Algorithms for Data Science
CSOR W4246

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Outline

Review of last lecture

- Probability theory review
  - Random variables
  - Linearity of expectation
- Expected running-time analysis of randomized Quicksort

Today

Data Compression

- Symbol codes
- Optimal lossless compression and prefix codes
- Trees and prefix codes
- The Huffman algorithm

Today

1. Symbol codes and optimal lossless compression
2. Prefix codes
3. The Huffman algorithm

Motivation

Data compression: find compact representations of data

Data compression standards

- `jpeg`, `gif` for image transmission
- `mp3` for audio content, `mpeg2` for video transmission
- utilities: `gzip`, `bzip2`

All of the above use the Huffman algorithm as a basic building block.

Data representation

- *Chromosome maps*: sequences of hundreds of millions of bases (symbols from $\{A, C, G, T\}$).
- **Goal:** store a chromosome map with 200 million bases.

How do we represent a chromosome map?

Data representation

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How do we represent a chromosome map?

- Encode every symbol that appears in the sequence separately by a fixed length binary string.
- Codeword $c(x)$ for symbol x: a binary string encoding x

Example code

- Alphabet $\mathcal{A} = \{A, C, G, T\}$ with 4 symbols
- Encode each symbol with 2 bits

alphabet symbol x	codeword $c(x)$
A	00
C	01
G	10
T	11

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- Input sequence: $ACGTAA$

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Output of encoding: the concatenation of the codewords for every symbol in the input sequence

Example

- Input sequence: $ACGTAA$
- Output: $c(A)c(C)c(G)c(T)c(A)c(A) = 000110110000$
- Total length of encoding $= 6 \cdot 2 = 12$ bits.

Symbol codes

Symbol code: a set of codewords where every input symbol is encoded separately.

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Examples of symbol codes

- $C_0 = \{00, 01, 10, 11\}$ is a symbol code for $\{A, C, G, T\}$.
- ASCII encoding system: every character and special symbol on the computer keyboard is encoded by a different 7-bit binary string.

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Remark 1.

C_0 and ASCII are fixed-length codes: each codeword has the same length.

Unique decodability

Decoding C_0?

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Decoding C_0

- read 2 bits of the output
- print the symbol corresponding to this codeword
- continue with the next 2 bits.

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Remark 2.

- *This decoding algorithm works for ASCII (replace 2 by 7)*
- C_0, *ASCII: distinct input sequences have distinct encodings*

Unique decodability

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Remark 2.

- *This decoding algorithm works for ASCII (replace 2 by 7)*
- *C_0, ASCII: distinct input sequences have distinct encodings*

Definition 1.

A symbol code is uniquely decodable if, for any two distinct input sequences, their encodings are distinct.

Lossless compression

- Lossless compression: compress and decompress without errors.
- Uniquely decodable codes allow for lossless compression.
- A symbol code achieves optimal lossless compression when it produces an encoding of minimum length among all uniquely decodable symbol codes.
- Huffman algorithm: provides a symbol code that achieves optimal lossless compression.

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Fixed-length vs variable-length codes

Chromosome map consists of 200 million bases as follows:

alphabet symbol x	frequency $freq(x)$
A	110 million
C	5 million
G	25 million
T	60 million

Fixed-length vs variable-length codes

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alphabet symbol x	frequency $freq(x)$
A	110 million
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- A appears much more often than the other symbols

$\Rightarrow$ It might be best to encode A with fewer bits

- Unlikely that the fixed-length encoding C_0 is optimal

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Prefix codes

Variable-length encodings

Code C_1

alphabet symbol x	codeword $c(x)$
A	0
C	00
G	10
T	1

Prefix codes

Variable-length encodings

Code C_1

alphabet symbol x	codeword $c(x)$
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- C_1 is not unique decodable! E.g., 101110: *how to decode it?*

Prefix codes

Variable-length encodings

Code C_2

alphabet symbol x	codeword $c(x)$
A	0
C	111
G	110
T	10

Prefix codes

Variable-length encodings

Code C_2

alphabet symbol x	codeword $c(x)$
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- C_2 is uniquely decodable
- In particular, no codeword is a prefix of another

Prefix codes

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Definition 2 (prefix codes).

A symbol code is a prefix code if no codeword is a prefix of another.

Decoding prefix codes

Decoding prefix codes?

Decoding prefix codes

Decoding prefix codes

1. scan the binary string from left to right until you've seen enough bits to match a codeword;
2. output the symbol corresponding to this codeword.
    - since no other codeword is a prefix of this codeword or contains it as a prefix, this sequence of bits cannot be used to encode any other symbol.
3. continue starting from the next bit of the bit string

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Thus prefix codes allow for

- unique decoding
- fast decoding (the end of a codeword is instantly recognizable)

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Thus prefix codes allow for

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Examples of prefix codes: C_0, C_2

Prefix codes and optimal lossless compression

- Decoding a prefix code is very fast

⇒ Would like to focus on prefix codes (rather than **all** uniquely decodable symbol codes) for achieving optimal lossless compression

- Information theory guarantees that
- So we can solely focus on prefix codes for optimal compression.

Compression gains from variable-length prefix codes

Chromosome map: *do we gain anything by using C_2 instead of C_0 when compressing the map of 200 million bases?*

Input

symbol x	$freq(x)$
A	110 million
C	5 million
G	25 million
T	60 million

Code C_0

x	$c(x)$
A	00
C	01
G	10
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Code C_2

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- C_0: 2 bits $\times$200 million symbols = 400 million bits
- C_2: $1 \cdot 110 + 3 \cdot 5 + 3 \cdot 25 + 2 \cdot 60 = 320$ million bits
- Improvement of 20% in this example

The general framework

Input:

- Alphabet $\mathcal{A} = \{a_1, \ldots, a_n\}$
- Set $P = \{p_1, \ldots, p_n\}$ of probabilities over $\mathcal{A}$

Output: binary prefix code C^* for $(\mathcal{A}, P)$ with

- codewords $c(a_i)$ with length ℓ_i, and
- expected length $L(C^*) = \sum_{a_i \in \mathcal{A}} p_i \cdot \ell_i$

such that $L(C^*)$ is minimum among all prefix codes.

Example

Chromosome example

Input

symbol x	$\Pr(x)$
A	$110/200$
C	$5/200$
G	$25/200$
T	$60/200$

Code C_0

x	$c(x)$
A	00
C	01
G	10
T	11

Code C_2

x	$c(x)$
A	0
C	111
G	110
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- $L(C_0) = 2$
- $L(C_2) = 1.6$
- *Coming up:* C_2 is the output of the Huffman algorithm, hence an optimal encoding for $(\mathcal{A}, P)$.

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Trees and prefix codes

- A binary tree T is a rooted tree such that each node that is not a leaf has at most two children.
- Binary tree for a prefix code: a branch to the left represents a 0 in the encoding and a branch to the right a 1.

Code C_0

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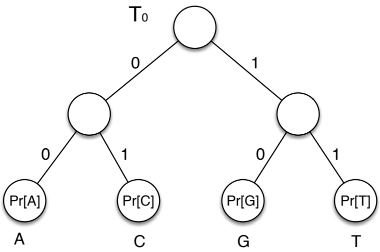

Binary tree for prefix code C_0

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- A binary tree T is a rooted tree such that each node that is not a leaf has at most two children.
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Code C_2

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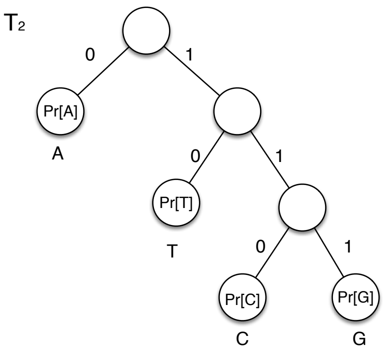

Binary tree for prefix code C_2

Properties of binary trees representing prefix codes

1. *Where do alphabet symbols appear in the tree?*
2. *What do codewords correspond to in the tree?*
3. *Consider the tree corresponding to the optimal prefix code. Can it have internal nodes with one child?*

Properties of binary trees representing prefix codes

1. Symbols must appear at the leaves of the tree T (*why?*)
   - $\Rightarrow$ T has n leaves.
2. Codewords are given by root-to-leaf paths; so $\ell_i = \mathrm{depth}_T(i)$.
   - $\Rightarrow$ **Expected length** of the prefix code corresponding to T

$$L(T) = \sum_{a_i \in \mathcal{A}} p_i \cdot \ell_i = \sum_{1 \leq i \leq n} p_i \cdot \mathrm{depth}_T(i).$$

3. Optimal tree must be full: all internal nodes must have exactly two children (*why?*)

More on optimal tree

Claim 1.

There is an optimal prefix code, with corresponding tree T^**, in which the two lowest frequency characters are assigned to leaves that are siblings in* T^* *at maximum depth.*

More on optimal tree

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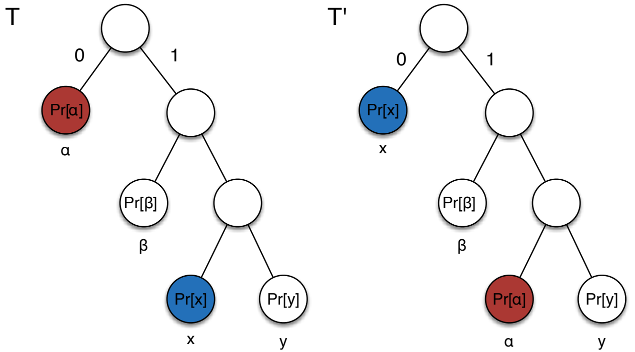
There is an optimal prefix code, with corresponding tree T^*, in which the two lowest frequency characters are assigned to leaves that are siblings in T^* at maximum depth.

Proof.

By an exchange argument: start with a tree for an optimal prefix code and transform it into T^*. □

Proof of Claim 1

- Let T be the tree for the optimal prefix code.
- Let α, β be the two symbols with the smallest probabilities, that is, $\Pr[\alpha] \leq \Pr[\beta] \leq \Pr[s]$ for all $s \in \mathcal{A} - \{\alpha, \beta\}$.
- Let x and y be the two siblings at maximum depth in T.



We want L(T) ≥ L(T′)

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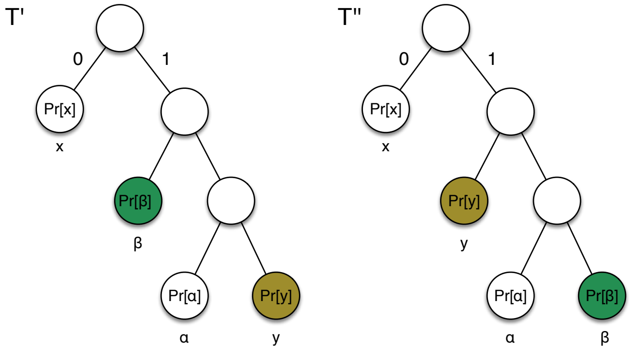

We want $L(T') \ge L(T'')$

How do the expected lengths of the two trees compare?

$$
\begin{aligned}
L(T) - L(T') &= \sum_{a_i \in \mathcal{A}} \Pr[a_i] \cdot \mathrm{depth}_T(i) - \sum_{a_i \in \mathcal{A}} \Pr[a_i] \cdot \mathrm{depth}_{T'}(i) \\
&= \Pr[\alpha] \cdot \mathrm{depth}_T(\alpha) + \Pr[x] \cdot \mathrm{depth}_T(x) \\
&\quad - \Pr[\alpha] \cdot \mathrm{depth}_{T'}(\alpha) - \Pr[x] \cdot \mathrm{depth}_{T'}(x) \\
&= \Pr[\alpha] \cdot \mathrm{depth}_T(\alpha) + \Pr[x] \cdot \mathrm{depth}_T(x) \\
&\quad - \Pr[\alpha] \cdot \mathrm{depth}_T(x) - \Pr[x] \cdot \mathrm{depth}_T(\alpha) \\
&= (\Pr[\alpha] - \Pr[x]) \cdot (\mathrm{depth}_T(\alpha) - \mathrm{depth}_T(x)) \geq 0
\end{aligned}
$$

- The third equality follows from the exchange.
- Similarly, exchanging β and y in T' yields $L(T') - L(T'') \geq 0$.
- Hence $L(T) - L(T'') \geq 0$.
- Since T is optimal, it must be $L(T) = L(T'')$.
- So T'' is also optimal.

The claim follows by setting T^* to be T''.

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Building the optimal tree

Claim 1 tells us how to build the optimal tree greedily!

1. Find the two symbols with the lowest probabilities
2. Remove them from the alphabet and replace them with a new meta-character with probability equal to the sum of their probabilities
   - **Idea:** this meta-character will be the parent of the two deleted symbols in the tree
3. Recursively construct the optimal tree using this process

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Huffman algorithm

```
Tree Huffman(𝒜, P)
  if |𝒜| = 2 then
      Encode one symbol using 0 and the other using 1
  end if
  Let α and β be the two symbols with the lowest probabilities
  Let ν be a new meta-character with probability Pr[α] + Pr[β]
  Let 𝒜₁ = 𝒜 − {α, β} + {ν}
  Let P₁ be the new set of probabilities over 𝒜₁
  T₁ = Huffman(𝒜₁, P₁)
  return T as follows: replace leaf node ν in T₁ by an internal
  node, and add two children labelled α and β below ν.
```

Remark 3.

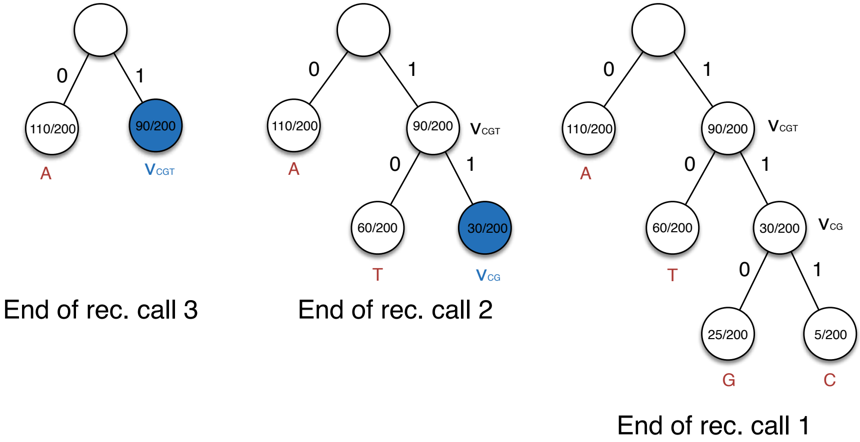
Output of `Huffman` *procedure is a binary tree* T*; the code for* $(\mathcal{A}, P)$ *is its corresponding prefix code.*

Example: recursive Huffman for chromosome map

Recursive call 1: $\texttt{Huffman}(\{A, C, G, T\}, \{\frac{110}{200}, \frac{5}{200}, \frac{25}{200}, \frac{60}{200}\})$

Recursive call 2: $\texttt{Huffman}(\{A, \nu_{CG}, T\}, \{\frac{110}{200}, \frac{30}{200}, \frac{60}{200}\})$

Recursive call 3: $\texttt{Huffman}(\{A, \nu_{CGT}\}, \{\frac{110}{200}, \frac{90}{200}\})$



Correctness

Proof: by induction on the size of the alphabet $n \geq 2$.

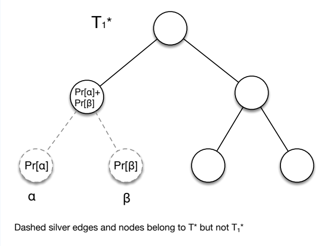
- **Base case.** For $n = 2$, Huffman is optimal.
- **Hypothesis.** Assume that Huffman returns the optimal prefix code for alphabets of n symbols.
- **Induction Step.** Let $\mathcal{A}$ be an alphabet of size $n + 1$, P the corresponding set of probabilities.

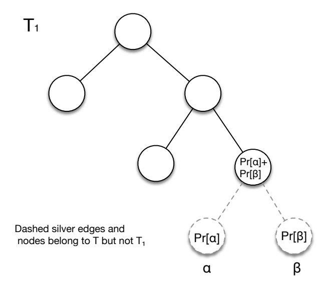
  Let T_1 be the optimal (by the hypothesis) tree returned by our algorithm for $(\mathcal{A}_1, P_1)$, where $\mathcal{A}_1, P_1, T_1$ as in the pseudocode. Let T be the final tree returned for $(\mathcal{A}, P)$ by our algorithm. We claim that T is optimal.

  We will prove the claim by contradiction; to this end, let T^* be the optimal tree with $L(T^*) < L(T)$.

Correctness

Claim 1 guarantees there is such an optimal tree T^* where α, β appear as siblings at maximum depth. Contracting α, β yields the tree T_1^* with expected length $L(T_1^*)$.





We have $L(T_1^*) \geq L(T_1)$ since T_1 is optimal for $(\mathcal{A}_1, P_1)$. But

$$L(T^*) = L(T_1^*) + (\Pr[\alpha] + \Pr[\beta]) \geq L(T_1) + (\Pr[\alpha] + \Pr[\beta]) = L(T)$$

hence contradicting optimality of T^*.

Implementation and running time

1. Straightforward implementation: $O(n^2)$ time
2. Store the alphabet symbols in a priority queue implemented as a binary min-heap with keys their probabilities
   - **Operations:** `Initialize` ($O(n)$), `Extract-min` ($O(\log n)$), `Insert` ($O(\log n)$)

   Total time: $O(n \log n)$ time

For an iterative implementation of Huffman, see your textbook.

Example: iterative Huffman for chromosome map

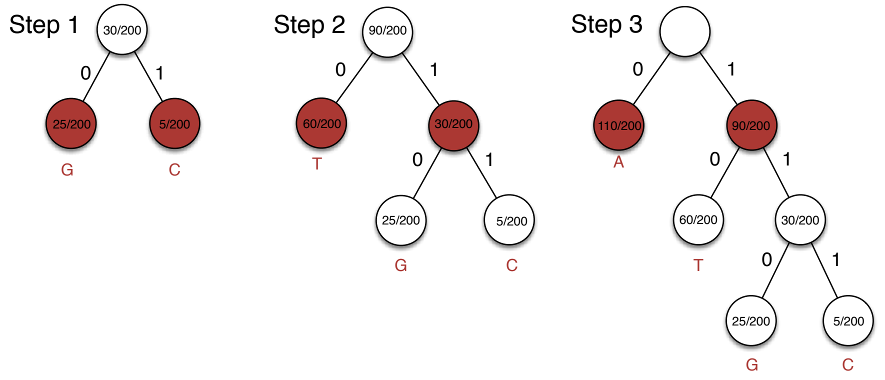
Input $(\mathcal{A}, P)$

symbol x	$\Pr(x)$
A	$110/200$
C	$5/200$
G	$25/200$
T	$60/200$

$\rightarrow$

Output code

symbol x	$c(x)$
A	0
C	111
G	110
T	10



Beyond Huffman coding

- Huffman algorithm provides an optimal symbol code
- Codes that encode larger blocks of input symbols might do better
- Storage on noisy media: *what if a bit of the output of the compressor is flipped?*
  - decompression cannot carry through
  - need error correction on top of compression