13/12/23

<u>Orthogonal matrix</u>

$$AA^T = A^TA = I \text{ (Identity matrix)}$$

$$A = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$AA^T = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Rank of a matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \neq 0$$

$\therefore$ Rank of matrix $= 2$.

$$\begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad \begin{aligned} C_3' &= C_3 - C_1 \\ C_4' &= C_4 - C_1 \end{aligned}$$

$$\approx \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} R_4' &= R_4 - R_1 \\ R_3' &= R_3 - R_1 \end{aligned}$$

$$\simeq \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} R_3' = R_3 - 3R_2 \\ R_4' = R_4 - R_2 \end{array}$$

$$\simeq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} C_3' = C_3 + 3C_2 \\ C_4' = C_4 + C_2 \end{array}$$

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \quad (\text{Rank} = 2)$$

HW

Q. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & 4 & -6 \\ 0 & 0 & 5 & -2 \\ 3 & 6 & 8 & -1 \end{bmatrix}$$

Eigen value

$|A - \lambda I| = 0 \quad \therefore \lambda =$ eigen value.

$AX = \lambda X \quad \therefore X =$ eigen vector.

$(A - \lambda I)X = 0$

$\therefore |A - \lambda I| = 0$

$$\# \quad \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\therefore \Rightarrow \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(1-\lambda)(-1-\lambda) - 3\right] + 2\left[-1-\lambda-1\right] + 3\left[3-1+\lambda\right]$$

$$= -(2-\lambda)\left[(1-\lambda)^2 - 3\right] \cancel{a} - 2(\lambda+2) + 3(2+\lambda)$$

$$= -(2-\lambda)\left[(1-\lambda)^2 - 3\right] + (\lambda+2)$$

$$= (\lambda+2)\left[-1(1-\lambda)^2 + 3 + 1\right]$$

$$= (\lambda+2)\left[-(\lambda^2 - 2\lambda + 1) + 4\right]$$

$$= (\lambda+2)(-\lambda^2 + 2\lambda + 3) = 0$$

$= \quad \lambda = -2 \quad \Big| \quad \lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow (\lambda+1)(\lambda-3) = 0$

$\Rightarrow \lambda = -1, +3$

Eigen vectors

Case 1, $\lambda = -2$

$AX = \lambda X$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$= \cancel{2x_1 - 2x_2 + 3x_3 + x_1 + x_2 + x_3 + x_1 + 3x_2 - x_3 =}$

$\Rightarrow 2x_1 - 2x_2 + 3x_3 = -2x_1$

$\Rightarrow 4x_1 - 2x_2 + 3x_3 = 0$ ——— (1)

$x_1 + x_2 + x_3 = -2x_2$

$\Rightarrow x_1 + 3x_2 + x_3 = 0$ ——— (2)

Solving (1) and (2) $\Rightarrow$

$$\frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14} \qquad \begin{pmatrix} -11 \\ -1 \\ 14 \end{pmatrix} \text{ or } \begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix}$$

Case 2) $\lambda = 1$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow 2x_1 - 2x_2 + 3x_3 = x_1 \Rightarrow x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + x_2 + x_3 = x_2$$

$$\Rightarrow x_1 + x_3 = 0$$

$$\frac{x_1}{-2} = \frac{x_2}{3-1} = \frac{x_3}{0+2}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{2} \qquad \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

<u>Case 3</u>

$\lambda = 3$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow 2x_1 - 2x_2 + 3x_3 = 3x_1 \Rightarrow -x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + x_2 + x_3 = 3x_2 \Rightarrow x_1 - 2x_2 + x_3 = 0$$

$$\frac{x_1}{-2+6} = \frac{-x_2}{-1-3} = \frac{x_3}{2+2}$$

$$= \frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{4} \qquad \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

X_1, X_2, X_3

$C_1 X_1 + C_2 X_2 + C_3 X_3 = 0 \Rightarrow C_1 = C_2 = C_3 = 0.$

Eg

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore C_1 X_1 + C_2 X_2 = 0$

$$C_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\left.\begin{aligned} \Rightarrow & C_1 + C_2 = 0 \\ \Rightarrow & 2C_1 + C_2 = 0 \\ & C_1 + C_2 = 0 \end{aligned}\right\} C_1 = 0,\ C_2 = 0.$$

$\therefore$ (So, there two vectors are independent.)

Property

If the eigen values are independent then eigen vectors will be independent.

λ_1, λ_2

X_1, X_2

$\therefore A X_1 = \lambda_1 X_1$

$A X_2 = \lambda_2 X_2$

$\therefore C_1 X_1 + C_2 X_2 = 0$

$\Rightarrow C_1 A X_1 + C_2 A X_2 = 0$

$\Rightarrow C_1 \lambda_1 X_1 + C_2 \lambda_2 X_2 = 0$

$C_1 \lambda_1 X_1 + C_2 \lambda_2 X_2 = 0$

$\Rightarrow \quad c_2 (\lambda_2 - \lambda_1) x_2 = 0$

$\therefore c_2 = 0 \quad \Big| \quad \because \lambda_2 - \lambda_1 \neq 0$

$x_2 \neq 0$

Similarly, $c_1 = 0$