Counting points on the representation variety

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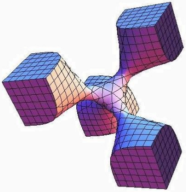


A representation space [CFLO].

Situating the representation variety

For a Riemann surface X and a reductive group G, consider the space of representations $\mathrm{Hom}(\pi_1(X), G) \subset G^r$.

There's an action $\mathrm{Hom}(\pi_1(X), G) \curvearrowleft G$ by conjugation.

$\rightsquigarrow$ We can quotient the representation space by G.

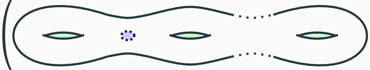
$\rightsquigarrow$ We obtain the orbit space $\mathrm{Hom}(\pi_1(X), G)/G$.

$$\begin{tikzcd}
 & \left\{\begin{matrix}\text{inequivalent reps.}\\ \pi_1(X) \to G\end{matrix}\right\} \arrow[d, equal] & \\
 & \mathrm{Hom}(\pi_1(X), G)/\!/G \arrow[dl, leftrightarrow, "\text{non-abelian Hodge theory}"'] \arrow[dr, leftrightarrow, "\text{Riemann-Hilbert correspondence}"] & \\
\underbrace{\mathrm{Higgs}_G(X)/\!/\mathcal{G}}_{\substack{\text{gauge theory,}\\ \text{integrable systems,}\\ \text{mirror symmetry}}} & & \underbrace{\mathrm{LocSys}_G(X)}_{\substack{\text{sheaf cohomology,}\\ \text{flat connections,}\\ D\text{-modules}}}
\end{tikzcd}$$

We need to define three pieces of data:

- $X :=$ once-punctured genus $g > 0$ compact orientable Riemann surface, which has the fundamental group

$$\Gamma := \frac{\langle x_1, y_1, \ldots, x_g, y_g, z \rangle}{[x_1, y_1] \ldots [x_g, y_g] z} = \pi_1\left(\quad \right).$$



- $G :=$ reductive group (split conn., conn. centre) over $\mathbb{F}_q$.
  Think $G = \mathrm{GL}_n$.

- $C := [s] =$ conjugacy class (s.s. and strongly regular) of G.
  Think $s = \mathrm{diag}(s_1, \ldots, s_n)$ with $s_i \neq s_j$.

The representation variety $\mathrm{R}(G, \Gamma, C)$ associated to this data is

$$\mathrm{R} := \left\{ (x_1, y_1, \ldots, x_g, y_g, z) \in G^{2g} \times C \;\middle|\; [x_1, y_1] \ldots [x_g, y_g] z = 1 \right\}.$$

We want to understand the topology of the representation variety. In particular, we seek an expression for the *E-polynomial* of R, denoted $E(\mathsf{R}; x, y) \in \mathbb{Z}[x, y]$.

For a complex variety X, the E-polynomial $E(\mathsf{X}; x, y)$ carries an abundunce of topological information:

(i) The dimension of X is half of the degree of $E(\mathsf{X}; x, y)$,

(ii) The Euler characteristic of X is $E(\mathsf{X}; 1, 1)$,

(iii) The # of (max'l dimension) irred. components of X is the leading coefficient of $E(\mathsf{X}; x, y)$.

Theorem [Katz]

Let X be a variety. Assume that $|\mathsf{X}(\mathbb{F}_q)| = P_{\mathsf{X}}(q)$ for some polynomial $P_{\mathsf{X}} \in \mathbb{Z}[q]$. Then $E(\mathsf{X}; x, y) = P_{\mathsf{X}}(xy)$.

Moral [Katz]

Just show $|\mathsf{X}(\mathbb{F}_q)|$ is a polynomial in q!

We may consider E as a function of one variable $q = xy$ and write $E(\mathsf{X}; q) = P_{\mathsf{X}}(q)$ instead. In this case, $\dim \mathsf{X} = \deg E(\mathsf{X}; q)$.

For example,

$$|\mathrm{GL}_2(\mathbb{F}_q)| = q^4 - q^3 - q^2 + q = P_{\mathrm{GL}_2}(q)$$

$$\text{dimension} = 4, \ \text{Euler characteristic} = 0,$$

$$\text{no. of irred. components} = 1.$$

Theorem [Frobenius 1896, Mednykh 1978]

$$|\mathbf{R}(\mathbb{F}_q)| = |C(\mathbb{F}_q)| \sum_{\chi \in \mathrm{Irr}(G(\mathbb{F}_q))} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-1} \chi(s).$$

$$\begin{matrix} \text{Understand} \\ \mathrm{Irr}(G(\mathbb{F}_q)) \end{matrix} \xrightarrow{\quad\text{mass formula}\quad} \begin{matrix} \text{Obtain } |\mathbf{R}(\mathbb{F}_q)| \\ \text{and } E(\mathbf{R}; q) \end{matrix}$$

This turns the problem of algebraic geometry into a problem of representation theory.

Theorems of Deligne-Lusztig, Curtis-Iwahori-Kilmoyer and Tits tell us that we need to look at:

- Stabiliser subgroups W_θ, where $W \curvearrowright \theta \in \mathrm{Irr}(T(\mathbb{F}_q))$, and
- The principal series representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \theta$.

One of the maximal tori of $G(\mathbb{F}_q) = \mathrm{GL}_n(\mathbb{F}_q)$ looks like

$$T(\mathbb{F}_q) = \begin{pmatrix} \mathbb{F}_q^\times & & \\ & \ddots & \\ & & \mathbb{F}_q^\times \end{pmatrix}$$

whose characters look like

$$\theta_{\alpha_1,\dots,\alpha_n} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} := \alpha_1(t_1)\cdots\alpha_n(t_n), \quad \alpha_i \in \mathrm{Irr}(\mathbb{F}_q^\times).$$

Recollections of representation theory

The Weyl group $W \simeq S_n$ acts via permutating the α_i's:

$$\sigma \cdot \theta_{\alpha_1,\dots,\alpha_n} := \theta_{\alpha_{\sigma(1)},\dots,\alpha_{\sigma(n)}}.$$

So their stabilisers look like

$$W_{\theta_{\alpha_1,\dots,\alpha_n}} \simeq S_{n_1} \times \cdots \times S_{n_r}, \quad \text{where } n_1 + \cdots + n_r = n.$$

The collection of all of these subgroups forms a lattice.

For GL_3, this is iso. to the lattice of set-partitions of $\{1,2,3\}$:

$$\begin{tikzcd}
 & \{1,2,3\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{1,2\}\sqcup\{3\} \arrow[dr, dash] & \{2\}\sqcup\{1,3\} \arrow[d, dash] & \{1\}\sqcup\{2,3\} \arrow[dl, dash] \\
 & \{1\}\sqcup\{2\}\sqcup\{3\} &
\end{tikzcd}$$

Each $\lambda \in \mathrm{Irr}(W_\theta)$ has an associated generic degree $D_\lambda \in \mathbb{Q}[q]$.

These capture important representation-theoretic data about the principal series representations $\mathcal{B}(\theta) := \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \theta$.

For example, if $G = \mathbf{GL}_3$ and $\theta = \theta_{\alpha,\alpha,\alpha}$, then $W_\theta \simeq S_3$.
Recall: Irreducible representations of $S_3 \longleftrightarrow$ Partitions of 3.

$$D_{1^3}(q) = q^3, \quad D_{2^1 1^1}(q) = q^2 + q, \quad D_{3^1}(q) = 1.$$

Then $W_\theta \simeq S_3$ has the Poincare polynomial

$$\sum_{w \in S_3} q^{\mathrm{len}(w)} = q^3 + 2(q^2 + q) + 1 = 1 \cdot D_{1^3}(q) + 2 \cdot D_{2^1 1^1}(q) + 1 \cdot D_{3^1}(q).$$

The coefficients tell you how $\mathcal{B}(\theta)$ decomposes!

$$\mathcal{B}(\theta) = V_1^{\oplus 1} \oplus V_2^{\oplus 2} \oplus V_3^{\oplus 1}.$$

Theorem [W., '22]

The E-polynomial for $\mathbf{R}$ is:

$$|C(\mathbb{F}_q)| \sum_{\substack{L \subseteq W \\ \text{refl.} \\ \text{subgp}}} \sum_{\zeta \in \mathrm{Irr}(L)} \dim(\zeta) \left(\frac{|G(\mathbb{F}_q)| P_L(q)}{P(q) D_\zeta(q)} \right)^{2g-1} \sum_{\substack{\theta \in \mathrm{Irr}(T(\mathbb{F}_q))/W \\ W_\theta = L}} \theta(s).$$

This broadens the current literature greatly - only three papers are known to deal with cases beyond GL_n.

Theorem [Hausel, Letellier, Rodriguez-Villegas, '11]

Suppose that $G = \mathrm{GL}_n$ and C is a 'generic' semisimple conjugacy class. Then

$$E(\mathbf{R}; q) = q^{\frac{1}{2} d_C} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{|Z(\mathrm{GL}_n(\mathbb{F}_q))|} \mathbb{H}_C(q^{1/2}, q^{-1/2}).$$

$$
\begin{tikzcd}
E(\mathsf{R}; q) & \text{HLRV's formula} \arrow[l, "G=\mathbf{GL}_n"'] \arrow[d, squiggly, "\text{Conjecture}"] \arrow[dl, dashed, <->, "??"'] \\
\text{Our formula} \arrow[u, "G=\text{reductive}"] \arrow[r, squiggly, dashed, "\textit{Conjectured}\ \text{conjecture}"'] & \text{Hodge polynomial } H(\mathsf{R}; x, y, t)
\end{tikzcd}
$$

References

[KNP] *Arithmetic geometry of character varieties with regular monodromy*, M. Kamgarpour, G. Nam, A. Puskás, 2022.

[Ball22] *Intersection cohomology of character varieties for punctured Riemann surfaces*, M. Ballandras, 2022.

[Cam17] *On the E-polynomial of parabolic Sp_{2n}-character varieties*, V. Cambò, 2017.

[HLRV] *Arithmetic harmonic analysis on character and quiver varieties*, T. Hausel, E. Letellier, F. Rodriguez-Villegas, 2012.

[HRV] *Mixed Hodge polynomials of character varieties*, T. Hausel, F. Rodriguez-Villegas, 2008.

[CFLO] Topology of Moduli Spaces of Free Group Representations in Real Reductive Groups, A. Casimiro, C. Florentino, S. Lawton, A. Oliveira, 2014.