TUTORIAL MII:
Two-parameter bifurcation analysis of fixed points and cycles in MATCONTM

May 11, 2016

We perform the two-parameter bifurcation analysis on a discrete-time predator-prey model using MatcontM.

$$\mathsf{PPModel} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax(1-x) - \dfrac{bxy}{1+\epsilon x} \\ \dfrac{dxy}{1+\epsilon x} \end{pmatrix} \tag{1}$$

The map (1) models the year-to-year dynamics of a prey-predator ecosystem. It describes how predators respond to changes in prey availability. The coordinates x and y represent the population densities of the prey and the predator, respectively, while a, b, d and ϵ are the parameters of the system. Parameters a and d determine the growth rate of the prey and predator; b determines the rate at which prey are consumed; ϵ acts as a limitation parameter on the growth of the predator population for increasing prey population density.

We determine the fixed points of system (1) by algebraically solving the fixed point equation. One trivial solution is the origin $F_1 = (0, 0)$, which means having neither prey nor predators. Taking $y^* = 0$, we get the solution $F_2 = (\frac{a-1}{a}, 0)$, the situation where no predators exist, only prey. For $x^* \neq 0$ and $y^* \neq 0$, we get the solution

$$F_3 = \left(\frac{1}{d-\epsilon}, \frac{d}{d-\epsilon} \left(\frac{a}{b} \left(1 - \frac{1}{d-\epsilon} \right) - \frac{1}{b} \right) \right). \tag{2}$$

We have F_1, F_2 and F_3 as the fixed points of map (1).

1 System specification

This map (1) has already been implemented as `PredatorPreyModel` in MatcontM. Select it using **System|Systembrowser** and highlighting `PredatorPreyModel`. Press **Edit** and ensure that all derivatives up to and including 5th order are generated symbolically. Press **OK**. Now you can actually load the system pressing **Load** button (see Figure 2).

Alternatively, you can implement the system yourself. Select **System|Systembrowser**, press **New** and enter the data shown in Figure 1. Here we use the name `PPModel`. Do not forget to generate all derivatives up to and including 5th order symbolically.

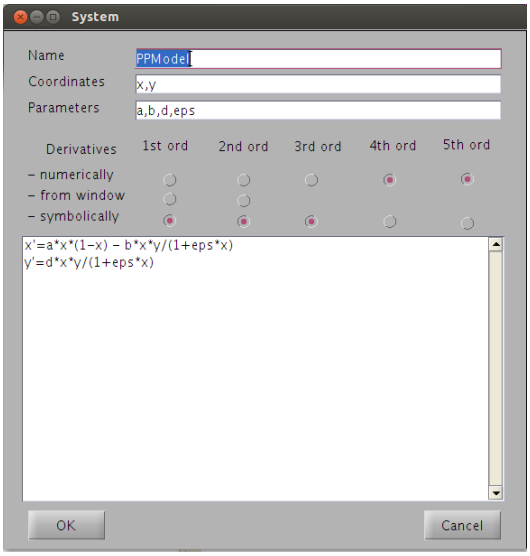


Figure 1: Entry for a new system. Press **OK** to add the system.

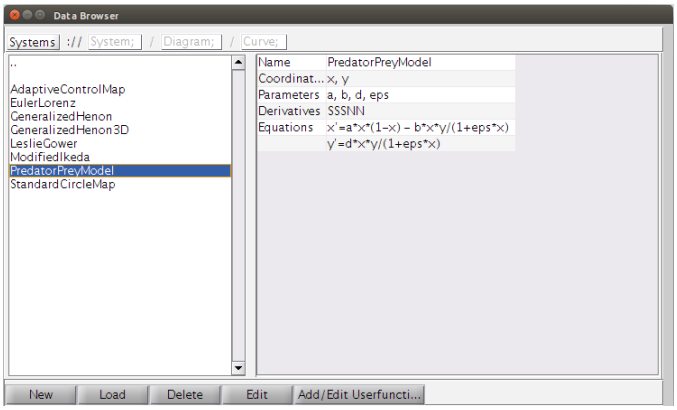


Figure 2: Press **Load** to select the highlighted system.

2 Continuation of fixed points

We will now compute a series of fixed-point curves.

Verify that the `Initial Point Type` is set to `Fixed Point` and the `Initializer` is set to `init_FPm_FPm`. Select the Starter window, activate parameter `a` and input $(0, 0)$ (fixed point F_1) with parameters $a = 0$, $b = 3$, $d = 3.5$ and eps $= 1$ (Figure 3). Open Plot2D window and with the layout as in Figure 4. It is also recommended to open the Numeric window.

Ensure that the numerical parameter values in the Continuer window are as follows: `InitStepSize = 0.01` while `MaxStepSize = 0.1`.

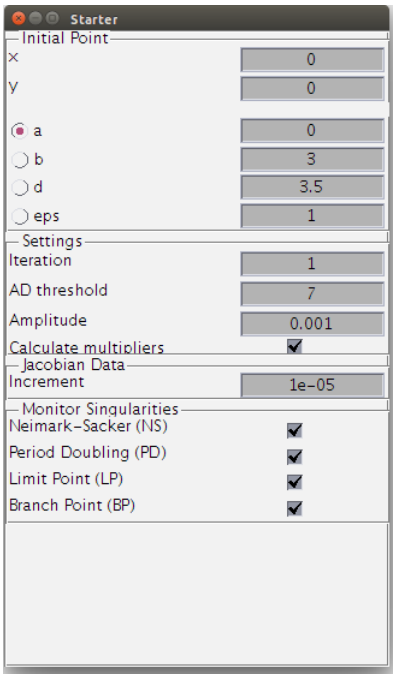


Figure 3

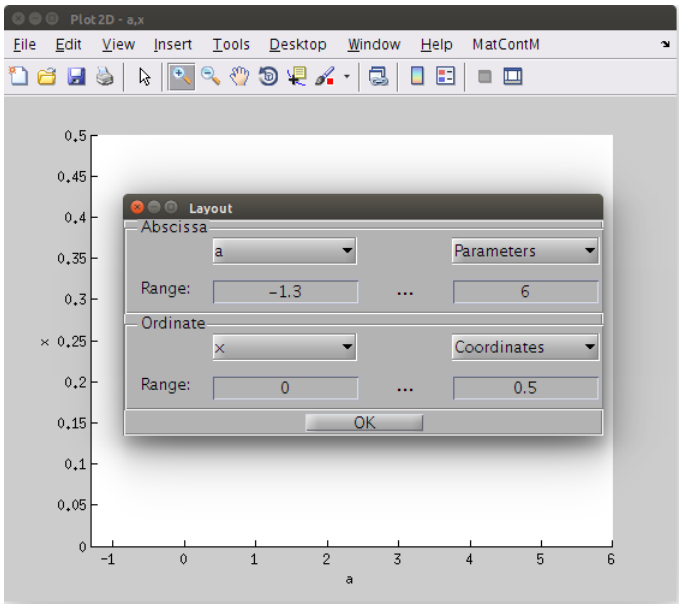


Figure 4

Select **Compute|Forward** in the main window and stop the computation after detecting the first `Branch Point (BP)`. This occurs at $a = 1$. A `BP` occurs when two fixed-point curves intersect. Select the `BP` point. The `Initial Point Type` in the main window is now set to `Branch Point`. The initializer `init_BPm_FPm` (selected by default) allows us to continue the intersecting curve of fixed points.

Select **Compute|Forward** and keep an eye on the Numeric window. Stop the computation after detecting the first `BP` point (at $a = \frac{4}{3}$). The fixed points of this new branch are F_2 fixed points. Select the latest `BP` point and select **Compute|Forward**. This branch consists of F_3 fixed points. Stop the computation after detecting a `NS` bifurcation at $a = 5$.

The resulting diagram is shown in Figure 5.

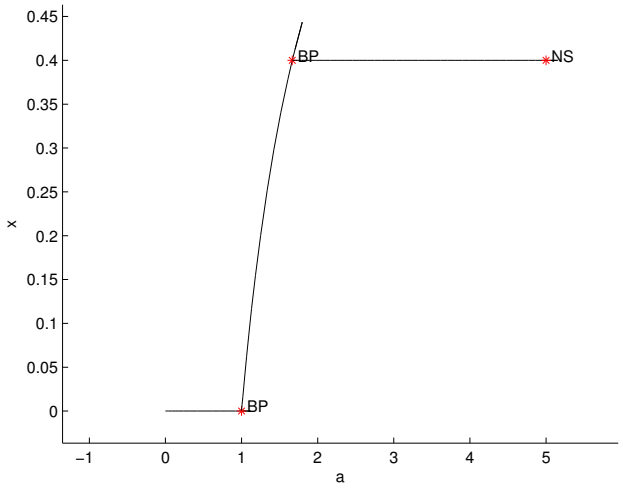


Figure 5

Section 5 demonstrates an alternative for obtaining this `NS` bifurcation.

3 Bifurcation curves

Select the `Neimark-Sacker (NS)` bifurcation. We will compute a number of bifurcation curves.

Before we start computing bifurcation curves, we will first make a change in our diagram settings. Curves are grouped together in what we call 'diagrams'. The current diagram in which the curves are stored is the default diagram named "`diagram`". We can create a new diagram using the context menu (right-click) in the `Current Curve` part of the main window, select **New Diagram** and enter the name `bifcurves`. All new curves will now be stored in that diagram.

The Neimark-Sacker bifurcation is now selected for continuation (see main window), a Neimark-Sacker bifurcation curve has been selected as curve (`init_NSm_NSm`). Go to the Starter window and select a and eps as the two active parameters (Figure 6).

Create a new plot2D window (you may close the other plot window(s)). We will visualize the (eps, a) parameter plane. Go to **MatcontM|Layout** in the plot menu of the Plot2D window. and select parameter `eps` for the Abscissa and parameter `a` for the Ordinate. A comfortable starting range for the abscissa is $[-5, 4]$ and for the ordinate we select $[-2, 18]$ (see Figure 7).

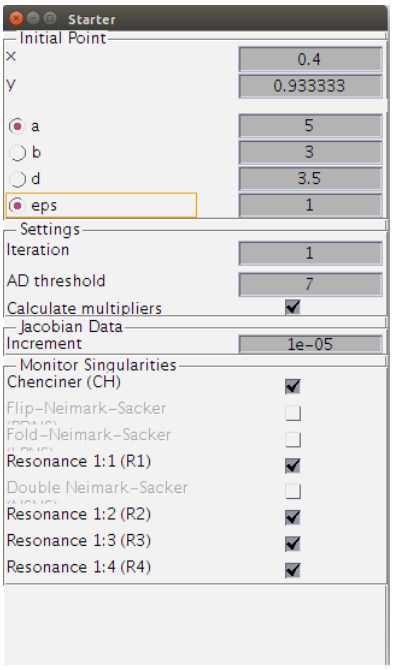


Figure 6

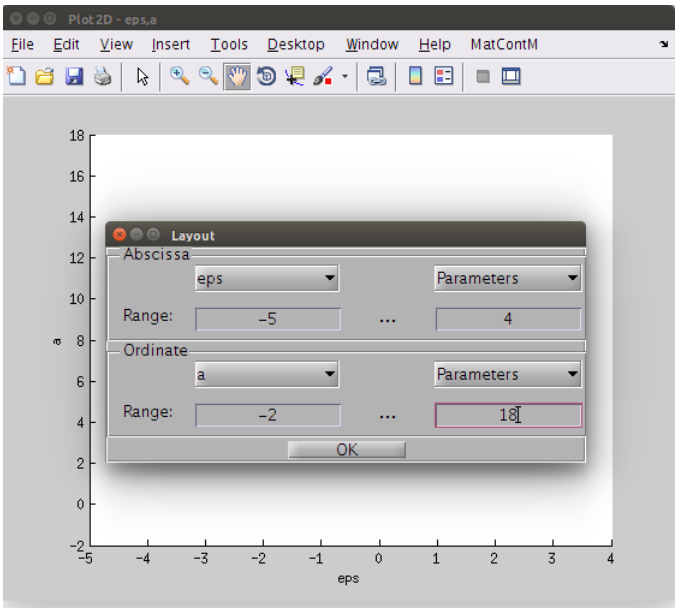


Figure 7

Start the continuation: Select **Compute**|**Forward** in the main window. You should detect several codimension 2 bifurcations. A `Resonance 1:4 (R4)` bifurcation, a `Resonance 1:3 (R3)` bifurcation and a `Resonance 1:2 (R2)` bifurcation (keep pressing **Resume** button). Select **Compute**|**Backward** and find a `Chenciner (CH)` bifurcation (see Figure 8).

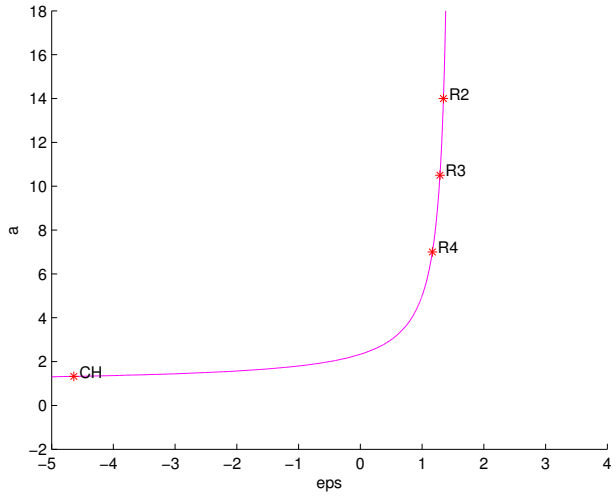


Figure 8

These codimension 2 bifurcations are *organizing centra*, allowing us to switch to other bifurcations curves. Select the `R2` bifurcation. You can do this by double-clicking on the graph or by using the **View Curve** browser. This codimension 2 bifurcation occurs when a Neimark-Sacker curve crosses a Period-doubling curve[1]. We will compute that `PD`-curve. Selecting the `R2` bifurcation allows us to select a number of branches seen in Figure 9.

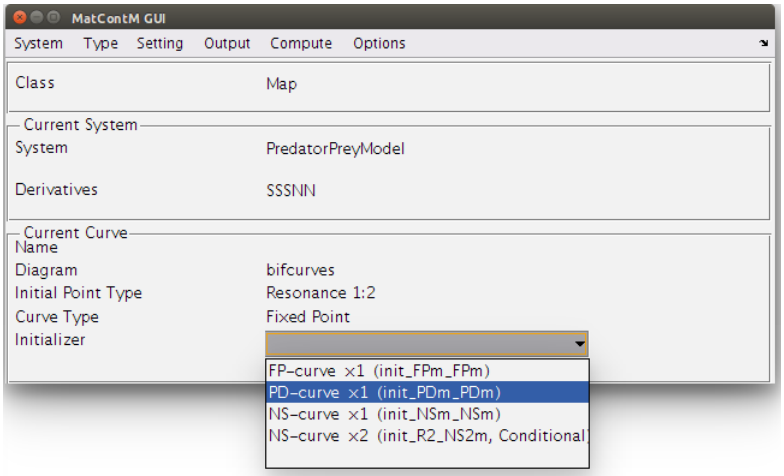


Figure 9

Select `PD-Curve x1` as initializer. The Starter data is inherited from the Neimark-Sacker curve, a and eps should still be selected as the two free parameters. Select **Compute|Forward** and find a `Fold-Flip (LPPD)` bifurcation; resume computations. Select **Compute|Backward** and the `R2` point gets rediscovered, after which you resume computations. The result is shown in Figure 10.

[1] At this point, the Neimark-Sacker curve turns into the *neutral saddle* curve.

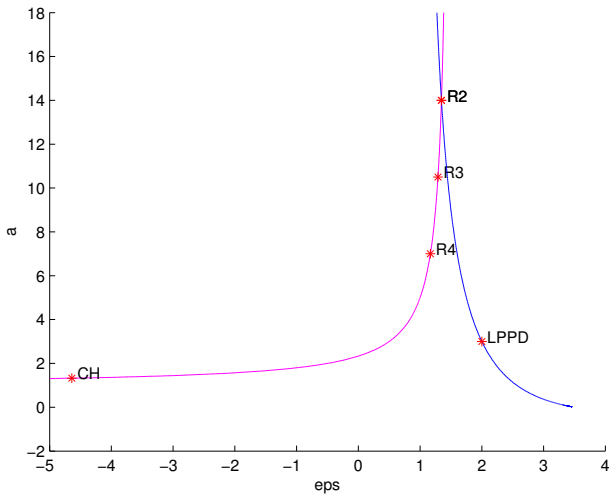


Figure 10

Select `R4` on the Neimark-Sacker curve. There are multiple branches available for continuation. Under the right condition, two `LP` curves of the fourth iterate ($\texttt{LP}^4$) can be computed.

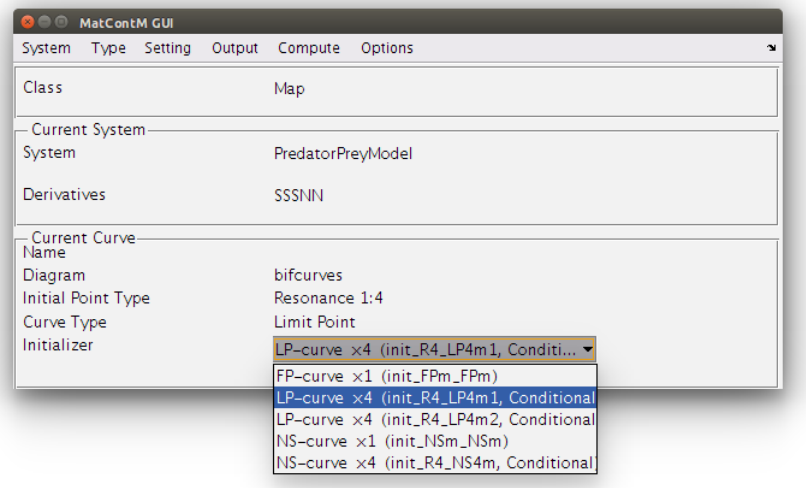


Figure 11

Select the init_R4_LP4m1 initializer, which sets up a conditional branch. In this situation, the condition is met and this bifurcation curve exists. If conditions are not met you will be warned once you start the continuation. Select **Compute|Forward**.

You should find two `LPPD` bifurcations, a `R1` bifurcation, one `Cusp (CP)` bifurcation and then the continuer will give up due to '*stepsize too small*'.

The result is shown in Figure 12, you can adjust the axis by using **MatcontM|Layout** (or by using the MATLAB *pan tool* in the toolbar).

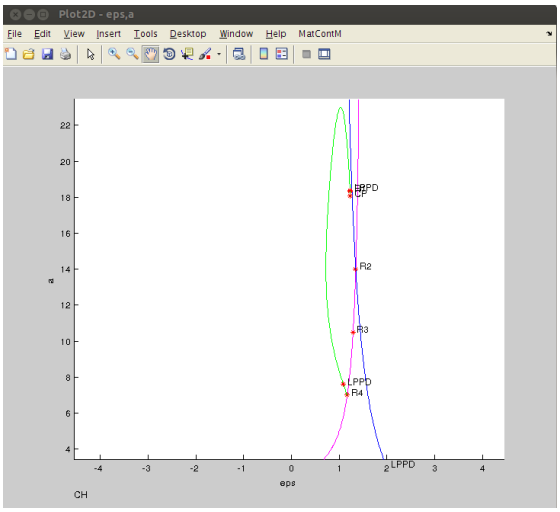


Figure 12: Computing the first LP4-curve from R4. The *pan tool* is selected in the toolbar.

Revisit the R4 bifurcation on the Neimark-Sacker curve. It should still be selected as the initial point in the main window. Choose init_R4_LP4m2 as initializer to compute the second LP^4 and select **Compute|Forward**.

You should find a LPPD point. After computation, you can extend this curve by selecting **Compute|Extend**. The continuation starts where it left off, the result is appended to the existing curve and is not stored in a new curve. If you extend several times you will find another LPPD bifurcation and a Resonance 1:1 (R1) bifurcation.

You can rescale your plot window with the MATLAB tools and the Layout window. When you change the scale of the graph, you might want to redraw all curves because the labels seem misplaced. By using the MatcontM menu in the Plot2D window, you can select **Redraw Diagram** to redraw all the curves in the current diagram. **Redraw Curve** only redraws the curve selected in the main window.

You can use these two draw buttons to visualize curves after they have been computed and stored. You just need to load the desired curve as current curve, open and configure a plot window and press **Redraw Curve/Diagram**.

The result is shown in Figure 13 as You should be able to see two green LP curves emanating from the R4 point on NS curve. This figure has been exported using the MATLAB save function after adjusting the scale.

We now select the first LPPD bifurcation, located on the first LP^4-curve. The easiest way is to double-click on the LPPD label. Make sure all tools are disabled. This bifurcation lies on a LP curve of the 4th iterated map. This bifurcation indicates presence of a

PD curve passing through the LPPD point. Select the PD-curve x1 initializer and perform a forward continuation, followed by a backwards continuation. This is the blue curve in Figure 13. The full picture of the `bifcurves` diagram is shown in Figure 14, after rescaling and removing unnecessary labels using the MATLAB *edit tool*.

The loop emanating from R3 is computed by double-clicking on R3 and computing NS-curve x3.

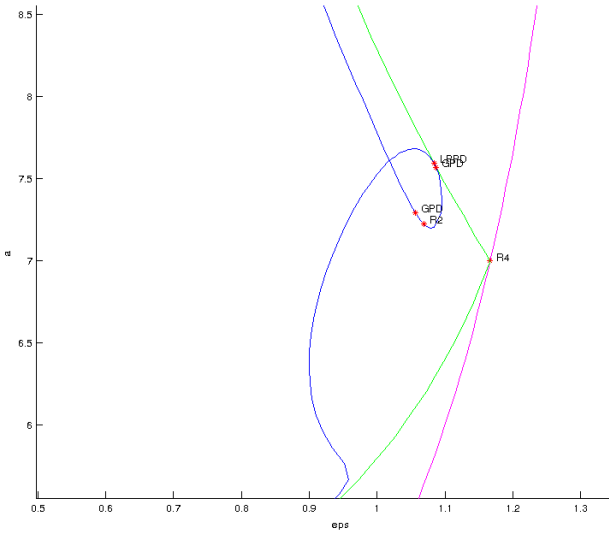


Figure 13

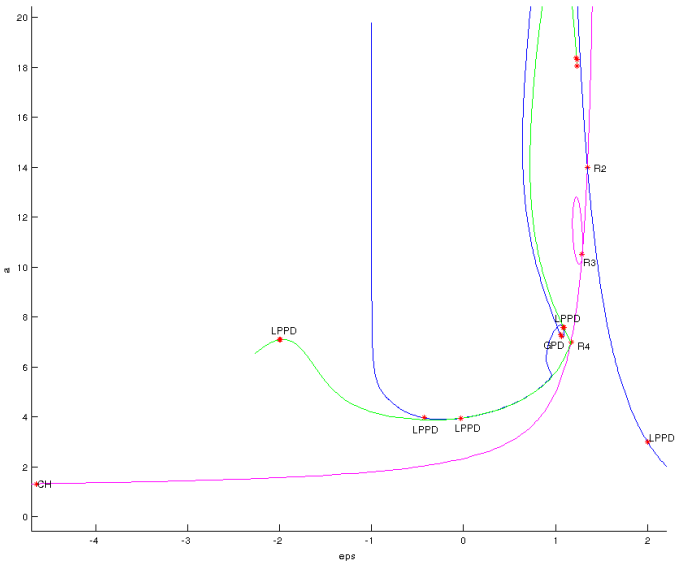


Figure 14: All the bifurcation curves in the diagram `bifcurves`

To create a three-dimensional plot, select **Output|Graphic|3D Plot**. Further select **MatcontM|Layout** in the Plot3D window and use the layout settings as in Figure 15. Select **MatcontM|Redraw Diagram** to draw all computed bifurcation curves. These are the curves located in the diagram `bifcurves`.

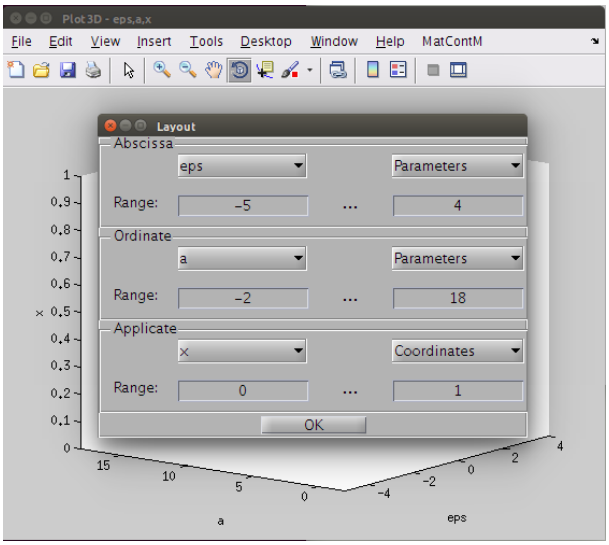


Figure 15

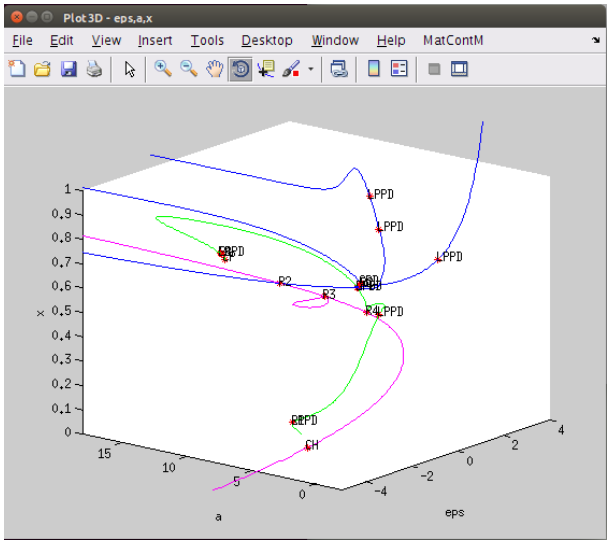


Figure 16: The *rotation tool* has been selected.

4 User functions

User functions are custom test functions added by the user to a system. The continuation algorithm evaluates these functions at each point on the computed curve and checks for sign changes in order to detect zeros. When a zero is detected, a bisection-like algorithm is used to determine the exact point where the zero occurs. These zero points are reported as special points. A test function can use the coordinates and parameters and returns a real value. Such functions can be used to detect situations specific to the system or to add flags for certain values.

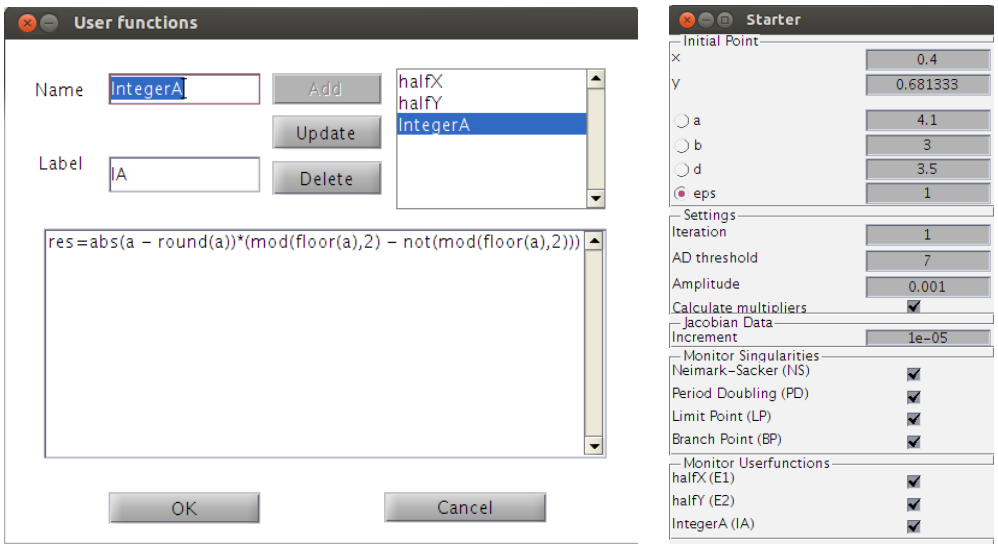


Figure 17: The User functions window and the Starter window after entering user functions.

We can add, edit and delete user functions by going to the System Browser by selecting **System|Systembrowser**. Select the system PPModel and click on the button **Add/Edit Userfunctions** below. The window for user functions appears (Figure 17). We can now add, edit and delete user functions for the selected system.

For example, we add 3 user functions. We wish to know when a coordinate (population density) reaches the value 0.5. We start by filling in the `Name` field, we will name our function `halfX`. We fill in the label by entering `E1`. The formula for our user function has to be entered in the form: `res=...` (no spaces after `res`). We can use the coordinates x and y and the parameters a, b, d and eps in our formulas.

We enter `res= x - 0.5`.

Click on the **Add** button to add the user function to the list in the right upper corner.

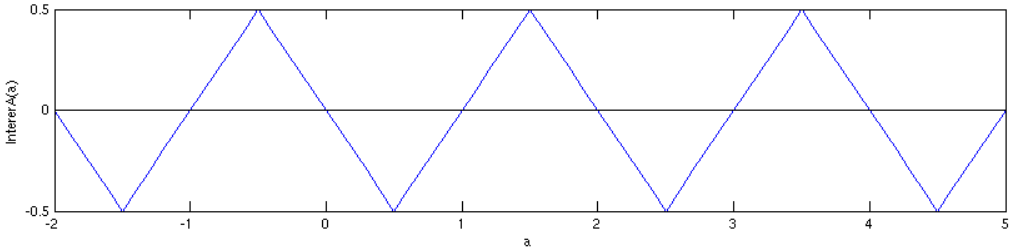
We name the second function `halfY`, the label is `E2` and the equation is `res=y - 0.5`. Press **Add**.

Our third function allows us to monitor when the parameter a has an integer value during computation.

We must design a function where $f(a) = 0$ if a is an integer. A sign change must occur in the zero point for it to be detected by the continuation algorithm. We can use the MATLAB built-in functions. We name the function IntegerA, give it label `IA` and use the equation:

```
res=abs(a - round(a))*(mod(floor(a),2) - not(mod(floor(a),2)))
```

The function values for the interval $[-2, 5]$ are shown in the following plot:



Press **Add**. Press **OK** to exit the window, the user functions are now specified.

You can always return to add, edit and delete user functions. You can edit a function by selecting it on the list, make changes, and press **Update**. Make sure the system is loaded as the `Current System` in the main window.

Now go to the Starter window (Figure 17). The user functions are now added to the sub panel named `Monitor Userfunctions`, you can enable them or disable them. They are disabled by default so enable them all.

Enabled user functions become available in the `Plot Layout` window and in the `Numeric` window (see Figure 18).

Using the settings from Figure 17 we compute a fixed-point curve by varying the parameter eps.

User function zeros are treated in the same way as the bifurcations, they are displayed on plots and they can be selected in the Data Window when viewing a curve. When a user function zero is selected as initial point for continuation, no initializers (branches) are available. This can be solved by declaring the type of the point, by selecting in the menu: **Type|Initial Point|**. The result of the computations are shown in Figure 19 and Figure 20.

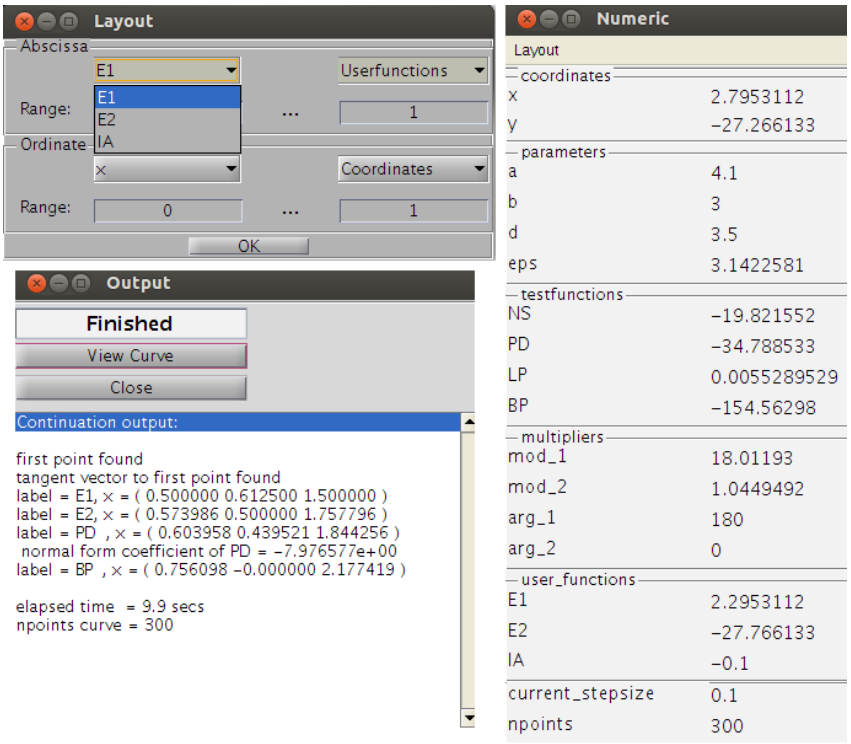


Figure 18: Numeric and Layout window with user functions. The continuer output displays the detection of user function zeros.

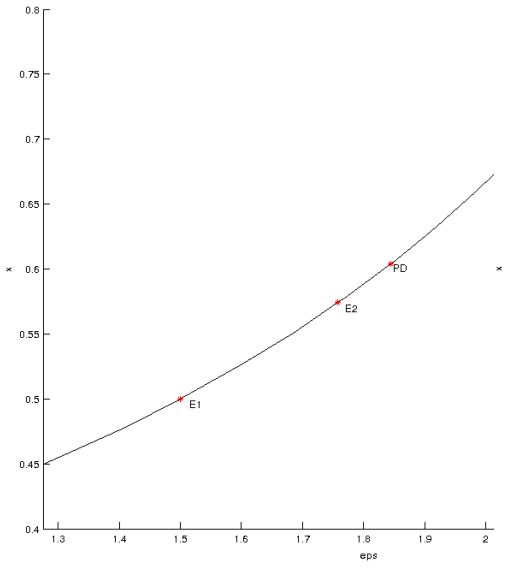


Figure 19

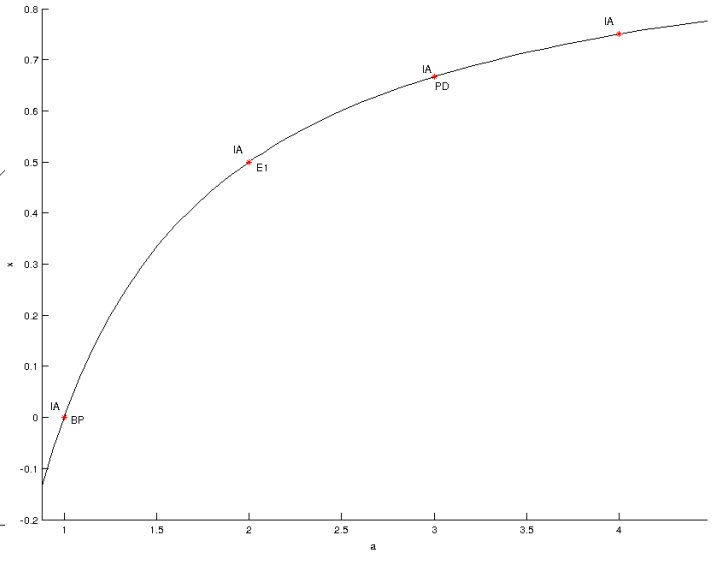


Figure 20

5 Configuring the Starter window from the MATLAB Command Window

Working with a graphical user interface has many advantages but also disadvantages. One of the difficulties is entering many numbers in the Starter window. Once the continuation has started and bifurcations are being detected, typing becomes minimal.

While the GUI of MatcontM is running, the MATLAB command line stays available. The `CommandLineInterface` tries to bridge the gap between GUI and command line, it allows for the transfer of data between the GUI and the command line workspace. We will show this by example for system (1).

First, select in the main window menu: **Type|Initial Point|Fixed Point**, to prepare for a fixed point continuation. The map `PPModel` (`PredatorPreyModel`) has three types of fixed points. F_1 and F_2 are simple to work with. F_3 is rather complicated. We will use the MATLAB *Anonymous Functions* features to make computing the coordinates easy for different sets of parameters.

```
>> F3_point = @(a,b,d,eps) [1/(d-eps) , (d/(d-eps))*((a/b)*(1-(1/(d-eps)))-(1/b))];
>> F3 = @(Param) F3_point(Param(1) , Param(2) , Param(3) , Param(4));
```

Note that you can copy-paste this if you are using a pdf viewer. We can now use the `F3_point`-function to compute the coordinates for any parameters or use the `F3`-function when the parameters are stored in an array.

```
>> param = [4.1,  3,  3.5,  1];
>> coord = F3(param)

coord =

    0.4000    0.6813
```

Make sure system (1) is loaded and the initial point type is set to `Fixed Point` and the initializer is set to `FP-Curve x1`. `Iteration` in the Starter window should be set to 1.

While the GUI is running, we execute in the MATLAB prompt:

```
>> cli = CommandLineInterface;
```

This creates a CommandLineInterface object that is stored in `cli`. This object is tethered to the current GUI session. If the GUI session is closed, `cli` becomes invalid. If the GUI is restarted a new `cli` object has to be created. You can use `cli` throughout the current session of the GUI.
We will transfer the data to the GUI:

```
>> cli.setPa(param);         %or:  cli.setParameters(param);
>> cli.setCo(coord);         %or:  cli.setCoordinates(coord);
>> %cli.setIP(coord, param) %alternative command, IP = InitialPoint
```

The parameters and coordinates in the Starter window should now be updated. Use `cli.getCurve()` to retrieve the raw continuation data after a computation.

If you select a as the free parameter and do a forward computation, you should find a Neimark-Sacker bifurcation at $a = 5$.

6 Additional problems

A. **Bifurcation curves of delayed logistic map** Revisit the discrete-time dynamical system generated by the map

$$\mathsf{DelLogMap} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} rx(1-y)+\epsilon \\ x \end{pmatrix}$$

studied in Tutorial MI. Use the located bifurcation points on the stored curves as initial data for the two-parameter continuation of these bifurcations to obtain Figure 21.

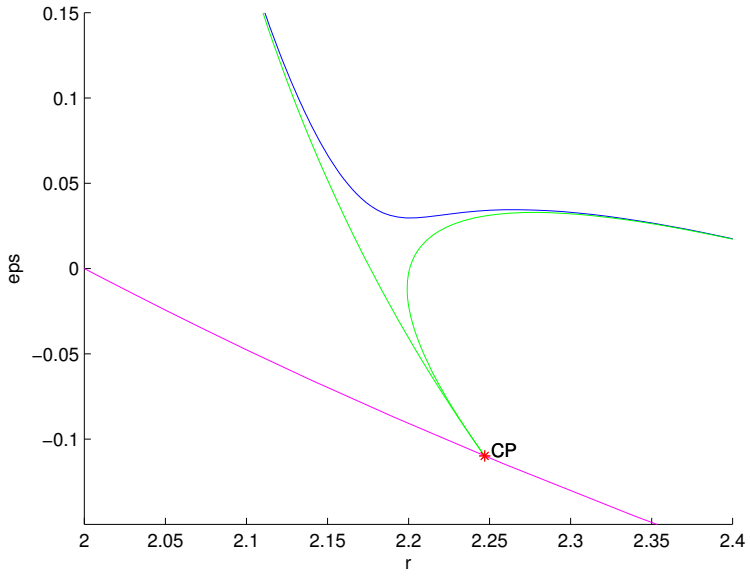


Figure 21: Bifurcation curves of DLM: NS (magenta), LPC for the period-7 cycles (green), and PD for the period-7 cycles (blue). Point labled by CP is actually close to the resonance 1:7 in the Neimark-Sacker curve NS.

B. **Discrete-time Lotka-Volterra model**

Consider the following simple prey-predator population model[2]

$$\begin{cases} x_{k+1} &= ax_k(1-x_k) - bx_k y_k, \\ y_{k+1} &= dx_k y_k \end{cases}$$

where (x_k, y_k) are the prey and predator densities in year k, and (a, b, d) are positive parameters. Study fixed points and cycles of the corresponding planar map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax(1-x) - bxy \\ dxy \end{pmatrix}.$$

1. Study the map for $b = 0.2$ and $(a, d) \in [0, 12] \times [0, 4]$ using MatContM:
   (i) Find by simulations a positive fixed point of the map at $a = 3.5$ and $d = 2.67$. Continue this fixed point w.r.t. parameter d and detect two codim 1 bifurcations of this point, namely: NS and PD.
   (ii) Continue the corresponding codim 1 bifurcation curves $NS^{(1)}$ and $PD^{(1)}$ in the (d, a)-plane and locate several codim 2 points, including strong resonances R2, R3, and R4 on the $NS^{(1)}$ curve.
   (iii) Starting from the detected R4 point, compute two $LPC^{(4)}$ bifurcation curves of 4-cycles rooted there.
   (iv) Compute the period-doubling $PD^{(4)}$ and Neimark-Sacker $NS^{(4)}$ bifurcation curves of 4-cycles located between the $LPC^{(4)}$ curves.
   Hint: To obtain starting points, first continue a stable period-4 cycle from one of the LPC points w.r.t. parameter a and stop the computation somewhere between the LPC-curves. Then continue the last stable 4-cycles w.r.t. parameter d to detect its NS and PD bifurcations.
2. Study the map analytically for positive (a, b, d):
   (i) Find all non-negative fixed points and derive explicit formulas for the $NS^{(1)}$ and $PD^{(1)}$ curves, as well as bifurcation curves of fixed points with at least one zero coordinate. What is the stability domain in the (d, a)-palne of the positive fixed point ?
   (ii) Compute analytically the normal form coefficients for the PD and NS bifurcations along the curves $NS^{(1)}$ and $PD^{(1)}$. Predict what happens when crossing these curves at generic points.

Are your numerical and analytical results in agreement with each other ?

[2]The map (1) reduces to his model for $\epsilon = 0$.