Subring

Necessary and sufficient condition

Let $S (\neq \phi)$ is a subset of a ring R. Then S is a subring of R iff

(i) $a - b \in S \;\forall\; a, b \in S$

(ii) $a \cdot b \in S \;\forall\; a, b \in S$

~~$(\mathbb{R}, +, \cdot)$ is a~~ s.

(1) $(\mathbb{Z}, +, \cdot)$ is a subring of $(\mathbb{Q}, +, \cdot)$ is a subring of $(\mathbb{R}, +, \cdot)$ is a subring of $(\mathbb{C}, +, \cdot)$

(2) $D[0, 1]$ is a subring of $C[0, 1]$

Note:-
A ring may have a multiplicative identity (unity) but ~~that~~ a subring may or may or not have a multiplicative identity.

eg> $(\mathbb{Z}, +, \cdot)$ has unity $= 1$
but its subring $(2\mathbb{Z}, +, \cdot)$ does not have any unity.

Examples

(i) Ring has unity but subring does not
$(\mathbb{Z}, +, \cdot)$ and subring $(2\mathbb{Z}, +, \cdot)$

(ii) Ring does not have unity but subring has.

$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$ and subring

$S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$

has identity $1_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$

(ii) Ring and sub-ring both have multiplicative identity but both are different. ie $1_R \neq 1_S$.

[Direct Product of Rings]

$$R_1 \times R_2 = \left[(a,b) : a \in R_1 \text{ and } b \in R_2 \right]$$

$$\Rightarrow (a,b) + (c,d) = (a+c, b+d)$$
$$(a,b) \cdot (c,d) = (ac, bd)$$

1st example: say we have ~~let~~ ring $R = \mathbb{Z} \times \mathbb{Z}$ and $1_R = (1,1)$

and subring $S = \mathbb{Z} \times [0]$ and $1_S = (1,0)$

2nd example

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right.$$

$$1_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

subring
$$S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$1_S = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

(iii) Ring R and subring S have same unity.

Notes:- If ring R is an integral domain, and has has a subring S.
Then if R and S have a unity (multiplicative identity) 1_R and 1_S respectively, then

$$1_R = 1_S.$$

Proof:-
Let R be an integral domain having identity 1_R and S be a subring of R having identity 1_S then we prove that $1_R = 1_S$

Let $x(\neq 0) \in S$ and $S \subseteq R$

So $x = 1_S \cdot x = 1_R \cdot x = x$

[as $x \in S$ and also $x \in R$.

So $x = 1_S \cdot x$ and $x = 1_R \cdot x$ also.]

$\therefore \quad (1_R - 1_S)\, x = 0$

Now x is non-zero i.e. $x \neq 0$ and R is an integral domain i.e. R has no divisor of zero

$\Rightarrow \quad 1_R - 1_S = 0$

$\Rightarrow \quad \underline{1_R = 1_S}$ (proved)

eg → Ring $R = (\mathbb{R}, +, \cdot)$ and subring $S = (\mathbb{Z}, +, \cdot)$ both have same multi^n identity element i.e 1.

i.e $1_R = 1_S = 1$.

Ideal

Let R be a ring and I be a ~~subring~~ non-empty subset of R, then I is called a left ideal of R if $(a-b) \in I$ and $ra \in I \ \forall \ a, b \in S$ and $r \in R$. and I is called a right ideal of R if $(a-b) \in I$ and $a \cdot r \in I \ \forall \ a, b \in S$ and $r \in R$.

Note.. An ideal is always a subring but converse may not be true.

PTO →

I is called an <u>ideal</u> if I is both left ideal as well as right ideal.

$\Rightarrow$ $(a-b) \in I$ $\forall$ $a, b \in I$

and $r.a \in I$ and $a \cdot r \in I$

$\forall$ $r \in R$.

<u>example of a subring but ~~not ideal~~.</u>

let ring $R = (\mathbb{Q}, +, \cdot)$ and subring $S = (\mathbb{Z}, +, \cdot)$

but note that S is not an ideal of R

$\frac{1}{2} \in R$ but $\frac{1}{2} \cdot 5 \notin S$

$5 \in S$

or $5 \cdot \frac{1}{2} \notin S$

$\therefore$ not an ideal.

<u>note:-</u>

In a commutative ring, left ideal and right ideal coincide. ie are equal

let $R = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$

$I_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$

<u>Example of a subring which is a left ideal but not a right ideal.</u>

let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$ and $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \in I_1$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} = \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \in I_1$

$\therefore I_1$ is left ideal $*$ - non-zero element

as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} a-x & b \\ c-y & d \end{pmatrix} \in I_1$ (calculate and find out)

To check if I_1 is also right ideal or not.

$$\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{bmatrix} * & 0 \\ k & 0 \end{bmatrix}$$

$\therefore$ not a right ideal

(calculate and see)

$\therefore$ I_1 is only left ideal not a right ideal.

<u>Example of subring which is right ideal but not a left ideal.</u>

let

$$R = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} ; \ a, b, c, d \in \mathbb{R} \right\}$$

$$I_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \ a, b \in \mathbb{R} \right\}$$

let $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in R$ $\quad \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in I_2$

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \in I_2$$

$\to$ calculate and find out

$\therefore$ right ideal

as $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} x-a & y-c \\ -b & -d \end{pmatrix} \in I_2$

But

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} ax & cy \\ bx & dy \end{bmatrix} \neq \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$$

$\notin I_2$

$\therefore$ not a left ideal

A ring R has identity 1_R and I is an ideal of ring R. Does $1_R \in I$?

If $1_R \in I$, then 'I' cannot be a proper ideal of ring R, it can ~~be~~ be trivial or improper ideal.

Proof :- ~~to show~~

(i) $I \subseteq R$ as I is a non-empty subset of R.

as I is ideal.

(ii) to show $R \subseteq I$

let $r \in R$ also

$\Rightarrow r = r.1_R \in I$ $\quad$ $r = 1_R . r \in I$

$\Rightarrow 1_R \in I$ $\quad$ $\Rightarrow 1_R \in I$ (again).

$\therefore R \subseteq I$ as for all $r \in R$ the above condition holds.

$\therefore I \subseteq R$ and $R \subseteq I \Rightarrow \underline{I = R}$

(proved).

Th:- A field F cannot have any non-zero proper ideal.

let F have two ideals $\{0\}$ and F itself.

$\{0\} \rightarrow$ trivial ~~ideal~~ ideal

$F \rightarrow$ improper ideal.

let $a (\neq) \in F$

so $a^{-1} \in F$ (by defn of 'field')

($aa^{-1} = a^{-1}a = 1_F$)

Therefore we write "a" as

$$a = 1_F\, a$$
$$= (a a^{-1}) a$$

Note:- $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$ are fields so they no ideal other than trivial and improper ideal.

But $\mathbb{Z}$ is not a field, so it has infinite ideals of form $n\mathbb{Z}$ where $n \in \mathbb{N}$

Prove that intersection of two ideals is always an ideal but union of two ideals need not be an ideal.

example:-

let ring $R = (\mathbb{Z}, +, \cdot)$

let $I_1 = 2\mathbb{Z}$

$I_2 = 3\mathbb{Z}$

Now, $2 \in I_1$ so $2 \in I_1 \cup I_2$

$3 \in I_2$ $\quad 3 \in I_1 \cup I_2$

but $2+3 = 5 \notin I_1 \cup I_2$

(4) Addition of two ideals form an ideal of the ring.

I_1 and I_2 be two ideals of R

$I_1 + I_2 = \{ a+b : a \in I_2$ and $b \in I_2 \}$

is an ideal.

(5) Product of two ideals may or may not be an Ideal.

I_1 and I_2 be two ideals of R

let $I_1 I_2 = \{ a_1 a_2 : a_1 \in I_1, a_2 \in I_2 \}$

then $I_1 I_2$ does not form an ideal.

let $a_1 a_2 \in I_1 I_2$

$c_1 c_2 \in I_1 I_2$

$a_1 a_2 + c_1 c_2$ may not be of form $x_1 x_2$

i.e may not belong to $I_1 I_2$

but let

finite sum

$I_1 I_2 = \{ \sum a_1 a_2 : a_1 \in I_1, a_2 \in I_2 \}$

then $I_1 I_2$ forms an ideal.

$\sum a_1 a_2 \in I_1 I_2$

$\sum c_1 c_2 \in I_1 I_2$

$\sum a_1 a_2 + \sum c_1 c_2 = \sum d_1 d_2 \in I_1 I_2$