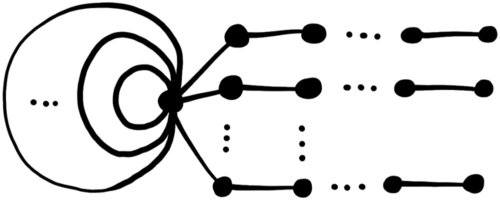
A purity conjecture for character varieties

Bailey Whitbread
University of Sydney



with S. Giannini, M. Kamgarpour, G. Nam

Character varieties

This talk is about two varieties:

$$X := \text{multiplicative character variety}$$

$$Y := \text{additive character variety}$$

X is built from reductive groups $G = \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$, etc.

Y is built from their Lie algebras $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$, etc.

Character varieties

This talk is about two varieties:

$$X := \text{multiplicative character variety}$$

$$Y := \text{additive character variety}$$

X is built from reductive groups $G = \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$, etc.

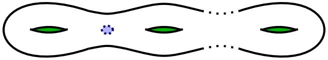
Y is built from their Lie algebras $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$, etc.

Main Theorem (GKNW '25)
$|X(\mathbb{F}_q)|$ and $|Y(\mathbb{F}_q)|$ are polynomials in q

Character varieties

$G :=$ reductive group over $\mathbb{F}_q$ $\qquad$ $\mathcal{C} :=$ conjugacy class in G

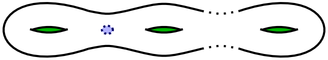
$\mathfrak{g} :=$ its Lie algebra over $\mathbb{F}_q$ $\qquad$ $\mathcal{O} :=$ adjoint orbit in $\mathfrak{g}$

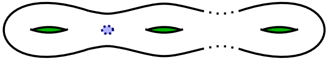
$\pi_1\Bigg($  $\Bigg) = \left\langle a_1, b_1, \ldots, a_g, b_g, c \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\rangle$

Character varieties

$G :=$ reductive group over $\mathbb{F}_q$ $\qquad$ $\mathcal{C} :=$ conjugacy class in G

$\mathfrak{g} :=$ its Lie algebra over $\mathbb{F}_q$ $\qquad$ $\mathcal{O} :=$ adjoint orbit in $\mathfrak{g}$

$\pi_1\Big($  $\Big) = \left\langle a_1, b_1, \ldots, a_g, b_g, c \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\rangle$

$\Big\{ f\colon \pi_1\big($  $\big) \to G \;\Big|\; f(c) \in \mathcal{C} \Big\} \Big/ G$

$$\Updownarrow$$

$$\left\{ (a_1, b_1, \ldots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\} \Big/ G$$

Character varieties

The multiplicative character variety is

$$\mathsf{X} := \left\{ (a_1, b_1, \ldots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\} \Big/ G$$

Character varieties

The multiplicative character variety is

$$X := \left\{ (a_1, b_1, \ldots, a_g, b_g, c) \in G^{2g} \times \mathcal{C} \;\middle|\; \prod_{i=1}^{g} [a_i, b_i]\, c = 1 \right\} \Big/ G$$

The additive character variety is

$$Y := \left\{ (x_1, y_1, \ldots, x_g, y_g, z) \in \mathfrak{g}^{2g} \times \mathcal{O} \;\middle|\; \sum_{i=1}^{g} [x_i, y_i] + z = 0 \right\} \Big/ G$$

Our purity conjecture

Examples of $|\mathsf{Y}(\mathbb{F}_q)|$:

$q^2 + 6q$

$q^4 + 6q^3 + 20q^2$

$q^8 + 6q^7 + 19q^6 + 45q^5 + 99q^4$

$q^6 + 2q^5 + 2q^4 + q^3$

$q^8 + 2q^7 + 4q^6 + 4q^5 + q^4$

$q^{12} + 2q^{11} + 3q^{10} + 5q^9 + \cdots$

Our purity conjecture

Examples of $|\mathsf{Y}(\mathbb{F}_q)|$:

$q^2 + 6q$

$q^4 + 6q^3 + 20q^2$

$q^8 + 6q^7 + 19q^6 + 45q^5 + 99q^4$

$q^6 + 2q^5 + 2q^4 + q^3$

$q^8 + 2q^7 + 4q^6 + 4q^5 + q^4$

$q^{12} + 2q^{11} + 3q^{10} + 5q^9 + \cdots$

Main Conjecture

(a) $|\mathsf{Y}(\mathbb{F}_q)| \in \mathbb{N}[q]$

(b) *The cohomology of* Y *is pure*

(c) *The cohomology of* Y *sits inside the cohomology of* X

Purity

$$H^*_{\mathrm{pure}}(\mathsf{Y}) \hookrightarrow H^*(\mathsf{Y}) \qquad H^*_{\mathrm{pure}}(\mathsf{X}) \hookrightarrow H^*(\mathsf{X})$$

Purity

$$H^*_{\mathrm{pure}}(Y) \hookrightarrow H^*(Y) \qquad H^*_{\mathrm{pure}}(X) \hookrightarrow H^*(X)$$

Theorem (Hausel–Letellier–Rodriguez-Villegas)

If $G = \mathrm{GL}_n$ then the cohomology of Y is pure

Purity

$$H^*_{\text{pure}}(Y) \hookrightarrow H^*(Y) \qquad H^*_{\text{pure}}(X) \hookrightarrow H^*(X)$$

Theorem (Hausel–Letellier–Rodriguez-Villegas)
If $G = \mathrm{GL}_n$ then the cohomology of Y is pure

Conjecture (Hausel–Letellier–Rodriguez-Villegas)
If $G = \mathrm{GL}_n$ then $H^*(Y) \simeq H^*_{pure}(X)$

Purity

$$H^*_{\mathrm{pure}}(\mathsf{Y}) \hookrightarrow H^*(\mathsf{Y}) \qquad H^*_{\mathrm{pure}}(\mathsf{X}) \hookrightarrow H^*(\mathsf{X})$$

Theorem (Hausel–Letellier–Rodriguez-Villegas)
If $G = \mathrm{GL}_n$ then the cohomology of Y is pure

Conjecture (Hausel–Letellier–Rodriguez-Villegas)
If $G = \mathrm{GL}_n$ then $H^*(\mathsf{Y}) \simeq H^*_{pure}(\mathsf{X})$

$$H^*_{\mathrm{pure}}(\mathsf{Y}) \xrightarrow{\sim} H^*(\mathsf{Y}) \overset{\sim}{\dashrightarrow} H^*_{\mathrm{pure}}(\mathsf{X}) \hookrightarrow H^*(\mathsf{X})$$

Kac polynomials

When $G = \mathrm{GL}_n$, the polynomials $|\mathsf{Y}(\mathbb{F}_q)|$ are Kac polynomials

$$\begin{matrix} \text{quiver } Q \\ \text{vector } \mathbf{v} \end{matrix} \quad \rightsquigarrow \quad K(q) := \#\left\{ \begin{matrix} \text{abs. indec. dim } \mathbf{v} \\ Q\text{-reps over } \mathbb{F}_q \end{matrix} \right\} \Big/ \text{iso.}$$

Kac polynomials

When $G = \mathrm{GL}_n$, the polynomials $|\mathsf{Y}(\mathbb{F}_q)|$ are Kac polynomials

$$\begin{array}{c}\text{quiver } Q \\ \text{vector } \mathbf{v}\end{array} \quad \rightsquigarrow \quad K(q) := \#\left\{\begin{array}{c}\text{abs. indec. dim } \mathbf{v} \\ Q\text{-reps over } \mathbb{F}_q\end{array}\right\}\Big/\text{iso.}$$

Theorem (Kac)
$K(q) \in \mathbb{Z}[q]$

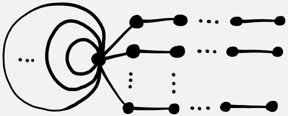
Kac polynomials

When $G = \mathrm{GL}_n$, the polynomials $|\mathsf{Y}(\mathbb{F}_q)|$ are Kac polynomials

$$\begin{matrix}\text{quiver } Q \\ \text{vector } \mathbf{v}\end{matrix} \quad \rightsquigarrow \quad K(q) := \#\left\{\begin{matrix}\text{abs. indec. dim } \mathbf{v} \\ Q\text{-reps over } \mathbb{F}_q\end{matrix}\right\}\Big/\text{iso.}$$

Theorem (Kac)
$K(q) \in \mathbb{Z}[q]$

Theorem (Crawley-Boevey)
If $G = \mathrm{GL}_n$ then Y is a quiver variety and $|\mathsf{Y}(\mathbb{F}_q)| = K(q)$



Kac polynomials

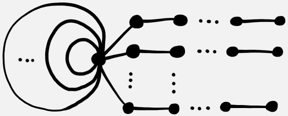
When $G = \mathrm{GL}_n$, the polynomials $|\mathsf{Y}(\mathbb{F}_q)|$ are Kac polynomials

$$\begin{array}{c}\text{quiver } Q\\ \text{vector } \mathbf{v}\end{array} \quad \rightsquigarrow \quad K(q) := \#\left\{\begin{array}{c}\text{abs. indec. dim } \mathbf{v}\\ Q\text{-reps over } \mathbb{F}_q\end{array}\right\}\Big/ \text{iso.}$$

Theorem (Kac)
$K(q) \in \mathbb{Z}[q]$

Theorem (HLRV)
$K(q) \in \mathbb{N}[q]$

Theorem (Crawley-Boevey)
If $G = \mathrm{GL}_n$ then Y is a quiver variety and $|\mathsf{Y}(\mathbb{F}_q)| = K(q)$



julia

```
               _
   _       _ _(_)_     |  Documentation: https://docs.julialang.org
  (_)     | (_) (_)    |
   _ _   _| |_  __ _   |  Type "?" for help, "]?" for Pkg help.
  | | | | | | |/ _` |  |
  | | |_| | | | (_| |  |  Version 1.11.1 (2024-10-16)
 _/ |\__'_|_|_|\__'_|  |  Official https://julialang.org/ release
|__/                   |

julia>
```

www.baileywhitbread.com