

# Codimension 2 bifurcations of periodic orbits in MatCont

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In this tutorial we show how to construct the bifurcation diagram in Figure 1, originally obtained in [1] based on a model in [2]. First recall the system:

$$\begin{cases} \dot{x} = r \left(1 - \frac{x}{K}\right) x - p(x, t)y, \\ \dot{y} = ep(x, t)y - dy, \end{cases} \quad (1)$$

where  $x$  and  $y$  are the numbers of individuals, respectively of prey and predator populations or suitable (but equivalent) measures of density or biomass. The parameters present in system (1) are the intrinsic growth rate  $r$ , the carrying capacity  $K$ , the efficiency  $e$  and the death rate  $d$  of the predator. The function  $p(x, t)$  is a functional response, for which the Holling type II is chosen, with constant attack rate  $a$  and half saturation  $b(t)$  that varies periodically with period one (year), i.e.

$$p(x, t) = \frac{ax}{b(t) + x}, \quad b(t) = b_0(1 + \varepsilon \cos 2\pi t).$$

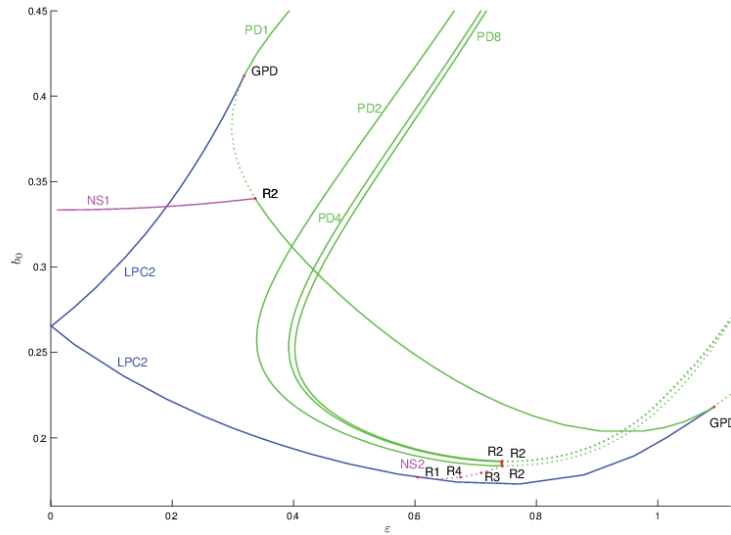


Figure 1: Bifurcation diagram of limit cycles in model (2). Blue are limit point of cycles bifurcations, green period doubling bifurcations and magenta Neimark-Sacker bifurcations. Continuous/dotted curves correspond to supercritical/subcritical bifurcations, respectively.

Instead of system (1), we consider the extended autonomous system

$$\begin{cases} \dot{x} = r \left(1 - \frac{x}{K}\right) x - \frac{axy}{b_0(1 + \epsilon u) + x}, \\ \dot{y} = e \frac{axy}{b_0(1 + \epsilon u) + x} - dy, \\ \dot{u} = u - 2\pi v - (u^2 + v^2)u, \\ \dot{v} = 2\pi u + v - (u^2 + v^2)v, \end{cases} \quad (2)$$

where the last two equations are decoupled from the first two and have a stable limit cycle with  $u(t) = \cos(2\pi t + \varphi)$ ,  $v(t) = \sin(2\pi t + \varphi)$ , and a phase shift  $\varphi$  depending on the initial conditions.

As a preliminary step, install Matlab on your computer. Then download MatCont 5.2 or higher version (see <http://sourceforge.net/projects/matcont/files/matcont>), unzip it and move the current Matlab folder to the one which contains the file `matcont.m`. Now start MatCont by typing `matcont` in the Matlab Command Window. (warning: depending on the operating system it can be necessary to first build the MEX-files from the c-files in the directory `Limitcycle`; see the documentation and readme - files).

## 1 Introduce the system in MatCont

First of all we have to create a new system. This can be done through the menu **Select -> System -> New**. We insert the system in the **System** window, and obtain the result shown in Figure 2. Note that the semicolons are optional. When pressing the OK button, the model is pre-computed (the selected derivatives

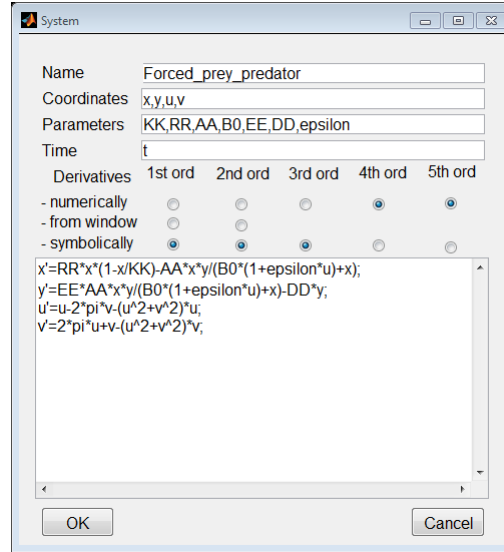


Figure 2: The **System** window for (2).

are computed symbolically), and we return to the MatCont principal window.

## 2 Find an initial solution

As a first step we have to find a periodic solution to which our continuation analysis can be applied. Therefore, we start a trajectory from a random point of our system, hoping to converge to a stable limit

cycle. When clicking the menu **Type -> Initial point -> Point**, two windows are opened, the **Starter** and the **Integrator** window. Put all the initial state and parameter values in the Starter window (fix the value of the state variables to  $x = y = 0.1$ , while  $u = 1$  and  $v = 0$  in order to have  $\varphi = 0$ , and the parameter values to  $K = 1, r = 6.28, a = 12.56, b_0 = 0.25, e = 1, d = 6.28, \varepsilon = 0.2$ ). Put 100 as **Interval** in the **Integrator** window. Figure 3 is obtained as a result.

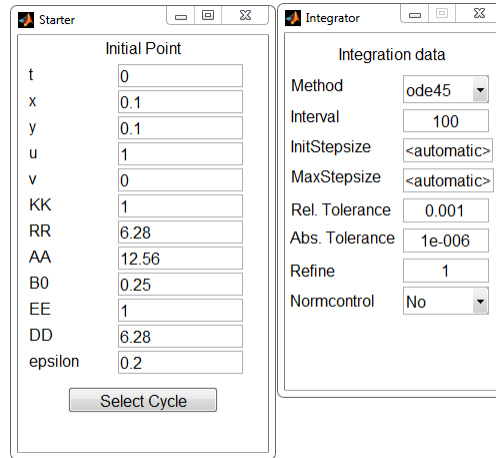


Figure 3: **Starter** and **Integrator** window.

Open a plot diagram, through the menu **Window -> Graphic -> 2Dplot** which (by default) visualizes the  $(x, x)$  plane. Change this into visualising the  $(x, y)$ – plane by the command **Layout -> Variables on axes**. By computing the trajectory making use of time-integration, through the menu **Compute -> Forward**, we obtain the result shown in Figure 4. The figure shows that the system has converged to a stable limit

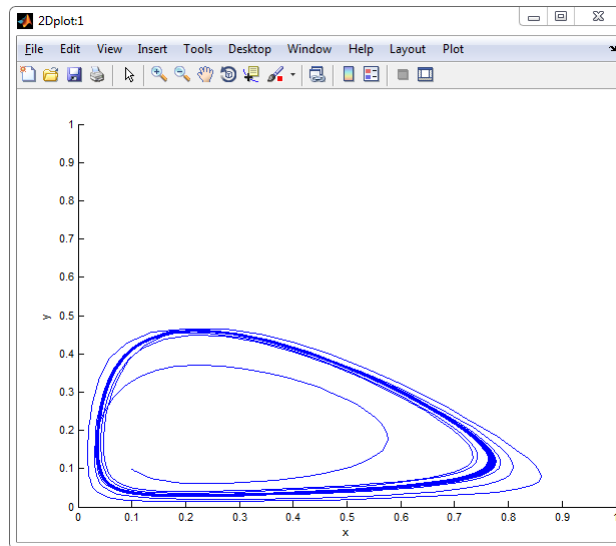


Figure 4: A trajectory of system (2).

cycle. We now want to select that invariant cycle in order to start a continuation. First we select a point on the limitcycle, through the MatCont menu **Select -> Initial Point** and then selecting the last point of the computed trajectory. Then we clear the plot, through the **Plot -> Clear** menu in the 2Dplot:1

window. We want to do an integration over a small interval, say 1, by changing the **Interval** field in the **Integrator** window, and compute a new trajectory through the **Compute -> Forward** menu in the **MatCont** window. In order to obtain the limit cycle, the trajectory should be extended once, through the **Compute -> Extend** menu in the **MatCont** window. We can then use the **Select Cycle** button in the **Starter** window, in order to select the obtained periodic solution. After having done that, the **Curve** field in the **MatCont** window automatically changes in a limit cycle continuation (namely **LC.LC(1)**), a new **Starter** window and the **Continuer** window are opened; it is now possible to do a limit cycle continuation.

### 3 Continue a periodic solution

Select the parameter  $b_0$  and the period of the limit cycle as free in the **Starter** window (note that the period is approximately equal to 2). Turn the detection of all singularities except BPC on and change the **MaxStepsize** in the **Continuer** window to 0.5 and **MaxNumPoints** to 30. The obtained windows are displayed in Figure 5. Start the continuation as usual with **Compute -> Forward** in the **MatCont** window.

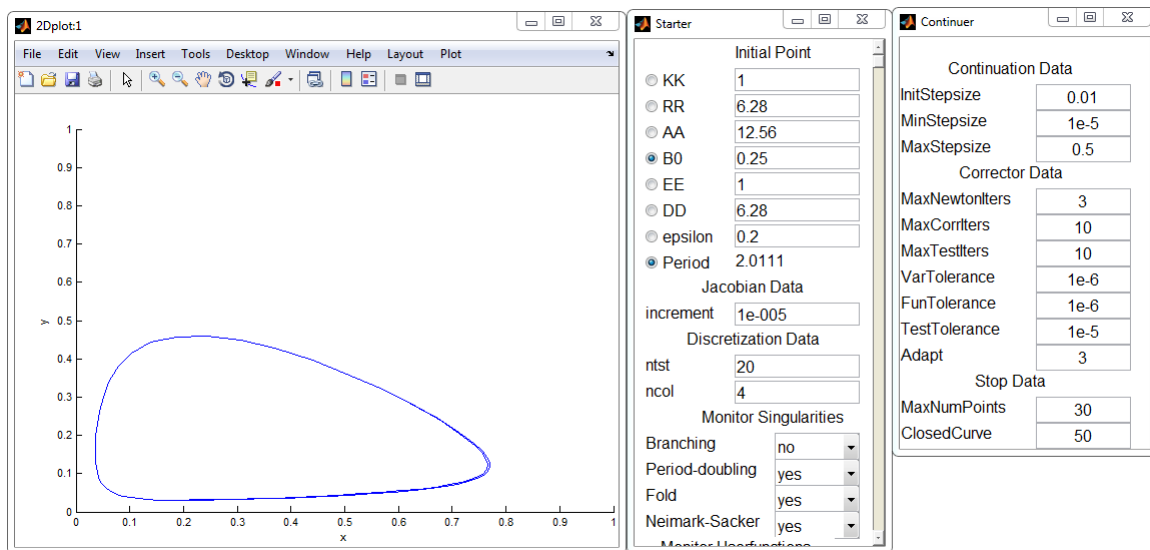


Figure 5: Ready for limit cycle continuation.

We detect the following bifurcation

Limit point cycle (period = 2.000000e+000, parameter = 3.399917e-001)  
Normal form coefficient = 6.547374e+007

This is the first point that we will use to compute the bifurcation diagram. We note that (in general) the normal form coefficients depend very much on the details of the computations and can be reliably used only if the bifurcation points are computed to high accuracy. Resume the computation; we detect another bifurcation

Limit point cycle (period = 2.000000e+000, parameter = 2.213011e-001)  
Normal form coefficient = 1.162697e+008

Save this curve as **LC1with2LPC** through the **Select -> Curve** menu in the **MatCont** window and the **Actions -> Rename** menu in the opened **Curves** window.

Change the variables on the axes of the **2Dplot:1** window through the menu **Layout -> Variables on axes** and select the two system parameters  $\varepsilon$  and  $b_0$  as plotting variables. Change the scale of the axes

through the menu **Layout -> Plotting region** into  $\varepsilon \in [0, 1.1]$  and  $b_0 \in [0.16, 0.45]$ . Select parameter  $\varepsilon$  as free in the **Starter** window, as shown in Figure 6 and start again a continuation (**Compute -> Forward** in the MatCont window). Note that the **Curve** name in the MatCont window is changed to **LC.LC(2)**. After some continuation steps we detect another bifurcation

Period Doubling (period = 2.000000e+000, parameter = 3.406025e-001)  
Normal form coefficient = -2.800698e-001

Depending on details of the computation, it can be necessary to compute backward instead of forward to find this point. Save this curve as **LC2with1PD** through the **Select -> Curve** menu in the MatCont window and the **Actions -> Rename** menu in the opened **Curves** window.

If we now redraw the bifurcation diagram (**Plot -> Redraw diagram** in the **2Dplot:1** window) we obtain the curves showed in Figure 6.

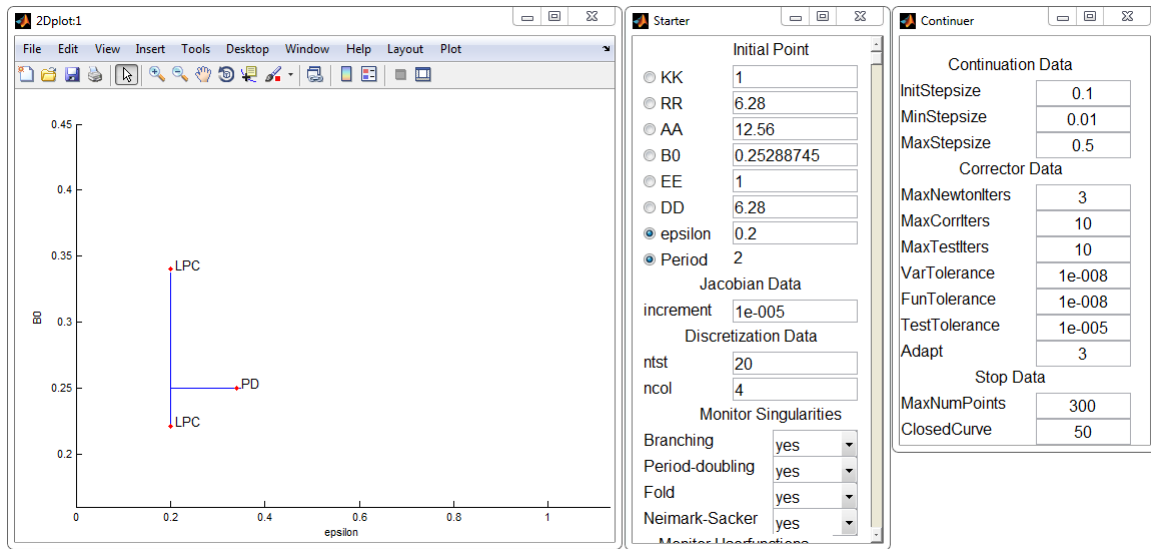


Figure 6: Ready for bifurcation continuation.

## 4 Compute the bifurcation diagram

Select the first Limit point of cycles bifurcation point in the limit cycle continuation curve **LC1with2LPC** as starting point of the continuation, through the **Select -> Initial point** menu in the MatCont window. The **Curve** field in the MatCont window automatically changes in a Limit point of cycles continuation (named **LPC.LPC(1)**). Select the parameters  $b_0$  and  $\varepsilon$  as free (active) in the **Starter** window, turn the monitoring of the singularities on and compute the bifurcation curve both forward and backward. One of these continuations may lead to a "stepsize too small" error message. Save the two obtained **LPC.LPC** curves as **LPC1for** and **LPC1back**, respectively (**Select -> Curve** and then **Actions -> Rename**).

Select the second Limit point of cycles bifurcation point of the limit cycle continuation curve **LC1with2LPC** as starting point of our continuation, through the **Select -> Initial point** menu in the MatCont window. The **Curve** field in the MatCont window now changes in **LPC.LPC(2)**. Select parameters  $b_0$  and  $\varepsilon$  as free (active) in the **Starter** window, turn the monitoring of the singularities on, and compute the bifurcation curve forward and backward. One of these computations may lead to a "stepsize too small" error message. Save the two obtained curves as **LPC2for** and **LPC2back**, respectively (**Select -> Curve** and then **Actions -> Rename**). Figure 7 shows the obtained curves.

Note that

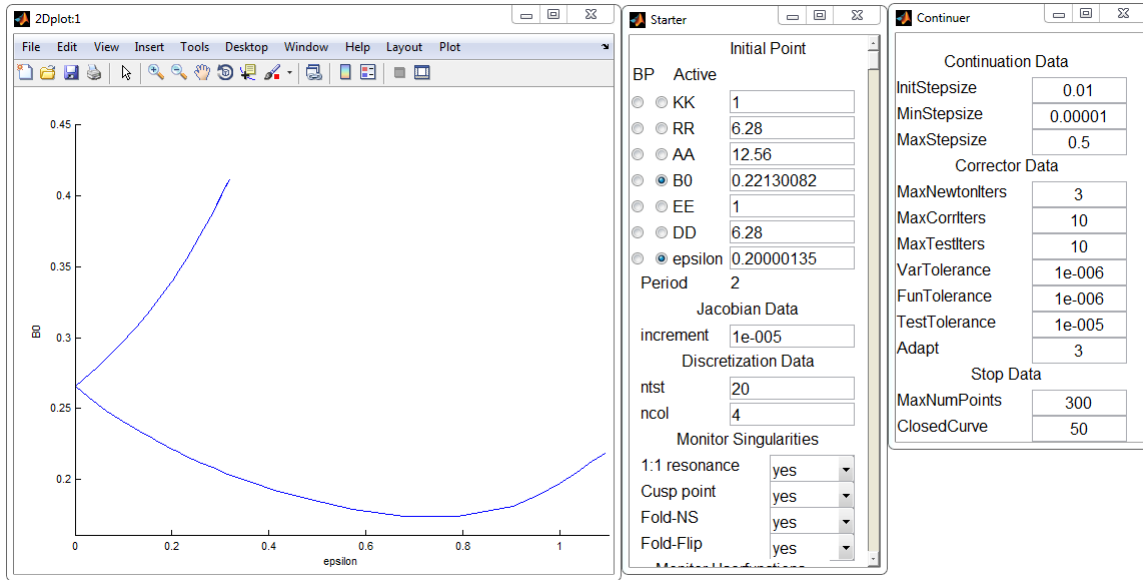


Figure 7: The limit point of cycle curves.

1. An error message "stepsize too small" pops up during both limit point of cycles continuations which in this case means that a codimension 2 point is encountered. Bifurcation theory allows us to say that those points are Generalized Period Doubling points (for a limit cycle with half of the period).
2. The two Limit point of cycles bifurcation curves touch at  $\varepsilon = 0$ . This is typical for periodically forced systems: that point is the resonant point in which the non-forced system has a Hopf bifurcation.
3. By analyzing the resonance 1:1 test function along the lower Limit point of cycles curve, a change of sign can be seen around  $\varepsilon = 0.6$ . The bifurcation is not detected since the change of sign is obtained by asymptotic behaviour and not by a simple zero-axis crossing.

A robust way to find the period doubling bifurcation continuation PD1 in Figure 8 is to start an orbit computation for  $b_0 = 0.4$ ,  $\varepsilon = 0.2$  with initial values  $x = 0.1$ ,  $y = 0.1$ ,  $u = 1$ , and  $v = 0$ . This orbit rapidly converges to a stable periodic orbit with period 1. We select a point on the limitcycle, through the MatCont menu **Select -> Initial Point** and then selecting the last point of the computed trajectory. Then we clear the plot, through the **Plot -> Clear** menu in the 2Dplot:1 window. We do an integration over the interval 1.1, by changing the **Interval** field in the **Integrator** window, and compute a new trajectory through the **Compute -> Forward** menu in the MatCont window. We can then use the **Select Cycle** button in the **Starter** window, in order to select the obtained periodic solution. After having done that, the **Curve** field in the MatCont window automatically changes in a limit cycle continuation (namely LC.LC(1)), a new **Starter** window and the **Continuer** window are opened; it is now possible to do a limit cycle continuation.

Select the period of the limit cycle (that now is equal to 1) and  $\varepsilon$  as free in the **Starter** window, turn the singularities detection on and continue the limit cycle backward and forward. We then find a period doubling point of the period 1 cycles for  $\varepsilon \approx 0.3050941$ .

Select (**Select -> Initial point**) this PD point, and prepare MatCont to continue the period-doubling bifurcation curve (**Type -> Curve -> Period doubling**). Select parameters  $b_0$  and  $\varepsilon$  as free in the **Starter** window, turn the detection of the singularities on and compute the period doubling bifurcation curve forward and backward. Save the two obtained curves (**Select -> Curve** and then **Actions -> Rename**). During the computations we obtain the following output

Generalized period doubling(period = 1.000000e+000, parameters = 2.183704e-001, 1.092944e+000)

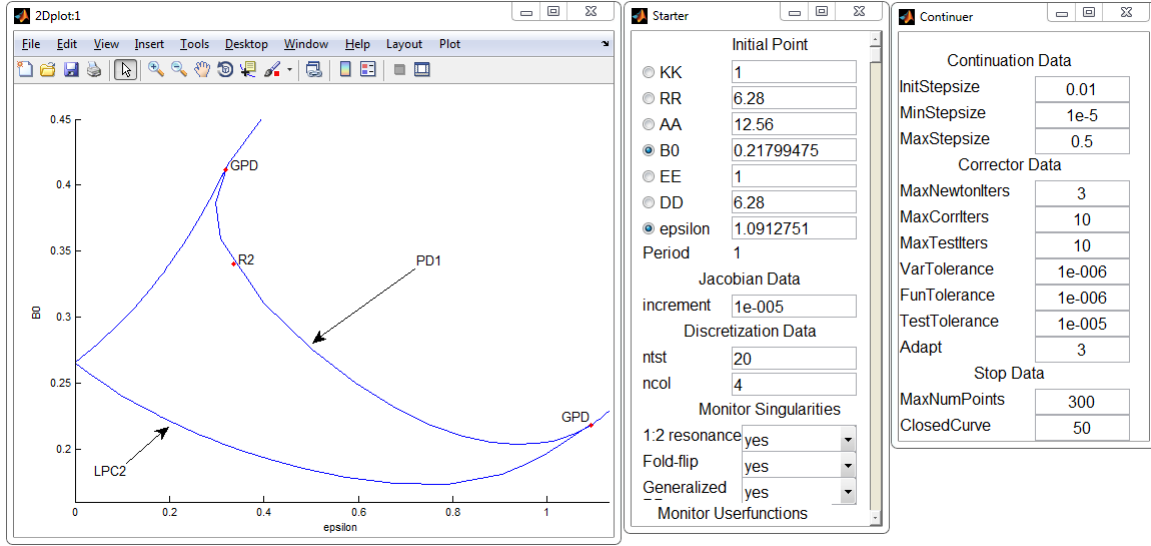


Figure 8: The period-doubling bifurcation curve PD1.

```
e = 4.155481e+001
1:2 resonance (period = 1.000000e+000, parameters = 3.401302e-001, 3.362215e-001)
(a,b)=(3.400446e+000, -1.292209e+001)
Generalized period doubling(period = 1.000000e+000, parameters = 4.117419e-001, 3.182304e-001)
e = -5.859315e+001
```

The result is presented in Figure 8. Notice that we found the two Generalized period-doubling bifurcation points. Rename the curve as PD1with2GPD1R2.

Select (Select -> Initial point) the resonance 1:2 point on the period doubling curve. Normal form analysis learns us that this point can be used as starting point for a Neimark-Sacker continuation of a period one limit cycle (Type -> Initial point -> Neimark-Sacker). The Curve type automatically switches to an NS continuation. Select  $b_0$  and  $\varepsilon$  as free parameters, turn the detection of the singularities on and compute (Compute -> Backward) the Neimark-Sacker bifurcation curve. Save the obtained curve (Select -> Curve and then Actions -> Rename), the result is shown in Figure 9.

We now come back to the result obtained in the last section in order to draw the last part of the bifurcation diagram. Select the period doubling bifurcation point of the second limit cycle continuation curve, i.e. LC2with1PD, and use it as starting point for a period doubling bifurcation continuation (Type -> Curve -> Period-doubling). Select  $b_0$  and  $\varepsilon$  parameters as free (in the Starter window), turn the codimension 2 bifurcation detection on, compute the bifurcation curve forward and backward, and save the obtained curves (Select -> Curve and Actions -> Rename). The result is presented in Figure 10. Along the backward continuation a resonance 1:2 point is found

```
1:2 resonance (period = 2.000000e+000, parameters = 1.838263e-001, 7.431601e-001)
(a,b)=(-6.555523e+001, 1.623663e+001)
```

Normal form analysis of this point learns us that it can be used as a starting point of a Neimark-Sacker bifurcation curve of the period 2 limit cycle and of a Neimark-Sacker bifurcation curve of a period 4 limit cycle. We now proceed in the same way as for the last resonance 1:2 point, i.e. select the point as the starting point (Select -> Initial point). Select it to be a Neimark-Sacker bifurcation curve (Type -> Initial point -> Neimark-Sacker), turn singularities detection on (in the Starter window), and do a backward continuation (Compute -> Backward).

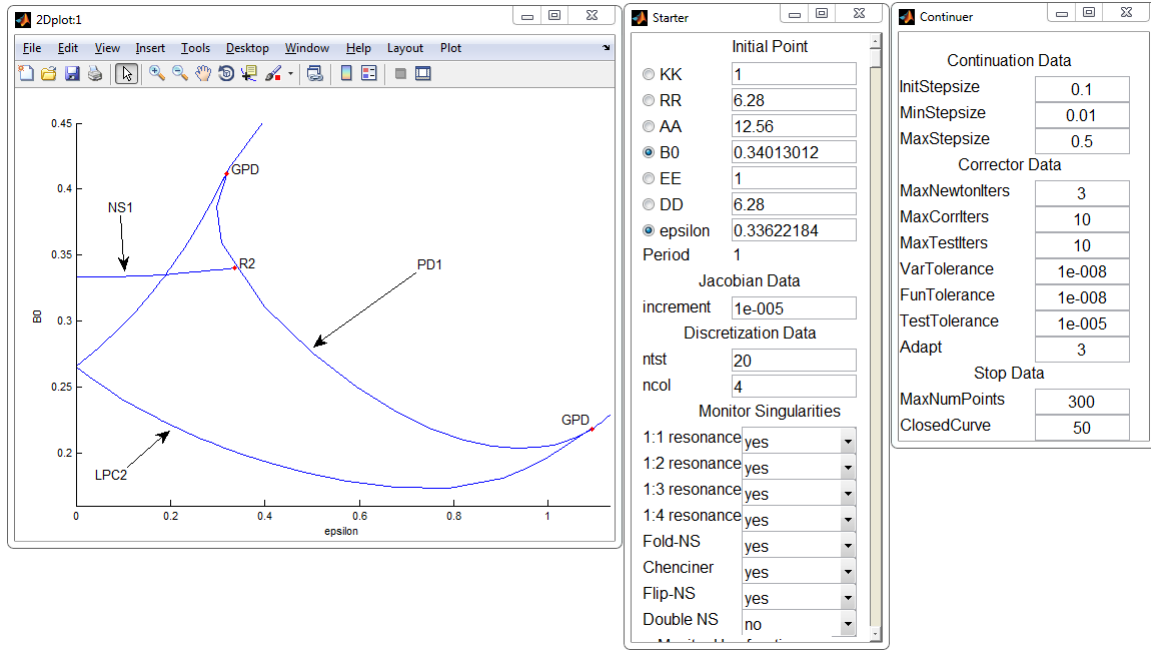


Figure 9: The Neimark-Sacker bifurcation curve NS1.

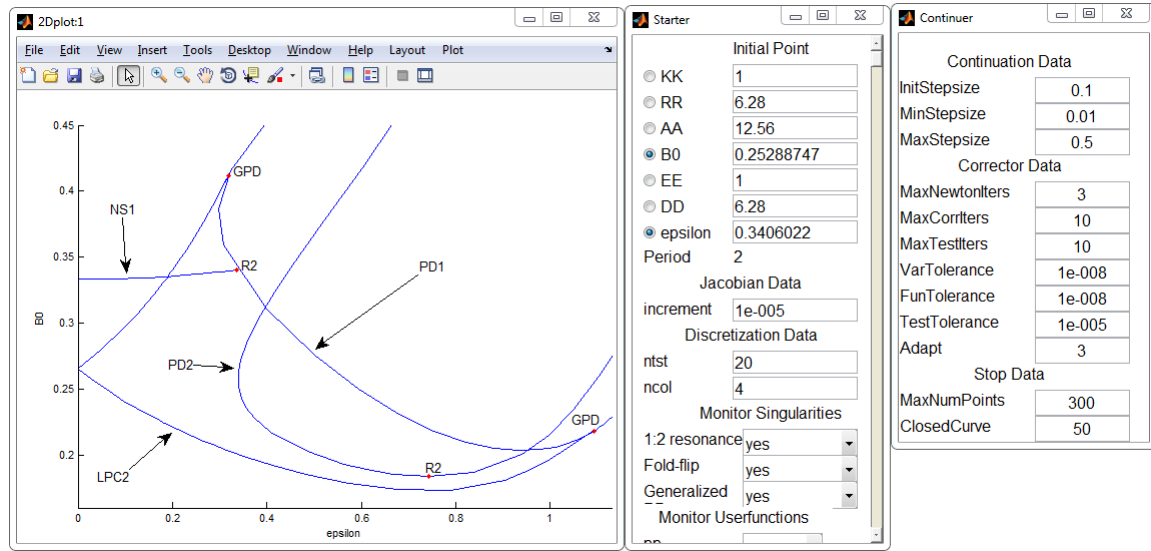


Figure 10: The period doubling curve PD2 of period 2 limit cycles.

Resonance 1:2 (period = 2.000000e+000, parameters = 1.838262e-001, 7.431608e-001)  
 $(a,b)=(-6.555444e+001, 1.623616e+001)$   
 Resonance 1:3 (period = 2.000000e+000, parameters = 1.798043e-001, 7.084321e-001)  
 $(b, \text{Re}(c))=(6.105853e+000 + (-1.785301e+000) i, 9.134561e+000)$   
 Resonance 1:4 (period = 2.000000e+000, parameters = 1.772743e-001, 6.740365e-001)  
 $(A,d)=(1.032142e-001+ (-7.518943e-001) i, 6.420366e+001+ (-9.214482e+001) i)$   
 Resonance 1:1 (period = 2.000000e+000, parameters = 1.773966e-001, 6.027878e-001)



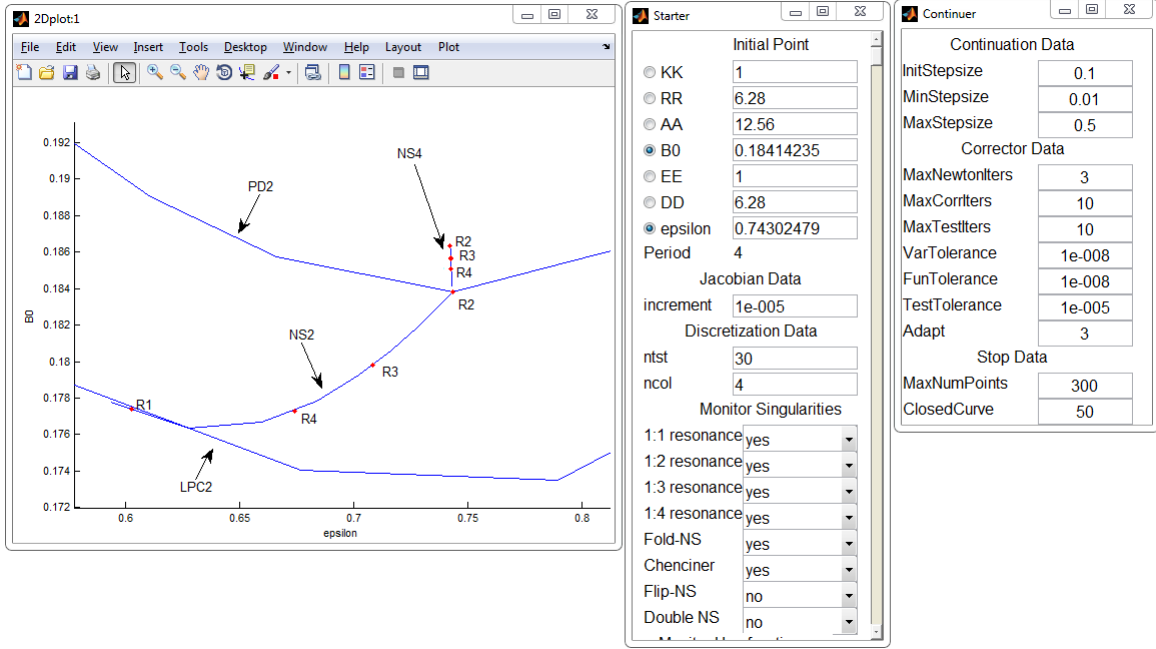


Figure 11: The secondary Neimark-Sacker bifurcation curve NS4 rooted at the resonance 1:2 point on the period doubling bifurcation curve of the period 2 limit cycle.

In order to obtain the second Neimark-Sacker curve we have to pass to the period-doubled solution and find the corresponding Neimark-Sacker bifurcation. So, after having selected the resonance 1:2 point from the period doubling bifurcation curve (Select -> Initial point), we change the point and curve type (Type -> Initial point -> Period doubling, Type -> Curve -> Limit Cycle). We select parameter  $b_0$  as free and turn on the detection of Neimark-Sacker bifurcations. We start the computation (Compute -> Forward) and after 3 continuation steps we find a point of the Neimark-Sacker curve

Neimark-Sacker (period = 4.000002e+000, parameter = 1.841332e-001)  
Normal form coefficient = -1.346407e-002

That point can be used (Select -> Initial point) as the starting point for the secondary Neimark-Sacker curve rooted at the resonance 1:2 point. Since we want to compute a longer trajectory, we increase the number of mesh intervals, changing the `ntest` field in the **Starter** window to 30. Denote  $b_0$  and  $\varepsilon$  as the free parameters, turn the detection of the singularities on and do the continuation (Compute -> Forward). We obtain

Resonance 1:4 (period = 4.000000e+000, parameters = 1.850697e-001, 7.426656e-001)  
(A,d)=(-1.339790e-002+ (-6.458158e-001) i, -4.166417e+002+ (-4.891725e+002) i)  
Resonance 1:3 (period = 4.000000e+000, parameters = 1.857095e-001, 7.424365e-001)  
(b,Re(c))=(4.043754e-001 + (1.143509e+001) i,-8.836545e+000)  
Resonance 1:2 (period = 4.000000e+000, parameters = 1.863586e-001, 7.422040e-001)  
(a,b)=(-2.684811e+002, -1.818726e+001)

and therefore find the bifurcation diagram presented in Figure 11.

After having saved the obtained bifurcation curve, we can use the last found resonance 1:2 point as a starting point of the period 4 period doubling bifurcation curve and of the Neimark-Sacker of period 8 limit cycles bifurcation curve. In this tutorial we only discuss the first curve; the procedure to obtain

all the other curves is similar to the one described in the previous cases. Select the resonance 1:2 point on the Neimark-Sacker of period 4 limit cycle bifurcation curve (Select -> Initial point). This point is a period-doubling bifurcation point (Type -> Initial point -> Period doubling) and we want to continue a period doubling bifurcation curve (Type -> Curve -> Period-doubling). Select  $b_0$  and  $\varepsilon$  as free parameters in the **Starter** window and turn the singularities detection on. Compute the bifurcation curve, we obtain

1:2 resonance (period = 4.000000e+000, parameters = 1.863585e-001, 7.422109e-001)  
(a,b)=(-9.217016e+002, 1.658100e+001)

and thus the bifurcation diagram shown in Figure 12.

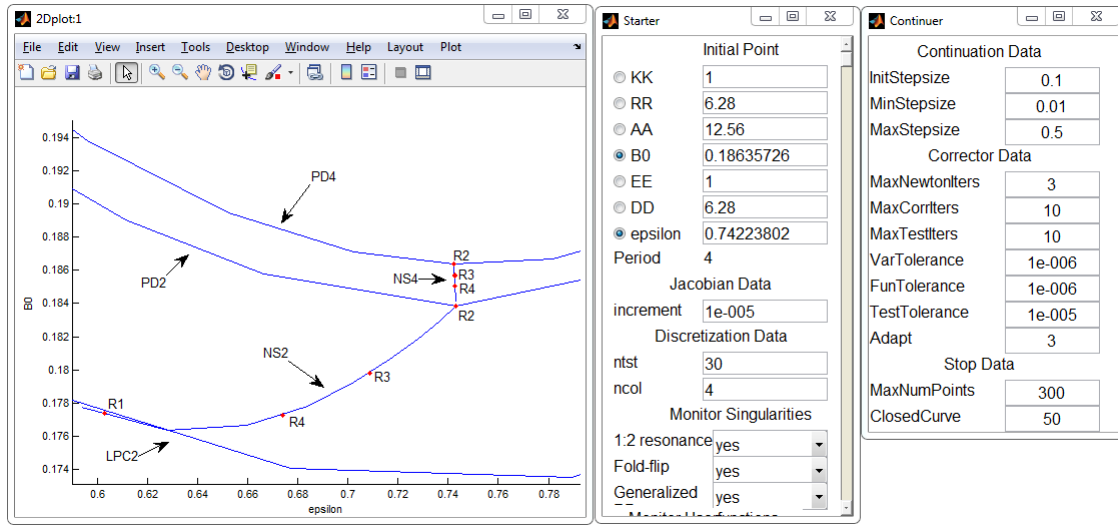


Figure 12: The period doubling bifurcation curve PD4 of the period 4 limit cycles.

Notice that in order to obtain the bifurcation diagram shown in Figure 1 the bifurcation curves should be computed with higher precision with respect to the ones used in this tutorial: here it has not been done in order to make this tutorial as easy as possible, but it is possible to achieve more accurate bifurcation curves by simply reducing the **MaxStepsize** parameter in the **Continuer** window.

## References

- [1] V. DE WITTE, F. DELLA ROSSA, W. GOVAERTS, AND YU. KUZNETSOV, *Numerical periodic normal-ization for codim 2 bifurcations of limit cycles - computational formulas, numerical implementation, and examples*, SIAM J. Appl. Dyn. Systems, (2013).
- [2] YU. KUZNETSOV, S. MURATORI, AND S. RINALDI, *Bifurcations and chaos in a periodic predator-prey model*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2 (1992), pp. 117–128.