Definitions

Bailey Wickham
MATH 412

November 25, 2020

1 Chapter 1

1.1 Maximum

Let $A \subset \mathbb{R}, m \in \mathbb{R}$. We say that the **maximum** of A is m if

1. for all $x \in A, m \geq x$
2. $m \in A$

1.2 Upper bound

Let A be a nonempty subset of $\mathbb{R}$, let $u \in \mathbb{R}$. u is an **upper bound** if for all $x \in A$ $x \leq u$

1.3 Supremum

Let A be a nenempty subset of $\mathbb{R}$, Let $l \in \mathbb{R}$. l is the **supremum** of A if for all $a \in A, a \leq l$ and for all $m \in \mathbb{R}$ if all $a \in A, a \leq m$ then $l \leq m$. l is the least upper bound.

1.4 Minimum

Let $A \subset \mathbb{R}, m \in \mathbb{R}$. We say that the **minimum** of A is m if

1. for all $x \in A, m \leq x$
2. $m \in A$

1.5 Axiom of Completeness

Every nonempty set of real numbers which is bounded above has a least upper bound.

1.6 Infimum

Let A be a nenempty subset of $\mathbb{R}$, Let $l \in \mathbb{R}$. l is the **infimum** of A if for all $a \in A, a \geq l$ and for all $m \in \mathbb{R}$ if all $a \in A, a \geq m$ then $l \geq m$. l is the least upper bound.

1.7 Bounded Above

A subset $A \subset \mathbb{R}$ is **bounded above** if there exists a supremum for A.

1.8 Bounded Below

A subset $A \subset \mathbb{R}$ is **bounded below** if there exists a infimum for A.

1.9 Bounded

A set is **bounded** if it is bounded below and bounded above

1.10 Bijection

A function f is a bijection if it is one to one and onto.

1.11 Cardinality

Sets A, B have the same cardinality if there exists a bijection between them.

1.12 Finite

Let A be a set, A is finite if there exists an $n \in \mathbb{N}$ s.t.A has card n

1.13 Countable

A set is countable if there is a bijection between $\mathbb{N}$ and A.

1.14 Sequence

A sequences is a function whose domain is $\mathbb{N}$

1.15 Convergence

A sequence (a_n) converges to a real number a if for every positive number ϵ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, then $|a_n - a| < \epsilon$

1.16 ϵ-Neighborhood

$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$

1.17 Topological Convergence

A sequence (a_n) converges to a if there exists an $n \in \mathbb{N}$ such that all terms after n are in an $\epsilon neighborhood$

1.18 Bounded

A sequence is bounded iff the set of all elements of the sequence is bounded.

1.19 Eventually

A sequence (a_n) is eventually in a set A if there exists an N s.t. $a_n \in A$ for all $n \geq N$

1.20 Frequently

A sequence (a_n) is eventually in a set A if for every N there exists an $n \geq N$ s.t. $a_n \in A$.

Chapter 2

2.4 The Monotone Convergence Theorem.

Increasing

A sequence is increasing if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$

Decreasing

A sequence is decreasing if for all $n \in \mathbb{N}$, $a_n \geq a_{n+1}$

Monotone

A sequence is monotone if it is increasing or decreasing.

Infinite Series

An infinite series is a expression in the form of:

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + ...$$

The sequence of partial sums (s_m) by $s_m = b_1 + b_2 + ... + b_m$. We say the series converges to B if the sequence of partial sums converges to B, denoted:

$$\sum_{n=1}^{\infty} b_n = B$$

Geometric Series

Let $r, a \in \mathbb{R}$. A geometric series is of the form

$$\sum_{n=1}^{\infty} ar^n$$

2.5 Subsequences

Subsequence

Let (a_n) be a sequence of real numbers and let $n_1 < n_2 ...$ be an increasing sequence of natural numbers. Then the sequence $(a_{n_1}, a_{n_2}, ...)$ is a subsequence of (a_n) and is denoted by (a_{n_k}).

2.6 The Cauchy Criterion

Cauchy Sequence

A sequence (a_n) is called a Cauchy sequence if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $m, n > N$ it follows that $|a_n - a_m| < \epsilon$

2.7 Series

Absolute and Conditional Convergence

if the series $\sum_{n=1}^{\infty} |a_n|$ converges we say the original series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If the original series converges but the series of absolute values diverges, then we say the sequence converges conditionally.

Rearagements

Let $\sum_{n=1}^{\infty} a_n$ be a series. A series $\sum_{n=1}^{\infty} b_n$ is called a rearrangement of the original series if there exists a bijection between the the series.

2 Intro to Topology

3.2 Open and Closed Sets

Open

A set $O \subseteq \mathbb{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.

Limit Point

A point x is a limit point of of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects A at some point other than x. Alternatively, x is a limit point if and only if $x = lim(a_n)$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

Isolated Point

A point $a \in A$ is an isolated point if it is not a limit point.

Closed Set

A set $F \subseteq \mathbb{R}$ is closed if it contains its limit points. Alternatively, a set is closed if and only if every Cauchy sequence contained in F has a limit that is also in F.

Closure

The closure of a set $A \subseteq \mathbb{R}$, denoted $\overline{A}$ is the set $A \cup L$ where L is the set of all limit points of A

3.3 Compact Sets

Compact

A set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a subsequence converging to a limit also in K. A set is Compact iff it is closed and bounded.

Bounded

A set $A \subseteq \mathbb{R}$ is bounded if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Open Cover

An open cover of A is a collection of open subsets of A such that their union is all of A. A finite subcover is a finite subcollection of open subsets of A that still contains A in the union.

3.4 Connected Sets

Seperated and Disconnected Sets

Two nonempty sets $A, B \subseteq \mathbb{R}$ are seperated if $\overline{A} \cap B = \overline{B} \cap A = \{\}$. A set is disconnected if it can be written as the union of two seperated sets.

3 Functional limits and continuity

4.2 Functional Limits

Functional Limit

Let $f : A \to \mathbb{R}$, and let c be a limit point in the domain of A we say that $lim_{x\to c}f(x) = L$ provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $|f(x) - L| < \epsilon$. Alternatively, if for every epsilon neighborhood $V_\epsilon(L)$ there exists a $V_\delta(c)$ such that for all $x \in V_\delta c$ different than c it follows that $f(x) \in V_\epsilon(L)$

4.3 Continous Functions

Continuity

A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $|f(x) - f(c)| < \epsilon$.

4.4 Continous Functions on Compact Sets

Uniform Continuity

Let $f : A \to \mathbb{R}$, f is uniformly continuous on A if for all $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

4.5 The IVT

IVP

A function has the intermediate value property on $[a, b]$ if for all $x < y$ in $[a, b]$, and all L between $f(x)andf(y)$it is always possible to find a point $c \in (x, y)$ where $f(c) = L$

4 Derivatives

5.2 Differentiability

Differentiability

Let $g : A \to \mathbb{R}$ be a function defined on an interval A. given $c \in A$, the derivative of g at c s defined by $g'(c) = lim_{x-c}\frac{g(x)-g(c)}{x-c}$ we say that g is differentiable at c, or on A if g is differentiable over all of A.

infinity

given $g : A \to \mathbb{R}$ and a limit point c of A, we say that $lim_{x\to c}g(x) = \infty$ if for every $M > 0$ there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) > M$.