Analysis of Algorithms, I

CSOR W4231.002

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Outline

1. Recap
2. Randomized Quicksort
3. Occupancy problems

Review of the last lecture

- Quicksort
  - Sorts in-place
  - Best-case running-time $O(n \log n)$
  - Worst-case running-time $\Theta(n^2)$
- Randomized algorithms
  - Randomized Quicksort
  - Discrete random variables, linearity of expectation

Pseudocode for randomized Quicksort

$\texttt{Randomized-Quicksort}(A, left, right)$

  if $|A| = 0$ **then** return $//A$ is empty
  end if
  $split = \texttt{Randomized-Partition}(A, left, right)$
  $\texttt{Randomized-Quicksort}(A, left, split - 1)$
  $\texttt{Randomized-Quicksort}(A, split + 1, right)$

$\texttt{Randomized-Partition}(A, left, right)$

  $b = \texttt{random}(left, right)$
  swap$(A[b], A[right])$
  return $\texttt{Partition}(A, left, right)$

Subroutine $\texttt{random}(i, j)$ returns a random number between i and j inclusive.

Expected running time analysis of randomized Quicksort

- Let $T(n)$ be the expected running time of `Randomized-Quicksort`.
- We want to bound $T(n)$.
- `Randomized-Quicksort` differs from `Quicksort` only in how they select their pivot elements.

$\Rightarrow$ We will analyze `Randomized-Quicksort` based on `Quicksort` and `Partition`.

Pseudocode for Partition

```
Partition(A, left, right)
   pivot = A[right]                              line 1
   split = left − 1                              line 2
   for  j = left to right − 1  do                line 3
       if A[j] ≤ pivot  then                     line 4
           swap(A[j], A[split + 1])              line 5
           split = split + 1                     line 6
       end if
   end for
   swap(pivot, A[split + 1])                     line 7
   return split + 1                              line 8
```

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   2. inside the for loop
      - let X be the total number of comparisons performed at line 4 in all calls to `Partition`
      - each comparison may require some further constant work (lines 5 and 6)

      $\Rightarrow$ total work inside the for loop in all calls to `Partition` is $O(X)$

Towards a bound for $T(n)$

The running time of `Randomized-Quicksort` is

$$O(n + X),$$

where X is the total number of comparisons performed by **all** `Partition` calls. To bound $T(n)$, we need analyze X.

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Fix any two input items. During the execution of the algorithm, they may be compared at most once.

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Fact 1.

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Proof.

Comparisons are only performed with the *pivot* of each `Partition` call. After `Partiton` returns, *pivot* is in its final location in the output and will not be part of the input to any future recursive call. □

Simplifying the analysis

- There are n numbers in the input, hence $\binom{n}{2} = \frac{n(n-1)}{2}$ distinct (unordered) pairs of input numbers.
- Fact 2 says that the algorithm will perform at most $\binom{n}{2}$ comparisons.
- *What is the expected number of comparisons?*

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- *What is the expected number of comparisons?*

To simplify the analysis

- relabel the input as $z_1, z_2, \ldots, z_n$, where z_i is the i-th smallest number.
- **assume** that all input numbers are distinct; thus $z_i < z_j$, for $i < j$.

Writing X as the sum of indicator random variables

Let X_{ij} be an indicator random variable such that

$$X_{ij} = \begin{cases} 1, & \text{if } z_i \text{ and } z_j \text{ are ever compared} \\ 0, & \text{otherwise} \end{cases}$$

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By linearity of expectation

$$E[X] = E\left[\sum_{1 \le i < j \le n} X_{ij}\right] = \sum_{1 \le i < j \le n} E[X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[X_{ij} = 1]$$

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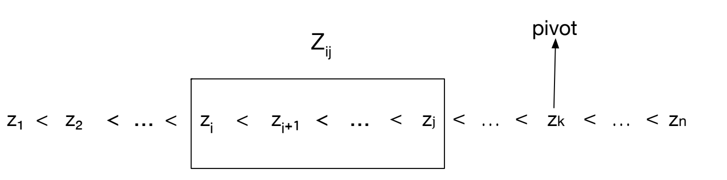
Goal: compute $\Pr[X_{ij} = 1]$, that is, the probability that two *fixed* items z_i and z_j are ever compared.

Fix two items z_i and z_j. When are they compared?

Notation: let $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$

Consider the initial call `Partition`$(A, 1, n)$. Assume it picks z_k **outside** Z_{ij} as *pivot* (see figure below).



1. z_i and z_j are **not** compared in this call *(why?)*.
2. All items in Z_{ij} will be greater (or smaller) than z_k, so they will **all be input to the same subproblem** after `Partition`$(A, 1, n)$ returns.

In the first `Partition` with $pivot \in Z_{ij} = \{z_i, \ldots, z_j\}$

The first `Partition` call that picks its *pivot* from Z_{ij} determines if z_i, z_j are ever compared. Three possibilities:

1. $pivot = z_i$

2. $pivot = z_j$

3. $pivot = z_\ell$, for some $i < \ell < j$

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The first `Partition` call that picks its *pivot* from Z_{ij} determines if z_i, z_j are ever compared. Three possibilities:

1. $pivot = z_i$

   z_i is compared with every element in $Z_{ij} - \{z_i\}$, thus with z_j too. z_i is placed in its final location in the output and will not appear in any future calls to `Partition`.

2. $pivot = z_j$

   z_j is compared with every element in $Z_{ij} - \{z_j\}$, thus with z_i too. z_j is placed in its final location in the output and will not appear in any future recursive calls.

3. $pivot = z_\ell$, for some $i < \ell < j$

   z_i and z_j are never compared *(why?)*

So z_i and z_j are compared when …

… either of them is chosen as *pivot* in that first Partition call that chooses its *pivot* element from Z_{ij}.

Now we can compute $\Pr[X_{ij} = 1]$:

$$\begin{aligned}\Pr[X_{ij} = 1] \quad = \quad & \Pr[z_i \text{ is chosen as } \textit{pivot} \text{ by the first } \texttt{Partition} \\ & \quad \text{that picks its } \textit{pivot} \text{ from } Z_{ij}, \ \ \textbf{or} \\ & \quad z_j \text{ is chosen as } \textit{pivot} \text{ by the first } \texttt{Partition} \\ & \quad \text{that picks its } \textit{pivot} \text{ from } Z_{ij}] \end{aligned} \tag{1}$$

The union bound

Suppose we are given a set of events $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, and we are interested in the probability that **any** of them happens.

Union bound: Given events $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, we have

$$\Pr\left[\bigcup_{i=1}^{n} \varepsilon_i\right] \leq \sum_{i=1}^{n} \Pr[\varepsilon_i].$$

Union bound for mutually exclusive events: Suppose that $\varepsilon_i \cap \varepsilon_j = \emptyset$ for each pair of events. Then

$$\Pr\left[\bigcup_{i=1}^{n} \varepsilon_i\right] = \sum_{i=1}^{n} \Pr[\varepsilon_i].$$

Computing the probability that z_i and z_j are compared

Since the two events in equation (1) are mutually exclusive, we obtain

$$\begin{aligned}
\Pr[X_{ij} = 1] &= \Pr[z_i \text{ is chosen as } \textit{pivot} \text{ by the first } \texttt{Partition} \\
&\qquad \text{call that picks its } \textit{pivot} \text{ from } Z_{ij}] \\
&+ \Pr[z_j \text{ is chosen as } \textit{pivot} \text{ by the first } \texttt{Partition} \\
&\qquad \text{call that picks its } \textit{pivot} \text{ from } Z_{ij}] \\
&= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}, \qquad (2)
\end{aligned}$$

since the set Z_{ij} contains $j - i + 1$ elements.

From $\Pr[X_{ij} = 1]$ to $E[X]$

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[X_{ij} = 1] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= 2 \sum_{i=1}^{n-1} \sum_{\ell=2}^{n-i+1} \frac{1}{\ell} \qquad (3)
\end{aligned}
$$

Note that $\sum_{\ell=1}^{k} \frac{1}{\ell} = H_k$ is the k-th harmonic number, such that

$$\ln k \leq H_k \leq \ln k + 1 \qquad (4)$$

Hence $\sum_{\ell=2}^{n-i+1} \frac{1}{\ell} \leq \ln(n-i+1)$. Substituting in (3), we get

$$E[X] \leq 2 \sum_{i=1}^{n-1} \ln(n-i+1) \leq 2 \sum_{i=1}^{n-1} \ln n = O(n \ln n)$$

From $E[X]$ to $T(n)$

- Equations (3), (4) also yield a lower bound of $\Omega(n \ln n)$ for $E[X]$ *(show this!)*.
- Hence $E[X] = \Theta(n \ln n)$. Then the expected running time of `Randomized-Quicksort` is

$$T(n) = \Theta(n \ln n)$$

Balls in bins problems

Occupancy problems: find the distribution of balls into bins when m balls are thrown independently and uniformly at random into n bins.

- Applications: analysis of randomized algorithms and data structures (e.g., hash table)

Q1: How many balls can we throw before it is more likely than not that some bin contains at least two balls?

In symbols: *find k such that*

$$\Pr[\exists \text{ bin with} \geq 2 \text{ balls after } k \text{ balls are thrown}] \geq 1/2$$

Easier to analyze the complement of this event

Easier to think about the probability that no two balls fall into the same bin. Since

$$\Pr[\exists \text{ bin with } \geq 2 \text{ balls}] = 1 - \Pr[\text{no two balls fall in the same bin}],$$

we can rephrase Q1 as follows.

Q1 (rephrased): Find k so that

$$\Pr[\text{no two balls fall in the same bin after } k \text{ balls are thrown}] < 1/2$$

Analysis

Consider one ball at a time.

- The 1st ball falls into some bin.
- The 2nd ball falls into a new bin w. prob. $1 - \frac{1}{n}$.
- The 3rd ball falls into a new bin (given that the first two balls fell into different bins) w. prob. $1 - \frac{2}{n}$.
- The m-th ball falls into a new bin (given that the first $m - 1$ balls fell into different bins) w. prob. $1 - \frac{m-1}{n}$.

The probability that all of these events occur simultaneously is

$$\prod_{k=1}^{m-1} \left(1 - \frac{k}{n}\right) \tag{5}$$

Application: the birthday paradox

Use $1 + x \leq e^x$ for all $x \geq 0$ to upper bound (5)

$$\prod_{k=1}^{m-1} e^{-k/n} = e^{-\sum_{k=1}^{m-1} k/n} = e^{-\frac{m(m-1)}{(2\cdot n)}} \approx e^{-\frac{m^2}{2n}} \tag{6}$$

Requiring $e^{-\frac{m^2}{2n}} < 1/2$ yields $m > \sqrt{n \cdot 2 \ln 2} = \Omega(\sqrt{n})$.

- **Application:** birthday paradox
  Assumption: For $n = 365$, each person has an independent and uniform at random birthday from among the 365 days of the year.
  Once 23 people are in a room, it is more likely than not that two of them share a birthday.

More balls-in-bins questions

- *Q2: What is the expected load of a bin after m balls are thrown?*
- *Q3: What is the expected #empty bins after m balls are thrown?*
- *Q4: What is the load of the fullest bin?*
- *Q5: What is the expected number of balls until* **every** *bin has at least one ball (Coupon Collector's Problem)?*

Expected load of a bin

Suppose that m balls are thrown independently and uniformly at random into n bins. Fix a bin j.

- Let X_{ij} be an indicator r.v. such that $X_{ij} = 1$ if and only if ball i falls into bin j. Then

$$E[X_{ij}] = \Pr[X_{ij} = 1] = \frac{1}{n}.$$

The total #balls in bin j is given by $X_j = \sum_{i=1}^{m} X_{ij}$. By linearity of expectation,

$$E[X_j] = \sum_{i=1}^{m} E[X_{ij}] = m/n.$$

Since bins are symmetric, the expected load of any bin is m/n.

Expected # empty bins

Suppose that m balls are thrown independently and uniformly at random into n bins. Fix a bin j.

- Let Y_j be an indicator r.v. such that $Y_j = 1$ if and only if bin j is empty.
- $\Pr[\text{ball } i \text{ does not fall in bin } j] = 1 - 1/n$
- $\Pr[\text{for all } i, \text{ ball } i \text{ does not fall in bin } j] = (1 - 1/n)^m$
- Hence $\Pr[Y_j = 1] = (1 - 1/n)^m$.

The number of empty bins is given by the random variable $Y = \sum_{j=1}^{n} Y_j$. By linearity of expectation

$$E[Y] = \sum_{j=1}^{n} E[Y_j] = \left(1 - \frac{1}{n}\right)^m \approx ne^{-m/n}$$

Max load in any bin, with high probability (case $m = n$)

Proposition 1.

When throwing n balls into n bins uniformly and independently at random, the maximum load in any bin is $\Theta(\ln n / \ln \ln n)$ with probability close to 1 as n grows large.

Two-sentence sketch of the proof.

1. Upper bound the probability that **any** bin contains more than k balls by a union bound: $\sum_{j=1}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \left(\frac{1}{n}\right)^{\ell} \left(1 - \frac{1}{n}\right)^{n-\ell}$.
2. Compute the smallest possible k^* such that the probability above is less than $1/n$ (which becomes negligible as n grows large).

□

Expected #balls until no empty bins

Suppose that we throw balls independently and uniformly at random into n bins, one at a time (the first ball falls at time $t = 1$).

- We call a throw a **success** if it lands in an empty bin.
- We call the sequence of balls starting after the $(j-1)$-st success and ending with the j-th success, the j-th **epoch**.
- Clearly the first ball is a **success**, hence ends epoch 1.
- Let η_2 be the #balls thrown in epoch 2.

$$\forall t \in \text{epoch } 2, \Pr[\text{ball } t \text{ in epoch } 2 \text{ is a } \textbf{success}] = \frac{n-1}{n}$$

- Similarly, let η_j be the #balls thrown in epoch j.

$$\forall t \in \text{epoch } j, \Pr[\text{ball } t \text{ in epoch } j \text{ is a } \textbf{success}] = \frac{n-j+1}{n}$$

At the end of the n-th epoch, each of the n bins has at least one ball.

Expected #balls until no empty bins (cont'd)

Let $\eta = \sum_{j=1}^{n} \eta_j$. We want

$$E[\eta] = E\left[\sum_{j=1}^{n} \eta_j\right] = \sum_{j=1}^{n} E[\eta_j]$$

- Each epoch is geometrically distributed with success probability $p_j = \frac{n-j+1}{n}$.
- Recall that the expectation of a geometrically distributed variable with success probability p is given by $1/p$.
- Thus $E[\eta_j] = \frac{1}{p_j} = \frac{n}{n-j+1}$.

Then

$$E[\eta] = \sum_{j=1}^{n} \frac{n}{n-j+1} = n \sum_{j=1}^{n} \frac{1}{j} = n(\ln n + O(1))$$

Probability review

- A sample space Ω consists of the possible outcomes of an experiment.
- Each point x in the sample space has an associated probability mass $p(x) \geq 0$, such that $\sum_{x \in \Omega} p(x) = 1$.
- Example experiment: flip a fair coin;
  $\Omega = \{heads, tails\}$; $\Pr[heads] = \Pr[tails] = 1/2$.
- We define an event $\mathcal{E}$ to be any subset of Ω, that is, a collection of points in the sample space.
- We define the probability of the event to be the sum of the probability masses of all the points in $\mathcal{E}$. That is,

$$\Pr[\mathcal{E}] = \sum_{x \in \mathcal{E}} p(x)$$