TUTORIAL MIII:

Computation of invariant manifolds and connecting orbits of saddle fixed points in MatcontM

May 18, 2016

This MatcontM tutorial is devoted to the numerical construction of 1D stable and unstable invariant manifolds of saddle fixed points of (orientation-preserving) maps

$$u \mapsto f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^m,$$

at fixed parameter values, and to the continuation of their intersections (i.e. orbits connecting saddles) with respect to one or two parameters.

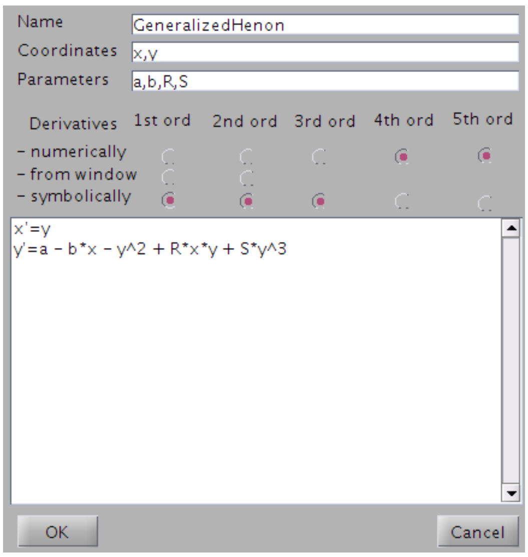
Notice that a "point" on a continuation curve will represent the entire orbit. We must first obtain an initial connecting orbit. Such an orbit lies in the intersection of the unstable manifold of a fixed point u_1 and the stable manifold of a fixed point u_2. In the case of a homoclinic orbit, u_1 and u_2 are the same point u_0. MatcontM can compute 1D stable and unstable manifolds and find intersection points between these manifolds. The intersection points can then serve as a representative of a connecting orbit.

The user can also provide MatcontM with an orbit that has been obtained through other means, by storing the orbit as a matrix in the Matlab workspace.

As an example, we will use a map called the *generalized Hénon map* (GHM):

$$\mathsf{GeneralizedHenon} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \alpha - \beta x - y^2 + Rxy + Sy^3 \end{pmatrix} \tag{1}$$

This system is implemented by default using the name `GeneralizedHenon`. You can re-enter the system in MatcontM using the information given in Figure 1.




```
name: GeneralizedHenon
Coordinates: x,y
Parameters: a,b,R,S
Equations:
x'=y
y'=a - b*x - y^2 + R*x*y + S*y^3
```

Figure 1: GHM specification in MatcontM

1 Computation of 1D invariant manifolds of a saddle

Once the system has been loaded we can start by computing a connecting orbit for continuation. We do this by changing the type of the initial point. We select in the main menu:

Type|Initial Point|Connecting Orbit (Figure 2). Once the type of the initial point has been set to an orbit, you will be able to select or compute a connecting orbit. You can choose between a homoclinic orbit (`HO-curve`) and a heteroclinic orbit (`HE-curve`) (See Figure 3). Make sure the homoclinic orbit option is selected.

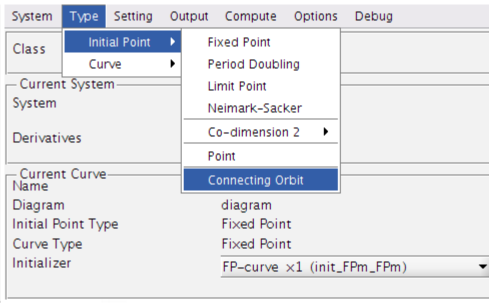


Figure 2: Set the type of the initial point to "Connecting Orbit"

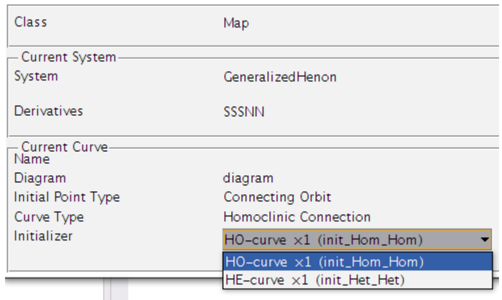


Figure 3: Select the homoclinic connection continuation (HO-Curve)

We select the Starter window. The `Initial Point` subpanel of the Starter window contains either zeros or a point that was selected on a curve of fixed points. This information is only used as default for computing manifolds and plays no further role in the continuation of connecting orbits, only the selected initial connecting orbit will be of importance. You will notice that a special panel has appeared in the Starter window. This panel will allow you to compute manifolds, find intersections, and select connecting orbits (see, e.g. Figure 8).

Before we start computing manifolds, we first have to enter the values of the parameters in the Starter window. Set $a = -0.4$, $b = 1.03$, $R = -0.1$ and $S = 0$ (Figure 4).

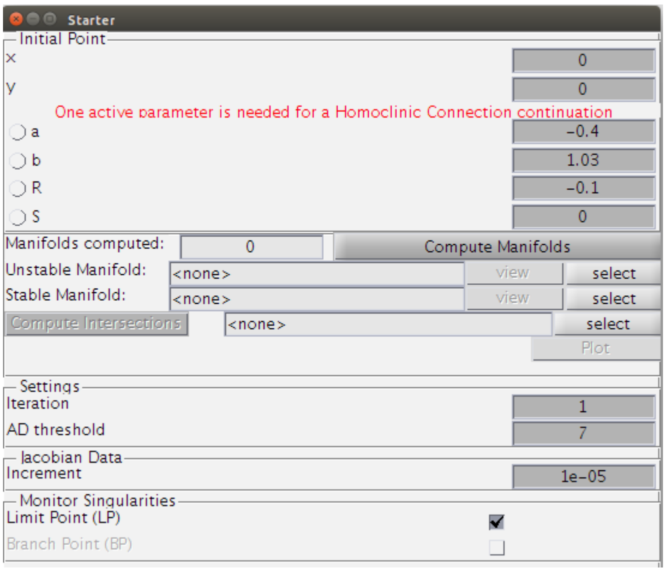


Figure 4: The Starter window after entering the parameter values.

Initialize computation of stable and unstable manifolds by pressing the **Compute Manifolds** button. A window for computing manifolds will appear (Figure 5). First you need to enter a fixed point there. Enter the point $(-1.621146385, -1.621146385)$. Press the **Select fixed point and proceed to configure** button.

We can now configure the algorithm used to compute the manifold. The most important setting defines whether we want to compute a stable or unstable manifold. We will first compute the unstable manifold: next to `function`, we select `Unstable Manifold`. We will adjust some other settings as well:

- set `nmax` to `4000` (maximum number of points in the manifold)
- set `distanceInit` to `-1e-08`
- set `deltaMax` to `0.01`

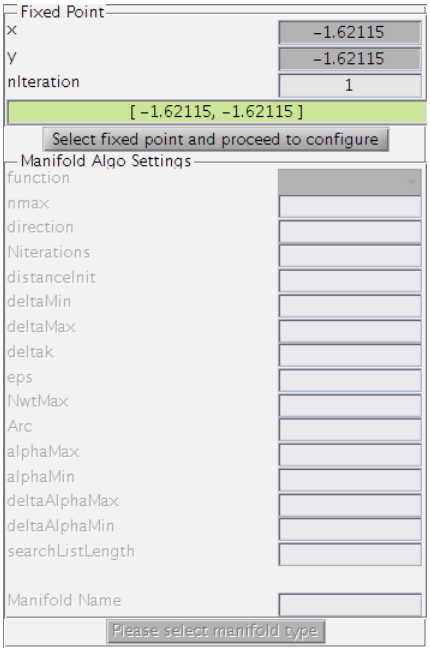


Figure 5: Manifold computing window after entering a fixed point

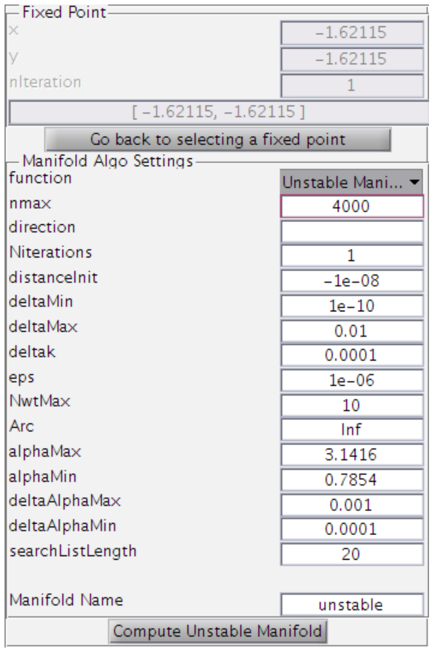


Figure 6: Manifold computing window after confirming the fixed point

The manifold window should now look like Figure 6. You can give the manifold a name, i.e. name the manifold "`unstable`" (Figure 6, see `Manifold Name`). If no name is given, a default name will be used. Press **Compute Unstable Manifold**. A window will appear that displays the output of the algorithm, close that window when the computing stops and go back to the manifold window (Figure 6).

Now compute the stable manifold, go to function and select `Stable Manifold`. Give this manifold the name "`stable`" and press **Compute Stable Manifold** button.

We have computed an unstable and a stable manifold of the same fixed point at given parameter values. Close the window in Figure 6 and return to the Starter window.

2 Finding intersections of stable and unstable manifolds

We have computed two manifolds: a stable and an unstable manifold. In the Starter window, go to the `Unstable Manifold` section and press the **select** button. A browser window pops up and lists the available unstable manifolds, we only have one available. We can select the manifold by double-clicking on the name in the list. Repeat the same process for `Stable Manifold` in the Starter window.

Once a stable and an unstable manifold are selected, the **Compute Intersections** button becomes available. We produce a list of intersections between the two manifolds by pressing this button. Afterwards, press the **select** button above the **Plot** button to select such an intersection as a candidate connecting orbit for continuation.

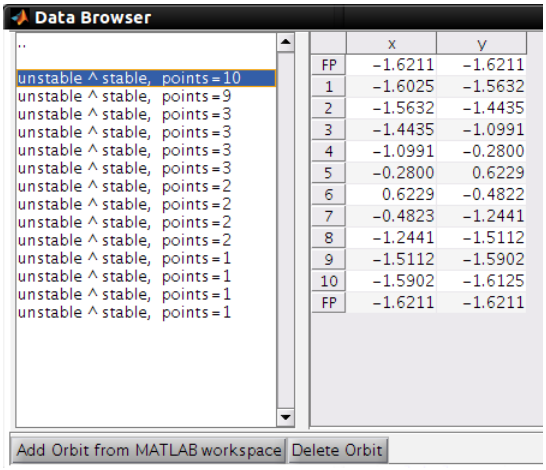


Figure 7: The intersection between the unstable and stable manifolds has produced multiple candidates. Select the connecting orbit by double-clicking.

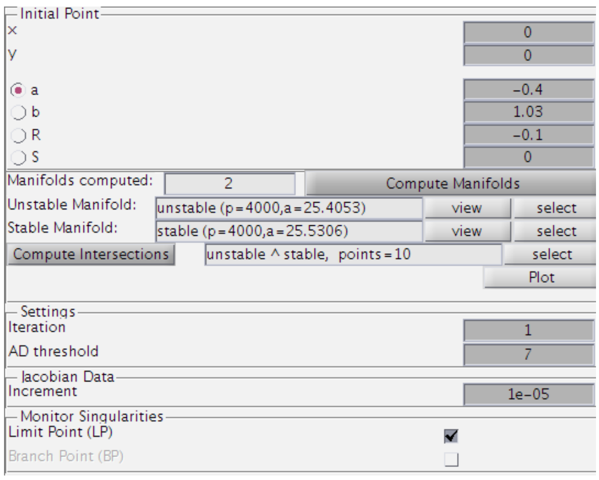


Figure 8: The Starter window after configuring a homoclinic continuation.

A browser window pops up that lists the available connecting orbits produced by the intersection (Figure 7). Select the longest intersection (with 10 points) by double-clicking. Once the connecting orbit has been selected, the Starter window should now look like in Figure 8. Select a as the free parameter. The values of x and y do not play a role in the continuation of a connecting orbit. The "initial point" is now the selected orbit.

The unstable manifold, stable manifold and connecting orbit can be plotted by pressing the **Plot** button (Figure 9). The MatcontM menu in the right upper corner can be used to fine tune the produced image (Figure 10).

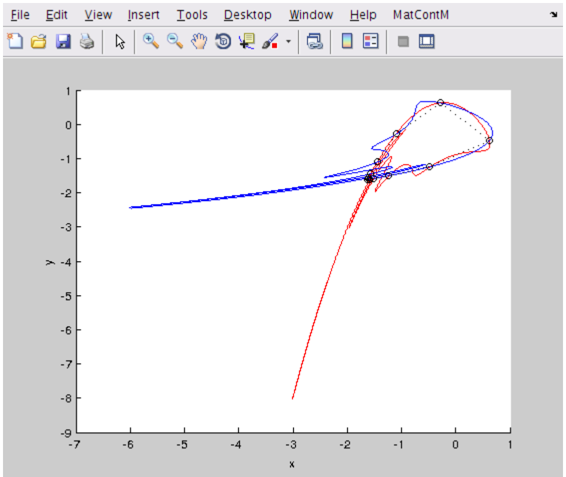


Figure 9: Stable and unstable manifolds with the connecting orbit.

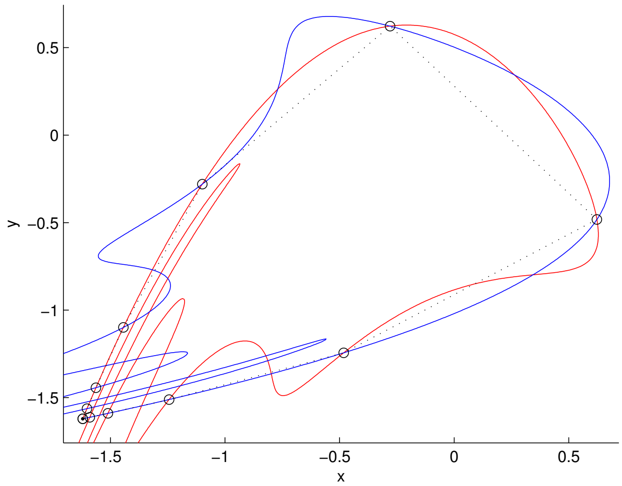


Figure 10: The plot can be manipulated by the tools in the MatcontM menu and afterwards exported to an image file.

3 Continuation of homoclinic orbit in one parameter

Start the continuation by going to the main window menu and selecting: **Compute**| **Forward**. Notice that each point on the continuation curve represents an orbit. You should find one bifurcation point on the curve (see Figure 11).

If you scroll right in the Output window you can see the parameter value of the detected tangency (last value of the continuation vector x, Figure 12). A homoclinic tangency is detected at $a = -0.54886101$.

Open a Plot2D window and configure the layout as seen in Figure 12. When you select

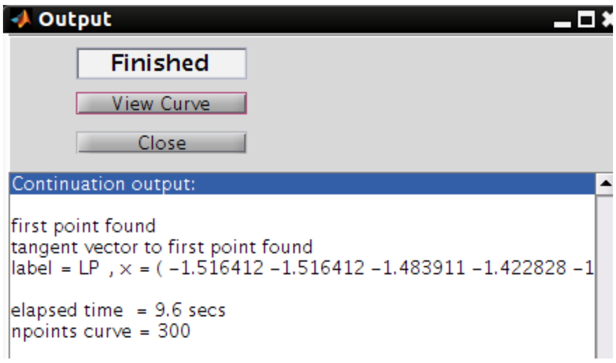


Figure 11: A homoclinic tangency is reported as LP.

the coordinate x, the coordinate of the fixed point in the homoclinic orbit is used. Draw the curve by selecting **MatcontM|Redraw curve** in the Plot2D window.

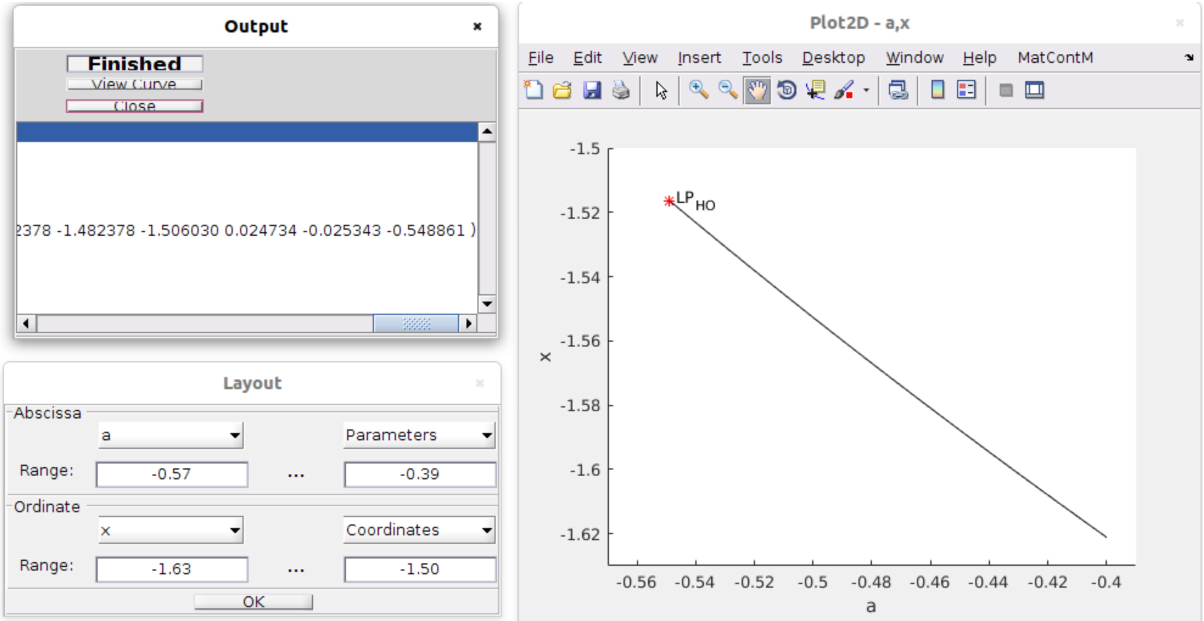


Figure 12

4 Continuation of homoclinic tangency in two parameters

Go back to the data browser and select the limit point that was detected in Figure 11.

We are now able to continue the homoclinic tangency starting from the detected limit point. We need to select two free parameters in the Starter window, parameters a and b. The homoclinic connecting orbit associated with the limit point is automatically selected: `LP_HO-points` (Figure 13).

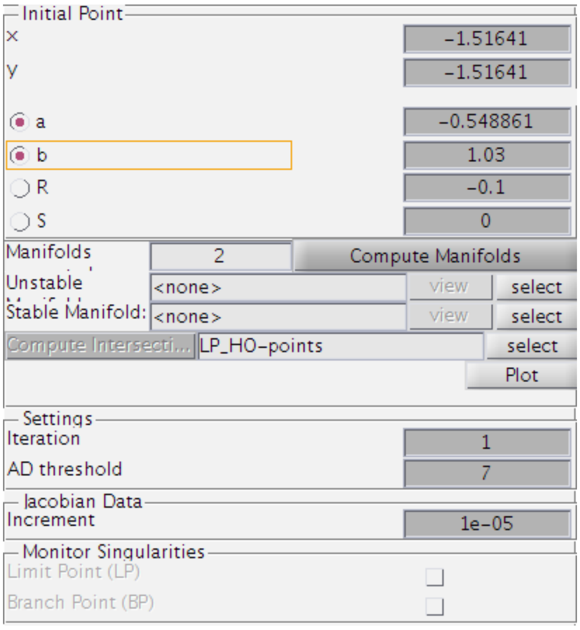


Figure 13: Starter window of a homoclinic tangency continuation.

In the Starter window, we can compute the stable and unstable manifold by pressing the **Compute Manifolds** button. We have used the same settings as in Figure 6. We do not need these manifolds for continuation, just for visualization. The default value for the fixed point for computing the manifolds has been selected to be the fixed point associated with `LP_HO-points`. Compute the stable and unstable manifold and select them in the Starter window. Press the **Plot** button to generate Figure 14. The connecting orbit `LP_HO` is represented by black circles and black dashed lines. The stable and unstable manifold of the saddle fixed point are also visualized. We do not need to compute an intersection because the orbit is already provided by `LP_HO-points`.

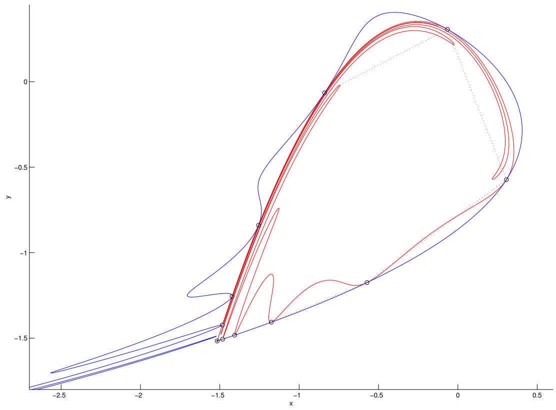


Figure 14: Homoclinic tangency.

We advise to set `MaxNumPoints` to 80 in the Continuer window to reduce the computation time. Now we are ready to start the homoclinic tangency continuation by selecting **Compute|Forward**. When the computation is finished, select **Compute|Backward**.

Open a Plot2D window to visualize the results of the computations in the parameter plane. Use the configuration seen in Figure 15. Select **MatcontM|Redraw diagram** in the plot window. The result is shown in Figure 16. Notice that the tangency curve (red) forms a wedge. This wedge actually consists of two homoclinic tangency curves (upper and lower) meeting tangentially at a point R1 corresponding to the strong resonance 1:1. The continuation process switched onto the other tangency curve near this codim 2 bifurcation point.

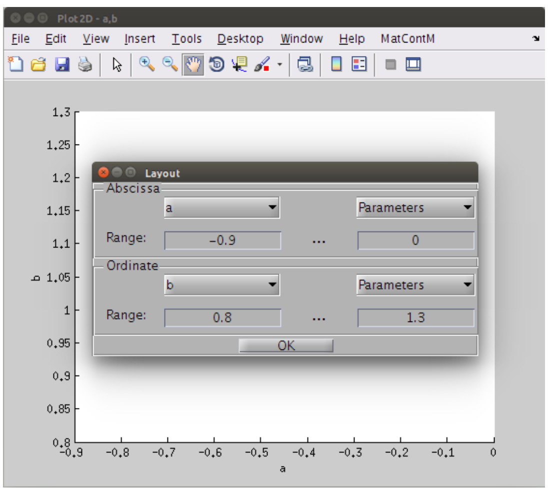


Figure 15

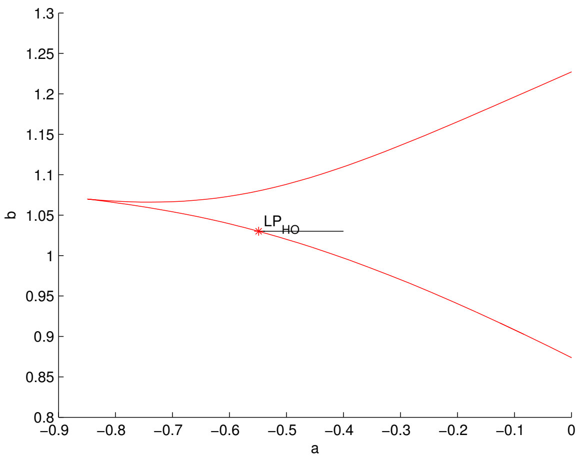


Figure 16

5 Continuation of a heteroclinic orbit obtained through the MATLAB workspace

We will now compute a heteroclinic bifurcation curve using a connecting orbit that was obtained outside MatcontM. For example, enter in the MATLAB command window:

```
C = [ 0.46661702380495,0.51950188769804 ,0.48317182408537, ...
       0.47311690109635 ,0.6123,0.841195,1.22990435,0.6419816, ...
       0.1347307,-0.14345697, -0.33379938281268,-0.38062126993587, ...
      -0.40424194997464,-0.41621339121235,-0.42230324914752, ...
      -0.42861702380495;0.46661702380495,0.37639405084666, ...
       0.43915841488454, 0.45696220552142, 0.2067,-0.276064, ...
      -1.3327006,-1.0930203,-0.7998431,-0.6233856,-0.49516248970159, ...
      -0.46255049612568,-0.44591635361118,-0.43743735402657, ...
      -0.43311132511221,-0.42861702380495];
```

[1]

The orbit is stored as a matrix under the variable name 'C' and is now available in the MATLAB workspace. We will now enter this orbit into MatcontM. First, make sure you have selected a "Connecting Orbit" as initial point type (Figure 2) and select the "HE-Curve" (Figure 3). Now open the Starter window and press the "select" button above the "plot" button. You will now see a list of the available connecting orbits for selection. You can add 'C' by clicking on the button below named "Add Orbit from MATLAB workspace".

Enter the variable name "C", the orbit now appears in the list and is available for selection. After selecting the orbit, the name will appear in the Starter window. For this example, make sure that iteration is set to 2 and b is selected as the free parameter. Set the values of parameter a to 0.3, b to -1.057, R to -0.5 and S to 0.

[1] You should be able to copy/past this into Matlab if you are using a pdf reader.

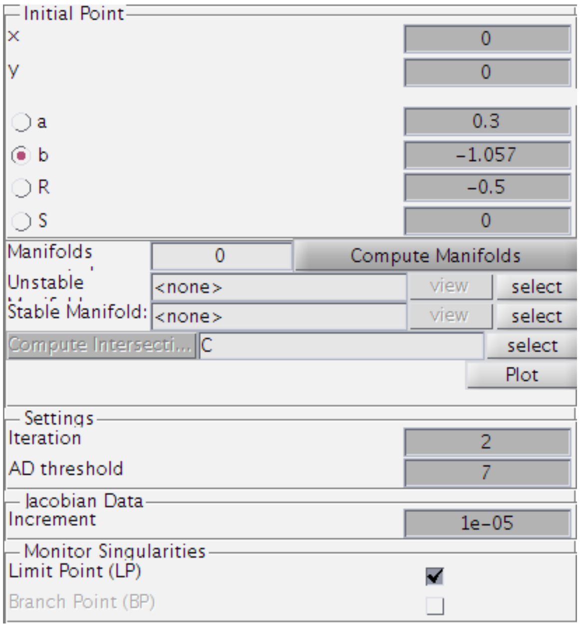


Figure 17: Starter window of a heteroclinic connection continuation.

The Starter window should now look like Figure 17. Proceed by selecting: **Compute|Backward**. You should detect at least one limit point. You can then select that point for a heteroclinic tangency continuation.

6 Additional Problems

A. Consider the following *Shear Map*:

$$\Psi = \psi \circ \varphi,$$

where

$$\varphi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda^u x \\ \lambda^s y \end{pmatrix}$$

and

$$\psi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - cf(x+y) \\ y + cf(x+y) \end{pmatrix}$$

with

$$f(z) = \begin{cases} 0 & \text{for } \; z \leq 1, \\ (z-1)^2 & \text{for } \; z > 1. \end{cases}$$

Fix $\lambda^s = 0.4$, $\lambda^u = 2.0$ in Ψ and compute the stable and unstable invariant manifolds of the origin for $c = 0.5$, $0.75, 0.9$, and 1.0. Find the critical parameter value c^* corresponding to a homoclinic tangency.

B. Consider the following planar *McMillan map*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x + \dfrac{2\mu y}{1+y^2} \end{pmatrix} \tag{2}$$

where $\mu > 1$, and its perturbation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x + \dfrac{2\mu y}{1+y^2} + \varepsilon(\beta x + \gamma y) \end{pmatrix} \tag{3}$$

where $\varepsilon > 0$ and (β, γ) are parameters. Study stable and unstable invariant manifolds of the saddle fixed point $(x, y) = (0, 0)$ of the maps, as well as intersections of these manifolds.

(i) Study the map (2) :

1. Prove that the set of points $(x, y) \in \mathbb{R}^2$ satisfying

$$x^2y^2 + x^2 + y^2 - 2\mu xy = C$$

is invariant w.r.t. map (2) for any real constant C. Illustrate this by numerical simulations in MatcontM.

2. Prove that the origin $(x, y) = (0, 0)$ is a saddle fixed point of (2) and compute its stable and unstable invariant manifolds to verify that they form a 'figure-of-eight'. This is a very degenerate case.

(ii) Study the map (3) for $\epsilon = 0.05$ and $\gamma = 1.9$ when $(\beta, \mu) \in [-0.5, 2.0] \times [1, 4.5]$ using MatContM:

1. For $(\beta, \mu) = (0.1, 2.0)$, compute and plot branches of the stable and unstable invariant manifolds of the saddle $(0, 0)$ that emanate from it into the positive quadrant. *Warning*: These manifolds eventually pass through other quadrants of the (x, y)-plane.
2. Locate with MatcontM intersection points of the invariant manifolds.
Hint: You have to accurately compute rather long branches of the manifolds to ensure that they intersent sufficiently many times near the saddle. For that, you may need to tune parameters of the algorithm.
3. Continue the obtained approximation of the primary homoclinic orbit to the saddle w.r.t. parameter β and detect two limit points. Verify that at the corresponding parameter values the stable and the unstable invariant manifolds of $(0, 0)$ are (almost) tangent.
4. Continue the found LPs w.r.t. parameters (β, μ) and describe the domain of existence of Poincaré homoclinic structure. What happens as $(\beta, \mu) \to (0, 1)$?

C. Compute in MatcontM the 1D-unstable invariant manifold of the saddle fixed point $(0, 0, 0)$ in the explicit *Euler scheme* for the Lorenz system:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + h\sigma(y - x) \\ y + h(rx - y - xz) \\ z + h(xy - bz) \end{pmatrix}$$

with $\sigma = 10, b = 8/3, r = 8.37$, and $h = 0.1$.