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Vector Space

A non-empty set V is said to form a vector space over a field F if —

(i) There is a binary composition $(+)$ on V called addition satisfying the condition:

(a) $\alpha + \beta \in V$ for all $\alpha, \beta \in V$

(b) $\alpha + \beta = \beta + \alpha \quad \forall \alpha, \beta \in V$.

(c) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \quad \forall \alpha, \beta, \gamma \in V$.

(d) There exist an element θ in V s.t. $\alpha + \theta = \alpha \quad \forall \alpha \in V$.

(e) for every α, there exists $-\alpha$ in V s.t. $\alpha + (-\alpha) = \theta$

and (ii) there is an external composition of F with V, called multiplication by real number satisfying condition

(f) $c\alpha \in V \quad \forall\, c \in F,\ \forall\, \alpha \in V$

(g) $c(d\alpha) = (cd)\alpha \quad \forall\, d \in F,\ \alpha \in V$

(h) $c(\alpha + \beta) = c\alpha + c\beta \quad \forall\, c \in F,\ \alpha, \beta \in V.$

(i) $(c + d)\alpha = c\alpha + d\alpha \quad \forall\, c, d \in F,\ \alpha \in V$

(j) $1 . \alpha = \alpha$, being identity element in F.

$\boxed{(V, R, +, \cdot)}$ is charging

Real Vector Space R^n

(a) Let V be the set of all ordered n-tuples $\{(a_1, a_2, \ldots, a_n) \mid a_i \in R\}$

Let + be a composition on V called addition, satisfying $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.

and an external composition of R with V called multiplication by real ~~number~~ is defined by $c(a_1, a_2 \ldots a_n) = (ca_1, ca_2 \ldots ca_n)$, $c \in R$.

Then, V is said to be real vector ~~number~~ space and it is denoted by R^n.

Real Vector Space (complex no.): C is the set of all complex number.

$\{a + ib \mid a, b \in R;\ i = \sqrt{-1}\}$

Addition is defined by $(a + ib) + (c + id) = a + c + i(b + d)$

and multiplication is defined by $c(c + ib) = cc + icb$, $c \in R$, then C is a real vector space.

Real vector space P_n (polynomial): Let V be the set of all real polynomials of degree $\le n$.

A real polynomial f of degree n is

$f = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ where $a_0, a_1, \ldots a_n$, ~~are~~ are real no.s.

~~real no.s~~

Addition is defined by:

$$\left(a_0 + a_1x + a_2x^2 + \dots + a_mx^m\right) + \left(b_0 + b_1x + \dots + b_nx^n\right);$$

$m < n$

$$= a_0 + b_0 + (a_1 + b_1)x + \dots (a_m + b_m)x^m + b_{m+1}x^{m+1} + \dots + b_nx^n.$$

OR,

$$(a_0 + a_1x + \dots + a_mx^m) + (b_0 + b_1x + \dots + b_nx^n) \quad ; \; m > n$$

$$= a_0 + b_0 + (a_1 + b_1)x + \dots + (a_n + b_n)x^n + a_{n+1}x^{n+1} + \dots + a_mx^m$$

OR

$$= (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_m + b_m)x^m \text{ if } n = m,$$

Multiplication is defined by $\Rightarrow$

$$c(a_0 + a_1x + \dots + a_nx^n) = ca_0 + ca_1x + \dots + ca_nx^n.$$

Q) In a vector space V find.

(i) $0.\alpha = \theta \;\; \forall \alpha \in V.$

(ii) $c.\theta = \theta \;\; \forall c \in F.$

(iii) $(-1)\alpha = -\alpha$

(iv) $c\alpha = \theta \Rightarrow c = 0$ or $\alpha = \theta$.

Proof: 0 is a zero element in F.

$0 + 0 = 0.$

$\Rightarrow (0 + 0)\alpha = 0.\alpha$

$\Rightarrow 0\alpha + 0\alpha = 0\alpha$

$\Rightarrow (-0\alpha + 0\alpha) + 0\alpha = -0\alpha + 0\alpha.$

$\Rightarrow \theta + 0\alpha = \theta$

$\Rightarrow 0\alpha = \theta.$

(ii) $\theta + \theta = \theta$.

$\Rightarrow c(\theta + \theta) = c\theta$

$\Rightarrow c\theta + c\theta = c\theta$

$\Rightarrow (-c\theta + c\theta) + c\theta = -c\theta + c\theta$

$\Rightarrow \theta + c\theta = \theta$

$\Rightarrow c\theta = \theta$.

(iii) where, $\theta = 0.\alpha = (1 + (-1))\alpha = 1\alpha + (-1)\alpha$

$\Rightarrow -\alpha + \theta = (-\alpha + \alpha) + (-1)\alpha$

$\Rightarrow -\alpha = \theta + (-1)\alpha = (-1)\alpha$.

(iv) Let $c\alpha = \theta$ where $c \neq 0$.

$\therefore c^{-1}$ exists in F.

Now, $c\alpha = \theta \Rightarrow c^{-1}(c\alpha) = c^{-1}\theta = \theta$.

$\Rightarrow (c^{-1}c)\alpha = \theta$; $\Rightarrow \alpha = \theta$.

Contrapositively, $c.\alpha = \theta$ and $\alpha \neq \theta \Rightarrow c = 0$.

~~Let $c.\alpha = \theta$ and $\alpha \neq \theta \Rightarrow c = 0$.~~

<u>Subspace</u>:- Let V be a vector space over a field F w.r.t. addition (+) and multiplication by elements of F (·). Let W be a non-empty element of V. If W form a vector space over a field F w.r.t. ~~addition~~ + and. then W is said to be a subspace V over a field F.

~~Result~~ <u>Remark</u>: A non-empty subset W of a vector space V over a field F is a subspace of V if (i) $\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$
(ii) $\alpha \in W, c \in F \Rightarrow c\alpha \in W$.

Ex:- Let S be the set of all solution of the system of eqn.

$a_{11}x + a_{12}x_2 + a_{13}x_3 = 0$.

$a_{21}x + a_{22}x_2 + a_{23}x_3 = 0$, $a_{ij} \in R$

Soln: $(0, 0, 0)$ is a soln of the system of eqn. therefore S is a non-empty s______ of R^3.

Let $\alpha = (u_1, u_2, u_3) \in S$, $\beta = (v_1, v_2, v_3) \in S$.

Now, $a_{11}u_1 + a_{12}u_2 + a_{13}u_3 = 0$ $\quad$ $a_{11}v_1 + a_{12}v_2 + a_{13}v_3 = 0$

$a_{21}\cancel{x}u_1 + a_{22}u_2 + a_{23}u_3 = 0$ $\quad$ $a_{21}v_1 + a_{22}v_2 + a_{23}v_3 = 0$

$\therefore \quad a_{11}(u_1+v_1) + a_{12}(u_2+v_2) + a_{13}(u_3+v_3) = 0.$

$a_{21}(u_1+v_1) + a_{22}(u_2+v_2) + a_{23}(u_3+v_3) = 0.$

This implies that $(u_1+v_1, u_2+v_2, u_3+v_3)$ is a solⁿ of the system of equation.

$\therefore \quad \alpha \in S, \ \beta \in S \implies \alpha+\beta \in S$ ——— (i)

Let c be a real no. then, $c\alpha \in S$ because (cu_1, cu_2, cu_3) is a solution of the system — (ii)

From (i) and (ii) it further that S is a solution of R^3.