Math - Vector Space

Ex: Determine the subspace of $\mathbb{R}^3$ spaned by the vectors $\alpha = (1,2,3)$, $\beta = (3,1,0)$. Examine if $\gamma = (2,1,3)$, $\delta = (-1,3,6)$ within the subspace.

$\Rightarrow$ $L(\alpha,\beta)$ is the set of vectors $\{c\alpha + d\beta\}$ where c, d are real numbers.

$$\therefore \{c\alpha + d\beta\} = c(1,2,3) + d(3,1,0)$$
$$= (c+3d,\ 2c+d,\ 3c)$$

If γ is in ~~the~~ $L(\alpha,\beta)$, it must satisfy—

$$(2,1,3) = (c+3d,\ 2c+d,\ 3c)$$

$$2c+d=1$$

$\therefore$ $c+3d=2$

$2c+d=1 \Rightarrow \cancel{2\cdot d + d = 0} \Rightarrow 2\cdot 1 + d = 1$

$3c = 3 \Rightarrow$ $\Rightarrow d = -1$

$\Rightarrow c = 1$

$\therefore$ $1 + 3\cdot(-1)$

$= 1 - 3$

$= -2 \neq 2$

$\therefore$ Equations are in consistant.

$\therefore$ $\gamma \notin L(\alpha,\beta)$

If $S \in L(\alpha, \beta)$ then

$(-1, 3, 6) = (c+3d, 2c+d, 3c)$

$\therefore c+3d = -1$	$\therefore c+3d = -1$	$\therefore c+3d = -1$
$2c+d = 3$	or, $2+3d = -1$	LHS
$3c = 6$	or, $3d = -3$	$= 2+3(-1)$
$\Rightarrow c = 2$	or, $d = -1$	$= 2-3$
		$= -1 =$ RHS

Ex. Let $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \alpha+\beta\}$ where $\alpha, \beta \in \mathbb{R}^n$. Show that $L(S) = L(T)$.

Since $S \subseteq T$, $L(S) \subseteq L(T)$ —— (i)

Again each element of T can be expressed as a linear combination of α, β.

So, by previous result $L(T) \subseteq L(S)$. —— (ii)

$\therefore$ So, by (i), (ii) we can write $L(S) = L(T)$

Linear Transformation (Linear Mapping):

Let U, W be vector space over field F. A mapping $f : U \to W$ is said to be linear transformation if it satisfies the following properties.

1. $f(\alpha+\beta) = f(\alpha) + f(\beta)$ for all $\alpha, \beta \in U$
2. $f(c\alpha) = cf(\alpha)$ for all $c \in F$, $\alpha, \beta \in U$

In particular $U = W$ the f is said to be linear operator on U.

Ex: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a_1, a_2, a_3) = (a_1, a_2)$

Show that T is a linear transformation.

$\Rightarrow$ Let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in \mathbb{R}^3$

then $T(\alpha+\beta) = T(a_1+b_1, a_2+b_2, a_3+b_3)$

$= (a_1+b_1, a_2+b_2)$

$= (a_1, a_2) + (b_1, b_2)$

$= T(\alpha) + T(\beta)$

$\therefore$ For all real c, $T(c\alpha) = T(ca_1, ca_2, ca_3)$

$= (ca_1, ca_2)$

$= c(a_1, a_2)$

$= cT(a_1, a_2)$

$\therefore$ T is linear transformation.

Ex: $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$T(a_1, a_2, a_3) = (a_1+1, a_2+1)$.

Show that T is linear mapping.

$\Rightarrow$ Let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in \mathbb{R}^3$

then $T(\alpha+\beta) = T[(a_1, a_2, a_3) + (b_1, b_2, b_3)]$

$= T[a_1+b_1, a_2+b_2, a_3+b_3]$

$= (a_1+b_1+1, a_2+b_2+1)$

$\therefore$ $T(\alpha) + T(\beta) = (a_1+1, a_2+1) + (b_1+1, b_2+1)$

$= (a_1+b_1+2, a_2+b_2+2)$

$\therefore T(\alpha+\beta) \neq T(\alpha) + T(\beta)$ $\therefore$ T is not linear mapping.

Ex: Define the linear operator T on $\mathbb{R}^3$ that maps the vectors, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ to $(1,1,1)$, $(0,1,-1)$ and $(1,2,0)$ respectively f is $T(1,1,-1)$, $T(2,2,-2)$.

Let $(x_1, x_2, x_3) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$$\therefore T(x_1, x_2, x_3) = x_1 T(1,0,0) + x_2 T(0,1,0) + x_3 T(0,0,1)$$

$$= x_1(1,1,1) + x_2(0,1,-1) + x_3(1,2,0)$$

$$= (x_1 + x_3,\ x_1 + x_2 + 2x_3,\ x_1 - x_2)$$

$T(1,1,-1) = (0,0,0)$

$T(2,2,-2) = (0,0,0)$

$x_1 + x_3 = 1$, $x_1 + x_2 + 2x_3 = 2$,

$x_1 - x_2 = 1$

$\therefore x_2 + x_3 = 0$

$\therefore x_1 + x_2 + 2x_3 = 1 \neq 2$