HW 4

Bailey Wickham
MATH 406

May 15, 2020

Problem 1

Define $T : \mathbb{C}^2 \to \mathbb{C}^3$ by

$$T(z_1, z_2) = (z_1 - 3z_2, z_1, 2z_1 + 5z_2)$$

Find T^*

Proof.

$$\begin{aligned}
\langle (x_1, x_2), T^*(y_1, y_2, y_3) \rangle &= \langle T(x_1, x_2), (y_1, y_2, y_3) \rangle \\
&= \langle (x_1 - 3x_2, x_1, 2x_1 + 5x_2), (y_1, y_2, y_3) \rangle \\
&= x_1 y_1 - 3x_2 y_1 + x_1 y_2 + 2x_1 y_3 + 5x_2 y_3 \\
&= x_1(y_1 + 2y_3 + y_2) - x_2(3y_1 + 5y_3) \\
\therefore T^*(y_1, y_2, y_3) &= (y_1 + y_2 + 2y_3, 3y_1 + 5y_3)
\end{aligned}$$

□

Problem 2

Let $P_2(\mathbb{R})$ be equipped by the usual polynomial inner product. Define $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(a + bx + cx^2) = bx$.

1. Show that T is not self adjoint
2. Show that $M(T)$ equals it's conjugate transpose. Does this violate 7.10?

Proof. 1. To show: $T^* \neq T$ or $\langle Tv, w \rangle \neq \langle v, Tw \rangle$
Take

$$\begin{aligned}
\langle 1, T(2x) \rangle &= \langle T(1), 2x \rangle \\
\langle 1, 2x \rangle &= \langle 0, x \rangle \\
1 &\neq 0
\end{aligned}$$

2. To show: $M(T) = M(T^*)$

$$M(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking the conjugate transpose, we get $M(T^*) = M(T)$. This is not a contradiction because theorem 7.10 is an if then, not an if and only if. Equal matricies do not imply an orthonormal basis.

$\square$

Problem 3

Let U be a subspace of the complex finite dimensional IPS V. Prove or disprove P_U must be self adjoint.

Proof. Let $z = x + y; v = u + w, u, x \in U, w, y \in U^{\perp}$. To show: $Pv = P^*v$, or $\langle Pv, z\rangle = \langle v, Pz\rangle$

$$\langle Pv, z\rangle =$$
$$\langle u, z\rangle$$
$$\langle u, x\rangle + \langle u, y\rangle \text{ u, y are orthogonal}$$
$$\langle u, x\rangle + 0$$
$$\langle u, x\rangle + \langle w, x\rangle$$
$$\langle v, x\rangle$$
$$\langle v, Pz\rangle$$
$$\therefore P_U \text{ is self adjoint}$$

$\square$

Problem 4

Discuss whether the set of self adjoint operators to a finite dimentional IPS is a subspace.

Proof. The sum of two self adjoint operators is always self adjoint. The zero map is a self adjoint operator. The only time when the the self adjoint operators don't form a subspace is under scalar multiplication over a complex field. They form a subspace over a real field. Axler 7.15 says T is self adjoint iff $Tv, v \in \mathbb{R}$, but under complex scalar multiplication, this isn't always true. $\square$

Problem Axler 3

Suppose $T \in L(V)$, prove U is invariant under T iff $U^\perp$ is invariant under T^*

Proof. Suppose U is invariant under T. Then $u \in U \implies Tu \in U$. Let $w \in U^\perp, u \in U$

$$\begin{gathered}0 = \langle u, w \rangle \\ 0 = \langle Tu, w \rangle \\ 0 = \langle u, T^* w \rangle \\ \therefore T^* u \in U^\perp\end{gathered}$$

So $U^\perp$ is invariant under T^*
Suppose $U^\perp$ is invariant under T^* Let $w \in U^\perp, u \in U$

$$\begin{gathered}0 = \langle Tu, w \rangle \\ 0 = \langle u, T^* w \rangle \\ \therefore w \in U^\perp \text{ using our hypothesis } \therefore u, Tu \in U \therefore\ U \text{ invariant under } T\end{gathered}$$

□

Problem Axler 4

Let $T \in L(V, W)$. Prove

1. T is injective iff T^* is surjective
2. T is surjective iff T^* is injective

Proof. 1. $\Longleftarrow$ Let T be injective. Axler 7.7 says that $nullT^* = (rangeT)^\perp$, so we want to show that $(rangeT)^\perp = 0$, implying that T^* is surjective. From my proof of 4 in last weeks homework, we can say that $(rangeT)^\perp = nullT$, and since T is injective $nullT = 0$, so T^* is surjective.
$\Longrightarrow$ Let T^* be surjective. Then $nullT = (rangeT^*)^\perp$. But T^* is surjective, so $(rangeT^*)^\perp = 0$.

$$\therefore nullT = 0 \therefore T \text{ is injective}$$

2. This proof follows from 1. by replacing T with T^* and T^* with T

□

Problem Axler 12

Suppose that T is normal and 3,4 are eigenvalues of T. Prove there exists a $v \in V$ s.t. $\|v\| = \sqrt{2}$and $\|Tv\| = 5$

Proof. Recall that distinct eigenvalues generate orthogonal eigenvectors for normal operators. Let u, v be the eigenvectors, take w to be their sum.

$$w = u + v$$
$$\|w\|^2 = (|a|\|u\|)^2 + (|b|\|v\|)^2 \text{ where u,v can be scaled to a } \|u\|, \|v\| \text{ of } 1$$
$$2 = a^2 + b^2$$

Now apply T.

$$w = u + v$$
$$Tw = Tu + Tv$$
$$Tw = |3|\|u\| + |4|\|v\|$$
$$\|Tw\|^2 = 3^2 + 4^2$$

□

Problem Axler 14

Suppose T is normal on V, $v, w \in V$ satisfy

$$\|v\| = \|w\| = 2, Tv = 3v, Tw = 4w$$

Show that $\|T(v + w)\| = 10$

Proof. Since v, w are eigenvectors, they are orthogonal, so we can use the pythagorean thm.

$$\|T(v, w)\|^2 = \|T(v) + T(w)\|^2$$
$$\|3v\|^2 + \|4w\|^2$$
$$9\|v\|^2 + 16\|w\|^2$$
$$9(4)^2 + 16(4)^2 = 10^2$$

□

Problem Axler 16

Suppose $T \in L(V)$ is normal. Prove

$$rangeT = rangeT^*$$

Proof. Using 7.20, we can see that anything T sends to 0, T^* sends to 0, so $nullT = nullT^*$. Using Axler 7.7 we can say that $(rangeT) = (nullT^*)^\perp$ and $rangeT^* = (nullT^*)^\perp$, so $rangeT = rangeT^*$ $\square$