$$x(t) = \text{Re} \sum_{n=1}^{\infty} C_n e^{in\omega_1 t} = \frac{1}{2} \sum_{n=1}^{\infty} (C_n e^{in\omega_1 t} + C_n^* e^{-in\omega_1 t})$$

By comparison with Eq. (1.2.6), we find

$$\begin{aligned} C_n = 2c_n &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) e^{-in\omega_1 t}\, dt \\ &= a_n - ib_n \end{aligned} \tag{13.2.9}$$

EXAMPLE 13.2.1

Determine the mean square value of a record of random vibration $x(t)$ containing many discrete frequencies.

Solution Because the record is periodic, we can represent it by the real part of the Fourier series:

$$\begin{aligned} x(t) &= \text{Re} \sum_{n=1}^{\infty} C_n e^{in\omega_0 t} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t}) \end{aligned}$$

where C_n is a complex number, and C_n^* is its complex conjugate. [See Eq. (13.2.9).] Its mean square value is

$$\begin{aligned} \overline{x^2} &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{1}{4} \sum_{n=1}^{\infty} (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t})^2\, dt \\ &= \lim_{T\to\infty} \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{C_n^2 e^{i2n\omega_0 t}}{i2n\omega_0 T} + 2C_n C_n^* + \frac{C_n^{*2} e^{-i2n\omega_0 t}}{-i2n\omega_0 T} \right)_0^T \\ &= \sum_{n=1}^{\infty} \frac{1}{2} C_n C_n^* = \sum_{n=1}^{\infty} \frac{1}{2} |C_n|^2 = \sum_{n=1}^{\infty} \overline{C_n^2} \end{aligned}$$

In this equation, $e^{\pm i2n\omega_0 t}$, for any t, is bounded between ± 1, and due to $T \to \infty$ in the denominator, the first and last terms become zero. The middle term, however, is independent of T. Thus, the mean square value of the periodic function is simply the sum of the mean square value of each harmonic component present.

■

13.3 FREQUENCY RESPONSE FUNCTION

In any linear system, there is a direct linear relationship between the input and the output. This relationship, which also holds for random functions, is represented by the block diagram of Fig. 13.3.1.

In the time domain, the system behavior can be determined in terms of the system impulse response $h(t)$ used in the convolution integral of Eq. (4.2.1).

$$y(t) = \int_0^t x(\xi) h(t - \xi)\, d\xi \tag{13.3.1}$$

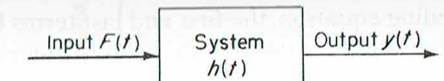

FIGURE 13.3.1. Input–output relationship of a linear system.

A much simpler relationship is available for the frequency domain in terms of the frequency response function $H(\omega)$, which we can define as the ratio of the output to the input under steady-state conditions, with the input equal to a harmonic time function of unit amplitude. The transient solution is thus excluded in this consideration. In random vibrations, the initial conditions and the phase have little meaning and are therefore ignored. We are mainly concerned with the average energy, which we can associate with the mean square value.

Applying this definition to a single-DOF system,

$$m\ddot{y} + c\dot{y} + ky = x(t) \tag{13.3.2}$$

let the input be $x(t) = e^{i\omega t}$. The steady-state output will then be $y = H(\omega)e^{i\omega t}$, where $H(\omega)$ is a complex function. Substituting these into the differential equation and canceling $e^{i\omega t}$ from each side, we obtain

$$(-m\omega^2 + ic\omega + k)H(\omega) = 1$$

The frequency response function is then

$$\begin{aligned} H(\omega) &= \frac{1}{k - m\omega^2 + ic\omega} \\ &= \frac{1}{k} \frac{1}{1 - (\omega/\omega_n)^2 + i2\zeta(\omega/\omega_n)} \end{aligned} \tag{13.3.3}$$

As mentioned in Chapter 3, we will absorb the factor $1/k$ in with the force. $H(\omega)$ is then a nondimensional function of ω/ω_n and the damping factor ζ.

The input–output relationship in terms of the frequency-response function can be written as

$$y(t) = H(\omega) F_0 e^{i\omega t} \tag{13.3.4}$$

where $F_0 e^{i\omega t}$ is a harmonic function.

For the mean square response, we follow the procedure of Example 13.2.1 and write

$$y = \tfrac{1}{2} F_0 (He^{i\omega t} + H^* e^{-i\omega t}) \tag{13.3.5}$$

Thus, by squaring and substituting into Eq. (13.2.4), we find the mean square value of y is

$$\begin{aligned} \overline{y^2} &= \frac{F_0^2}{4} \lim_{T\to\infty} \frac{1}{T} \int_0^T (H^2 e^{i2\omega t} + 2HH^* + H^* e^{-i2\omega t})\, dt \\ &= \frac{F_0^2}{2} H(\omega) H^*(\omega) = \overline{F_0^2} |H(\omega)|^2 \end{aligned} \tag{13.3.6}$$

In the preceding equation, the first and last terms become zero because of $T \to \infty$ in the denominator, whereas the middle term is independent of T. Equation (13.3.6) indicates that the mean square value of the response is equal to the mean square excitation multiplied by the square of the absolute values of the frequency response function. For excitations expressed in terms of Fourier series with many frequencies, the response is the sum of terms similar to Eq. (13.3.6).

EXAMPLE 13.3.1

A single-DOF system with natural frequency $\omega_n = \sqrt{k/m}$ and damping $\zeta = 0.20$ is excited by the force

$$F(t) = F\cos\tfrac{1}{2}\omega_n t + F\cos\omega_n t + F\cos\tfrac{3}{2}\omega_n t$$

$$= \sum_{m=1/2,1,3/2} F\cos m\omega_n t$$

Determine the mean square response and compare the output spectrum with that of the input.

Solution The response of the system is simply the sum of the response of the single-DOF system to each of the harmonic components of the exciting force.

$$x(t) = \sum_{m=1/2,1,3/2} |H(m\omega)| F\cos(m\omega_n t - \phi_m)$$

where

$$|H(\tfrac{1}{2}\omega_n)| = \frac{1/k}{\sqrt{9/16 + (0.20)^2}} = \frac{1.29}{k}$$

$$|H(\omega_n)| = \frac{1/k}{\sqrt{4(0.20)^2}} = \frac{2.50}{k}$$

$$|H(\tfrac{3}{2}\omega_n)| = \frac{1/k}{\sqrt{25/16 + 9(0.20)^2}} = \frac{0.72}{k}$$

$$\phi_{1/2} = \tan^{-1}\frac{4\zeta}{3} = 0.083\pi$$

$$\phi_1 = \tan^{-1}\infty = 0.50\pi$$

$$\phi_{3/2} = \tan^{-1}\frac{-12\zeta}{5} = -0.142\pi$$

Substituting these values into $x(t)$, we obtain the equation

$$x(t) = \frac{F}{k}[1.29\cos(0.5\omega_n t - 0.083\pi)$$
$$+ 2.50\cos(\omega_n t - 0.50\pi)]$$
$$+ 0.72\cos(1.5\omega_n t + 0.142\pi)]$$

The mean square response is then

$$\overline{x^2} = \frac{F^2}{2k^2}[(1.29)^2 + (2.50)^2 + (0.72)^2]$$

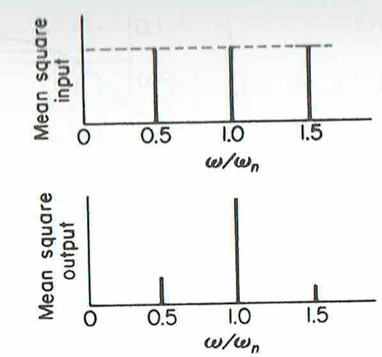

FIGURE 13.3.2. Input and output spectra with discrete frequencies.

Figure 13.3.2 shows the input and output spectra for the problem. The components of the mean square input are the same for each frequency and equal to $F^2/2$. The output spectrum is modified by the system frequency-response function. ■

13.4 PROBABILITY DISTRIBUTION

By referring to the random time function of Fig. 13.4.1, what is the probability of its instantaneous value being less than (more negative than) some specified value x_1? To answer this question, we draw a horizontal line at the specified value x_1 and sum the time intervals Δt_i during which $x(t)$ is less than x_1. This sum divided by the total time then represents the fraction of the total time that $x(t)$ is less than x_1, which is the probability that $x(t)$ will be found less than x_1.

$$P(x_1) = \text{Prob}[x(t) < x_1]$$
$$= \lim_{t\to\infty}\frac{1}{t}\sum \Delta t_i \tag{13.4.1}$$

If a large negative number is chosen for x_1, none of the curve will extend negatively beyond x_1, and, hence, $P(x_1 \to -\infty) = 0$. As the horizontal line corresponding to x_1 is moved up, more of $x(t)$ will extend negatively beyond x_1, and the fraction of the total

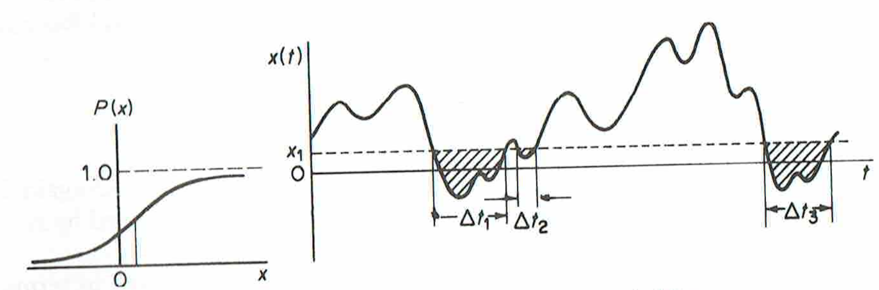

FIGURE 13.4.1. Calculation of cumulative probability.