Isomorphism theorems.

First isomorphism theorem.

$$f: G \longrightarrow G'$$

$$\ker f = \{ x \in G : f(x) = e_{G'} \}$$ normal subgroup of G

$$\operatorname{Im} f = \{ f(x) : x \in G \} \subseteq G'$$

$$G/\ker f = \{ a \ker f : a \in G \}$$
$$= \{ ak : a \in G, k \in \ker f \}$$

Let G and G' be two groups and $f: G \to G'$ be an epimorphism. Then $G/\ker f \cong G'$.

If $f: G \to G'$ is homomorphism only then

$$G/\ker f \cong \operatorname{Im} f.$$ where $\operatorname{Im} f \subseteq G'$

if mapping is surjective (epimorphism) then $\operatorname{Im} f = G'$

$$\therefore G/\ker f \cong G'$$

Proof:-

we define a mapping $\phi: G/\ker f \longrightarrow G'$ by

$$\phi(a \ker f) = f(a)$$ such that $a \in G$, $a/\ker f \in G/\ker f$.

(i) to prove that ϕ is well defined.

let $a, b \in G$ such that $a \ker f = b \ker f$

we know that

if $aH = bH$ where aH and bH are two left cosets on H.

$\Rightarrow$ ~~abb~~ $a^{-1}b \in H$.

now $a\ker f = b \ker f$

$\Rightarrow a^{-1}b \in \ker f$

$\Rightarrow f(a^{-1}b) = e_{G'}$ (by definition of $\ker f$)

Since f is homomorphism $\Rightarrow f(ab) = f(a)\cdot f(b)$

$\Rightarrow f(a^{-1})f(b) = e_{G'}$

$\Rightarrow [f(a)]^{-1} f(b) = e_{G'}$

Multiplying by $f(a)$ on both sides

$f(a)[f(a)]^{-1} f(b) = e_{G'}\cdot f(a)$

$e_{G'}\cdot f(b) = e_{G'}(f(a))$

$\Rightarrow f(b) = f(a)$. $\therefore$ well-defined.

$\Rightarrow \varphi(a\ker f) = \varphi(b\ker f)$

(ii) to prove that φ is ~~a homomorphism~~ bijective.

Since φ is well-defined.

one-one $\varphi(a\ker f) = \varphi(b\ker f)$

$\Rightarrow a\ker f = b\ker f$ conversely

$\therefore \varphi$ is injective

onto to show that every element in G' has pre-image in $G/\ker f$.

let $y \in G'$

Now we know $f: G \to G'$ is epimorphism (given)

$\Rightarrow$ there exists $x \in G$ such that $f(x) = y \in G'$

Now

$$y = f(x) = \phi(x \ker f) \quad \text{by } \phi \text{ mapping}$$

$\Rightarrow$ ϕ is surjective as for any element y in G' there exist $(x \ker f)$ in $G/\ker f$ such that $y = \phi(x \ker f)$.

$\Rightarrow$ ϕ is onto.

$\Rightarrow$ ϕ is bijective.

(iii) to prove that ϕ is homomorphism.

to show that

$$\phi(a \ker f \cdot b \ker f) = \phi(a \ker f) \cdot \phi(b \ker f)$$

Now

$$\phi(a \ker f \cdot b \ker f) = \phi[(ab) \ker f] \quad \text{[multiplying two cosets gives a coset]}$$

$$= f(ab) \quad \text{[by defn of } \phi]$$

$$= f(a) \cdot f(b) \quad \text{[since } f \text{ is homomorphism]}$$

$$= \phi(a \ker f) \cdot \phi(b \ker f)$$

$\Rightarrow$ ϕ is a homomorphism

$\therefore$ ϕ is an isomorphism.

Problem 1

Show that

(i) $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

(ii) $S_n/A_n \cong \mathbb{Z}_2$

(iii) $GL(n,\mathbb{R})/SL(n,\mathbb{R}) \cong \mathbb{R}^*$

(i) Define a mapping $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ by

$$\varphi(a) = [a] \quad \text{for all } a \in \mathbb{Z}$$

we prove the above mapping bijective (one-one and onto)

$\Rightarrow$ we say φ is an epimorphism.

$\therefore$ by first isomorphic theorem

$$\frac{\mathbb{Z}}{\ker\varphi} \cong \mathbb{Z}_n$$

Next, we prove that $n\mathbb{Z}$ is $\ker\varphi$.

identity in $\mathbb{Z}_n$

$$\begin{aligned} \ker\varphi &= \{x \in \mathbb{Z} : \varphi(x) = [x] = [0]\} \\ &= \{x \in \mathbb{Z} : x - 0 = nt \text{ for all } t \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z} : x = nt\} \\ &= n\mathbb{Z} \end{aligned}$$

$\Rightarrow \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ (proved).

(ii) Prove:-

$$S_n/A_n \cong \mathbb{Z}_2$$

Define a mapping

$$\varphi : S_n \longrightarrow \mathbb{Z}_n$$

$$\varphi(\sigma) = [0] \text{ if } \sigma \text{ is even}$$

$$= [1] \text{ if } \sigma \text{ is odd.}$$

We have already proved previously that φ is epimorphism (surjective mapping & homomorphism)

$\therefore$ By first isomorphism theorem,

$$S_n/\ker\varphi \cong \mathbb{Z}_2.$$

Now we prove A_n is $\ker \varphi$

$$\ker\varphi = \{\sigma \in S_n : \varphi(\sigma) = [0]\}$$ identity of $\mathbb{Z}_2$

$$= \{\sigma \in S_n : \sigma \text{ is even}\}$$

$$= A_n.$$

$\therefore$ A_n is $\ker\varphi$

$\Rightarrow$ $S_n/\ker\varphi \cong \mathbb{Z}_2$ (proved)

Note:- $|A_n| = \dfrac{n!}{2}$

Proof

We know $\left|S_n/A_n\right| = |\mathbb{Z}_2|$ since $S_n/\ker\varphi \cong \mathbb{Z}_2$ is bijective

$$= \frac{|S_n|}{|A_n|} = |\mathbb{Z}_2| \quad \text{[By Lagrange's theorem]}$$

$\Rightarrow \quad |A_n| = \dfrac{|S_n|}{|Z_2|}$

$\Rightarrow \quad |A_n| = \dfrac{n!}{2}$ (proved)

(iii) Prove $\quad GL(n, \mathbb{R}) / SL(n, R) \cong R^*$

we define

$\varphi : GL(n, \mathbb{R}) \longrightarrow SL(n, R)$ s.t

$\varphi(A) = \det A \quad \forall \ A \in GL(n, R)$

we have proved that φ is epimorphism. (surjective & homomorphism)

to prove onto

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & \cdots \\ 0 & 1 & - & - & - \\ 0 & 0 & 1 & \cdots & \\ \vdots & & - & - & 1 \end{pmatrix} \in GL(n, R)$$

$\det A = \alpha \in R.$

$\therefore$ onto

By 1st isomorphism theorem,

$$\frac{GL(n, \mathbb{R})}{\ker\varphi} \cong R^*$$

To prove $SL(n, \mathbb{R})$ is $\ker\varphi$

$\ker\varphi = \{A \in GL(n, R) : \varphi(A) = 1\}$

$\Rightarrow \ker\varphi = \{ \det(A) = 1, \ A \in GL(n, \mathbb{R})\}$

$= SL(n, \mathbb{R})$

$\therefore \dfrac{GL(n, R)}{SL(n, R)} \cong R^*$

Th:- Every epimorphism from a group $(\mathbb{Z}, +)$ on itself is an isomorphism.

let $f : \mathbb{Z} \to \mathbb{Z}$ be an epimorphism

we know that,
A group homomorphism is a monomorphism (injective mapping) iff $\ker f$ contains $\{e_G\}$

ie $\ker f = \{e_G\}$

given f is an homomorphism

$\Rightarrow$ $\ker f$ is a normal subgroup of $\mathbb{Z}$ (domain)

and $\operatorname{Im} f$ is a subgroup of $\mathbb{Z}$ (co-domain)

~~then~~ ~~that~~ we know $\ker f = n\mathbb{Z}$ for $n \in \mathbb{N}$ or $n = 0$

now, if possible
let $\ker f = n\mathbb{Z}$ for some $n \in \mathbb{N}$

by first isomorphism theorem,

$$\mathbb{Z}/\ker f \cong \mathbb{Z}$$

$$\Rightarrow \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}$$

$$\Rightarrow \mathbb{Z}_n \cong \mathbb{Z} \quad (\text{contradition})$$

Since $\mathbb{Z}_n$ is finite group but $\mathbb{Z}$ is infinite,
Since their cardinalities are different,
$\therefore$ no bijective mapping can exist
$\therefore$ not an isomorphism

$\Rightarrow$ only possibility $\Rightarrow$ $\ker f \neq n\mathbb{Z}$ for $n \in \mathbb{N}$

but $\ker f = \underline{0}$ ie $e_{\mathbb{Z}}$

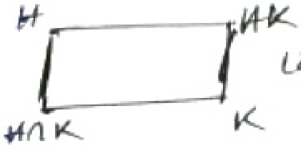
$\therefore$ it is a monomorphism.

Hence it is an isomorphism.

2nd isomorphism theorem (Proof using 1st isomorphism theorem)

Let G be a group and H, K are two subgroups of G such that K is normal in G.

Then $H/H\cap K \cong HK/K$



Lattice diagram

$H\cap K$ is normal to H $\Rightarrow$ $H/H\cap K$ exist
K is normal to HK $\Rightarrow$ HK/K exist.
and both are isomorphic.

$$\varphi : H \to HK/K$$
by $\varphi(h) = hK \quad \forall\, h\in H,\ k\in K$

prove that φ is bijective.

Now $\ker\varphi = \{h\in H : \varphi(h) = e_G\}$

$\ker\varphi = \{h\in H : hK = e_G\}$

$\ker\varphi = \{h\in H : h = k^{-1}\}$

$hk = e_G$
$hkk^{-1} = ek^{-1}$
$h = ek^{-1}$
$h = k^{-1}$

$\ker\varphi = \{h\in H : h\in K$ as $k^{-1}\in K$ only and $h = k^{-1}\}$

$\ker\varphi = H\cap K$

~~without using 1st isomorphism theorem~~

$\therefore$ By 1st isomorphism theorem
$$H/H\cap K \cong HK/K$$

3rd isomorphism theorem

Let G be a group and H, K be two normal subgroups of G such that $H \subseteq K$ (H is contained in K)

then $\dfrac{G/H}{K/H} \cong G/K$

Using 1st isomorphism theorem,

$$\phi : G/H \to G/K$$

such that $\phi(aH) = aK$ for all $aH \in G/H$

(we show that the above mapping is epimorphism. and then we show $\ker\phi = K/H$.)

To prove $\ker\phi = K/H$

$$\ker\phi = \{ xH \in G/H \text{ st. } \phi(xH) = \underbrace{K}_{\text{(identity of } K/H)} \}$$

$$= \{ xH \in G/H : xK = K \}$$

$$= \{ xH \in G/H : x \in K \}$$

$$= K/H.$$

$\Rightarrow \dfrac{G/H}{\ker\phi} \cong G/K$ (by 1st isomorphism theorem)

$\Rightarrow \dfrac{G/H}{K/H} \cong G/K$

Direct product

External direct product

Let G_1 and G_2 be two groups

considering $G_1 \times G_2 = \{(a,b) : a \in G_1, b \in G_2\}$

$(a,b)(c,d) = (\cancel{ac}, \cancel{bd})$ $\quad a, c \in G_1$, $\quad b, d \in G_2$

$(a \circ c, b * d)$ $\longleftrightarrow$ external direct product

where '$\circ$' is operation of G_1

and '$*$' is operation of G_2

eg $G_1 = R$

$G_2 = R$

$G_1 \times G_2 = R \times R = R^2 = \{(x,y) : x, y \in R\}$

[Prove $Z_m \times Z_n \cong Z_{mn}$ iff $\gcd(m,n) = 1$]

eg $Z_2 \times Z_3 \cong Z_6$

Internal direct product

let G be a group and H, K be two normal subgroups of G. If $G = HK$ and $H \cap K = \{e_G\}$

then $G \cong H \times K$

$K_4 = \{e, a, b, c\}$

$H = \{e, a\}$ ⟵ normal subgroups of K_4

$K = \{e, b\}$

$HK = \{e, a\} * \{e, b\}$

$HK = \{e, a, b, ab\}$

$H \cap K = \{e\}$

$\Rightarrow K_4 \cong H \times K$

Note

① If G is a finite cyclic group of order n then $G \cong Z_n$

② If G is infinite cyclic group then $G \cong Z$

$|G| = 6$ $\quad G \cong \mathbb{Z}_6$ or S_3

$|G| = 4$ $\quad G \cong \mathbb{Z}_4$ (~~infinite~~ cyclic)

$G \cong K_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ (non-cyclic)