HW 1

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Problem 1

Let V be a Real inner product space. Show that if $u, v \in V$ have the same norm, then $u + v$, $u - v$ are orthogonal.

Proof. First, we are in a Real product space, so our terms commute. We also know that $\langle u, u\rangle = \langle v, v\rangle$. We will show inner product of $u + v, u - v$ is 0 to show they are orthogonal

$$\begin{gathered}
\langle u+v, u-v\rangle \\
\langle u, u-v\rangle + \langle v, u-v\rangle \\
\langle u, u\rangle - \langle v, v\rangle + \langle u, v\rangle - \langle u, v\rangle \\
\text{here we know } \langle u, u\rangle = \langle v, v\rangle \text{ because } u, v \text{ have the same norm} \\
\therefore 0 \\
\therefore u+v, u-v \text{ are orthogonal}
\end{gathered}$$

□

Problem 2

Give examples of u, v and an inner product space V where

$$\|u+v\| = \|u\|^2 + \|v\|^2$$

despite the fact u is not orthogonal to v.

Proof. Take our inner product space to be $\mathbb{C}$ under the normal inner product, $u = 1, v = 1 + i$. Our vectors are not orthogonal, $\langle u, v\rangle = 1 - i$.

$$\begin{aligned}
\|2+i\|^2 &= \|1\|^2 + \|1+i\|^2 \\
\sqrt{4-1}^2 &= \sqrt{1}^2 + \sqrt{2}^2 \\
3 &= 3
\end{aligned}$$

□

Problem 3

Let the vector space $\mathcal{P}_2(\mathbb{R})$ be equipped with an inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx$$

Find all polynomials that are orthogonal to the polynomial x. Is this infinite? Is this a subspace?

Proof. To get all orthogonal polynomials, we want to set our inner product equal to 0.

$$\begin{aligned}
0 &= \int_0^1 x(a_0 + a_1 x + a_2 x^2)dx \\
0 &= \int_0^1 a_0 x + a_1 x^2 + a_2 x^3 dx \\
0 &= \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_2}{4}x^4 \Big|_0^1 \\
0 &= \frac{a_0}{2} + \frac{a_1}{3} + \frac{a_2}{4}
\end{aligned}$$

This is an infinite collection by varying a_0, a_1, a_2. This is dim 2 subspace. $\square$

Problem 4

Let V be an IPS equipped with $\langle \cdot, \cdot \rangle$, and let $T \in \mathcal{L}(V)$. Show that

$$\langle u, v\rangle_{cousin} = \langle Tu, Tv\rangle$$

is an inner product on V iff T is injective

Proof. $\implies$ Let $\langle u, v\rangle_{cousin}$ be an inner product. We want to show that 0 is the only vector in the null space. Assume $Tu = 0$. Then $\langle Tu, Tu\rangle = 0$ because $\langle \cdot, \cdot \rangle$ is an inner product. This is equivalent to $\langle u, u\rangle_{cousin} = 0$. Since $\langle \cdot, \cdot \rangle_{cousin}$ is an inner product,

$$\therefore u = 0$$
$$\therefore T \text{ is injective}$$

$\impliedby$ Let T be injective. We must show $\langle \cdot, \cdot \rangle_{cousin}$ is an inner product.
positivity: Take $\langle v, v\rangle_{cousin} = \langle Tv, Tv\rangle$. $\langle Tv, Tv\rangle \geq 0$ Since $\langle \cdot, \cdot \rangle$ is an inner product. Therefore $\langle \cdot, \cdot \rangle$ satisfies positivity.
definiteness: Take $\langle v, v\rangle_{cousin} = 0$, therefore $\langle Tv, Tv\rangle = 0$ and $Tv = 0$ since T is injective, $v = 0$, and $\langle \cdot, \cdot \rangle_{cousin}$ is definite.

The following properties follow the same structure as the first two and do not require the injectivity of T.

$$\therefore \langle \cdot, \cdot \rangle_{cousin} \text{ is an inner product.}$$

□

Problem Axler 5

Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ Prove that $T - \sqrt{2}I$ is invertible.

Proof. We know $\sqrt{2}$ is an eigenvalue of T iff $null(T - \lambda I) \neq 0$, taking the contrapositive, $null(T - \lambda I) = 0$ iff $\sqrt{2}$ is not an eigenvalue. Assume $\sqrt{2}$ is an eigenvalue. Then

$$\|Tv\| \leq \|v\|$$
$$\|\sqrt{2}v\| \leq \|v\|$$
$$\sqrt{2}\|v\| \leq \|v\|$$

Which is a contradiction. Therefore $\sqrt{2}$ is not an eigenvalue, and $T - \sqrt{2}I$ is invertible. □

Problem Axler 8

Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove $u = v$

Proof. Take $u = v, u - v = 0$ if we take our norm, substituting using our assumptions,

$$\|u - v\| = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle$$
$$\|u - v\| = 0$$

□

Problem Axler 11

Prove $16 \leq (a + b + c + d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d})$

Proof. Since we are in $\mathbb{R}$, we can take the Cauchy-Schwarz Inequality example in 6.17.a.

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$$
$$4^2 \leq (a + b + c + d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d})$$

□

Problem Axler 19

Prove $\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$

Proof. Take our proof for Axler 20 and set the complex component to 0. $\square$

Problem Axler 20

Prove $\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$

Proof. I did this on paper and it took about a page, this is the summarized version.

$$\langle u, v\rangle = \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle + \langle u+vi, u+vi\rangle i + \langle u-vi, u-vi\rangle i}{4}$$
$$\langle u, v\rangle = \frac{4\langle u, v\rangle + 2\langle u, vi\rangle i + 2\langle iv, u\rangle i}{4}$$
$$\langle u, v\rangle = \frac{4\langle u, v\rangle}{4}$$

$\square$