1/10/23 $S \neq \rho$

ρ is an equivalence relation

$[a]_\rho = \{x \in S : a \rho x \text{ holds}\}$

Q $\rho = a \equiv b \pmod n$ $a, b \in \mathbb{Z}$

$[a]_\rho = \{x \in \mathbb{Z}, a \equiv x \pmod n\}$

$= \{a - x \text{ is div by } n\}$

$= \{x \in \mathbb{Z}, x = a + nt\}$

$[0] = \{x \in \mathbb{Z}, x = 0 + nt$ i.e $x = nt$ x is div by $n\}$

$[1] = \{x \in \mathbb{Z}, x = 1 + nt$ i.e leaves remainder 1 on dividing by $n\}$

$[2] = \{x \in \mathbb{Z}, x$ leaves remainder 2, on dividing by $n\}$

$\vdots$

$[n-1]$

Quotient Group

Let G be a group and H be a normal subgroup of G.

$G/H = \{aH : a \in G\}$ if $(G, \cdot)$

$= \{a+H : a \in G\}$ if $(G, +)$

$(aH) * (bH) = (ab)H \quad \forall\ aH, bH \in G/H$

$(G/H, *)$ is a quotient group

eg for $(\mathbb{Z}, +)$, $(n\mathbb{Z}, +)$ is a normal subgroup.

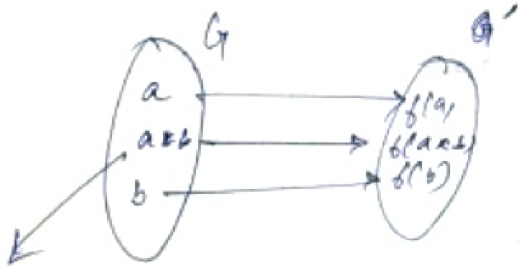
$\Rightarrow \mathbb{Z}/n\mathbb{Z} = \{a + n\mathbb{Z} : a \in \mathbb{Z}\}$

quotient group

Structure preserving mappings :- morphisms.

↳ Homomorphisms & Isomorphisms.

Let $f : (G, *) \longrightarrow (G', \circ)$



~~a * b =~~ if $f(a * b) = f(a) \circ f(b)$ then homomorphism for all $a, b \in G$

eg - $f : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}_n, +)$

$f(a) = [a] \ \forall \ a \in \mathbb{Z}$

~~for $f(\mathbb{Z}, +)$~~

$f(a+b) = [a+b] = [a] + [b]$

$= f(a) + f(b) \ \forall \ a, b \in \mathbb{Z}$

Trivial Homomorphisms.

eg. $f(a) = e_G$

for $a, b \in G$

$f(a) = e_G$

$f(b) = e_G$

Since $a, b \in G$

$\Rightarrow ab \in G$ (closure property)

$f(ab) = e_G$

$f(ab) = f(a) \cdot f(b)$ as $e_G = e_G \cdot e_G$

$(S_n, \circ)$

→ Symmetric group. (collection of bijective mappings)

Let there be a mapping

$f : (S_n, \circ) \to (\mathbb{Z}_2, +)$ $\qquad Z_n = \{[0], [1]\}$

$f(\sigma) = 0$ if σ is even

$= 1$ if σ is odd

Let $\sigma_1, \sigma_2 \in S_n$

To check if $f(\sigma_1 \circ \sigma_2) = f(\sigma_1) + f(\sigma_2)$

then homomorphism.

Case 1

σ_1 and σ_2 are even permutations

Then $\sigma_1 \circ \sigma_2$ is also even.

$f(\sigma_1 \circ \sigma_2) = [0]$

Now

$= [0] + [0]$

$= \underset{\text{even}}{f(\sigma_1)} + \underset{\text{even}}{f(\sigma_2)}$

$\therefore$ condition satisfied.

Case 2.

σ_1 and σ_2 are odd permutations.

Then $\sigma_1 \circ \sigma_2$ is ~~also~~ even

$f(\sigma_1 \circ \sigma_2) = [0]$

$f(\sigma_1) = [1]$

$f(\sigma_2) = [1]$

$f(\sigma_1) + f(\sigma_2) = [1] + [1] = [2]$

$= [0]$

$\therefore f(\sigma_1 \circ \sigma_2) = f(\sigma_1) + f(\sigma_2)$

$\therefore$ condition satisfied.

case 3

σ_1 is ~~odd~~ even and σ_2 is ~~even~~ odd

then $\sigma_1 \circ \sigma_2$ is odd

$f(\sigma_1 \circ \sigma_2) = \text{odd} = [1]$

$f(\sigma_1) = [0]$

$f(\sigma_2) = [1]$

$f(\sigma_1) + f(\sigma_2) = [0] + [1] = [1]$

$\therefore f(\sigma_1 \circ \sigma_2) = f(\sigma_1) + f(\sigma_2)$

$\therefore$ condition satisfied

$\therefore$ homomorphism exists.

case 4

σ_1 is odd and σ_2 is even.

similar as case 3.

~~General linear group.~~

$f: (GL_n(R), \cdot) \longrightarrow (R^*, \cdot)$ ie $R - \{0\}$

$f(A) = \det(A) \quad \forall A \in GL_n(R)$

$f(AB) =$ ~~det(A) · det(A)~~ $\qquad A, B \in GL(n, R)$

$= \det(AB) \qquad \therefore \det(A) \neq 0, \ \det(B) \neq 0$

$= \det A \cdot \det B$

$= f(A) \cdot f(B)$

$\therefore$ homomorphism.

$]s[2]$

Proof

Homomorphism $G \longrightarrow G'$

(i) $f(e_G) \to e_{G'}$ $\quad e_G \in G$ and $e_{G'} \in G'$

(ii) $f(a^n) = [f(a)]^n \quad \forall\, n \in \mathbb{N}$

(iii) $f(a^{-1}) = [f(a)]^{-1}$

Let: $f: G \longrightarrow G'$ be a homomorphism
then f is called a monomorphism if f is injective
f is called is called a epimorphism if f is surjective
and f is called an isomorphism if f is bijective.

i.e. $G \cong G'$

If two groups are homomorphic,
then Image $\mathrm{Im} f = \{ f(x),\ x \in G \} \longrightarrow$ subset of codomain
Kernel $\mathrm{Kn} f = \{ x \in G : f(x) = e_{G'} \}$ $\longrightarrow$ subset of G (domain)

Prove
(i) $\mathrm{Im} f$ is a subgroup of G'
(ii) $\mathrm{Kn} f$ is a normal subgroup of G

then we can construct quotient group $(G/\mathrm{Kn} f)$

Thm $\mathrm{Kn} f$ is a normal subgroup of G

subgroup test $a, b \in \mathrm{Kn} f$

then $f(a) = e_G = f(b)$ by defn of kernel

$f(ab^{-1}) = f(a) \cdot f(b^{-1})$
$= f(a) \cdot [f(b)]^{-1}$
$= e_G \cdot [e_G]^{-1}$
$= e_G \cdot e_G = e_G \quad \therefore\ ab^{-1} \in \mathrm{Kn} f$

normality test
$h \in H$
$g \in G$
$ghg^{-1} \in H$

Subgroup test:
$a, b \in H$
$\Rightarrow ab^{-1} \in H$

$\therefore$ $Ker f$ is a subgroup of G.

normal subgroup test. $g \in G$ and $h \in Ker f$

Since, $h \in Ker f$

$\Rightarrow f(h) = e_G$ by definition of kernel.

Now $f(ghg^{-1}) = f(g) \cdot f(h) \cdot f(g^{-1})$

$= f(g) \cdot f(h) \cdot [f(g)]^{-1}$

$= f(g) \cdot e_G \cdot (f(g))^{-1}$

$=$

$= f(g) \cdot [f(g)]^{-1}$

$= e_G$

$\therefore$ $ghg^{-1} \in Ker f$

$\therefore$ <u>$Ker f$ is normal subgroup of G</u>

<u>To check if a group homomorphism is injective or not.</u>

Th Let G and G' be two groups and $f: G \longrightarrow G'$ be a homorphism. then

f is injective iff $Ker f$ contains identity element

(converse is also true)

one-one $f: A \to B$

$f(a) = f(b)$

$\Rightarrow a = b$

Proof. Let f be injective

Let $x \in Ker f$

then $f(x) = e_{G'}$

Since f is homomorphic

$f(x) = e_{G'} = f(e_G)$ $\Rightarrow$ since one-one $x = e_G$

$\Rightarrow \quad \ker f = \{e_G\}$

Conversely,

supposing $\ker f$ contains e_G to show that f is one-one

let $a, b \in G$ be such that $f(a) = f(b)$

$[f(b)]^{-1} f(a)\ f(b) \cdot [f(b)]^{-1}$

$[f(b)]^{-1} f(a) = e_{G'}$

$f(b^{-1}) f(a) = e_{G'}$

$f(ab^{-1}) = e_{G'}$

$ab^{-1} \in \ker f$

$\therefore \ker f = \{e_G\}$

now

$ab^{-1} = e_G$

$ab^{-1}b = e_G b = b$

$a = b$

let $f : \mathbb{Z} \to \mathbb{Z}_n$

$f(a) = [a] \quad \forall\, a \in \mathbb{Z}$

$\ker f = ?$

$= \{x \in \mathbb{Z},\ x \text{ is divisible by } n\}$

$= \{nt\}$

$f : S_n \to \mathbb{Z}_n$

$f(\sigma) = [0]$ σ is even

$f(\sigma) = [1]$ σ is odd

$\ker f = A_n$ ↳ even permutation

$f : GL(R) \to R^*$

$f(A) = \det A$

$A \in GL(R)$

$\det A = 1$

$\ker f = SL_n(R)$

Isomorphism theorems