Vector Space

A nonempty set V is said to form a vector space over the field F if i) there is a binary composition $(+)$ on V, called addition satisfying the conditions

1. $\alpha+\beta \in V$ for all $\alpha, \beta \in V$
2. $\alpha+\beta = \beta+\alpha$ for all $\alpha, \beta \in V$
3. $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ for all $\alpha, \beta, \gamma \in V$
4. There exists an element θ in V s.t. $\alpha+\theta = \alpha$ for all $\alpha \in V$
5. for each α in V there exists an element $-\alpha$ in V s.t. $\alpha+(-\alpha)=\theta$

and ii) There is an external composition of F with V, called multiplication by real no. satisfying the conditions

6. $c\alpha \in V$ for all $c \in F$, all $\alpha \in V$
7. $c(d\alpha) = (cd)\alpha$ for all $c, d \in F$, all $\alpha \in V$
8. $c(\alpha+\beta) = c\alpha + c\beta$ for all $c \in F$, $\alpha, \beta \in V$
9. $(c+d)\alpha = c\alpha + d\alpha$ for all $c, d \in F$, $\alpha \in V$
10. $1\alpha = \alpha$, 1 being the identity element in F.

Examples Real vector space $\mathbb{R}^n$

Let V be the set of all ordered n-tuples $\{(a_1, a_2, \cdots, a_n) / a_i \in \mathbb{R}\}$

Let $+$ be a composition on V called addition, defined by

$(a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$ and

an external composition of $\mathbb{R}$ with V, called multiplication by real numbers be defined by

$$c(a_1, a_2, \cdots, a_n) = (ca_1, ca_2, \cdots, ca_n), \; c \in \mathbb{R}$$

Then V is said to be real vector space and it is denoted by $\mathbb{R}^n$

Real vector space $\mathbb{C}$ $\mathbb{C}$ is the set of all complex nos.

$\{a+ib / a, b \in \mathbb{R}, i=\sqrt{-1}\}$. Let $+$ be a composition on $\mathbb{C}$ called addition defined by $(a+ib)+(c+id) = a+c+i(b+d)$ and an external composition of $\mathbb{R}$ with $\mathbb{C}$ called multiplication by a real no. be defined by

$c(a+ib) = ca + icb$, $c \in R$. Then C is a real vector space.

3. Real vector space P_n. Let V be the set of all real polynomials of degree $\leq n$. A real polynomial f of degree r is $f = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r$ where $a_0, a_1, \ldots, a_r$ are real numbers.

Let $+$ be a composition on V called addition defined by

$$(a_0 + a_1 x + \cdots + a_m x^m) + (b_0 + b_1 x + \cdots + b_r x^r)$$
$$= (a_0+b_0) + (a_1+b_1)x + \cdots + (a_m+b_m)x^m + b_{m+1}x^{m+1} + \cdots + b_r x^r, \quad m < r$$
$$= (a_0+b_0) + (a_1+b_1)x + \cdots + (a_r+b_r)x^r + b_{r+1}x^{r+1} + \cdots + b_m x^m, \quad r < m$$
$$= (a_0+b_0) + (a_1+b_1)x + \cdots + (a_m+b_m)x^m \quad \text{if } r = m.$$

and an external composition of $\mathbb{R}$ with V, called multiplication of polynomials by real nos, be defined by

$$c(a_0 + a_1 x + \cdots + a_r x^r) = ca_0 + (ca_1)x + \cdots + (ca_r)x^r,$$ Then

V is a real vector space and it is denoted by P_n

Result :- In a vector space V over a field F,

i) $0.\alpha = \theta$ for all $\alpha \in V$

ii) $c.\theta = \theta$ for all $c \in F$

iii) $(-1)\alpha = -\alpha$

iv) $c\alpha = \theta \Rightarrow c = 0$ or $\alpha = \theta$.

Proof :- 0 is a zero element in F.

$$0 + 0 = 0$$
$$\Rightarrow (0+0)\alpha = 0.\alpha$$
$$\Rightarrow 0\alpha + 0\alpha = 0.\alpha$$
$$\Rightarrow -0.\alpha + (0.\alpha + 0.\alpha) = -0.\alpha + 0.\alpha$$
$$\Rightarrow (-0.\alpha + 0.\alpha) + 0.\alpha = \theta$$
$$\Rightarrow \theta + 0.\alpha = \theta \Rightarrow 0.\alpha = \theta.$$

ii) θ is a ~~ze~~ zero element in V $\therefore \theta + \theta = \theta$

$$\Rightarrow c(\theta + \theta) = c.\theta \Rightarrow c.\theta + c.\theta = c.\theta$$
$$\Rightarrow -c\theta + (c\theta + c\theta) = -c\theta + c.\theta \Rightarrow (-c\theta + c\theta) + c\theta = \theta$$

$\Rightarrow \theta + c.\theta = \theta \Rightarrow c.\theta = \theta$.

iii) we have $\theta = 0.\alpha = [1+(-1)].\alpha$

$= 1\alpha + (-1)\alpha$

or, $-\alpha + \theta = -\alpha + (\alpha + (-1)\alpha)$

or, $-\alpha = (-\alpha+\alpha) + (-1)\alpha = \theta + (-1)\alpha = (-1)\alpha$

iv) Let $c\alpha = \theta$ and $c \neq 0$

$\therefore c^{-1}$ exists in F

Now, $c\alpha = \theta \Rightarrow c^{-1}(c\alpha) = c^{-1}\theta = \theta$

$\Rightarrow (c^{-1}c)\alpha = c^{-1}\theta \Rightarrow 1.\alpha = c^{-1}\theta = \theta \Rightarrow \alpha = \theta$

Contrapositively, $c.\alpha = \theta$ and $\alpha \neq \theta \Rightarrow c = 0$

$\therefore c\alpha = \theta \Rightarrow$ either $c = 0$ or, $\alpha = \theta$.

<u>Subspace</u>:- Let V be a vector space over a field F with respect to addition $(+)$ and multiplication by elements of $F(\cdot)$

Let W be a nonempty subset of V. If W forms a vector space over F w.r.to $+$ and $\cdot$ then W is said to be a subvector space or subspace of V

Result:- A non-empty subset W of a vector space V over a field F is a subspace of V iff

i) $\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$

ii) $\alpha \in W, c \in F \Rightarrow c\alpha \in W$

<u>Ex</u>. Let S be the set of all solutions of the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0, \quad a_{ij} \in \mathbb{R}$$

Sol^n: $(0,0,0)$ is a solution of the system of equations. Therefore S is a nonempty subset of $\mathbb{R}^3$.

Let $\alpha = (u_1, u_2, u_3) \in S$, $\beta = (v_1, v_2, v_3) \in S$

Then $a_{11}u_1 + a_{12}u_2 + a_{13}u_3 = 0$, $a_{11}v_1 + a_{12}v_2 + a_{13}v_3 = 0$

$a_{21}u_1 + a_{22}u_2 + a_{23}u_3 = 0$, $a_{21}v_1 + a_{22}v_2 + a_{23}v_3 = 0$

$\therefore$ $a_{11}(u_1+v_1) + a_{12}(u_2+v_2) + a_{13}(u_3+v_3) = 0$

$a_{21}(u_1+v_1) + a_{22}(u_2+v_2) + a_{23}(u_3+v_3) = 0$

This implies that $(u_1+v_1, u_2+v_2, u_3+v_3)$ is a solution of the system of equations.

Therefore, $\alpha \in S, \beta \in S \Rightarrow \alpha+\beta \in S$ .... (i)

Let c be a real no. Then $c\alpha \in S$ because (cu_1, cu_2, cu_3) is a solution of the system. ..... (ii)

From i) and ii) it follows that S is a subspace of $\mathbb{R}^3$.

Ex. Let S be the set of all real ordered triplets

$$\{(x,y,z) \mid x^2+y^2 = z^2\}$$

Soln Then S is a nonempty subset of $\mathbb{R}^3$ because S contains the triplets $(0,0,0)$. Let us examine if S is a subspace of $\mathbb{R}^3$

Let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in S$

Then, $a_i, b_i \in \mathbb{R}$ and $a_1^2 + a_2^2 = a_3^2$, $b_1^2 + b_2^2 = b_3^2$

Now, $\alpha+\beta = (a_1+b_1, a_2+b_2, a_3+b_3)$

$\alpha+\beta$ may not belong to S because

$(a_1+b_1)^2 + (a_2+b_2)^2 \neq (a_3+b_3)^2$.

For example, let $\alpha = (3,-4,5)$, $\beta = (-3,4,5)$. Then,

$\alpha \in S, \beta \in S$ but $\alpha+\beta = (0,0,10) \notin S$

$\therefore$ S is not a subspace of $\mathbb{R}^3$.

Ex Let V be a vector space over a field F and let α be a fixed vector of V. Then the set $W = \{c\alpha, c \in F\}$ forms a subspace of V.

Remark

The intersection of two subspaces of a vector space V over a field F is a subspace of V.

Proof :- Let W_1 and W_2 be two subspaces of V. $W_1 \cap W_2$ is not empty because $\theta \in W_1 \cap W_2$

Let α_1 and α_2 be elements of $W_1 \cap W_2$

Then $\alpha_1, \alpha_2 \in W_1$ and $\alpha_1, \alpha_2 \in W_2$

Since W_1 is a subspace of V, i) $\alpha_1 + \alpha_2 \in W_1$
and ii) $c\alpha_1 \in W_1$, c being a scalar in F

Since W_2 is a subspace of V i) $\alpha_1 + \alpha_2 \in W_2$
ii) $c\alpha_1 \in W_2$, c being a scalar in F

Therefore $\alpha_1 + \alpha_2 \in W_1 \cap W_2$ and $c\alpha_1 \in W_1 \cap W_2$.

This implies that $W_1 \cap W_2$ is a subspace of V.

Remark :-

The union of two spaces of V is not in general a subspace of V.

Proof :- Let $\alpha, \beta \in V$ and let $W_1 = \{c\alpha / c \in F\}$, $W_2 = \{c\beta / c \in F\}$

Then W_1 and W_2 are two subspaces of V.

Let $\xi \in W_1$ and $\eta \in W_2$

Then $\xi = r\alpha$, $\eta = s\beta$ for some scalars $r, s \in F$

Now, $\xi + \eta \,(= r\alpha + s\beta)$ is not in general, an element of W_1 or W_2

so, $\xi + \eta \notin W_1 \cup W_2$ in general

Therefore $W_1 \cup W_2$ may not be a subspace of V

Remark :- If U, W be two subspaces of a vector space V over a field F, then the subset $\{u + w \mid u \in U, w \in W\}$ forms a subspace of V.

Proof :- Let S be a subset $\{u + w\}$ where $u \in U$ and $w \in W$

Let α_1 and α_2 be elements of S

Therefore $\alpha_1 = u_1 + w_1$ for some $u_1 \in U, w_1 \in W$

$\alpha_2 = u_2 + w_2$ for some $u_2 \in U, w_2 \in W$

Now, $\alpha_1 + \alpha_2 = u_1 + u_2 + (w_1 + w_2)$

$= u_3 + w_3$ where $u_3 = u_1 + u_2 \in U, w_3 = w_1 + w_2 \in W$

Therefore $\alpha_1 + \alpha_2 \in S$

Let c be a scalar in F

Then $c\alpha_1 = c(u_1 + w_1) = cu_1 + cw_1$

$= u' + w'$ where $u' = cu_1 \in U, w' = cw_1 \in W$

Therefore $c\alpha_1 \in S$. So, S is a subspace of V

Note:- This subspace is called the linear sum of the subspaces U and W and is denoted by $U + W$.

Result This subspace $U + W$ is the smallest subspace of V containing the subspaces U and W.

Proof: Let P be any subspace of V containing the subspaces U and W. We shall prove that $U + W$ is a subset of P.

Let α be an element of $U + W$

$\therefore \alpha = u_1 + w_1$ where $u_1 \in U$ and $w_1 \in W$

Since $U \subseteq P$, $u_1 \in P$ and since $W \subseteq P$, $w_1 \in P$

Since P is a subspace of V and $u_1, w_1 \in P$ so, $u_1 + w_1 \in P$

$\therefore \alpha \in P$. Thus $\alpha \in U + W \Rightarrow \alpha \in P$ $\therefore U + W \subseteq P$. Therefore $U + W$ is the smallest subspace containing U and V.

Result! Let S be a finite set of vectors in a vector space V over a field F. The set of all linear combinations of the vectors in S forms a subspace of V and this is the smallest subspace containing S

Proof: Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the elements of S. Let W be the set of all linear combinations of $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then $W = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n\}$

subsets of a vector space V

$c_1, c_2, \cdots, c_n \in F$. W is a nonempty subset of V since $\alpha_1 \in W$.
Let $\alpha, \beta \in W$. Then $\alpha = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n$ for some scalars $r_i \in F$, $\beta = s_1\alpha_1 + s_2\alpha_2 + \cdots + s_n\alpha_n$ for some scalars $s_i \in F$

$\therefore \alpha + \beta = (r_1+s_1)\alpha_1 + (r_2+s_2)\alpha_2 + \cdots + (r_n+s_n)\alpha_n$
$= t_1\alpha_1 + t_2\alpha_2 + \cdots + t_n\alpha_n$ where $t_i = r_i + s_i \in F$

$\therefore \alpha + \beta \in W$, ... (i)

Let c be a scalar in F, then $c\alpha = (cr_1)\alpha_1 + (cr_2)\alpha_2 + \cdots + (cr_n)\alpha_n = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n$ where $d_i = cr_i \in F$

$\therefore c\alpha \in W$ .... (ii)

From (i) & (ii) it follows that W be a subspace of V.
Let P be any subspace containing the set S. Let $\xi \in W$.
Then $\xi = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$ for some scalars $x_i \in F$.
Since P is a subspace containing α_i and $x_i \in F$, $x_i\alpha_i \in P \ \forall i$
Since P is a subspace and $x_1\alpha_1, x_2\alpha_2, \cdots, x_n\alpha_n \in P$. Therefore $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n \in P$, $\therefore W \subseteq P$
So, W is a smallest subspace containing S.

Def: The smallest subspace containing a finite set S of vectors of a vector space V is said to be the linear span of S and is denoted by L(S)

Result: If S and T be two finite subset of vectors of a vector space V and $S \subseteq T$, then $L(S) \subseteq L(T)$.

Proof: Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and ξ be a vector belonging to L(S). Therefore $\xi = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n$ for some scalars $r_i \in F$
Now $\alpha_1 \in S \Rightarrow \alpha_1 \in T$ $\therefore \alpha_1 \in L(T)$ and since L(T) is a subspace, $r_1\alpha_1 \in L(T)$, $r_2\alpha_2 \in L(T)$, $\cdots$, $r_n\alpha_n \in L(T)$. $\therefore \xi \in L(T)$
Thus $\xi \in L(S) \Rightarrow \xi \in L(T)$, $\therefore L(S) \subseteq L(T)$.

Result: If S and T be two finite subsets of a vector space V over a field F, and each element of T be a linear combination of the vectors of S then $L(T) \subseteq L(S)$.

Proof: $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $T = \{\beta_1, \beta_2, \cdots, \beta_m\}$ and Let

$\beta_i = c_{i1}\alpha_1 + c_{i2}\alpha_2 + \cdots + c_{in}\alpha_n$ for some c_{ij} in F. Let

$\xi = p_1\beta_1 + p_2\beta_2 + \cdots + p_m\beta_m$ be an arbitrary element of T

Then ξ can be expressed as $d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n$, $d_i \in F$

$\therefore \xi \in L(T) \Rightarrow \xi \in L(S)$

$\therefore L(T) \subseteq L(S)$.

Ex: Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1,2,3)$, $\beta = (3,1,0)$. Examine if $\gamma = (2,1,3)$, $\delta = (-1,3,6)$ are in the subspace.

Soln: $L(\alpha, \beta)$ is the set of vectors $\{c\alpha + d\beta\}$ where c, d are real numbers, $c\alpha + d\beta = c(1,2,3) + d(3,1,0)$

$= (c+3d, 2c+d, 3c)$

If $\gamma \in L(\alpha, \beta)$, then there must be real numbers c and d such that $(2,1,3) = (c+3d, 2c+d, 3c)$

$\therefore c + 3d = 2,\ 2c + d = 1,\ 3c = 3$

These equations are inconsistent and so, γ is not in $L(\alpha, \beta)$

If $\delta \in L(\alpha, \beta)$, then there must be real numbers c and d such that $(-1, 3, 6) = (c+3d, 2c+d, 3c)$

$\therefore c + 3d = -1,\ 2c + d = 3,\ 3c = 6$

These equations are consistent, giving $c = 2$, $d = -1$

$\therefore \delta = 2(1,2,3) - 1(3,1,0)$ showing that $\delta \in L(\alpha, \beta)$

Ex: Let $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \alpha+\beta\}$ where $\alpha, \beta \in \mathbb{R}^n$

Show that $L(S) = L(T)$

Sol^n: Since $S \subseteq T$, $L(S) \subseteq L(T)$ .... (1)

Again, each element of T can be expressed as linear combination of α and β, the elements of S. So by the previous result, $L(T) \subseteq L(S)$. So from (1), we can write $L(S) = L(T)$.

Linear Transformation (linear mapping)

Let V and W be vector spaces over a field F. A mapping $f: V \longrightarrow W$ is said to be a linear transformation if it satisfies the following properties:

1. $f(\alpha+\beta) = f(\alpha) + f(\beta)$ for all $\alpha, \beta \in V$
2. $f(c\alpha) = cf(\alpha)$ for all $c \in F$, all $\alpha \in V$

In particular if $W = V$, then f is said to be a linear operator on V.

Ex. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $T(a_1, a_2, a_3) = (a_1, a_2)$

Sol^n: Let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in \mathbb{R}^3$

Then, $T(\alpha+\beta) = T(a_1+b_1, a_2+b_2, a_3+b_3)$

$= (a_1+b_1, a_2+b_2) = (a_1, a_2) + (b_1, b_2)$

$= T(\alpha) + T(\beta)$

For all real c, $T(c\alpha) = T(ca_1, ca_2, ca_3)$

$= (ca_1, ca_2) = c(a_1, a_2) = cT(\alpha)$

$\therefore$ T is a linear transformation.