PS 4

Exercise 1

a) $\Pr(y_i = 1 | x_i) = \Lambda(x_i'\beta)$

$$\underset{k\times 1}{x_i} = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ik} \end{pmatrix} ; \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$$

$$\Lambda(z) = \frac{e^z}{1+e^z} ; \quad 1-\Lambda(z) = \frac{1}{1+e^z}$$

$$\ell_i(\beta) = y_i \ln p_i + (1-y_i)\ln(1-p_i)$$

$$= y_i \ln \Lambda(x_i'\beta) + (1-y_i)\ln(1-\Lambda(x_i'\beta))$$

$$= y_i\left(x_i'\beta - \ln(1+\exp(x_i'\beta))\right)$$

$$+ (1-y_i)\left(-\ln(1+\exp(x_i'\beta))\right)$$

$$= y_i x_i'\beta - \ln(1+\exp(x_i'\beta))$$

$$\mathcal{L}(\beta) = \sum_{i=1}^{N} \left\{ y_i x_i'\beta - \ln(1+\exp(x_i'\beta)) \right\}$$

$$S(\beta) = \sum_{i=1}^{N} \left\{ y_i x_i' - \frac{1 \cdot \exp(x_i'\beta)}{1+\exp(x_i'\beta)} x_i' \right\}'$$

$$= \sum_{i=1}^{N} \left\{ y_i x_i - \underbrace{\frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)}}_{= \Lambda(x_i'\beta)} x_i \right\}$$

FOC: $S(\beta) \overset{!}{=} 0$

$$\sum_{i=1}^{N} \left(y_i - \Lambda(x_i'\beta)\right) x_i \overset{!}{=} 0$$

$$\mathrm{Avar}(\hat{\beta}) = \frac{1}{N} A_0^{-1} = -\frac{1}{N} E(H_i(\beta_0))^{-1}$$

$$= -E(H(\beta_0))^{-1}$$

$$H(\beta) = \frac{\partial (S(\beta)')}{\partial \beta} = \frac{\partial}{\partial \beta}\left(\sum \left(y_i - \Lambda(x_i'\beta)\right) x_i\right)$$

$$= \left| \; \frac{\partial \Lambda(z)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\exp(z)}{1+\exp(z)}\right) \right.$$

$$= \frac{\exp(z)(1+\exp(z)) - \exp(z)\exp(z)}{(1+\exp(z))^2}$$

$$= \frac{\exp(z)}{(1+\exp(z))^2} = \underbrace{\frac{1}{1+\exp(z)}}_{= 1-\Lambda(z)} \underbrace{\frac{\exp(z)}{1+\exp(z)}}_{= \Lambda(z)}$$

$$= \left(1-\Lambda(z)\right) \Lambda(z)$$

$$H(\beta) = \frac{\partial}{\partial \beta}\left(\sum \left(y_i - \Lambda(x_i'\beta)\right) x_i \right)$$

$$= - \sum \left(1-\Lambda(x_i'\beta)\right)\Lambda(x_i'\beta)\, x_i x_i'$$

$$x_i' x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \end{pmatrix} \left(x_{i1}\ x_{i2} \dots\right) = \begin{pmatrix} x_{i1}^2 & \dots \\ x_{i1}x_{i2} & \ddots \end{pmatrix}$$

$$Avar(\hat{\beta}) = E\left(\sum (1-\Lambda_i)\Lambda_i\, x_i x_i' \right)^{-1}$$