HW 2

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MATH 406

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Problem 1

Find the Gram-Schmidt basis for $(0, 1, 4), (2, 0, 8), (3, 6, 0)$

Proof.

$$e_1 = \frac{1}{\sqrt{17}}(0, 1, 4),$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\| v_2 - \langle v_2, e_1 \rangle e_1 \|} = \frac{(2, 0, 8) - \frac{32}{\sqrt{17}}(\frac{1}{\sqrt{17}}(0, 1, 4))}{\|(2, 0, 8) - \frac{32}{17}(0, 1, 4)\|} = \frac{(2, \frac{-32}{17}, \frac{8}{17})}{\sqrt{\frac{2244}{289}}}$$

$$e_3 = \frac{v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2}{\| v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2 \|} =$$

$$\frac{(3, 6, 0) - \frac{6}{\sqrt{17}}(\frac{1}{\sqrt{17}}(0, 1, 4)) - \frac{-90}{17\sqrt{\frac{2244}{289}}}(\frac{(2, \frac{-32}{17}, \frac{8}{17})}{\sqrt{\frac{2244}{289}}}}{\|(3, 6, 0) - \frac{6}{\sqrt{17}}(\frac{1}{\sqrt{17}}(0, 1, 4)) - \frac{-90}{17\sqrt{\frac{2244}{289}}}(\frac{(2, \frac{-32}{17}, \frac{8}{17})}{\sqrt{\frac{2244}{289}}}\|}$$

□

Problem 2

Show that if v_1, v_2, v_3 are orthonormal in V, then applying Gram-Schmidt has no effect.

Proof. We can prove this for any $v_1 \dots v_n$ without much extra work, so we will do that! Since v_1 is already orthonormal, $e_1 = v_1$. Our basis is orthonormal, so any $v_i, i \neq 1$ will be orthogonal to e_1 meaning the inner product of $\langle v_{i \neq 1}, e_1 \rangle = 0$, This reduces e_2 to:

$$e_2 = \frac{v_2 - 0e_1}{\|v_2 - 0e_1\|} = v_2$$

We can then induct this procedure, knowing that our $v_1 \dots v_n$ are already orthonormal. This reduces the Gram-Schmidt procedure to:

$$e_j = \frac{v_j - 0e_1 - \cdots - 0e_{j-1}}{\|v_j - 0e_1 - \cdots - 0e_{j-1}\|} = v_j$$

for any $1 < j \leq n$

$$\therefore \text{Each iteration of GS leaves our vectors unchanged}$$

□

Problem 3

Say we have applied GS to turn the linearly independent list v_1, v_2, v_3, v_4 into the orthonormal list e_1, e_2, e_3, e_4, prove

$$span(v_1, v_2, v_3) = span(e_1, e_2, e_3)$$

Proof. Let GS be the Gram-Schmidt procedure and take

$$e_1, e_2, e_3 = GS(v_1, v_2, v_3)$$

Then $span(e_1, e_2, e_3) = span(v_1, v_2, v_3)$ by definition of the Gram-Schmidt procedure. Now take $e_1, e_2, e_3, e_3 = GS(v_1, v_2, v_3, v_4)$. By our inductive definition of GS, e_1, e_2, e_3 are unchanged and our basis has been expanded to $e_1 \dots e_4$.

$$\therefore span(e_1, e_2, e_3) = span(v_1, v_2, v_3)$$

□

Problem Axler 2

Suppose that $e_1, \dots, e_m$ is an orthonormal list of vectors in V. Let $v \in V$, prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

iff $v \in span(e_1, \dots, e_m)$

Proof. $\implies$ Let $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$ There are two cases: $v \in span(e_1, \dots, e_m)$. If this is true, we are done. So for contradiction, assume $v \notin span(e_1, \dots, e_m)$. We can then extend our list to $e_1, \dots, e_m, v$, where v can be made of a linear combination of these vectors. Apply GS to our new list to get $GS(e_1, \dots, e_m, v) = e_1, \dots, e_k$ where $k > m$ or else v would have been in the span. Taking the norm and squaring it, we get:

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_k\rangle|^2 \neq |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

which is a contradiction of our assumption.

$\Longleftarrow$ Let $v \in span(e_1, \ldots, e_m)$ Then $v = a_1e_1 + \cdots + a_me_m$ and by 6.30 in Axler, $v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$. Taking the norm as in Axler 6.25 and squaring it:

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

□

Problem Axler 7

Find a polynomial $q \in P_2(\mathbb{R})$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x)dx$$

for every $p \in P_2(\mathbb{R}$

Proof. We are being asked to find the Riesz representation, $\phi(v) = p(\frac{1}{2} = \int_0^1 p(x)q(x)dx$. We need to find a $q \in P_2(\mathbb{R})$. We have a orthonormal basis for P_2 of $e_1 = 1, e_2 = 2\sqrt{3}(x - \frac{1}{2}), e_3 = 6\sqrt{5}(x^2 - x + \frac{1}{6})$. The Riesz representation theorem says that we can find our $q(x)$ by

$$q = \overline{\phi(e_1)}e_1 + \cdots + \overline{\phi(e_n)}e_n$$
$$q = -\frac{3}{2} + 15x + -15x^2$$

□

Problem Axler 12

Suppose V is a finite dim inner product space, with $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. Prove there exists c, $\|v\|_1 \leq c\|v\|_2$

Proof. Take $e_1, \ldots, e_n$ to be our orthonormal basis under $\langle\cdot,\cdot\rangle_2$. Let $c = max(\|e_1\|_1 \ldots \|e_n\|_1)$. The reasoning behind this is that if we scale every element by the largest possibility, then we should get an inequality. Taking norms, we get:

$$\|v\|_2^2 = \|a_1\|^2 e_1 + \cdots + \|a_n\|^2 e_n \text{ and}$$
$$\|v\|_1^2 = \|a_1 e_1 + \cdots + a_n e_n\|_1^2$$
$$\text{and using the triangle inequality}$$
$$\|v\|_1^2 \leq (\|a_1 e_1\|_1 + \ldots \|a_n e_n\|_1)^2$$
$$\text{and knowing that } c \text{ is the max the norm of any } e_i \text{could possibly be, distribute}$$
$$\|v\|_1^2 \leq (c\|a_1 e_1\|_2 + \ldots c\|a_n e_n\|_2)^2$$
$$\|v\|_1^2 \leq c^2 \|v\|_2^2$$
$$\|v\|_1 \leq c\|v\|_2$$
$$\therefore \text{a } c \text{ exists such that the norms differ by a constant factor}$$

□

Problem Axler 14

Suppose $e_1, \ldots e_n$ is an orthonormal basis for V, and $v_1, \ldots, v_n$ are vectors such that

$$\|e_j - v_j\| < \frac{1}{\sqrt{n}}$$

Proof. We want to show linear independence. To do this, set $a_1 v_1 + \cdots + a_n v_n = 0$ and $a_1 e_1, \ldots, a_n e_n$ with the same coefficients, where at least one $a_j \neq 0$. Take the sum and the triangle inequality:

$$\begin{aligned}\|a_1 v_1 + \ldots a_n v_n - a_1 e_n + \cdots + a_n e_n\| &\leq \|a_1 v_1 + \ldots a_n v_n\| - \|a_1 e_1 + \cdots + a_n e_n\| \\ &= (a_1 v_1 + \cdots + a_n v_n) - (a_1 e_1 + \cdots + a_n e_n) \\ &= a_1(v_1 - e_1) + \cdots + a_n(v_1 - e_1)\end{aligned}$$

□

Problem Axler 16

Working this one out on paper, will try to finish this weekend.