and

$$x_2 + x_4 + x_6 = 10, \qquad x_2, x_4, x_6 > 0. \tag{4}$$

The number of integer solutions for Eq. (3) equals the number of integer solutions for

$$y_1 + y_3 + y_5 + y_7 = 1, \qquad y_1, y_3, y_5, y_7 \geq 0.$$

This number is $\binom{4+1-1}{1} = \binom{4}{1} = 4$. Similarly, for Eq. (4), the number of solutions is $\binom{3+7-1}{7} = \binom{9}{7} = 36$. Consequently, by the rule of product there are $4 \cdot 36 = 144$ arrangements of five E's and 10 O's that determine seven runs, the first run starting with E.

The seven runs may also have the first run starting with an O and the last run ending with an O. So now let w_1 count the number of O's in the first run, w_2 the number of E's in the second run, w_3 the number of O's in the third run, . . . , and w_7 the number of O's in the seventh run. Here we want the number of integer solutions for

$$w_1 + w_3 + w_5 + w_7 = 10, \qquad w_1, w_3, w_5, w_7 > 0$$

and

$$w_2 + w_4 + w_6 = 5, \qquad w_2, w_4, w_6 > 0.$$

Arguing as above, we find that the number of ways to arrange five E's and 10 O's, resulting in seven runs where the first run starts with an O, is $\binom{4+6-1}{6}\binom{3+2-1}{2} = \binom{9}{6}\binom{4}{2} = 504$.

Consequently, by the rule of sum, the five E's and 10 O's can be arranged in $144 + 504 = 648$ ways to produce seven runs.

EXERCISES 4

1. In how many ways can 10 (identical) dimes be distributed among five children if (a) there are no restrictions? (b) each child gets at least one dime? (c) the oldest child gets at least two dimes?

2. In how many ways can 15 (identical) candy bars be distributed among five children so that the youngest gets only one or two of them?

3. Determine how many ways 20 coins can be selected from four large containers filled with pennies, nickels, dimes, and quarters. (Each container is filled with only one type of coin.)

4. A certain ice cream store has 31 flavors of ice cream available. In how many ways can we order a dozen ice cream cones if (a) we do not want the same flavor more than once? (b) a flavor may be ordered as many as 12 times? (c) a flavor may be ordered no more than 11 times?

5. **a)** In how many ways can we select five coins from a collection of 10 consisting of one penny, one nickel, one dime, one quarter, one half-dollar, and five (identical) Susan B. Anthony dollars?

b) In how many ways can we select n objects from a collection of size $2n$ that consists of n distinct and n identical objects?

6. Answer Example 32, where the 12 symbols being transmitted are four A's, four B's, and four C's.

7. Determine the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 32,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 4$ **b)** $x_i > 0, \quad 1 \leq i \leq 4$

c) $x_1, x_2 \geq 5, \quad x_3, x_4 \geq 7$

d) $x_i \geq 8, \quad 1 \leq i \leq 4$ **e)** $x_i \geq -2, \quad 1 \leq i \leq 4$

f) $x_1, x_2, x_3 > 0, \quad 0 < x_4 \leq 25$

8. In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among three student helpers if each helper wants at least one donut of each kind?

9. Columba has two dozen each of n different colored beads. If she can select 20 beads (with repetitions of colors allowed) in 230,230 ways, what is the value of n?

10. In how many ways can Lisa toss 100 (identical) dice so that at least three of each type of face will be showing?

11. Two n-digit integers (leading zeros allowed) are considered equivalent if one is a rearrangement of the other. (For example, 12033, 20331, and 01332 are considered equivalent five-digit integers.) (a) How many five-digit integers are not equivalent? (b) If the digits 1, 3, and 7 can appear at most once, how many nonequivalent five-digit integers are there?

12. Determine the number of integer solutions for

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 5$

b) $x_i \geq -3, \quad 1 \leq i \leq 5$

13. In how many ways can we distribute eight identical white balls into four distinct containers so that (a) no container is left empty? (b) the fourth container has an odd number of balls in it?

14. a) Find the coefficient of v^2w^4xz in the expansion of $(3v + 2w + x + y + z)^8$.

b) How many distinct terms arise in the expansion in part (a)?

15. In how many ways can Beth place 24 different books on four shelves so that there is at least one book on each shelf? (For any of these arrangements consider the books on each shelf to be placed one next to the other, with the first book at the left of the shelf.)

16. For which positive integer n will the equations

(1) $x_1 + x_2 + x_3 + \cdots + x_{19} = n$, and

(2) $y_1 + y_2 + y_3 + \cdots + y_{64} = n$

have the same number of positive integer solutions?

17. How many ways are there to place 12 marbles of the same size in five distinct jars if (a) the marbles are all black? (b) each marble is a different color?

18. a) How many nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + x_3 + \cdots + x_7 = 37$, $x_1 + x_2 + x_3 = 6$?

b) How many solutions in part (a) have $x_1, x_2, x_3 > 0$?

19. How many times is the **print** statement executed for the following program segment? (Here, i, j, k, and m are integer variables.)

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      for m := 1 to k do
        print (i * j) + (k * m)
```

20. In the following program segment, i, j, k, and *counter* are integer variables. Determine the value that the variable *counter* will have after the segment is executed.

```
counter := 10
for i := 1 to 15 do
  for j := i to 15 do
    for k := j to 15 do
      counter := counter + 1
```

21. Find the value of *sum* after the given program segment is executed. (Here i, j, k, *increment*, and *sum* are integer variables.)

```
increment := 0
sum := 0
for i := 1 to 10 do
  for j := 1 to i do
    for k := 1 to j do
      begin
        increment := increment + 1
        sum := sum + increment
      end
```

22. Consider the following program segment, where i, j, k, n, and *counter* are integer variables and the value of n (a positive integer) is set prior to this segment.

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    for k := 1 to j do
      counter := counter + 1
```

We shall determine, in two different ways, the number of times the statement

```
counter := counter + 1
```

is executed. (This is also the value of *counter* after execution of the program segment.) From the result in Example 39, we know that the statement is executed $\binom{n+3-1}{3} = \binom{n+2}{3}$ times. For a fixed value of i, the **for** loops involving j and k result in $\binom{i+1}{2}$ executions of the counter increment statement. Consequently, $\binom{n+2}{3} = \sum_{i=1}^{n} \binom{i+1}{2}$. Use this result to obtain a summation formula for

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2.$$

23. a) Given positive integers m, n with $m \geq n$, show that the number of ways to distribute m identical objects into n distinct containers with no container left empty is

$$C(m-1, m-n) = C(m-1, n-1).$$

b) Show that the number of distributions in part (a) where each container holds at least r objects ($m \geq nr$) is

$$C(m-1+(1-r)n, n-1).$$

24. Write a computer program (or develop an algorithm) to list the integer solutions for

a) $x_1 + x_2 + x_3 = 10, \quad 0 \leq x_i, \quad 1 \leq i \leq 3$

b) $x_1 + x_2 + x_3 + x_4 = 4, \quad -2 \leq x_i, \quad 1 \leq i \leq 4$

25. Consider the 2^{19} compositions of 20. (a) How many have each summand even? (b) How many have each summand a multiple of 4?

26. Let n, m, k be positive integers with $n = mk$. How many compositions of n have each summand a multiple of k?

27. Frannie tosses a coin 12 times and gets five heads and seven tails. In how many ways can these tosses result in (a) two runs of heads and one run of tails; (b) three runs; (c) four runs;

(d) five runs; (e) six runs; and (f) equal numbers of runs of heads and runs of tails?

28. a) For $n \geq 4$, consider the strings made up of n bits — that is, a total of n 0's and 1's. In particular, consider those strings where there are (exactly) two occurrences of 01. For example, if $n = 6$ we want to include strings such as 010010 and 100101, but not 101111 or 010101. How many such strings are there?

b) For $n \geq 6$, how many strings of n 0's and 1's contain (exactly) three occurrences of 01?

c) Provide a combinatorial proof for the following: For $n \geq 1$,

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even.} \end{cases}$$

5 The Catalan Numbers (Optional)

In this section a very prominent sequence of numbers is introduced. This sequence arises in a wide variety of combinatorial situations. We'll begin by examining one specific instance where it is found.

EXAMPLE 42

Let us start at the point (0, 0) in the xy-plane and consider two kinds of moves:

$$\text{R: } (x, y) \rightarrow (x+1, y) \qquad \text{U: } (x, y) \rightarrow (x, y+1).$$

We want to know how we can move from (0, 0) to (5, 5) using such moves — one unit to the right or one unit up. So we'll need five R's and five U's. At this point we have a situation like that in Example 14, so we know there are $10!/(5!\,5!) = \binom{10}{5}$ such paths. But now we'll add a twist! In going from (0, 0) to (5, 5) one may touch but *never* rise above the line $y = x$. Consequently, we want to include paths such as those shown in parts (a) and (b) of Fig. 9 but not the path shown in part (c).

The first thing that is evident is that each such arrangement of five R's and five U's must start with an R (and end with a U). Then as we move across this type of arrangement — going from left to right — the number of R's at any point must equal or exceed the number of U's. Note how this happens in parts (a) and (b) of Fig. 9 but not in part (c). Now we can solve the problem at hand if we can count the paths [like the one in part (c)] that go from (0, 0) to (5, 5) but rise above the line $y = x$. Look again at the path in part (c) of Fig. 9. Where does the situation there break down for the first time? After all, we start with the requisite R — then follow it by a U. So far, so good! But then there is a second U and, at this (first) time, the number of U's exceeds the number of R's.

Now let us consider the following transformation:

$$\text{R, U, U, } \vdots \text{ U, R, R, R, U, U, R} \leftrightarrow \text{R, U, U, } \vdots \text{ R, U, U, U, R, R, U.}$$

What have we done here? For the path on the left-hand side of the transformation, we located the first move (the second U) where the path rose above the line $y = x$. The moves up to and including this move (the second U) remain as is, but the moves that follow are interchanged — each U is replaced by an R and each R by a U. The result is the path on the right-hand side of the transformation — an arrangement of four R's and six U's, as seen in part (d) of Fig. 9. Part (e) of that figure provides another path to be avoided; part (f) shows what happens when this path is transformed by the method described above. Now suppose we start with an arrangement of six U's and four R's, say

$$\text{R, U, R, R, U, U, U, } \vdots \text{ U, U, R.}$$