad T,$$

with derived subgroups

$$[L_1,L_1] = \left\{\begin{pmatrix} \mathrm{SL}_2 & \\ & 1 \end{pmatrix}\right\}, \quad [L_2,L_2] = \left\{\begin{pmatrix} 1 & \\ & \mathrm{SL}_2 \end{pmatrix}\right\}, \quad [T,T] = 1.$$

Clearly, a necessary condition for C to be generic is $abc = 1$ because $[L_1,L_1]$, $[L_2,L_2]$ and $[T,T]$ are contained in SL_3. Thus, we assume $c = (ab)^{-1}$, and observe three facts: $S \in [L_1,L_1]$ if and only if $ab = 1$, $S \in [L_2,L_2]$ if and only if $a = 1$ and $S \in [T,T]$ if and only if $a = b = 1$. Thus, a necessary and sufficient condition for C to be generic is $S = \mathrm{diag}(a,b,(ab)^{-1})$ with $ab \neq 1$ and $a \neq 1 \neq b$.

On the other hand, a necessary and sufficient condition for S to be strongly regular is $a \neq b \neq c \neq a$ because S being strongly regular means $C_G(S) = T$ [Ste65]. If $c = (ab)^{-1}$ then $a \neq c$ if and only if $a^2b \neq 1$ and $b \neq c$ if and only if $ab^2 \neq 1$.

Therefore, C is strongly regular and generic if and only if it is represented by

$$S = \mathrm{diag}(a,b,(ab)^{-1}) \text{ with } a \neq 1,\ b \neq 1,\ ab \neq 1,\ a^2b \neq 1 \text{ and } ab^2 \neq 1,$$

C is strongly regular and but not generic if and only if it is represented by

$$S = \mathrm{diag}(a,b,c) \text{ with } a \neq b \neq c \neq a \text{ and either } ab = 1,\ bc = 1,\ a = 1 \text{ or } c = 1,$$

C is generic but not strongly regular if and only if it is represented by

$$S = \mathrm{diag}(a,b,(ab)^{-1}) \text{ with } ab \neq 1,\ a \neq 1 \text{ and either } a = b,\ a^2b = 1 \text{ or } ab^2 = 1,$$

and C is neither strongly regular nor generic if and only if it is represented by

$$S = \mathrm{diag}(a,b,c) \text{ with either } ab = 1,\ c = 1,\ bc = 1 \text{ or } a = 1 \text{ and either } a = b,\ b = c \text{ or } a = c.$$

Since $G = \mathrm{GL}_3$, this can be compared with [HLRV11, Definition 2.1.1].

As another example, it is proven in [Nam23, Lemma 53,Lemma 60] that if G equals SO_5 or the semisimple group of adjoint type G_2 then the conjugacy class of a strongly regular $S \in T$ is generic.

5.2 Conjugacy classes can be chosen generically

In this section, we show generic collections of strongly regular conjugacy classes exist. To this end, let $\mathcal{L}$ be the set of proper Levi subgroups of G containing T. It is important to note $\mathcal{L}$ is a finite set only depending on the root datum of G (cf. §3.1). Moreover, let $\underline{S}$ denote the tuple $(S_1,\dots,S_n)$ in T^n. Recall from §3.4 that W acts on $T(\mathbb{F}_q)$ by $w \cdot S := \dot{w}S\dot{w}^{-1}$. Then for each tuple $\underline{w} := (w_1,\dots,w_n)$ in W^n, denote by $\underline{w}\cdot\underline{S}$ the product $(w_1 \cdot S_1)\cdots(w_n \cdot S_n)$ in T.

Proposition 38. *If $S_1,\dots,S_n \in T$ satisfy*

$$\underline{w}\cdot\underline{S} \in [G,G] \setminus \bigcup_{L\in\mathcal{L}} [L,L]$$

for all $\underline{w} \in W^n$ then the collection $(C_1,\dots,C_n)$ is generic.

Note such S_i obviously satisfy $S_1\cdots S_n \in [G,G]$ (cf. Assumption 1).

Proof. For want of a contradiction, assume the collection is not generic; i.e., $X_1\cdots X_n \in [L,L]$ for some proper Levi L of G containing T and for some $X_i \in C_i \cap L$. Write $X_i = g_iS_ig_i^{-1}$ for some $g_i \in G$. Then T and $g_iTg_i^{-1}$ are maximal tori inside of L so they are conjugate by some $l_i \in L$. That is, $l_iTl_i^{-1} = g_iTg_i^{-1}$, meaning $g_i^{-1}l_i$ normalises T. Thus, we can write $g_i = l_i\dot{w}_i$ for some $l_i \in L$ and $w_i = \dot{w}_iT \in W$; i.e., we can write $X_i = l_i(w_i.S_i)l_i^{-1}$. Then

$$\begin{aligned} X_1\cdots X_n &= l_1(w_1\cdot S_1)l_1^{-1}\cdots l_n(w_n\cdot S_n)l_n^{-1} \\ &= [l_1,w_1\cdot S_1](w_1\cdot S_1)[l_2,w_2\cdot S_2](w_2\cdot S_2)\cdots[l_n,w_n\cdot S_n](w_n\cdot S_n), \end{aligned}$$

where the second equality is obtained by inserting $(w_i\cdot S_i)^{-1}(w_i\cdot S_i)$ after $l_i(w_i\cdot S_i)l_i^{-1}$. Notice $w_i\cdot S_i$ lies in L since $X_i = l_i(w_i\cdot S_i)l_i^{-1}$ does, so the above expression is a product of elements in L and $[L,L]$. By assumption, this product lies in $[L,L]$, so the product $\underline{w}\cdot\underline{S}$ does too, which is a contradiction. □

The following proposition guarantees such a collection of strongly regular S_i exist:

Proposition 39. *There exist strongly regular $S_1, \dots, S_n \in T$ satisfying the condition of Proposition 38.*

Proof. Let

$$A := \bigcap_{\underline{w} \in W^n} \left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [G,G] \right\} \quad \text{and} \quad B := \bigcup_{L \in \mathcal{L}} \bigcup_{\underline{w} \in W^n} \left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [L,L] \right\}.$$

Such S_i existing is the same as finding a tuple of strongly regular elements in the set

$$\bigcap_{\underline{w} \in W^n} \left(\left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [G,G] \right\} \setminus \bigcup_{L \in \mathcal{L}} \left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [L,L] \right\} \right).$$

Basic set identities imply this set is equal to $A \setminus B$. Of course, $A \setminus B$ is non-empty if $\dim(A) > \dim(B)$ which we verify below. □

Lemma 40. *Let*

$$A := \bigcap_{\underline{w} \in W^n} \left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [G,G] \right\} \quad \textit{and} \quad B := \bigcup_{L \in \mathcal{L}} \bigcup_{\underline{w} \in W^n} \left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [L,L] \right\}.$$

Then $\dim(A) > \dim(B)$.

Proof. We claim

$$\dim(A) = (n-1)\dim(T) + \operatorname{rank}[G,G]$$

and

$$\dim(B) = (n-1)\dim(T) + \max_{L \in \mathcal{L}} \operatorname{rank}[L,L].$$

The inequality $\dim(A) > \dim(B)$ follows since $\mathcal{L}$ is finite and $\operatorname{rank}[G,G] > \operatorname{rank}[L,L]$ for all $L \in \mathcal{L}$ by Proposition 22. To compute $\dim(A)$ and $\dim(B)$, note

$$\underline{w} \cdot \underline{S} = (\dot{w}_1 S_1 \dot{w}_1^{-1}) \cdots (\dot{w}_n S_n \dot{w}_n^{-1}) = [\dot{w}_1, S_1] S_1 [\dot{w}_2, S_2] S_2 \cdots [\dot{w}_n, S_n] S_n.$$

Therefore $\underline{w} \cdot \underline{S} \in [G,G]$ if and only if $S_1 \cdots S_n \in [G,G]$ and $\underline{w} \cdot \underline{S} \in [L,L]$ if and only if $S_1 \cdots S_n \in [L,L]$. Then we can simplify:

$$\left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [G,G] \right\} = \left\{ \underline{S} \in T^n \;\middle|\; S_1 \cdots S_n \in [G,G] \right\},$$
$$\left\{ \underline{S} \in T^n \;\middle|\; \underline{w} \cdot \underline{S} \in [L,L] \right\} = \left\{ \underline{S} \in T^n \;\middle|\; S_1 \cdots S_n \in [L,L] \right\}.$$

The result follows after proving the latter has dimension $(n-1)\dim(T) + \operatorname{rank}[L,L]$. To this end, fix $S_1, \dots, S_n \in T$ and suppose $S_1 \cdots S_n \in [L,L]$. Since $L = [L,L]Z(L)$, we decompose $S_i = a_i b_i$, where $a_i \in T \cap [L,L]$ and $b_i \in T \cap Z(L) = Z(L)$. Then

$$S_1 \cdots S_n = a_1 \cdots a_n b_1 \cdots b_n$$

so $b_1 \cdots b_n$ lies in $Z(L) \cap [L,L]$ since $a_1, \dots, a_n$ lies in $[L,L]$ and $S_1 \cdots S_n$ lies in $[L,L]$.

Observe we are free to choose $a_1, \dots, a_n \in T \cap [L,L]$ and $b_1, \dots, b_{n-1} \in Z(L)$ as long as b_n lies in $(Z(L) \cap [L,L])(b_1 \cdots b_{n-1})^{-1}$. Therefore

$$\dim\left(\left\{\underline{S} \in T^n \,\middle|\, S_1 \cdots S_n \in [L,L]\right\}\right) = \underbrace{n \dim(T \cap [L,L])}_{a_1,\dots,a_n} + \underbrace{(n-1)\dim(Z(L))}_{b_1,\dots,b_{n-1}} + \underbrace{\dim(Z(L) \cap [L,L])}_{b_n}.$$

The last term is zero since $[L,L]$ is semisimple, and we can rewrite the first two terms as

$$(n-1)\underbrace{(\dim(T \cap [L,L]) + \dim(T \cap Z(L)))}_{\dim(T)} + \underbrace{\dim(T \cap [L,L])}_{\operatorname{rank}[L,L]}. \qquad \square$$

We will see later in §6.4 that our counting formula for the character variety does not actually depend on $\mathcal{C}$, provided the conjugacy classes are chosen generically.

5.3 Stabilisers are finite étale

In this section, we prove every point on the representation variety has finite étale stabilisers under the G/Z-action when conjugacy classes are chosen generically.

Define the following sets:

(i) Let $\mathcal{I}$ be the set of isolated pseudo-Levi subgroups of G containing T, and

(ii) Let $\mathcal{Z}$ be the subgroup of G generated by the centres of the isolated pseudo-Levi subgroups in $\mathcal{I}$.

Note $\mathcal{I}$ is finite, cf. Proposition 17. A key observation is the following:

Lemma 41. *The group $\mathcal{Z}$ is an abelian group containing the centre Z of G, and $\mathcal{Z}/Z$ is finite.*

Proof. Each isolated pseudo-Levi $L \in \mathcal{I}$ has a centre contained in T:

$$Z(L) = C_L(L) \subseteq C_G(L) \subseteq C_G(T) = T.$$

Then $\mathcal{Z} \subseteq T$ so $\mathcal{Z}$ is abelian. Next, G is isolated in G so $\mathcal{Z}$ contains Z. Lastly, by Proposition 20, the quotient $Z(L)/Z$ is finite. Therefore $\mathcal{Z}/Z$ is generated by finitely many elements in the abelian group T/Z, and these elements have finite order, so $\mathcal{Z}/Z$ is finite. □

Now fix a point on the representation variety, denoted $p = (A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_n)$. The group $\mathcal{Z}/Z$ is of use due to the following:

Proposition 42. *The stabiliser $\mathrm{Stab}_{G/Z}(p)$ lies in $g\mathcal{Z}g^{-1}/Z$ for some $g \in G$.*

Proof. As subsets of G/Z, we have $\mathrm{Stab}_{G/Z}(p) = \mathrm{Stab}_G(p)/Z$. Fixing $h \in \mathrm{Stab}_G(p)$, it suffices to show $h \in g\mathcal{Z}g^{-1}$ for some $g \in G$. Since h stabilises p, we know $A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_n$ all lie in $C_G(h)$ which implies the inclusion

$$X_1 \cdots X_n = ([A_1, B_1] \cdots [A_g, B_g])^{-1} \in [C_G(h), C_G(h)].$$

Now $h \in C_G(X_1) = gTg^{-1}$ for some $g \in G$ so $C_G(h)$ is a pseudo-Levi subgroup of G containing gTg^{-1}. In other words, $g^{-1}C_G(h)g$ is a pseudo-Levi subgroup of G containing T. Furthermore, $C_G(h)$ must be isolated in G. If it were not then $C_G(h) \subset L$ for some proper Levi subgroup L of G, meaning $X_1 \cdots X_n \in [L, L]$ which contradicts genericity of $\mathcal{C}$. Thus, $g^{-1}C_G(h)g$ is isolated too. It is straightforward to verify $h \in Z(C_G(h))$ so $h \in g^{-1}\mathcal{Z}g$. □

This establishes finiteness of the stabiliser $\mathrm{Stab}_{G/Z}(p)$, but we can say more:

Proposition 43. *If $L \in \mathcal{I}$ then the finite group $Z(L)/Z$ is étale.*

Proof. One checks by hand using Table 2.1 and Table 3.1 that if $\mathrm{char}(\mathbb{F}_q)$ is very good for G then it is very good for every $L \in \mathcal{I}$. Therefore $\mathrm{char}(\mathbb{F}_q)$ is never a torsion prime for any $L \in \mathcal{I}$ (cf. [Ste75]), so each $Z(L)/Z$ is of order prime to $\mathrm{char}(\mathbb{F}_q)$ and therefore étale by [Mil17, Corollary 11.31]. □

Corollary 44. *The stabiliser $\mathrm{Stab}_{G/Z}(p)$ is finite étale.*

Chapter 6

Formulas for the character sum $S_\tau(q)$

Let $S_1, \ldots, S_n \in T$ be strongly regular elements representing conjugacy classes $C_1, \ldots, C_n$ and let τ be a G-type. Recall from §4.2 the definition of the character sum

$$S_\tau(q) := \sum_{\chi \in \mathscr{T}^{-1}(\tau)} \prod_{i=1}^{n} \chi(S_i).$$

We saw in §4.2 that determining $|\mathbf{R}(\mathbb{F}_q)|$ amounts to determining $S_\tau(q)$. In this chapter, we give several formulas for $S_\tau(q)$, first without a generic choice of conjugacy classes, and then with one.

This chapter is the most technical chapter appearing in this thesis. In particular, in view of the definition of polynomial count in §1.3, it is not enough to conclude that $S_\tau(q)$ is a polynomial; we must prove the polynomial we obtain is stable under base change. The main point of this chapter is we obtain a simple and stable formula for $S_\tau(q)$ when conjugacy classes are chosen generically.

To elucidate the important ideas, we evaluate $S_\tau(q)$ in two steps: when there is only one puncture, and when there are multiple punctures. In both cases, we will see an auxiliary sum of $T(\mathbb{F}_q)$-character values appearing in our formulas for $S_\tau(q)$. Following methods developed in [KNP23], we explain how to evaluate these auxiliary sums in §6.3 and how they simplify in the generic setting in §6.4.

6.1 A formula for $S_\tau(q)$: once-punctured case

Let $\tau = [L, \rho]$ be a G-type; recall that this means L is an endoscopy group of G containing T and ρ is a principal unipotent character of $L(\mathbb{F}_q)$. In this section, we explain the evaluation of $S_\tau(q)$ when there is one conjugacy class containing a strongly regular $S \in T$. In this case, $S_\tau(q)$ is given by

$$S_\tau(q) = \sum_{\chi \in \mathscr{T}^{-1}(\tau)} \chi(S).$$

Define the auxiliary sum

$$\alpha_{L,S}(q) := \sum_{\substack{\theta \in T(\mathbb{F}_q)^\vee \\ W_\theta = W(L)}} \theta(S).$$

This section is dedicated to proving the following formula for $S_\tau(q)$:

Proposition 45. *Under the assumptions of this section, our formula for $S_\tau(q)$ is*

$$S_\tau(q) = \dim(\tilde{\rho})\frac{|[L]|}{|W|}\sum_{w\in W}\alpha_{L,w\cdot S}(q),$$

where $\tilde{\rho}$ is the $W(L)$-character associated to ρ and $[L]$ is the W-orbit of L.

Recall W acts on $T(\mathbb{F}_q)$ by $w\cdot S := \dot{w}S\dot{w}^{-1}$, where $\dot{w}$ is any lift of w (i.e., $w = \dot{w}T \in W$). The proof of Proposition 45 is centered around a result of Deligne–Lusztig which was used in §4.2 to reformulate Frobenius' formula:

Proposition 46 (Corollary 7.6 of [DL76])**.** *If $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))$ and $S \in T$ is strongly regular then*

$$\chi(S) = \sum_{\theta\in T(\mathbb{F}_q)^\vee}\langle\chi, R_T^G\,\theta\rangle\,\theta(S).$$

In particular, $\chi(S) = 0$ unless χ is a principal series character.

We are ready to prove Proposition 45:

Proof. Recall from §3.4 that irreducible summands in $R_T^G\,\theta$ are in bijection with characters of W_θ with their multiplicities given by the corresponding W_θ-character's dimension. Given a character $\phi \in \mathrm{Irr}(W_\theta)$, denote the corresponding irreducible summand in $R_T^G\,\theta$ by $\chi_{\theta,\phi}$. Recall W acts on $T(\mathbb{F}_q)^\vee$ by $(w\cdot\theta)(S) := \theta(w\cdot S)$. Then using Proposition 46, we compute

$$\chi_{\theta,\phi}(S) = \dim(\phi)\sum_{w\in W/W_\theta}(w\cdot\theta)(S) = \frac{\dim(\phi)}{|W_\theta|}\sum_{w\in W}\theta(w\cdot S).$$

If $\chi_{\theta,\phi}$ has type $\tau = [L,\rho]$ then the dual of $\check{G}_\theta$, which we denote by G_θ, is an endoscopy group of G containing T. Moreover, G_θ lies in the W-orbit of L, and $\phi \in \mathrm{Irr}(W_\theta)$ is paired with $\rho \in \mathrm{Uch}(L(\mathbb{F}_q))$ according to the bijections

$$\mathrm{Irr}(W_\theta) \longleftrightarrow R_{\check{T}}^{\check{G}_\theta}\,1 \longleftrightarrow R_T^L\,1 \subseteq \mathrm{Uch}(L(\mathbb{F}_q)).$$

Denote by $\tilde{\rho}$ the character in $\mathrm{Irr}(W(L))$ corresponding to $\rho \in \mathrm{Uch}(L(\mathbb{F}_q))$. Then

$$S_\tau(q) = \sum_{\chi\in\mathscr{T}^{-1}(\tau)}\chi(S) = \sum_{\substack{[\theta]\in T(\mathbb{F}_q)^\vee/W\\ G_\theta\in[L]}}\chi_{\theta,\tilde{\rho}}(S).$$

Note $G_\theta \in [L]$ if and only if $W_\theta \in [W(L)]$ [DL76, Theorem 5.13]. This means

$$S_\tau(q) = \sum_{\substack{[\theta]\in T(\mathbb{F}_q)^\vee/W\\ W_\theta\in[W(L)]}}\chi_{\theta,\tilde{\rho}}(S).$$

We write this as a sum over all $\theta \in T(\mathbb{F}_q)^\vee$, rather than W-orbits $[\theta] \in T(\mathbb{F}_q)^\vee/W$. To do so, notice $\chi_{\theta,\tilde{\rho}} = \chi_{w.\theta,\tilde{\rho}}$ since $R_T^G\,\theta \simeq R_T^G\,w\cdot\theta$ by Proposition 25. Therefore

$$S_\tau(q) = \frac{|W(L)|}{|W|}\sum_{\substack{\theta\in T(\mathbb{F}_q)^\vee\\ W_\theta\in[W(L)]}}\chi_{\theta,\tilde{\rho}}(S).$$

Lastly, we can replace the condition $W_\theta \in [W(L)]$ with the condition $W_\theta = W(L)$. We do so by averaging over the orbit size $|[W(L)]| = |W|/|N_W(W(L))| = |[L]|$ (cf. [Car72, Lemma 34]), giving

$$S_\tau(q) = |W(L)|\frac{|[L]|}{|W|}\sum_{\substack{\theta\in T(\mathbb{F}_q)^\vee \\ W_\theta = W(L)}} \chi_{\theta,\tilde{\rho}}(S).$$

We substitute in our formula for $\chi_{\theta,\tilde{\rho}}(S)$, giving

$$S_\tau(q) = \dim(\tilde{\rho})\frac{|[L]|}{|W|}\sum_{w\in W}\sum_{\substack{\theta\in T(\mathbb{F}_q)^\vee \\ W_\theta = W(L)}} \theta(w\cdot S) = \dim(\tilde{\rho})\frac{|[L]|}{|W|}\sum_{w\in W}\alpha_{L,w\cdot S}(q).$$ □

6.2 A formula for $S_\tau(q)$: multi-punctured case

In this section, we explain the evaluation of $S_\tau(q)$ when there is a collection $\mathcal{C} = (C_1,\ldots,C_n)$ containing strongly regular $S_i \in T$. In this case, $S_\tau(q)$ is given by

$$S_\tau(q) = \sum_{\chi\in\mathscr{T}^{-1}(\tau)} \chi(S_1)\cdots\chi(S_n).$$

In this section, we prove the following formula for $S_\tau(q)$:

Proposition 47. *Under the assumptions of this section, our formula for $S_\tau(q)$ is*

$$S_\tau(q) = \frac{\dim(\tilde{\rho})^n}{|W(L)|^{n-1}}\frac{|[L]|}{|W|}\sum_{\underline{w}\in W^n}\alpha_{L,\underline{w}\cdot\underline{S}}(q).$$

Proof. As in the once-punctured case, fix $\tau = [L,\rho]$ and recall $\chi \in \mathscr{T}^{-1}(\tau)$ if and only if $\chi = \chi_{\theta,\tilde{\rho}}$ for some $\theta \in T(\mathbb{F}_q)^\vee$ with $G_\theta \in [L]$. Therefore

$$S_\tau(q) = \sum_{\substack{[\theta]\in\check{T}(\mathbb{F}_q)/W \\ G_\theta\in[L]}} \chi_{\theta,\tilde{\rho}}(S_1)\cdots\chi_{\theta,\tilde{\rho}}(S_n) = |W(L)|\frac{|[L]|}{|W|}\sum_{\substack{\theta\in\check{T}(\mathbb{F}_q) \\ W_\theta = W(L)}} \chi_{\theta,\tilde{\rho}}(S_1)\cdots\chi_{\theta,\tilde{\rho}}(S_n).$$

Let $\underline{S}$ denote the tuple $(S_1,\ldots,S_n)$ in T^n and, for each $\underline{w} := (w_1,\ldots,w_n) \in W^n$, let $\underline{w}\cdot\underline{S}$ denote the product $(w_1\cdot S_1)\cdots(w_n\cdot S_n)$ in T. Then, from the once-punctured case, we know

$$\chi_{\theta,\tilde{\rho}}(S_1)\cdots\chi_{\theta,\tilde{\rho}}(S_n) = \frac{\dim(\tilde{\rho})^n}{|W_\theta|^n}\sum_{w_1,\ldots,w_n\in W}\theta(\underline{w}\cdot\underline{S}).$$

Plugging this into the expression for $S_\tau(q)$ above completes the proof. □

6.3 A formula for the auxiliary sum $\alpha_{L,S}(q)$

Given an endoscopy group L of G containing T, we have seen in §6.1 and §6.2 that the auxiliary sum

$$\alpha_{L,S}(q) := \sum_{\substack{\theta\in T(\mathbb{F}_q)^\vee \\ W_\theta = W(L)}} \theta(S)$$

plays a key role in the determination of $S_\tau(q)$.

In this section, we give a method to evaluate $\alpha_{L,S}(q)$ following [KNP23, §5]. We emphasise a new idea appearing in this thesis: genericity of conjugacy classes yields a simple formula for $\alpha_{L,S}(q)$, written in terms of isolated endoscopy groups. We explain this simplification in §6.4. In this section, we simplify the auxililary sum as much as we can without assuming genericity of conjugacy classes.

Our evaluation of $\alpha_{L,S}(q)$ centers around an application of Möbius inversion. Consider the partially ordered set P of endoscopy groups of G containing T, ordered by inclusion of their root systems, which **by abuse of notation we write as inclusion of the endoscopy groups**. Then P comes with Möbius function $\mu\colon P \times P \to \mathbb{Z}$. Define the sum

$$\Delta_{L,S}(q) := \sum_{\substack{\theta \in T(\mathbb{F}_q)^\vee \\ W_\theta \supseteq W(L)}} \theta(S),$$

so the Möbius inversion formula [Sta12, §3.7] yields

$$\alpha_{L,S}(q) = \sum_{L' \supseteq L} \mu(L,L')\,\Delta_{L',S}(q),$$

where the sum is over all endoscopy groups L' of G containing L.

We turn our attention to evaluating $\Delta_{L,S}(q)$, starting with an application of Pontryagin duality:

Proposition 48 (Lemma 26 of [KNP23])**.** *Let $f\colon A \to B$ be a surjective homomorphism of finite abelian groups and $f^\vee\colon B^\vee \to A^\vee$ be the Pontryagin dual $f^\vee(\varphi) = \varphi \circ f$. For each $a \in A$, we have*

$$\sum_{\theta \in f^\vee(B^\vee)} \theta(a) = \begin{cases} |B|, & \text{if } f(a) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

To alleviate notation, let $k = \mathbb{F}_q$. We apply this result to the (Pontryagin dual of the) natural map

$$f_L\colon T(k) \to \frac{T(k)}{T(k) \cap [L(k),L(k)]}.$$

Corollary 49 (Corollary 27 and Proposition 28 of [KNP23])**.** *We have*

$$\Delta_{L,S}(q) = \begin{cases} |\check{T}(k)^{W(L)}|, & \text{if } f_L(S) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\Delta_{L,S}(q)$ is zero unless $S \in [L(k),L(k)]$.

In light of the above, we must understand the fixed points $\check{T}(k)^{W(L)}$. To this end, let

$$\pi_0^L := |\pi_0(\check{T}^{W(L)})(k)|$$

where $\pi_0(\check{T}^{W(L)})$ is the component group of $\check{T}^{W(L)}$. We recall precisely what this means now (cf. [KNP23, §4]). Recall the cocharacter lattice $\check{X}$ admits an action of $W(L)$, and define the $W(L)$-coinvariants of $\check{X}$ by

$$\check{X}_{W(L)} := \check{X}/\langle x - w\cdot x \mid x \in \check{X},\ w \in W(L)\rangle,$$

so that $\check{T}^{W(L)} = \operatorname{Spec} k[\check{X}_{W(L)}]$. Then $\check{X}_{W(L)}$ is an abelian group with torsion part $\operatorname{Tor}(\check{X}_{W(L)})$, and the group of components is the k-group scheme $\pi_0(\check{T}^{W(L)}) := \operatorname{Spec} k[\operatorname{Tor}(\check{X}_{W(L)})]$.

The following proposition explains why we introduced the group of components:

Proposition 50 (§4 of [KNP23]). *Let L be an endoscopy group of G containing T. Then*

(i) $\check{T}(k)^{W(L)} \simeq Z(\check{L})$ *as k-group schemes,*

(ii) *We have* $|\check{T}(k)^{W(L)}| = |\pi_0(\check{T}^{W(L)})(k)| \times |(\check{T}^{W(L)})^\circ(k)|$, *and*

(iii) *We have* $|(\check{T}^{W(L)})^\circ(k)| = |Z(\check{L})^\circ(k)| = (q-1)^{\mathrm{rank}(Z(L))}$.

Corollary 51. *The sums $\Delta_{L,S}(q)$, $\alpha_{L,S}(q)$ and $S_\tau(q)$ are polynomials in q.*

As stated at the beginning of this chapter, we must ensure these polynomials are stable under base change. That is, if k'/k is a finite extension, we must ensure the above polynomials do not change.

There are two things that may go wrong if we replace k with k':

(i) First, the integer $|\pi_0(\check{T}^{W(L)})(k)|$ may change. This is because $\pi_0(\check{T}^{W(L)})$ is an étale group scheme and the associated action of $\mathrm{Gal}(\bar{k}/k)$ may not be trivial; i.e., the Galois action may 'hide' k'-points which appear after base change to k'. Since $\pi_0(\check{T}^{W(L)})$ is finite, this issue is resolved by choosing k large enough in the first place so that the Galois action is trivial.

(ii) Second, in view of Corollary 49, we may have $S \notin [L(k), L(k)]$ but $S \in [L(k'), L(k')]$, which means the polynomial $\Delta_{L,S}(q)$ may change. This is because the inclusion $[L(k), L(k)] \hookrightarrow [L,L](k)$ may be strict (cf. [KNP23, §5.2.1]). Since there are only finitely many endoscopy groups of G containing T, we can resolve this issue by choosing k large enough in the first place so that this behavior does not occur.

These two observations are why we say 'potentially polynomial count' in the theorems in §2.1. The main point of this section is, after choosing k large enough as explained above, we have a formula for $\alpha_{L,S}(q)$ which is stable under base change:

Corollary 52. *Let L be an endoscopy group of G containing L. If $S \in T$ is strongly regular then*

$$\alpha_{L,S}(q) = \sum_{\substack{L' \supseteq L \\ S \in [L'(k), L'(k)]}} \mu(L,L')\, \pi_0^{L'}\, |Z(L')^\circ(k)|,$$

where the sum is over all endoscopy groups L' of G containing L satisfying $S \in [L'(k), L'(k)]$.

6.4 A simple formula for $S_\tau(q)$: generic case

We have not yet used our generic assumption. We do so now, providing a major simplification of the auxiliary sum $\alpha_{L,S}(q)$ (and hence of $S_\tau(q)$) via the following proposition:

Proposition 53. *Suppose L is an endoscopy group of G containing T.*

(i) *An element of G lies in $[L,L]$ if and only if it lies in $[L(k'), L(k')]$ for some finite extension k' of k,*

(ii) *A generic element S lies in $[L,L]$ if and only if L is isolated with respect to G, and*

(iii) *If L is isolated with respect to G then $|Z(L)^\circ(k)| = |Z(k)|$.*

Proof. *(i)* If $x \in [L(k'),L(k')]$ for some finite extension k'/k then $x \in [L,L](k') \cap G(k)$ so $x \in [L,L](k)$. On the other hand, if $x \in [L,L](k)$ then $x \in [L,L](\bar{k})$ so $x \in [L(\bar{k}),L(\bar{k})]$. In other words, there exists $a_1,b_1,\dots,a_r,b_r \in L(\bar{k})$ such that $x = [a_1,b_1]\dots[a_r,b_r]$. However, $L(\bar{k})$ equals the union of all $L(k')$ as k' ranges over all finite extensions of k, proving the result.

(ii) If L is not isolated with respect to G then there exists a proper Levi M of G containing L. Since S is generic, we must have $S \notin [M,M]$ so $S \notin [L,L]$. On the other hand, if L is isolated with respect to G then Proposition 21 says $T \cap [L,L]$ and $T \cap [G,G]$ are equal. The latter contains S so $S \in [L,L]$.

(iii) This follows from Proposition 20 and connectedness of Z. □

Corollary 54. *If $S \in T$ is strongly regular and generic then*

$$\alpha_{L,S}(q) = |Z(\mathbb{F}_q)| \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L,L')\, \pi_0^{L'},$$

where the sum is over all isolated endoscopy groups L' of G containing L.

Corollary 55. *If $\mathcal{C} = (C_1,\dots,C_n)$ is a generic collection of strongly regular conjugacy classes then*

$$S_\tau(q) = |Z(\mathbb{F}_q)|\, \dim(\tilde{\rho})^n |[L]| \left(\frac{|W|}{|W(L)|}\right)^{n-1} \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L,L')\pi_0^{L'}.$$

Proof. This follows from Proposition 47. □

Corollary 56. *If S is generic then $S_\tau(q)$ is independent of S.*

We conclude this section by using our formula to compute two important examples of $S_\tau(q)$:

(i) If τ is the G-type $[G,\mathrm{triv}]$ then $\dim(\tilde{\rho}) = 1$, $|[L]| = 1$ and $|W(L)| = |W|$. Moreover, the only endoscopy group of G containing G is G, so we only need to compute $\mu(G,G) = 1$ and

$$\pi_0^G = |\pi_0(\check{T}^W)| = |\pi_0(Z(\check{G})|.$$

Therefore

$$S_\tau(q) = |\pi_0(Z(\check{G})|\,|Z(\mathbb{F}_q)|.$$

(ii) We saw in §4.3 that choosing a GL_n-type $\tau = [L,\rho]$ is the same as choosing a tuple of partitions $(\lambda_1,\dots,\lambda_r)$ with $n = |\lambda_1| + \cdots + |\lambda_r|$; the endoscopy L is of the form $\mathrm{GL}_{|\lambda_1|} \times \cdots \times \mathrm{GL}_{|\lambda_r|}$ and the unipotent character ρ is of the form $\lambda_1 \otimes \cdots \otimes \lambda_r$. Then

(a) $\dim(\tilde{\rho}) = \dim(\tilde{\lambda_1}) \times \cdots \times \dim(\tilde{\lambda_r})$ and $\dim(\tilde{\lambda_i})$ is the dimension of the $S_{|\lambda_i|}$-character labelled by λ_i, given by the well-known hook length formula [Mac95, I, 7.]

$$\dim(\tilde{\lambda_i}) = \frac{|\lambda_i|!}{\prod_{h \in H(\lambda_i)} h}$$

where $H(\lambda_i)$ is the multi-set of hook lengths of the partition λ_i, *(b)* The Weyl group $W(L)$ is isomorphic to $S_{|\lambda_1|} \times \cdots \times S_{|\lambda_r|}$,

(c) The orbit size $|[L]|$ equals

$$\frac{|W|}{|N_W(W(L))|} = \frac{n!}{|N_{S_n}(S_{|\lambda_1|} \times \cdots \times S_{|\lambda_r|})|} = \frac{n!}{\prod_i m_i!\,(i!)^{m_i}},$$

where m_i equals the number of λ_j's equal to i [DH93, p. 1545],

(d) The only isolated endoscopy group of GL_n containing T is GL_n itself,

(e) $\mu(L, \mathrm{GL}_n) = (-1)^{r-1}(r-1)!$ [DH93, Theorem 1], and

(f) The centre of GL_n is connected so $\pi_0^{\mathrm{GL}_n} = 1$.

Therefore

$$S_\tau(q) = (-1)^{r-1}(r-1)!\,\frac{|\lambda_1|! \times \cdots \times |\lambda_r|!}{\prod_i m_i!\,(i!)^{m_i}} \left(\frac{n!}{\prod_{h \in H(\lambda_1) \cup \cdots \cup H(\lambda_r)} h}\right)^n (q-1).$$

This is the regular case of [HLRV11, Theorem 4.3.1 (1)].

Chapter 7

Proofs of main results

Recall the character variety is the GIT quotient

$$\mathbf{X} := \mathbf{R}/\!\!/(G/Z) = \mathbf{R}/\!\!/G$$

and the character stack is the quotient stack

$$\mathfrak{X} := [\mathbf{R}/(G/Z)].$$

In this chapter, we prove our main results about $\mathfrak{X}$ and $\mathbf{X}$ when $\mathcal{C}$ is a collection of strongly regular conjugacy classes (which are sometimes chosen generically). Specifically, we

(i) Prove $\mathfrak{X}$ is potentially rational count and calculate its counting function in Theorem 57,

(ii) Calculate the dimension and number of components of $\mathfrak{X}$ in Theorem 59,

(iii) Prove $\mathfrak{X}$ and $\mathbf{X}$ have the same point-count when conjugacy classes are generic in Theorem 60,

(iv) Give the simplified counting polynomial of $\mathfrak{X}$ and $\mathbf{X}$ in Theorem 62,

(v) Calculate the dimension and number of components of $\mathbf{X}$ in Theorem 63,

(vi) Calculate the Euler characteristic of $\mathbf{X}$ in Theorem 64, Theorem 65 and Theorem 66, and

(vii) Prove the counting polynomial of $\mathbf{X}$ is palindromic in Theorem 68.

7.1 Counting functions for $\mathfrak{X}$

In this section, we prove Theorem 2, restated here in detail:

Theorem 57. *The character stack $\mathfrak{X}$ is potentially rational count with counting function*

$$\|\mathfrak{X}\|(q) = \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q)$$

with notation given below. Moreover, if $g \geq 1$ then $\mathfrak{X}$ is potentially polynomial count.

In the above theorem, we have:

(i) $\|Z\|(q) = |Z(\mathbb{F}_q)| = (q-1)^{\dim(Z)}$ is the counting polynomial of the centre of G,

(ii) $\|T\|(q) = |T(\mathbb{F}_q)| = (q-1)^{\dim(T)}$ is the counting polynomial of the maximal split torus T,

(iii) $\mathcal{T}(G)$ is the set of types of G, i.e., the W-orbits of pairs (L,ρ) where L is an endoscopy group of G containing T and ρ is a principal unipotent character of $L(\mathbb{F}_q)$,

(iv) For a type $\tau = [L,\rho]$, we have

$$\|\tau\|(q) := q^{|\Phi(G)^+|-|\Phi(L)^+|}\frac{\|L\|(q)}{\|\rho\|(q)}$$

and

$$S_\tau(q) = \frac{\dim(\tilde{\rho})^n}{|W(L)|^{n-1}}\frac{|[L]|}{|W|}\sum_{\underline{w}\in W^n}\alpha_{L,\underline{w}\cdot\underline{S}}(q),$$

where:

(a) $\Phi(G)^+$ and $\Phi(L)^+$ are the positive roots of G and L, respectively,

(b) $\|L\|(q) = |L(\mathbb{F}_q)|$ is the counting polynomial of L,

(c) $\|\rho\|(q) = \rho(1)$ is the degree polynomial of ρ,

(d) $W(L)$ is the Weyl group of L,

(e) $\tilde{\rho}$ is the character of $W(L)$ corresponding to the principal unipotent character ρ of $L(\mathbb{F}_q)$,

(f) $[L]$ is the W-orbit of L arising from the W-action on Φ,

(g) $\underline{w}\cdot\underline{S} := (w_1\cdot S_1)\cdots(w_n\cdot S_n)$ with $w\cdot S := \dot{w}S\dot{w}^{-1}$,

(h) $\alpha_{L,S}(q) = \sum_{\substack{L'\supseteq L\\ S\in[L'(k),L'(k)]}} \mu(L,L')\,\pi_0^{L'}\,|Z(L')^\circ(k)|$,

(i) μ is the Möbius function on the poset of endoscopy groups of G containing T, and

(j) $\pi_0^{L'} = |\pi_0(\check{T}^{W(L')})(k)|$ is the number of components of the finite étale k-group scheme $\check{T}^{W(L')}$.

Proof. Proposition 30 and Proposition 47 imply

$$|\mathfrak{X}(\mathbb{F}_q)| = \frac{\|Z\|(q)}{\|T\|(q)^n}\sum_{\tau\in\mathcal{T}(G)}\|\tau\|(q)^{2g-2+n}S_\tau(q),$$

where $S_\tau(q)$ is given by the formula above. The functions $\|Z\|(q)$, $\|T\|(q)$, $\|\tau\|(q)$ and $S_\tau(q)$ are polynomials, and we explained in §6.3 why this formula becomes stable under base change after passing to a large enough field extension, so $\mathfrak{X}$ is potentially rational count. Moreover, if $g\geq 1$ then $\|T\|(q)^n$ divides $\|\tau\|(q)^{2g-2+n}$ [GM20, Remark 2.3.27] so $\mathfrak{X}$ is potentially polynomial count. □

7.2 Dimension and components of $\mathfrak{X}$

In this section, we determine the degree and leading coefficient of $\|\mathfrak{X}\|(q)$. Ultimately, we need the following result. Suppose $S \in T$ and L and L' are endoscopy groups of G containing T. Then define

$$Q_{L,L',S}(q) := P_{W(L)}(q)^{2g-2+n}\Delta_{L',S}(q),$$

where $P_{W(L)}(q)$ is the Poincaré polynomial of $W(L)$ and we recall from §6.3 that

$$\Delta_{L,S}(q) := \sum_{\substack{\theta \in T(\mathbb{F}_q)^\vee \\ W_\theta \supseteq W(L)}} \theta(S).$$

Proposition 58 (Proposition 33 of [KNP23]). *Suppose $2g-2+n \geq 2$. Then $\deg Q_{L,L',S}$ is maximal if and only if $L = L' = G$.*

Later in §7.5, by choosing conjugacy classes generically, we lower this to $2g-2+n \geq 1$ in accordance with Assumption 4. We now prove Theorem 5, restated here:

Theorem 59. *If $2g-2+n \geq 2$ then the character stack is non-empty of dimension*

$$\dim(\mathfrak{X}) = (2g-2+n)\dim(G) + 2\dim(Z) - n\,\mathrm{rank}(G)$$

with number of components equal to

$$|\pi_0(\mathfrak{X})| = |\pi_0(Z(\check{G}))|$$

where $Z(\check{G})$ is the centre of the Langlands dual group $\check{G}$.

Proof. We claim only $\tau = [G, \mathrm{triv}]$ contributes to the top degree of $\|\mathfrak{X}\|$. Assuming the claim, calculating the degree and leading coefficient of

$$\frac{\|Z\|(q)}{\|T\|(q)^n}\|\tau\|(q)^{2g-2+n}S_\tau(q)$$

when $\tau = [G, \mathrm{triv}]$ is straightforward and completes the proof. To prove the claim, fix a G-type $\tau = [L, \rho]$. In view of §7.1, the degree of $S_\tau(q)$ does not depend on ρ. Moreover, from §3.3, we have

$$\deg\|\tau\| = |\Phi(G)^+| + \dim(T) + \deg P_{W(L)} - \deg\|\rho\|$$

which is maximised if and only if $\rho = \mathrm{triv}$ [GM20, Proposition 4.5.9]. This means the only types contributing to the top degree of $\|\mathfrak{X}\|(q)$ are of the form $\tau = [L, \mathrm{triv}]$. Assuming $\tau = [L, \mathrm{triv}]$, we have

$$\|\tau\|(q) = q^{|\Phi(G)^+|}(q-1)^{\dim(T)}P_{W(L)}(q).$$

From §7.1 and §6.3, we have

$$S_\tau(q) = \sum_{\underline{w} \in W^n}\sum_{L' \supseteq L} C(G, L, L')\,\Delta_{L', \underline{w}\cdot\underline{S}}(q)$$

where the sum is over all endoscopy groups L' of G containing L and $C(G, L, L')$ is some rational number dependent only on the root datum of G, L and L' and not on q. Therefore

$$\|\tau\|(q)^{2g-2+n}S_\tau(q) = F(q)\sum_{\underline{w} \in W^n}\sum_{L' \supseteq L} C(G, L, L')\,\underbrace{P_{W(L)}(q)^{2g-2+n}\,\Delta_{L', \underline{w}\cdot\underline{S}}(q)}_{Q_{L,L',\underline{w}\cdot\underline{S}}(q)}$$

where $F(q)$ is independent of L. Applying Proposition 58 completes the proof of the claim. $\square$

7.3 $\mathfrak{X}$ and $\mathbf{X}$ have the same point-count

In this section, we prove Theorem 7, restated here:

Theorem 60. *If $\mathcal{C}$ is generic then:*

(i) *G/Z acts on $\mathbf{R}$ with finite étale stabilisers,*

(ii) *$\mathbf{R}$ is smooth and equidimensional,*

(iii) *$\mathfrak{X}$ is a smooth Deligne-Mumford stack,*

(iv) *$\mathbf{X}$ is the coarse moduli space for $\mathfrak{X}$, and*

(v) *$\mathfrak{X}$ and $\mathbf{X}$ have the same number of points over finite fields.*

Proof. *(i)* This is Corollary 44.

(ii) This follows from an application of the Regular Value Theorem and is already known when the G/Z-action is free [KNP23, Theorem 1]. In our setting, the G/Z-action is not necessarily free, but the aforementioned proof still works with one modification. The key difference is G/Z acting freely on $\mathbf{R}$ implies $\mathrm{Stab}_G(p) = Z$ for each $p \in \mathbf{R}$ (because $\mathrm{Stab}_{G/Z}(p) = \mathrm{Stab}_G(p)/Z$) and this is used in [KNP23, §2.1.7] to conclude $\mathrm{Lie}(\mathrm{Stab}_G(p)) = \mathrm{Lie}(Z) = 0$. In our setting, we have $\mathrm{Lie}(\mathrm{Stab}_G(p)/Z) = \mathrm{Lie}(\mathrm{Stab}_{G/Z}(p)) = 0$ by Corollary 44.

(iii) In view of [Ols16, Remark 8.3.4], Corollary 44 implies $\mathfrak{X}$ is a Deligne–Mumford stack. Since $\mathbf{R}$ and G/Z are smooth, the smoothness of $\mathfrak{X}$ follows, cf. [Ols16, §8.2].

(iv) To identify the coarse moduli space, we show all orbits of the G/Z-action on $\mathbf{R}$ are closed (this implies, for every algebraically closed field K, the map $\mathfrak{X}(K) \to \mathbf{X}(K)$ is bijective, cf. [Ols16, Definition 11.1.1]). Observe the action map

$$G/Z \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}, \quad (gZ, p) \mapsto (g \cdot p, p)$$

is proper by [MFK94, Proposition 0.8] and Corollary 44. Proper maps are closed so the image of $G/Z \times \{x\}$ is closed. This image is $\mathrm{Orb}_{G/Z}(x) \times \{x\}$ so the orbit $\mathrm{Orb}_{G/Z}(x)$ is closed.

(v) Let $\mathfrak{I} := \mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$ be the inertia stack associated to $\mathfrak{X}$. A key result attributed to Serre [Beh91, Corollary 2.3.4] says if the morphism $\mathfrak{I} \to \mathfrak{X}$ is étale then $|\mathfrak{X}(\mathbb{F}_q)| = |\mathbf{X}(\mathbb{F}_q)|$. This means it suffices to show the fibres of $\mathfrak{I} \to \mathfrak{X}$ are étale (cf. [Noo04, §2]), and these are the stabiliser group schemes $\mathrm{Stab}_{G/Z}(p)$ where $p \in \mathbf{R}$, which were shown to be étale in Corollary 44. □

Corollary 61. *If $\mathcal{C}$ is generic then*

$$|\mathbf{X}(\mathbb{F}_q)| = |\mathfrak{X}(\mathbb{F}_q)| = \frac{|\mathbf{R}(\mathbb{F}_q)|}{|(G/Z)(\mathbb{F}_q)|} = \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathfrak{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q).$$

7.4 Counting polynomials for X

In this section, we prove Theorem 8, restated here in detail:

Theorem 62. *If $\mathcal{C}$ is generic then* **X** *is potentially polynomial count with counting polynomial*

$$\|\mathbf{X}\|(q) = \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q).$$

Above, the notation is the same as Theorem 57, except

$$S_\tau(q) = |Z(\mathbb{F}_q)| \dim(\tilde{\rho})^n |[L]| \left(\frac{|W|}{|W(L)|} \right)^{n-1} \nu(L),$$

where

$$\nu(L) := \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L') \pi_0^{L'}$$

is a sum over all isolated endoscopy groups L' of G containing L.

Proof. Corollary 55, Theorem 57 and Corollary 61 imply

$$|\mathbf{X}(\mathbb{F}_q)| = \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q).$$

The functions $\|Z\|(q)$, $\|T\|(q)$, $\|\tau\|(q)$ and $S_\tau(q)$ are polynomials, and we explained in §6.3 why this formula becomes stable under base change after passing to a large enough field extension, so **X** is potentially polynomial count. □

7.5 Dimension and components of X

In this section, we prove Theorem 9, restated here:

Theorem 63. *If $\mathcal{C}$ is generic then the character variety is non-empty of dimension*

$$\dim(\mathbf{X}) = (2g-2+n)\dim(G) + 2\dim(Z) - n\,\mathrm{rank}(G)$$

with number of components equal to

$$|\pi_0(\mathbf{X})| = |\pi_0(Z(\check{G}))|,$$

where $Z(\check{G})$ is the centre of the Langlands dual group $\check{G}$.

Proof. We claim only $\tau = [G, \mathrm{triv}]$ contributes to the top degree of $\|\mathbf{X}\|$. The degree of $S_\tau(q)$ is always $\dim(Z)$ by Corollary 55, so we just maximise $\deg \|\tau\|$. From §3.3, this equals

$$\deg \|\tau\| = |\Phi(G)^+| - |\Phi(L)^+| + \dim(L) - \deg \|\rho\| = |\Phi(G)^+| + \dim(T) + \deg P_{W(L)} - \deg \|\rho\|$$

and is maximised if and only if $\tau = [G, \mathrm{triv}]$ [GM20, Proposition 4.5.9]. The degree and leading coefficient of $\|\mathbf{X}\|$ follows, noting that if $\tau = [G, \mathrm{triv}]$ then the leading coefficient of

$$\frac{\|Z\|(q)}{\|T\|(q)^n} \|\tau\|(q)^{2g-2+n} S_\tau(q)$$

is $\pi_0^G = |\pi_0(Z(\check{G}))|$. □

7.6 Euler characteristic

In this section, we prove Theorem 10 which concerns the Euler characteristic of $\mathbf{X}$.

Theorem 64. *Suppose $\mathcal{C}$ is generic. If either $g > 1$, or $g > 0$ and $\dim(Z) > 0$, then $\chi(\mathbf{X}) = 0$.*

Proof. We rewrite

$$\|\mathbf{X}\|(q) = \|Z\|(q) \sum_{\tau \in \mathcal{T}(G)} \|\tau\|(q)^{2g-2} \left(\frac{\|\tau\|(q)}{\|T\|(q)} \right)^n S_\tau(q).$$

We saw in the proof of Theorem 57 that $\|T\|(q)$ divides $\|\tau\|(q)$ so $\|Z\|(q)$, $\|\tau\|(q)^{2g-2}$, $\frac{\|\tau\|(q)}{\|T\|(q)}$ and $S_\tau(q)$ are all polynomials as long as $g \geq 1$. If $g > 1$ then $2g-2 > 0$ and one checks $\|\tau\|(1)^{2g-2} = 0$ so $\chi(\mathbf{X}) = \|\mathbf{X}\|(1) = 0$. Similarly, if $g = 1$ and $\dim(Z) > 0$ then $\|Z\|(1) = 0$ so $\chi(\mathbf{X}) = \|\mathbf{X}\|(1) = 0$. □

Theorem 65. *Suppose $\mathcal{C}$ is generic. If $g = 1$ and $\dim(Z) = 0$ then*

$$\chi(\mathbf{X}) = |W|^{n-1} \sum_L |W(L)|\, |\mathrm{Irr}(W(L))|\, \nu(L),$$

where the sum is over all endoscopy groups L of G containing T, and

$$\nu(L) := \sum_{\substack{G \supseteq L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L')\, \pi_0^{L'}.$$

In the definition of $\nu(L)$, the sum is over all isolated endoscopy groups of G containing L.

Proof. Expanding the sum over τ and rearranging yields

$$\|\mathbf{X}\|(q) = \sum_{[L]} q^{|\Phi(G)^+| - |\Phi(L)^+|} \left(\frac{\|L\|(q)}{\|T\|(q)} \right)^n |[L]| \left(\frac{|W|}{|W(L)|} \right)^{n-1} \nu(L) \sum_\rho \left(\frac{\dim(\tilde{\rho})}{\|\rho\|(q)} \right)^n$$

where the first sum is over all endoscopy groups L of G containing T and the second sum is over all principal unipotent characters of $L(\mathbb{F}_q)$. In view of §3.3, we have

$$\frac{\|L\|(q)}{\|T\|(q)} = q^{|\Phi(L)^+|} P_{W(L)}(q)$$

which equals $|W(L)|$ when evaluated at $q = 1$. We also have $\|\rho\|(1) = \dim(\tilde{\rho})$ [GM20, p. 231] so $\sum_\rho (\frac{\dim(\tilde{\rho})}{\|\rho\|(q)})^n$ evaluated at $q = 1$ equals $|\mathrm{Irr}(W(L))|$. Therefore

$$\|\mathbf{X}\|(1) = \sum_{[L]} |W(L)|^n |[L]| \left(\frac{|W|}{|W(L)|} \right)^{n-1} \nu(L)\, |\mathrm{Irr}(W(L))|.$$

□

Theorem 66. *Suppose $\mathcal{C}$ is generic. If $g = 0$ and $n \geq 3$ then*

$$\chi(\mathbf{X}) = \frac{1}{(2r)!} \frac{d^{2r}}{dq^{2r}} \bigg|_{q=1} \xi(q),$$

where $2r := 2\dim(T) - 2\dim(Z)$ is twice the semisimple rank of G, and

$$\xi(q) := q^{|\Phi(G)^+|(n-2)} \sum_L \nu(L) \left(\frac{|W|}{|W(L)|} \right)^{n-1} \sum_\rho \dim(\tilde{\rho})^n \left(\frac{P_{W(L)}(q)}{\|\rho\|(q)} \right)^{n-2}.^{1}$$

In the definition of $\xi(q)$, the first sum is over all endoscopy groups L of G containing T, and the second sum is over all principal unipotent characters of $L(\mathbb{F}_q)$.

[1] All derivatives of $\xi(q)$ exist at $q = 1$ because it is a rational function defined at $q = 1$.

Proof. Suppose $g = 0$ and $n \geq 3$. Expanding the sum over τ and rearranging yields

$$\|\mathbf{X}\|(q)(q-1)^{2r} = \xi(q).$$

Differentiating $2r$ times and evaluating at $q = 1$ yields the Euler characteristic. $\square$

Some examples of $\chi(\mathbf{X})$ are provided in §A.7 and §A.8. It would be interesting to understand the function $\xi(q)$ as it governs the Euler characteristic of character varieties associated to punctured spheres. We do not know any interpretation of this function.

7.7 Palindromicity

In this section, we prove Theorem 11 which says $\|\mathbf{X}\|(q)$ is palindromic. Before doing so, we must relate the types of χ and its Alvis–Curtis dual $D_G(\chi)$:

Proposition 67. *Suppose $\chi \in R_T^G\theta$ has type $\tau = [L,\rho]$ and ρ is matched with $\phi \in \mathrm{Irr}(W(L))$ according to Proposition 26. Then $D_G(\chi)$ has type $[L, D_L(\rho)]$ where $D_L(\rho)$ is matched with $\phi \otimes \mathrm{sgn} \in \mathrm{Irr}(W(L))$. Hence, there is an involution*

$$D_G \colon \mathcal{T}(G) \to \mathcal{T}(G), \quad [L,\rho] \mapsto [L, D_L(\rho)].$$

Proof. This is a consequence of Proposition 27. In particular, if χ is a summand in $R_T^G\theta$ then $D_G(\chi)$ is too. Therefore, if χ has type $[L,\rho]$ then $D_G(\chi)$ has type $[L,\rho']$ where the unipotent character ρ'. Specifically, if ρ is matched with ϕ according to Proposition 26 then ρ' is matched with $\phi \otimes \mathrm{sgn}$. This is an involution since $\mathrm{sgn} \otimes \mathrm{sgn} = \mathrm{triv}$. $\square$

We are now ready to prove Theorem 11, restated here:

Theorem 68. *If $\mathcal{C}$ is generic then $\|\mathbf{X}\|$ is a palindromic polynomial; i.e.,*

$$\|\mathbf{X}\|(q) = q^{\dim(\mathbf{X})}\|\mathbf{X}\|(1/q).$$

Proof. We have $\dim(\mathbf{X}) = (2g-2+n)\dim(G) - 2\dim(Z) + n\,\mathrm{rank}(G)$ so we just need to understand $\|\mathbf{X}\|(1/q)$. Using formulas from §3.3, the following identities are straightforward:

$$\begin{aligned}
\|Z\|(1/q) &= (-1)^{\dim(Z)}\, q^{-\dim(Z)}\, \|Z\|(q), \\
\|T\|(1/q) &= (-1)^{\dim(T)}\, q^{-\dim(T)}\, \|T\|(q), \\
\|L\|(1/q) &= (-1)^{\dim(T)}\, q^{-\dim(T)-3|\Phi(L)^+|}\, \|L\|(q).
\end{aligned}$$

Next, by Proposition 27 and Proposition 67, if $\tau = [L,\rho]$ is a G-type then $D_G(\tau) = [L, D_L(\rho)]$ and

$$\|\tau\|(1/q) = (-1)^{\dim(T)}\, q^{-\dim(T)-|\Phi(G)|}\, \|D_G(\tau)\|(q).$$

Lastly, recall from Corollary 55 the formula

$$S_\tau(q) = |Z(\mathbb{F}_q)|\, \dim(\tilde{\rho})^n\, |[L]| \left(\frac{|W|}{|W(L)|}\right)^{n-1} \nu(L).$$

Then $S_{D_G(\tau)}(q) = S_\tau(q)$ since $\dim(\tilde{\rho} \otimes \mathrm{sgn}) = \dim(\tilde{\rho})$. Therefore

$$S_\tau(1/q) = (-1)^{\dim(Z)} q^{-\dim(Z)} S_{D_G(\tau)}(q),$$

allowing us to compute

$$\|\mathbf{X}\|(1/q) = q^{-\dim(\mathbf{X})} \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}(G)} \|D_G(\tau)\|(q)^{2g-2+n} S_{D_G(\tau)}(q).$$

The result follows since $D_G\colon \mathcal{T}(G) \to \mathcal{T}(G)$ is an involution and therefore a bijection. □

7.8 Consistency checks

In this section, we prove $\|\mathbf{X}\|(q) = 0$ when $(g,n) = (0,1)$ or $(0,2)$ using our formula for $\|\mathbf{X}\|(q)$ in Theorem 62. This follows from the following observation:

Lemma 69. *Suppose L is an endoscopy group of G containing T and recall the sum*

$$\nu(L) := \sum_{\substack{G \supseteq L' \supseteq L \\ L' \text{ isolated}}} \mu(L,L')\, \pi_0^{L'}$$

over all isolated endoscopy groups L' of G containing L. Then the sum

$$\sum_{G \supseteq L \supseteq T} \nu(L)$$

over all endoscopy groups L of G containing T is equal to zero.

Proof. We rearrange

$$\sum_{G \supseteq L \supseteq T} \nu(L) = \sum_{\substack{G \supseteq L' \supseteq L \\ L' \text{ isolated}}} \pi_0^{L'} \sum_{T \subseteq L \subseteq L'} \mu(L,L').$$

But the sum

$$\sum_{T \subseteq L \subseteq L'} \mu(L,L')$$

over all endoscopy groups L of G containing T and contained in L' is always zero. This is because sums of the form $\sum_{\substack{x \in P \\ x \leq m}} \mu(x,m)$ where m is the maximal element of a finite poset P are always zero. In our case, P is the poset of all endoscopy groups of G containing T and contained in L'. □

Proposition 70. *If $(g,n) = (0,1)$ or $(0,2)$ then $\|\mathbf{X}\|(q) = 0$.*

Proof. If $(g,n) = (0,1)$ then

$$\|\mathbf{X}\|(q) = \frac{\|Z\|(q)}{\|T\|(q)} \sum_{\tau \in \mathcal{T}(G)} \frac{S_\tau(q)}{\|\tau\|(q)}.$$

Expanding the sum over τ gives

$$\|\mathbf{X}\|(q) = \frac{\|Z\|(q)^2}{\|T\|(q) q^{|\Phi(G)^+|}} \sum_L \frac{q^{|\Phi(L)^+|} \nu(L)}{\|L\|(q)} \sum_\rho \dim(\tilde{\rho}) \|\rho\|(q),$$

where the first sum is over all endoscopy groups L of G containing T, and the second sum is over principal unipotent characters ρ of $L(\mathbb{F}_q)$. By Proposition 26, the expression $\sum_\rho \dim(\tilde{\rho})\|\rho\|(q)$ is equal to $\dim(R_T^L 1)$ which equals

$$P_{W(L)}(q) = \frac{\|L\|(q)}{q^{|\Phi(L)^+|}\|T\|(q)}.$$

Therefore

$$\|\mathbf{X}\|(q) = \frac{\|Z\|(q)^2}{\|T\|(q)^2 q^{|\Phi(G)^+|}} \sum_L \nu(L) = 0$$

by Lemma 69. The case $(g,n) = (0,2)$ is handled the same way but the expression $\sum_\rho \dim(\tilde{\rho})\|\rho\|(q)$ is replaced with $\sum_\rho \dim(\tilde{\rho})^2$ which equals $|W(L)|$ since it is the sum of the squared dimensions of the irreducible characters of a finite group [EGH+11, Theorem 4.1.1]. $\square$

Appendix A

Examples

In this appendix, we give several examples of counting polynomials and Euler characteristics of character varieties using our main theorems from §2 and §7.

We state some notation used in the following tables. The ith cyclotomic polynomial is denoted Φ_i; in particular

$$\Phi_1 = q-1,\ \Phi_2 = q+1,\ \Phi_3 = q^2+q+1,\ \Phi_4 = q^2+1 \text{ and } \Phi_6 = q^2-q+1.$$

The copies of A_1 with longer and shorter roots are denoted A_1 and A_1', respectively. The unipotent characters in type A_{r-1} are labelled by partitions; in particular, r^1 is the trivial character and 1^r is the Steinberg character. The unipotent characters in type B_2 are labelled using Lusztig's symbols [GM20, §4.4]; in particular, $\binom{2}{}$ is the trivial character and $\binom{0\,1\,2}{1\,2}$ is the Steinberg character. The unipotent characters in type G_2 are in the notation of [Car93]; in particular, $\phi_{1,0}$ is the trivial character and $\phi_{1,6}$ is the Steinberg character.

A.1 $\|\mathbf{X}\|$ when $G = \mathrm{GL}_2$

The following table contains the data required to compute $\|\mathbf{X}\|(q)$ using Theorem 62:

$\tau = [L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\lvert L(\mathbb{F}_q)\rvert$	$\rho(1)$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	π_0^L	$\nu(L)$	$\|\tau\|(q)$	$S_\tau(q)$
$[\mathrm{GL}_2, 2^1]$	1	$q\Phi_1^2\Phi_2$	1	1	2	1	1	1	$q\Phi_1^2\Phi_2$	Φ_1
$[\mathrm{GL}_2, 1^2]$	1	$q\Phi_1^2\Phi_2$	q	1	2	1	1	1	$\Phi_1^2\Phi_2$	Φ_1
$[T, \mathrm{triv}]$	0	Φ_1^2	1	1	1	1		-1	$q\Phi_1^2$	$-2^{n-1}\Phi_1$

Table A.1: The three GL_2-types.

From the table, we have

$$\begin{aligned}\|\mathbf{X}\|(q) &= \frac{1}{\Phi_1^{2n-2}}\Big[\underbrace{\Phi_1(q\Phi_1^2\Phi_2)^{2g-2+n}}_{[\mathrm{GL}_2,2^1]}+\underbrace{\Phi_1(\Phi_1^2\Phi_2)^{2g-2+n}}_{[\mathrm{GL}_2,1^2]}+\underbrace{-2^{n-1}\Phi_1(q\Phi_1^2)^{2g-2+n}}_{[T,1^1]}\Big]\\ &= \underbrace{q^{2g-2+n}\Phi_1^{4g-3}\Phi_2^{2g-2+n}}_{[\mathrm{GL}_2,2^1]}+\underbrace{\Phi_1^{4g-3}\Phi_2^{2g-2+n}}_{[\mathrm{GL}_2,1^2]}+\underbrace{-2^{n-1}q^{2g-2+n}\Phi_1^{4g-3}}_{[T,1^1]}.\end{aligned}$$

A.2 $\|\mathbf{X}\|$ when $G = \mathrm{GL}_3$

The following table contains the data required to compute $\|\mathbf{X}\|(q)$ using Theorem 62:

$\tau = [L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\lvert L(\mathbb{F}_q)\rvert$	$\rho(1)$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	π_0^L	$\nu(L)$	$\|\tau\|(q)$	$S_\tau(q)$
$[\mathrm{GL}_3, 3^1]$	3	$q^3\Phi_1^3\Phi_2\Phi_3$	1	1	6	1	1	1	$q^3\Phi_1^3\Phi_2\Phi_3$	Φ_1
$[\mathrm{GL}_3, 2^1 1^1]$	3	$q^3\Phi_1^3\Phi_2\Phi_3$	$q\Phi_2$	2	6	1	1	1	$q^2\Phi_1^3\Phi_3$	$2^n\Phi_1$
$[\mathrm{GL}_3, 1^3]$	3	$q^3\Phi_1^3\Phi_2\Phi_3$	q^3	1	6	1	1	1	$\Phi_1^3\Phi_2\Phi_3$	Φ_1
$[A_1, 2^1]$	1	$q\Phi_1^3\Phi_2$	1	1	2	3		-1	$q^3\Phi_1^3\Phi_2$	$-3^n\Phi_1$
$[A_1, 1^2]$	1	$q\Phi_1^3\Phi_2$	q	1	2	3		-1	$q^2\Phi_1^3\Phi_2$	$-3^n\Phi_1$
$[T, \mathrm{triv}]$	0	Φ_1^3	1	1	1	1		2	$q^3\Phi_1^3$	$\frac{1}{3}6^n\Phi_1$

Table A.2: The six GL_3-types.

From the table, we have

$$
\begin{aligned}
\|\mathbf{X}\|(q) &= \frac{1}{\Phi_1^{3n-3}}\Bigg[\underbrace{\Phi_1(q^3\Phi_1^3\Phi_2\Phi_3)^{2g-2+n}}_{[\mathrm{GL}_3,3^1]} + \underbrace{2^n\Phi_1(q^2\Phi_1^3\Phi_3)^{2g-2+n}}_{[\mathrm{GL}_3,2^2 1^1]} + \underbrace{\Phi_1(\Phi_1^3\Phi_2\Phi_3)^{2g-2+n}}_{[\mathrm{GL}_3,1^3]} \\
&\qquad + \underbrace{-3^n\Phi_1(q^3\Phi_1^3\Phi_2)^{2g-2+n}}_{[A_1,2^1]} + \underbrace{-3^n\Phi_1(q^2\Phi_1^3\Phi_2)^{2g-2+n}}_{[A_1,1^2]} + \underbrace{\frac{1}{3}6^n\Phi_1(q^3\Phi_1^3)^{2g-2+n}}_{[T,\mathrm{triv}]}\Bigg] \\
&= \underbrace{q^{6g-6+3n}\Phi_1^{6g-3}\Phi_2^{2g-2+n}\Phi_3^{2g-2+n}}_{[\mathrm{GL}_3,3^1]} + \underbrace{2^n q^{4g-4+2n}\Phi_1^{6g-3}\Phi_3^{2g-2+n}}_{[\mathrm{GL}_3,2^2 1^1]} + \underbrace{\Phi_1^{6g-3}\Phi_2^{2g-2+n}\Phi_3^{2g-2+n}}_{[\mathrm{GL}_3,1^3]} \\
&\qquad + \underbrace{-3^n q^{6g-6+3n}\Phi_1^{6g-3}\Phi_2^{2g-2+n}}_{[A_1,2^1]} + \underbrace{-3^n q^{4g-4+2n}\Phi_1^{6g-3}\Phi_2^{2g-2+n}}_{[A_1,1^2]} + \underbrace{\frac{1}{3}6^n q^{6g-6+3n}\Phi_1^{6g-3}}_{[T,\mathrm{triv}]}.
\end{aligned}
$$

A.3 $\|\mathbf{X}\|$ when $G = \mathrm{PGL}_2$

The following table contains the data required to compute $\|\mathbf{X}\|(q)$ using Theorem 62:

$\tau = [L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\lvert L(\mathbb{F}_q)\rvert$	$\rho(1)$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	π_0^L	$\nu(L)$	$\|\tau\|(q)$	$S_\tau(q)$
$[\mathrm{PGL}_2, 2^1]$	1	$q\Phi_1\Phi_2$	1	1	2	1	2	2	$q\Phi_1\Phi_2$	2
$[\mathrm{PGL}_2, 1^2]$	1	$q\Phi_1\Phi_2$	q	1	2	1	2	2	$\Phi_1\Phi_2$	2
$[T, \mathrm{triv}]$	0	Φ_1	1	1	1	1	1	-2	$q\Phi_1$	-2^n

Table A.3: The three PGL_2-types.

From the table, we have

$$\begin{aligned}\|\mathbf{X}\|(q) &= \frac{1}{\Phi_1^n}\left[\underbrace{2(q\Phi_1\Phi_2)^{2g-2+n}}_{[\mathrm{PGL}_2,2^1]}+\underbrace{2(\Phi_1\Phi_2)^{2g-2+n}}_{[\mathrm{PGL}_2,1^2]}+\underbrace{-2^n(q\Phi_1)^{2g-2+n}}_{[T,1^1]}\right]\\ &= \underbrace{2q^{2g-2+n}\Phi_1^{2g-2}\Phi_2^{2g-2+n}}_{[\mathrm{PGL}_2,2^1]}+\underbrace{2\Phi_1^{2g-2}\Phi_2^{2g-2+n}}_{[\mathrm{PGL}_2,1^2]}+\underbrace{-2^nq^{2g-2+n}\Phi_1^{2g-2}}_{[T,1^1]}.\end{aligned}$$

A.4 $\|\mathbf{X}\|$ when $G = \mathrm{PGL}_3$

The following table contains the data required to compute $\|\mathbf{X}\|(q)$ using Theorem 62:

$\tau = [L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\lvert L(\mathbb{F}_q)\rvert$	$\rho(1)$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	π_0^L	$\nu(L)$	$\|\tau\|(q)$	$S_\tau(q)$
$[\mathrm{PGL}_3,3^1]$	3	$q^3\Phi_1^2\Phi_2\Phi_3$	1	1	6	1	3	3	$q^3\Phi_1^2\Phi_2\Phi_3$	3
$[\mathrm{PGL}_3,2^1 1^1]$	3	$q^3\Phi_1^2\Phi_2\Phi_3$	$q\Phi_2$	2	6	1	3	3	$q^2\Phi_1^2\Phi_3$	$3\cdot 2^n$
$[\mathrm{PGL}_3,1^3]$	3	$q^3\Phi_1^2\Phi_2\Phi_3$	q^3	1	6	1	3	3	$\Phi_1^2\Phi_2\Phi_3$	3
$[A_1,2^1]$	1	$q\Phi_1^2\Phi_2$	1	1	2	3		-3	$q^3\Phi_1^2\Phi_2$	-3^{n+1}
$[A_1,1^2]$	1	$q\Phi_1^2\Phi_2$	q	1	2	3		-3	$q^2\Phi_1^2\Phi_2$	-3^{n+1}
$[T,\mathrm{triv}]$	0	Φ_1^2	1	1	1	1		6	$q^3\Phi_1^2$	6^n

Table A.4: The six PGL_3-types.

From the table, we have

$$\begin{aligned}
\|\mathbf{X}\|(q) = \frac{1}{\Phi_1^{2n}}\Big[& \underbrace{3(q^3\Phi_1^2\Phi_2\Phi_3)^{2g-2+n}}_{[\mathrm{PGL}_3,3^1]} + \underbrace{3\cdot 2^n(q^2\Phi_1^2\Phi_3)^{2g-2+n}}_{[\mathrm{PGL}_3,2^21^1]} + \underbrace{3(\Phi_1^2\Phi_2\Phi_3)^{2g-2+n}}_{[\mathrm{PGL}_3,1^3]} \\
& + \underbrace{-3^{n+1}(q^3\Phi_1^2\Phi_2)^{2g-2+n}}_{[A_1,2^1]} + \underbrace{-3^{n+1}(q^2\Phi_1^2\Phi_2)^{2g-2+n}}_{[A_1,1^2]} + \underbrace{6^n(q^3\Phi_1^2)^{2g-2+n}}_{[T,\mathrm{triv}]}\Big] \\
= & \underbrace{3q^{6g-6+3n}\Phi_1^{4g-4}\Phi_2^{2g-2+n}\Phi_3^{2g-2+n}}_{[\mathrm{PGL}_3,3^1]} + \underbrace{3\cdot 2^n q^{4g-4+2n}\Phi_1^{4g-4}\Phi_3^{2g-2+n}}_{[\mathrm{PGL}_3,2^21^1]} + \underbrace{3\Phi_1^{4g-4}\Phi_2^{2g-2+n}\Phi_3^{2g-2+n}}_{[\mathrm{PGL}_3,1^3]} \\
& + \underbrace{-3^{n+1}q^{6g-6+3n}\Phi_1^{4g-4}\Phi_2^{2g-2+n}}_{[A_1,2^1]} + \underbrace{-3^{n+1}q^{4g-4+2n}\Phi_1^{4g-4}\Phi_2^{2g-2+n}}_{[A_1,1^2]} + \underbrace{6^n q^{6g-6+3n}\Phi_1^{4g-4}}_{[T,\mathrm{triv}]}.
\end{aligned}$$

A.5 $\|\mathbf{X}\|$ when $G = \mathrm{SO}_5$

The following table contains the data required to compute $\|\mathbf{X}\|(q)$ using Theorem 62:

$\tau = [L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\lvert L(\mathbb{F}_q)\rvert$	$\rho(1)$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	π_0^L	$\nu(L)$	$\|\tau\|(q)$	$S_\tau(q)$
$[\mathrm{SO}_5, \binom{2}{}]$	4	$q^4\Phi_1^2\Phi_2^2\Phi_4$	1	1	8	1	2	2	$q^4\Phi_1^2\Phi_2^2\Phi_4$	2
$[\mathrm{SO}_5, \binom{0\,1}{2}]$	4	$q^4\Phi_1^2\Phi_2^2\Phi_4$	$\frac{1}{2}q\Phi_4$	1	8	1	2	2	$2q^3\Phi_1^2\Phi_2^2$	2
$[\mathrm{SO}_5, \binom{1\,2}{0}]$	4	$q^4\Phi_1^2\Phi_2^2\Phi_4$	$\frac{1}{2}q\Phi_4$	1	8	1	2	2	$2q^3\Phi_1^2\Phi_2^2$	2
$[\mathrm{SO}_5, \binom{0\,2}{1}]$	4	$q^4\Phi_1^2\Phi_2^2\Phi_4$	$\frac{1}{2}q\Phi_2^2$	2	8	1	2	2	$2q^3\Phi_1^2\Phi_4$	2^{n+1}
$[\mathrm{SO}_5, \binom{0\,1\,2}{1\,2}]$	4	$q^4\Phi_1^2\Phi_2^2\Phi_4$	q^4	1	8	1	2	2	$\Phi_1^2\Phi_2^2\Phi_4$	2
$[A_1 \times A_1, 2^1 \otimes 2^1]$	2	$q^2\Phi_1^2\Phi_2^2$	1	1	4	1	4	2	$q^4\Phi_1^2\Phi_2^2$	2^n
$[A_1 \times A_1, 2^1 \otimes 1^2]$	2	$q^2\Phi_1^2\Phi_2^2$	q	1	4	1	4	2	$q^3\Phi_1^2\Phi_2^2$	2^n
$[A_1 \times A_1, 1^2 \otimes 2^1]$	2	$q^2\Phi_1^2\Phi_2^2$	q	1	4	1	4	2	$q^3\Phi_1^2\Phi_2^2$	2^n
$[A_1 \times A_1, 1^2 \otimes 1^2]$	2	$q^2\Phi_1^2\Phi_2^2$	q^2	1	4	1	4	2	$q^2\Phi_1^2\Phi_2^2$	2^n
$[A_1, 2^1]$	1	$q\Phi_1^2\Phi_2$	1	1	2	2		-4	$q^4\Phi_1^2\Phi_2$	$-2 \cdot 4^n$
$[A_1, 1^2]$	1	$q\Phi_1^2\Phi_2$	q	1	2	2		-4	$q^3\Phi_1^2\Phi_2$	$-2 \cdot 4^n$
$[A_1', 2^1]$	1	$q\Phi_1^2\Phi_2$	1	1	2	2		-2	$q^4\Phi_1^2\Phi_2$	-4^n
$[A_1', 1^2]$	1	$q\Phi_1^2\Phi_2$	q	1	2	2		-2	$q^3\Phi_1^2\Phi_2$	-4^n
$[T, \mathrm{triv}]$	0	Φ_1^2	1	1	1	1		8	$q^4\Phi_1^2$	8^n

Table A.5: The fourteen SO_5-types.

A.6 $\|\mathbf{X}\|$ when $G = G_2$

The following table contains the data required to compute $\|\mathbf{X}\|(q)$ using Theorem 62:

$\tau = [L, \rho]$	$\lvert\Phi(L)^+\rvert$	$\lvert L(\mathbb{F}_q)\rvert$	$\rho(1)$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	π_0^L	$\nu(L)$	$\|\tau\|(q)$	$S_\tau(q)$
$[G_2, \phi_{1,0}]$	6	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1	1	12	1	1	1	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1
$[G_2, \phi'_{1,3}]$	6	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{3}q\Phi_3\Phi_6$	1	12	1	1	1	$3q^5\Phi_1^2\Phi_2^2$	1
$[G_2, \phi''_{1,3}]$	6	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{3}q\Phi_3\Phi_6$	1	12	1	1	1	$3q^5\Phi_1^2\Phi_2^2$	1
$[G_2, \phi_{2,1}]$	6	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{6}q\Phi_2^2\Phi_3$	2	12	1	1	1	$6q^5\Phi_1^2\Phi_6$	2^n
$[G_2, \phi_{2,2}]$	6	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{2}q\Phi_2^2\Phi_6$	2	12	1	1	1	$2q^5\Phi_1^2\Phi_3$	2^n
$[G_2, \phi_{1,6}]$	6	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	q^6	1	12	1	1	1	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1
$[A_2, 3^1]$	3	$q^3\Phi_1^2\Phi_2\Phi_3$	1	1	6	1	3	2	$q^6\Phi_1^2\Phi_2\Phi_3$	2^n
$[A_2, 2^1 1^1]$	3	$q^3\Phi_1^2\Phi_2\Phi_3$	$q\Phi_2$	2	6	1	3	2	$q^5\Phi_1^2\Phi_3$	4^n
$[A_2, 1^3]$	3	$q^3\Phi_1^2\Phi_2\Phi_3$	q^3	1	6	1	3	2	$q^3\Phi_1^2\Phi_2\Phi_3$	2^n
$[A_1 \times A'_1, 2^1 \otimes 2^1]$	2	$q^2\Phi_1^2\Phi_2^2$	1	1	4	3	2	1	$q^6\Phi_1^2\Phi_2^2$	3^n
$[A_1 \times A'_1, 2^1 \otimes 1^2]$	2	$q^2\Phi_1^2\Phi_2^2$	q	1	4	3	2	1	$q^5\Phi_1^2\Phi_2^2$	3^n
$[A_1 \times A'_1, 1^2 \otimes 2^1]$	2	$q^2\Phi_1^2\Phi_2^2$	q	1	4	3	2	1	$q^5\Phi_1^2\Phi_2^2$	3^n
$[A_1 \times A'_1, 1^2 \otimes 1^2]$	2	$q^2\Phi_1^2\Phi_2^2$	q^2	1	4	3	2	1	$q^4\Phi_1^2\Phi_2^2$	3^n
$[A_1, 2^1]$	1	$q\Phi_1^2\Phi_2$	1	1	2	3		-4	$q^6\Phi_1^2\Phi_2$	$-2\cdot 6^n$
$[A_1, 1^2]$	1	$q\Phi_1^2\Phi_2$	q	1	2	3		-4	$q^5\Phi_1^2\Phi_2$	$-2\cdot 6^n$
$[A'_1, 2^1]$	1	$q\Phi_1^2\Phi_2$	1	1	2	3		-2	$q^6\Phi_1^2\Phi_2$	-6^n
$[A'_1, 1^2]$	1	$q\Phi_1^2\Phi_2$	q	1	2	3		-2	$q^5\Phi_1^2\Phi_2$	-6^n
$[T, \mathrm{triv}]$	0	Φ_1^2	1	1	1	1		12	$q^6\Phi_1^2$	12^n

Table A.6: The eighteen G_2-types.

A.7 $\chi(\mathbf{X})$ when $g = 1$ and $\dim(Z) = 0$

In this section, we compute the Euler characteristic of $\mathbf{X}$ when $g = 1$ and $\dim(Z) = 0$. By Theorem 65, the Euler characteristic is given by

$$\chi(\mathbf{X}) = |W|^{n-1} \sum_{L} |W(L)|\, |\mathrm{Irr}(W(L))|\, \nu(L),$$

where the sum is over all endoscopy groups L of G containing T. We can simplify

$$\chi(\mathbf{X}) = |W|^{n-1} \sum_{[L]} |[L]|\, |W(L)|\, |\mathrm{Irr}(W(L))|\, \nu(L),$$

where the sum is now over all W-orbits of endoscopy groups of G containing T.

A.7.1 $G = \mathrm{SO}_5$

The endoscopy groups of G containing T are $\mathrm{SO}_5, A_1 \times A_1, A_1, A_1'$ and T (up to the W-action).

$[L]$	$\lvert[L]\rvert$	$W(L)$	$\lvert W(L)\rvert$	$\lvert\mathrm{Irr}(W(L))\rvert$	$\nu(L)$
$[\mathrm{SO}_5]$	1	D_8	8	5	2
$[A_1 \times A_1]$	1	$S_2 \times S_2$	4	4	2
$[A_1]$	2	S_2	2	2	-4
$[A_1']$	2	S_2	2	2	-2
$[T]$	1	1	1	1	8

Table A.7: Calculations for $\chi(\mathbf{X})$ when $G = \mathrm{SO}_5$ and $g = 1$.

Using the formula, we have $\chi(\mathbf{X}) = 72 \times 8^{n-1} = 9 \times 2^{3n}$.

A.7.2 $G = G_2$

The endoscopy groups of G containing T are $G_2, A_2, A_1 \times A_1, A_1, A_1'$ and T (up to the W-action).

$[L]$	$\lvert[L]\rvert$	$W(L)$	$\lvert W(L)\rvert$	$\lvert\mathrm{Irr}(W(L))\rvert$	$\nu(L)$
$[G_2]$	1	D_{12}	12	6	1
$[A_2]$	1	S_3	6	3	2
$[A_1 \times A_1']$	2	$S_2 \times S_2$	4	4	1
$[A_1]$	2	S_2	2	2	-4
$[A_1']$	2	S_2	2	2	-2
$[T]$	1	1	1	1	12

Table A.8: Calculations for $\chi(\mathbf{X})$ when $G = G_2$ and $g = 1$.

Using the formula, we have $\chi(\mathbf{X}) = 104 \times 12^{n-1} = 8 \times 13 \times 12^{n-1}$.

A.8 $\chi(\mathbf{X})$ when $g = 0$ and $n \geq 3$

In this section, we provide examples of the Euler characteristic of $\mathbf{X}$ when $g = 0$ and $n \geq 3$. The Euler characteristics are summarised in the following table:

G	$\chi(\mathbf{X})$
GL_2	$2^{n-4}(n-1)(n-2)$
GL_3	$2^{n-5}3^{n-3}(n-1)(n-2)(9n^2-27n+16)$
GL_4	$2^{3n-9}3^{n-4}(n-1)(n-2)(108n^4-648n^3+1350n^2-1129n+324)$
SO_5	$2^{3n-8}(n-1)(n-2)(11n^2-33n+19)$
G_2	$2^{2n-7}3^{n-3}(n-1)(n-2)(207n^2-621n+350)$

Table A.9: Calculations for $\chi(\mathbf{X})$ when $g = 0$ and $n \geq 3$.

By Theorem 66, the Euler characteristic is given by

$$\chi(\mathbf{X}) = \frac{1}{2r!} \frac{d^{2r}}{dq^{2r}}\bigg|_{q=1} \xi(q),$$

where $2r := 2\dim(T) - 2\dim(Z)$ is twice the semisimple rank of G and

$$\xi(q) = q^{|\Phi(G)^+|(n-2)} \sum_{[L]} \nu(L)\, |[L]| \left(\frac{|W|}{|W(L)|}\right)^{n-1} r_L(q),$$

where the sum is over all W-orbits of endoscopy groups of G containing T and

$$r_L(q) := \sum_{\rho} \dim(\tilde{\rho})^n \left(\frac{P_{W(L)}(q)}{\|\rho\|(q)}\right)^{n-2},$$

where the sum is over all principal unipotent characters of $L(\mathbb{F}_q)$. Calculations of $r_L(q)$ are below:

L	$r_L(q)$
T	1
A_1	$\Phi_2^{n-2} + (\frac{\Phi_2}{q})^{n-2}$
$A_1 \times A_1$	$r_{A_1}(q)^2$
A_2	$(\Phi_2\Phi_3)^{n-2} + (\frac{\Phi_2\Phi_3}{q^3})^{n-2} + 2^n(\frac{\Phi_3}{q})^{n-2}$
B_2	$(\Phi_2^2\Phi_4)^{n-2} + (\frac{\Phi_2^2\Phi_4}{q^4})^{n-2} + 2(\frac{2\Phi_2^2}{q})^{n-2} + 2^n(\frac{2\Phi_4}{q})^{n-2}$
G_2	$(\Phi_2^2\Phi_3\Phi_6)^{n-2} + (\frac{\Phi_2^2\Phi_3\Phi_6}{q^6})^{n-2} + 2(3\frac{\Phi_2^2}{q})^{n-2} + 2^n(6\frac{\Phi_6}{q})^{n-2} + 2^n(2\frac{\Phi_3}{q})^{n-2}$
A_3	$(\Phi_2^2\Phi_3\Phi_4)^{n-2} + (\frac{\Phi_2^2\Phi_3\Phi_4}{q^6})^{n-2} + 2^n(\frac{\Phi_2^2\Phi_3}{q^2})^{n-2} + 3^n(\frac{\Phi_2^2\Phi_4}{q})^{n-2} + 3^n(\frac{\Phi_2^2\Phi_4}{q^3})^{n-2}$

Table A.10: Calculations for $r_L(q)$. The ith cyclotomic polynomial is denoted Φ_i; in particular, $\Phi_2 = q+1$, $\Phi_3 = q^2+q+1$, $\Phi_4 = q^2+1$ and $\Phi_6 = q^2-q+1$.

A.8.1 $G = \mathrm{GL}_2$

In this case, $\xi(q)$ equals

$$q^{(n-2)}\left[r_{A_1}(q) - 2^{n-1} r_T(q)\right].$$

Differentiating $2r = 2\dim(T) - 2\dim(Z) = 2$ times, evaluating at $q = 1$ and dividing by $(2r)!$ gives

$$\chi(\mathbf{X}) = 2^{n-4}(n-1)(n-2).$$

A.8.2 $G = \mathrm{GL}_3$

In this case, $\xi(q)$ equals

$$q^{3(n-2)}\left[r_{A_2}(q) - 3^n r_{A_1}(q) + 2\times 6^{n-1} r_T(q)\right].$$

Differentiating $2r = 2\dim(T) - 2\dim(Z) = 4$ times, evaluating at $q = 1$ and dividing by $(2r)!$ gives

$$\chi(\mathbf{X}) = 2^{n-5}3^{n-3}(n-1)(n-2)(9n^2 - 27n + 16).$$

This agrees with [HLRV11, (1.5.8)].

A.8.3 $G = \mathrm{GL}_4$

In this case, $\xi(q)$ equals

$$q^{6(n-2)}\left[r_{A_3}(q) - 4^n r_{A_2}(q) - 3\times 6^{n-1} r_{A_1\times A_1}(q) + 12^n r_{A_1}(q) - 6\times 24^{n-1} r_T(q)\right].$$

Differentiating $2r = 2\dim(T) - 2\dim(Z) = 6$ times, evaluating at $q = 1$ and dividing by $(2r)!$ gives

$$\chi(\mathbf{X}) = 2^{3n-9}3^{n-4}(n-1)(n-2)(108n^4 - 648n^3 + 1350n^2 - 1129n + 324).$$

A.8.4 $G = \mathrm{SO}_5$

In this case, $\xi(q)$ equals

$$q^{4(n-2)}\left[2r_{B_2}(q) + 2^n r_{A_1\times A_1}(q) - 4^{n+1} r_{A_1}(q) - 4^n r_{A_1}(q) + 8^n r_T(q)\right].$$

Differentiating $2r = 2\dim(T) - 2\dim(Z) = 4$ times, evaluating at $q = 1$ and dividing by $(2r)!$ gives

$$\chi(\mathbf{X}) = 2^{3n-8}(n-1)(n-2)(11n^2 - 33n + 19).$$

A.8.5 $G = G_2$

In this case, $\xi(q)$ equals

$$q^{6(n-2)}\left[r_{G_2}(q) + 2^n r_{A_2}(q) + 3^n r_{A_1\times A_1}(q) - 2\times 6^n r_{A_1}(q) - 6^n r_{A_1}(q) + 12^n r_T(q)\right].$$

Differentiating $2r = 2\dim(T) - 2\dim(Z) = 4$ times, evaluating at $q = 1$ and dividing by $(2r)!$ gives

$$\chi(\mathbf{X}) = 2^{2n-7}3^{n-3}(n-1)(n-2)(207n^2 - 621n + 350).$$

Appendix B

Counting polynomials in Julia

Our expression for the counting polynomial $\|\mathbf{X}\|(q)$ involves well-known representation-theoretic data. There are several computer algebra systems which calculate this data, allowing one to quickly and automatically compute $\|\mathbf{X}\|(q)$. These systems include the Chevie system in GAP and Julia [GHL$^+$96] and the Magma computer algebra system [BCP97].

In this appendix, we explain how to compute $\|\mathbf{X}\|(q)$ using the Chevie system. In particular, we calculate the table given in §A.6 which directly yields the counting polynomial of the associated G_2-character variety.

The ideas found in this chapter have been developed into a package for Julia found at

`https://github.com/baileywhitbread/CharacterVarieties.jl.`

B.1 Calculating pseudo-Levi subgroups

Once the Chevie package is loaded, we can calculate some familiar objects from representation theory:

```
using Chevie;
G = coxgroup(:G,2);
uc = UnipotentCharacters(G);
```

The command `UnipotentCharacters(G)` returns a dictionary containing, among other things, the unipotent characters of $G(\mathbb{F}_q)$.

Rather than calculate the endoscopy groups of G containing T, we work in the dual $\check{G}$ and calculate the pseudo-Levi subgroups of $\check{G}$ containing $\check{T}$. This is because we have access to a convenient function `sscentralizer_reps` which returns a list of representatives of centralisers of semisimple elements. This is used together with the functions `reflection_subgroup` and `orbits` which allow us to calculate all desired pseudo-Levi subgroups:

```
G_dual = rootdatum(simplecoroots(G),simpleroots(G));
pseudo_levi_orbit_reps = reflection_subgroup.(Ref(G_dual),sscentralizer_reps(G_dual));
pseudo_levi_orbits = orbits(G_dual,pseudo_levi_orbit_reps);
```

We collect these pseudo-Levi subgroups into a single list and create another list of the isolated-pseudo Levi subgroups:

```
pseudo_levis = [];
isolated_pseudo_levis = [];
for pseudo_levi_orbit in pseudo_levi_orbits
        for pseudo_levi in pseudo_levi_orbit
                append!(pseudo_levis,[pseudo_levi])
                if length(gens(pseudo_levi)) == length(gens(G)) # Isolated iff no. of simples equal
                        append!(isolated_pseudo_levis,[pseudo_levi])
                end
        end
end
```

B.2 Calculating $\nu(L)$

We also need to calculate the function $\nu(L)$ for endoscopy groups of G containing T. Recall from §62 the definition

$$\nu(L) := \sum_{L'} \mu(L,L')\, \pi_0^{L'},$$

where the sum is over all isolated endoscopy groups L' of G containing L, μ is the Möbius function on the poset of endoscopy groups of G containing T, $\pi_0^{L'} = |\pi_0(\check{T}^{W(L')})|$ is the number of components of $\check{T}^{W(L')}$. Written in terms of pseudo-Levi subgroups L of $\check{G}$ containing $\check{T}$, we instead have

$$\nu(L) = \sum_{L'} \mu(L,L')\, \pi_0^{L'},$$

where the sum is over all isolated pseudo-Levi subgroups L' of $\check{G}$ containing L, μ is the Möbius function on the poset of pseudo-Levi subgroups of $\check{G}$ containing $\check{T}$, $\pi_0^{L'} = |\pi_0(T^{W(L')})|$ is the number of components of $T^{W(L')}$. In view of this formula, we define a few helper functions.

First, we need a way of checking whether pseudo-Levi subgroups contain each other, so that we can implement the Möbius function. Given a pseudo-Levi subgroup L the function `inclusion` returns a list of its roots. For instance, `inclusion(G)` returns `[1,2,...,12]`, representing the twelve roots of G_2. We pair this with Julia's built-in function `issubset`:

```
function subset(L,M)
        return issubset(inclusion(L),inclusion(M))
end

function equal(L,M)
        return inclusion(L) == inclusion(M)
end
```

Next, we need to compute π_0^L. Luckily, the function `algebraic_center(L)` returns a dictionary of information about $Z(L)$ and the key `.AZ` returns the component group of $Z(L)$. Thus, we define:

```
function pi0(L)
        return length(algebraic_center(L).AZ)
end
```

We are now ready to implement the Möbius function:

```
function mob(A,B,poset)
        if equal(A,B)
                return 1
        elseif subset(A,B)
                mob_value = 0
                for element in poset
                        if subset(A,element) && subset(element,B) && !equal(element,B)
                                mob_value += mob(A,element,poset)
                        end
                end
                return (-1)*mob_value
        else
                error("First argument is not a subset of the second argument")
        end
end
```

Finally, we can implement the $\nu(L)$ function:

```
function nu(L)
        nu_value = 0
        for isolated_pseudo_levi in isolated_pseudo_levis
                if subset(L,isolated_pseudo_levi)
                        nu_value += mob(L,isolated_pseudo_levi,pseudo_levis)*pi0(isolated_pseudo_levi)
                end
        end
        return nu_value
end
```

B.3 Calculating G-types and their associated data

We calculate the tables given in §A.1, §A.2, §A.5 and §A.6 using the array `type_data = Array{Any}(nothing,0,8)`:

```
for pseudo_levi in pseudo_levi_orbit_reps
        pseudo_levi_order_poly = generic_order(pseudo_levi,Pol(:q));
        pseudo_levi_positive_root_size = Int(length(roots(pseudo_levi))/2);
        pseudo_levi_orbit_size = length(orbits(G_dual,pseudo_levi));
        pseudo_levi_weyl_size = length(pseudo_levi);
        pseudo_levi_nu = nu(pseudo_levi);
        pseudo_levi_uc = UnipotentCharacters(pseudo_levi);
        pseudo_levi_uc_names = charnames(pseudo_levi_uc,limit=true);
        pseudo_levi_uc_degree_polys = degrees(pseudo_levi_uc);
        for i in 1:length(pseudo_levi_uc)
                if Int(pseudo_levi_uc_degree_polys[i](1)) != 0 # Check unipotent char. is principal
                        type_row = Array{Any}(nothing,1,0);
                        global type_row = hcat(type_row,[(pseudo_levi,pseudo_levi_uc_names[i])]);
                        global type_row = hcat(type_row,[pseudo_levi_positive_root_size]);
                        global type_row = hcat(type_row,[pseudo_levi_uc_degree_polys[i]]);
                        global type_row = hcat(type_row,[pseudo_levi_order_poly]);
                        global type_row = hcat(type_row,[Int(pseudo_levi_uc_degree_polys[i](1))]);
                        global type_row = hcat(type_row,[pseudo_levi_weyl_size]);
                        global type_row = hcat(type_row,[pseudo_levi_orbit_size]);
                        global type_row = hcat(type_row,[pseudo_levi_nu]);
                        global type_data = vcat(type_data,type_row);
                end
        end
end
```

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