Problem 2: Periodic epidemic model

Consider the following epidemic model

$$\begin{cases} \dot{S} & = & \mu - \mu S - \beta(t) S I, \\ \dot{E} & = & \beta(t) S I - (\mu + \alpha) E, \\ \dot{I} & = & \alpha E - (\mu + \gamma) I, \end{cases} \tag{1}$$

that describes the spread of a non-lethal disease in a large population. Here the fractions of susceptible (S), exposed (E), and infective (I) individuals are the state variables[1], while (μ, α, γ) are positive parameters. It is assumed that contact rate $\beta(t)$ is periodic in time with period 1(year), namely

$$\beta(t) = \beta_0(1 + \delta \cos(2\pi t)),$$

where $\beta_0 > 0$ is the mean contact rate and $\delta \geq 0$ is the degree of seasonality.

The aim is to study with MatCont existence and stability of period-1,-2, and -3 cycles in (1) for fixed

$$\mu = 0.02, \quad \alpha = 35.842, \quad \gamma = 100,$$

corresponding to measles, when $(\delta, \beta_0) \in [0, 0.6] \times [0, 6000]$.

Try to do this first without using suggestions from the next page.

[1] The value $R = 1 - S - E - I$ gives the fraction of recovered (permanently immune) individuals. Thus, (1) is usually called the SEIR-model.

1. Consider an equivalent autonomous system

$$\begin{cases} \dot{S} &=& \mu - \mu S - \beta_0(1+\delta u)SI, \\ \dot{E} &=& \beta_0(1+\delta u)SI - (\mu+\alpha)E, \\ \dot{I} &=& \alpha E - (\mu+\gamma)I, \\ \dot{u} &=& u - 2\pi v - (u^2+v^2)u, \\ \dot{v} &=& 2\pi u + v - (u^2+v^2)v, \end{cases}$$

and introduce new variables

$$\begin{cases} s &=& \ln S, \\ e &=& \ln E, \\ \imath &=& \ln I, \end{cases}$$

to better handle very small values of S, E, and I.

2. Find by simulations a period-1 cycle in the $(s, e, \imath, u, v)$-space at $\delta = 0$ and $\beta_0 = 5000$. (*Hint*: Use a stiff integration `Method`, e.g. `ode23s`.)

3. Continue the found period-1 cycle w.r.t. δ and find its period-doubling (PD) bifurcation.

4. Starting from the obtained PD-point, compute the period-doubling curve $PD^{(1)}$ in the (δ, β_0)-plane and locate two different degenerate period-doubling (DPD) points. Report the parameter values corresponding to these codim 2 points.

5. Starting from the PD-point, continue the period-2 cycle and find two limit point of cycles (LPC) bifurcations.

   Use the found LPC points to compute the $LPC^{(2)}_{1,2}$ bifurcation curves in the (δ, β_0)-plane and locate a cusp point of cycles (CPC) where they meet. Report the parameter values corresponding to this codim 2 bifurcation point. (*Hint*: MatCont will not detect CPC in this case, so use the numerical output to locate it approximately.)

   What are the other end-points of these curves ?

6. Compute the bifurcation curve $PD^{(2)}$ where the period-2 cycle exhibits a period-doubling bifurcation. (*Hint*: To locate a point on this curve, continue w.r.t. β_0 a branch of the period-2 cycles, starting from the PD-point found in Step 3.)

7. Compute a curve $LPC^{(3)}$ where a stable and an unstable period-3 cycles are born via the LPC-bifurcation. (*Hint*: To locate a stable period-3 cycle, simulate the system at $(\delta, \beta_0) = (0.1, 1200)$.)

   Compute a curve $PD^{(3)}$ corresponding to the period-doubling bifurcation of the stable period-3 cycle.

8. Classify cycles existing in various domains in the (δ, β_0)-plane to the left from the curves $PD^{(2)}$ and $PD^{(3)}$.