Convex Minimization Problems

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Source code

1. Introduction

We study problems that can be transformed into the problem of minimizing a convex function over a convex set, that is, finding $\min_{\eta \in Q} f(\eta)$ where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $Q \subseteq \mathbb{R}^n$ is a convex set. Convex minimization problems are common in many areas of applied mathematics. Here, we focus on solving differential equations, which are turned into minimization problems by considering their variational formulation.

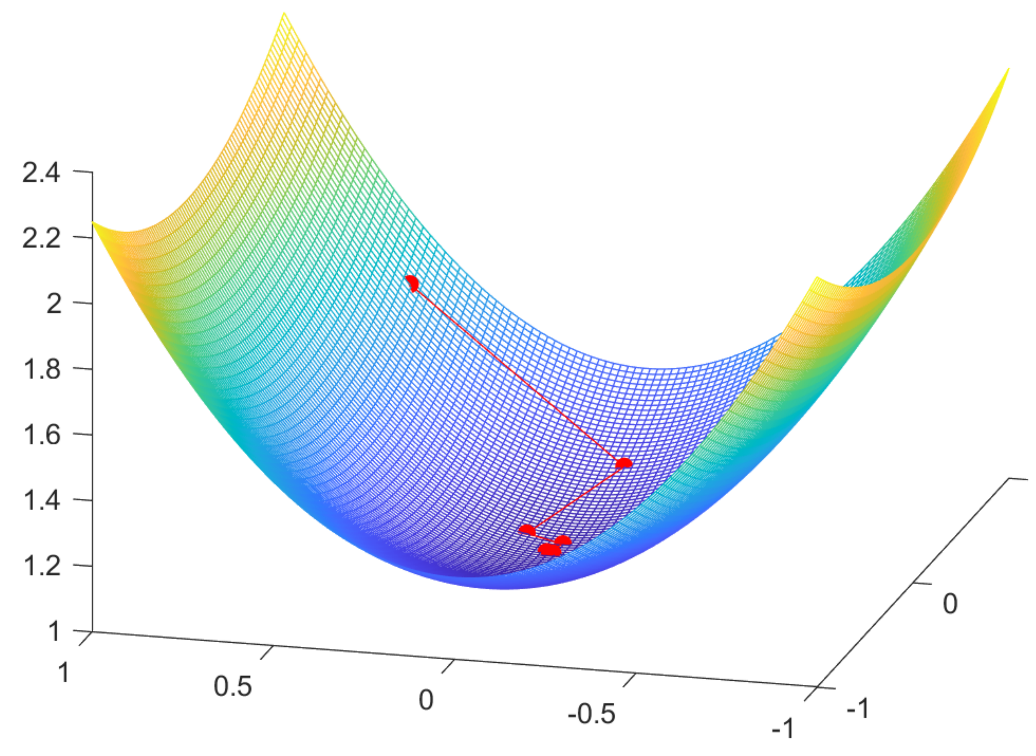

Figure 1: *An iterative algorithm for minimizing* $f(x, y) = (x/2)^2 + y^2 + 1$.

2. An overview of minimization algorithms

We consider four iterative methods of the form $\xi_{k+1} = \xi_k + \alpha_k d_k$ where $\xi_0 \in \mathbb{R}^n$ is an initial approximation, $d_k \in \mathbb{R}^n$ is some search direction, and $\alpha_k \in \mathbb{R}$ is the step length.

- Gradient method with optimal step length, $d_k = -\nabla f(\xi_k)$, $\alpha_k = \alpha_k^{opt}$.
- Newton's Method, $d_k = -H_k^{-1} \nabla f(\xi_k)$, $\alpha_k = 1$.
- Quasi Newton's Method, $d_k = -H_k^{-1} \nabla f(\xi_k)$, $\alpha_k = \alpha_k^{opt}$.
- Generalized conjugate gradient, $\alpha_k = \alpha_k^{opt}$, $d_{k+1} = -\nabla f(\xi_{k+1}) + \beta_k d_k$, $\beta_k = \frac{\nabla f(\xi_{k+1}) \cdot \nabla f(\xi_{k+1})}{\nabla f(\xi_k) \cdot \nabla f(\xi_k)}$, and $d_0 = -\nabla f(\xi_0)$.

α_k^{opt} is the optimal step length in the sense that

$$f(\xi_k + \alpha_k^{opt} d_k) = \min_{\alpha \geq 0} f(\xi_k + \alpha d_k)$$

and $H_k \in \mathbb{R}^{n \times n}$ is the Hessian matrix of f. When $Q = \mathbb{R}^n$ the problem is called *unconstrained* and these methods work directly. When $Q \neq \mathbb{R}$, the problem is *constrained* and it is possible to modify the minimization algorithms to search only in Q. However, we choose to consider instead a method of penalization, so that we instead minimize the function $f(x) + p(x)$ where $p(x)$ is zero whenever $x \in Q$ and large when x is outside of Q.

3. An unconstrained problem

An example of an unconstrained problem appears when we wish to solve the differential equation

$$\begin{aligned} u'' + (1+x)u &= e^x, \quad x \in (0,1), \\ u(0) &= u(1) = 0. \end{aligned}$$

This is transformed into the variational formulation of finding $u \in V$ such that $A(u, v) = L(v)$ for all $v \in V$ where V is some Hilbert space whose definition we omit, $A(u, v) = \int_0^1 u'v' + (1+x)uv \, dx$, and $L(v) = \int_0^1 e^x v \, dx$. This problem and the original equation is now equivalent to minimizing the function $F(v) = A(v, v)/2 - L(v)$ for $v \in V$ by the Lax-Milgram theorem.

The space V is discretized with finite elements resulting in a finite-dimensional space V_h on which we want to minimize F. Thus, we have an unconstrained minimization problem.

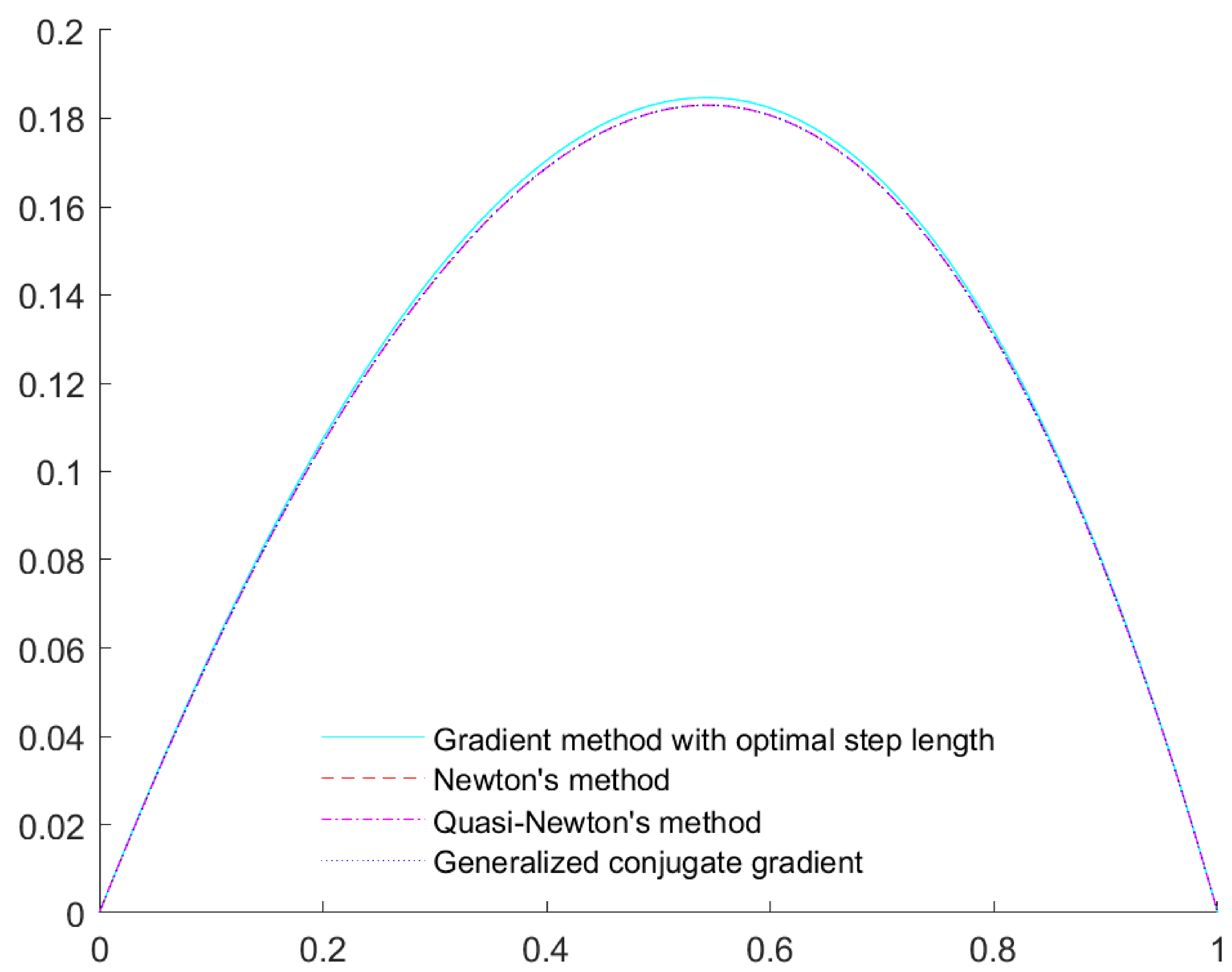

Figure 2: *The solution* u *generated using all four methods.*

4. Membrane deformed by obstacle

Consider a one-dimensional membrane with its endpoints fixed is deflected under the presence of an obstacle. The membrane satisfies the Laplace equation, implying that we want to minimize the function $F(v) = A(v, v)/2 - L(v)$ where $A(u, v) = \int_0^1 u'v' \, dx$ and $L(v) = 0$. However, the obstacle also forces the side condition $u \geq \psi$ on our solution, so our minimization problem is the constrained problem

$$\min_{v \in K} F(v)$$

where $K = \{v \in V : v \geq \psi\}$ and V is similar to the space used in the last section.

To solve this problem, we again approximate the space V by a finite-dimensional space V_h using the finite element method. Finite-dimensional approximation of K is obtained by defining $K_h = V_h \cap K$. The penalization was given by $p(x) = R \cdot \sum_j \max\{0, \psi_j - x_j\}^2$ where R is a constant.

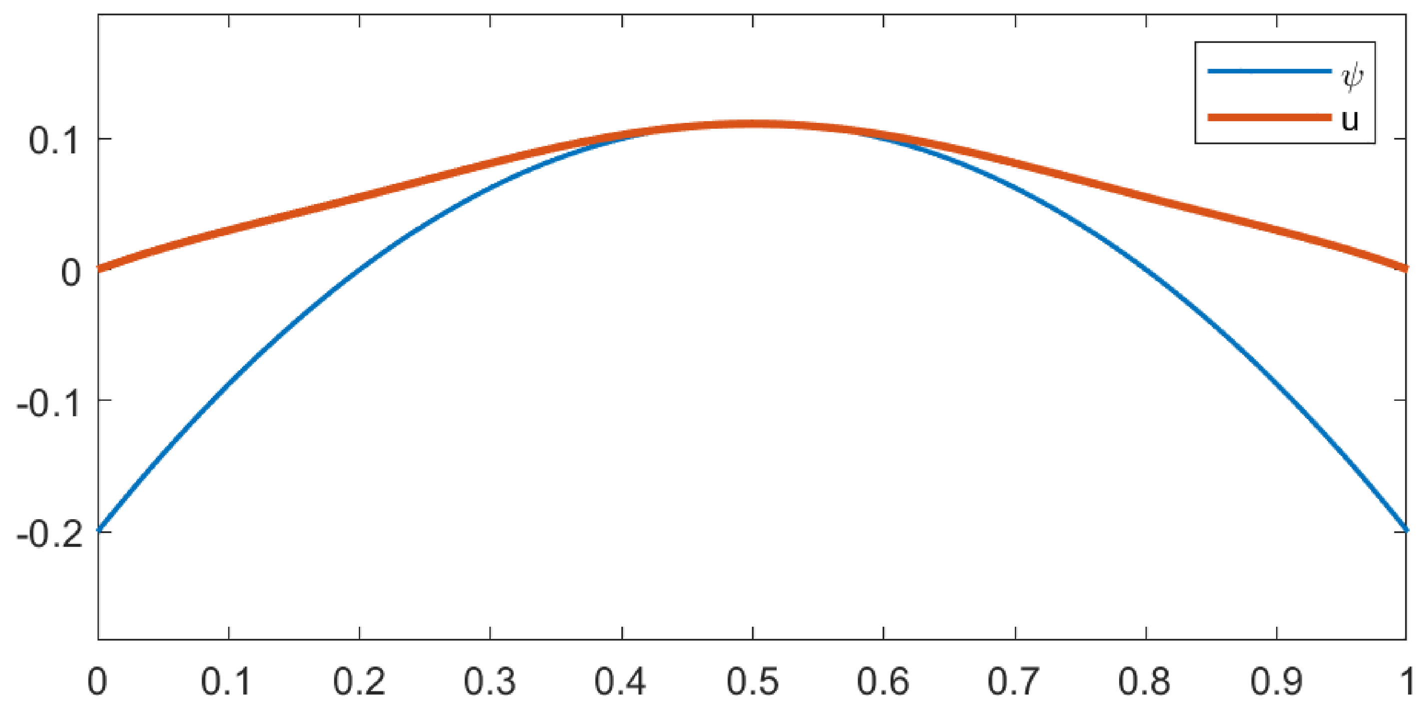

Figure 3: *The membrane* u *deformed by an obstacle, which is given by the function* ψ. *The gradient method was used.*

References

[1] Claes Johnson, *Numerical solution of partial differential equations by the finite element method*. Dover Publications Inc (2009), ISBN: 9780486469003, Chapter 13.

[2] Penalty method - Wikipedia, `https://en.wikipedia.org/wiki/Penalty_method`, [Online; accessed 17-April-2025]