Skew Symmetric determinant

29/11/23

- Any even order skew symmetric determinant is always Perfect square.

$$\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2$$

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = 0 - a\begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 & e \\ -b & -d & f \\ -c & -e & 0 \end{vmatrix}$$

$$+ c\begin{vmatrix} -a & 0 & d \\ -b & -d & 0 \\ -c & -e & -f \end{vmatrix}$$

$$= -a\left[-af^2 - dcf + ebf\right] + b\left[-aef + e(-be - cd)\right]$$

$$+ c\left[-adf + d(bc - cd)\right]$$

$$= a^2f^2 + adcf - aebf - abef + b^2e^2 - becd + acdf$$
$$- becd + c^2d^2$$

$= a^2f^2 + 2acdf - 2acbf - \cancel{2b}\,2acbf - 2becd + b^2c^2 + c^2d^2$

$= (af - be + cd)^2$

Adjoint or adjugate of a determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$\text{Cofactor} = (\text{minor})(-1)^{i+j}$$

$$\boxed{\Delta' = \Delta^2}$$

$$\Delta\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_4 + c_2C_4 & + a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

$$\therefore \Delta' = \frac{\Delta^3}{\Delta} = \Delta^2.$$

$$\therefore \boxed{\Delta' = \Delta^{n-1}}$$

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \frac{1}{a_1} \begin{vmatrix} a_1 A_1 & B_1 & C_1 \\ a_1 A_2 & B_2 & C_2 \\ a_1 A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \frac{1}{a_1} \begin{vmatrix} a_1 A_1 + b_1 B_1 + c_1 C_1 & B_1 & C_1 \\ a_1 A_2 + b_1 B_2 + c_2 C_2 & B_2 & C_2 \\ a_1 A_3 + b_1 B_3 + c_3 C_3 & B_3 & C_3 \end{vmatrix} \quad C_1' \to C_1 + b_1 C_2 + c_1 C_3$$

$$= \frac{1}{a_1} \begin{vmatrix} 0 & B_1 & C_1 \\ 0 & B_2 & C_2 \\ 0 & B_3 & C_3 \end{vmatrix}$$

$$= 0$$

Cramer's Rule

$$a_1 x_1 + b_1 y_1 + c_1 z_1 = d_1$$
$$a_2 x_1 + b_2 y_1 + c_2 z_1 = d_2$$
$$a_3 x_1 + b_3 y_1 + c_3 z_1 = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$x_1 \Delta = \begin{vmatrix} x_1 a_1 & b_1 & c_1 \\ x_1 a_2 & b_2 & c_2 \\ x_1 a_3 & b_3 & c_3 \end{vmatrix}$$

$$c_1' = c_1 + c_2 y_1 + c_3 z_1$$

$$= \begin{vmatrix} x_1 a_1 + b_1 y_1 + c_1 z_1 & b_1 & c_1 \\ x_1 a_2 + b_2 y_1 + c_2 z_1 & b_2 & c_2 \\ x_1 a_3 + b_3 y_1 + c_3 z_1 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta x$$

$$\therefore \quad x_1 = \frac{\Delta x}{\Delta}$$

$$y_1 = \frac{\Delta y}{\Delta}$$

$$z_1 = \frac{\Delta z}{\Delta}$$

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Q. Solve using Cramer rule

$$3x + y + 2z = 3$$
$$x - 5y - z = 5$$
$$2x + 3y + 2z = 0$$

$$= \begin{vmatrix} 3 & 1 & 2 \\ 1 & -5 & -1 \\ 2 & 3 & +2 \end{vmatrix} = 3(-10+3) - 1(2+2) + 2(3+10)$$

$$= -21 - 4 + 26$$

$$= 1$$

$$\Delta x = \begin{vmatrix} 3 & 1 & 2 \\ 5 & -5 & -1 \\ 0 & 3 & 2 \end{vmatrix} = 3(-10+3) - 1(10) + 2(15)$$

$$= -21 - 10 + 30$$

$$= -1$$

$$\therefore x = \frac{\Delta x}{\Delta} = -1$$

$$\Delta y = \begin{vmatrix} 3 & 3 & 2 \\ 1 & 5 & -1 \\ 2 & 0 & 2 \end{vmatrix} = 3(10) - 3(2+2) + 2(0-10)$$

$$= \cancel{30 - 36 - 20}$$

$$= \cancel{26}\ 30 - 12 - 20$$

$$= -2$$

$$y = \frac{\Delta y}{\Delta} = \frac{-2}{1} = -2$$

$$z_1 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & -5 & 5 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= 3(0-15) + 1(0-10) + 3(3+10)$$

$$= -45 + 10 + 39$$

$$= \cancel{-16}\ 4$$

$$\therefore z = \frac{\Delta_z}{\Delta} = 4$$

$$\therefore (x=-1,\ y=-2,\ z=4)$$

$\begin{array}{r} 13 \\ 3 \\ \hline 39 \end{array}$ $\begin{array}{r} 55 \\ 39 \\ \hline 16 \end{array}$

Evaluate – $\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & b^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix}$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

$$= \{-c(-ba) + b(ca)\}^2$$

$$= (2abc)^2$$

$$= 4a^2b^2c^2.$$

Any square matrix can be exp... sum of symmetric and skew symmetric matrix

$$A = \frac{1}{2}\left(\underbrace{A + A^T}_{\text{Symmetric}} + \underbrace{A - A^T}_{\text{Skew symmetric}}\right)$$

$$\begin{cases} (A+A^T)^T = A^T + A = A + A^T \Rightarrow \text{So it is symmetric} \\ (A-A^T)^T = A^T - A = -(A-A^T) \Rightarrow \text{So it is skew symmetric} \end{cases}$$

$\therefore$ Hence it is proved (proved).

$A = P + Q$

$A^T = P^T + Q^T = P - Q$

$2P = A + A^T$

$\Rightarrow P = \frac{1}{2}(A + A^T)$ Similarly $Q = \frac{1}{2}(A - A^T)$

$\therefore$ P, Q are unique.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad adj(A) = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A \cdot \operatorname{adj} A = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \Delta \cdot I$$

$$\therefore \quad A \cdot \frac{\operatorname{adj}(A)}{\Delta} = I$$

$AB = BA = I$

Then $B = A^{-1}$

$$\therefore \quad \boxed{A^{-1} = \frac{\operatorname{adj}(A)}{\Delta}}$$

Matrix

$A = [a_{ij}]$ $\quad\quad \Delta = \det(A)$

If $a_{ij} = a_{ji} \Rightarrow A$ is symmetric

Adjoint of A

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$A' = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$\therefore A \cdot \text{adj}(A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \Delta I_3$$

$$= A \cdot \frac{\text{adj}(A)}{\Delta} = I_3$$

$$\therefore \boxed{A^{-1} = \frac{\text{adj}(A)}{\Delta}}$$

P. $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ $\quad adj(A) = \begin{bmatrix} 1 & -2 & -1 \\ -5 & 8 & 6 \\ 3 & -5 & -4 \end{bmatrix}$

$$\Delta = 2(2-1) - 3(4+1) + 4(2+1)$$

$$= 2 - 15 + 12$$

$$= -1$$

$$\therefore A^{-1} = \frac{adj(A)}{\Delta} = \begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}$$

P. Solve,

$$x - y + 2z = 1$$

$$x + y + z = 2$$

$$2x - y + z = 5$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & +1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad adj(A) = \begin{bmatrix} 2 & -1 & -3 \\ +1 & -3 & 1 \\ -3 & -1 & 2 \end{bmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\therefore \{AX = D\}$$

$det(A) = -5$

$$A^{-1}AX = A^{-1}D$$

$$\therefore \boxed{X = A^{-1}D}$$

$$A^{-1} = \frac{1}{-5}\begin{bmatrix} 2 & -1 & -3 \\ 1 & -3 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad X = \frac{-1}{-5}\begin{bmatrix} 2 & -1 & -3 \\ 1 & -3 & 1 \\ -3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore \quad x = 3$

$y = 0$

$z = -1$

Q. $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ Show that $A^3 = A^{-1}$

Hint: Calculate A^4 as $A^2 \times A^2$ which is equal to I.

$A[A^3] = A \cdot A^{-1}$

$\Rightarrow A^4 = I$

$\therefore \boxed{A^3 = A^{-1}}$