HW 0

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Problem 1

Kronecker's Theorem. *Let F be a field and $p \in F[x]$ a irreducible polynomial. Then there exsts an extention field E of F and an element $a \in E$ such that $p(a) = 0$*

Proof. First, we claim that $E = F[x]/\langle p(x)\rangle$ is that extention field. To show that E is an extention field of F, we define

$$\varphi : F \to F[x]/\langle p(x)\rangle.$$

We can quickly check that φ is a ring homomorphism. We know E contains the elements of F, so we must now check that φ is one-to-one. To check that φ is one-to-one, assume that $\varphi(a) = \varphi(b)$, where $a + p(x) = \varphi(a), b + p(x) = \varphi(b)$. We know $ker\varphi$ is a prime ideal with $a, b \in ker\varphi$, so $a - b \in ker\varphi$. Since $ker\varphi$ is prime, $a - b = c(x)p(x)$ for some polynomial $c(x)$.

$$\therefore a - b = 0$$
$$\therefore a = b$$
$$\therefore \varphi \text{ is one-to-one over } F$$

Now we must show that $p(x)$ has a root in E. We claim $\alpha = x + p(x)$ is the root in E.

$$\begin{aligned}
p(x) &= a_0 + a_1x \dots a_nx^{n-1} \\
p(\alpha) &= a_0 + a_1(x + \langle p(x)\rangle) + \dots (a_nx^n + \langle p(x)\rangle) \\
p(\alpha) &= a_0 + a_1x + \dots + a_nx^n + \langle p(x)\rangle \\
p(\alpha) &= 0 + \langle p(x)\rangle
\end{aligned}$$

$$\therefore p(x) \text{ has a root in } E \text{ and E is an extention field containing } F$$

$\square$