Baker Potts

NAME 6098

Homework #1 Spring 2013

1. The transfer functions of four LTI-SISO systems are given below.

(a) $G_1(s) = \dfrac{1}{s^2(s+1)}$ (b) $G_2(s) = \dfrac{(s-1)}{s^2(s+1)}$

(c) $G_3(s) = \dfrac{(s+0.1)}{s(s^2+0.2s+1.01)}$ (d) $G_4(s) = \dfrac{(s-0.1)}{s(s^2+0.2s+1.01)}$

Plot the root loci of these transfer functions. Confirm your plot by using appropriate computational aids.

2. Consider the LTI system with description in state space as follows.

$$\left.\begin{array}{l}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ \mathbf{y} = \mathbf{C}\mathbf{x}\end{array}\right\}, \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

(i) Determine the transfer function matrix of the system; the system poles and decide stability; the system zeros.

See problems A-11-1, A-11-2, and A-11-3

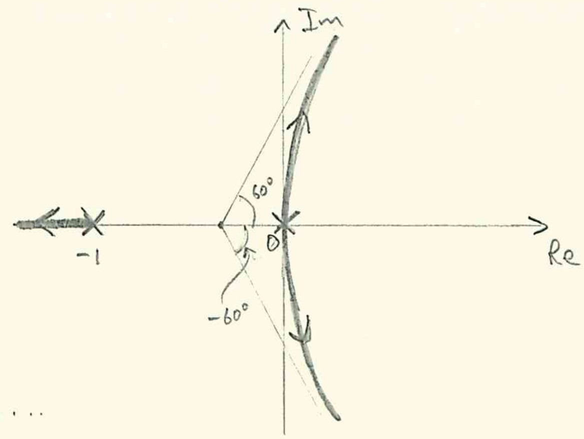
(ii) Determine the matrices $\mathbf{A}_0$, $\mathbf{B}_0$ and $\mathbf{C}_0$ of the system's state-space realization in *companion canonical form.*

(iii) Calculate the gain matrix $\mathbf{F}$ of the *full-state feedback control law* of the system in companion canonical form state-space realization so that the *closed-loop* system poles are: 0, –1/3, –1, –2. see Fig 12.

(iv) Determine the zero-input response of the closed-loop system, by using the relationship $Y(s) = \mathbf{C}_0(s\mathbf{I} - \mathbf{A}_0 - \mathbf{B}_0\mathbf{F})^{-1}\mathbf{x}_0$ and the inverse Laplace transform, for initial conditions $\mathbf{x}_0 = [0 \quad 0 \quad 0 \quad 1]^T$. Is the closed-loop system observable? Briefly justify your answer.

see section 11-7

(1) (a) $G_1(s) = \dfrac{1}{s^2(s+1)}$



poles: $p_1 = 0$, $p_2 = 0$, $p_3 = -1$

zeros: none

- root loci must meet angle and magnitude conditions

$$\angle G_1(s) = \pm 180^\circ(2k+1) \quad k = 0, 1, 2, \ldots$$

$$|G_1(s)| = 1$$

- determine root loci on real axis

  - test pt on positive real axis

$$\angle s = \theta_1 = \theta_2 = 0^\circ$$

$$\angle s+1 = \theta_3 = 0^\circ$$

$$\angle G_1(s) = -\theta_1 - \theta_2 - \theta_3 = 0^\circ \neq \pm 180^\circ(2k+1)$$

  - test pt b/w 0 and -1

$$\angle s = \theta_1 = \theta_2 = 180^\circ \quad \angle s+1 = \theta_3 =$$

$$-180^\circ - 180^\circ - 0^\circ \neq \pm 180^\circ(2k+1)$$

  - test pt b/w -1 and $-\infty$

$$\angle s = \theta_1 = \theta_2 = 180^\circ \quad \angle s+1 = \theta_3 = 180^\circ$$

$$3(-180^\circ) = \pm 180(2k+1) \checkmark$$ $\Rightarrow$ see shaded line < -1 above

- determine asymptotes of root loci

angles of asymptotes $= \dfrac{\pm 180^\circ(2k+1)}{n-m}$ $\quad$ $n \equiv$ # of finite poles $= 3$, $\quad m \equiv$ # of finite zeros $= 0$

$$= \pm 60^\circ(2k+1) \Rightarrow 60^\circ, 180^\circ, -60^\circ$$

  - find pt of intersection w/ real axis

using eq. 6-13 from Ogata's "Modern Control Engineering":

$$s = -\frac{(p_1 + p_2 + p_3) - \cancel{(z_1 + \ldots + z_m)}^{=0}}{n-m}$$

$$s = -\frac{(0 + 0 - 1)}{3 - 0} = \frac{1}{3} \Rightarrow s = -\frac{1}{3}$$

- find the breakaway and/or break-in pts.

  - breakaway pts correspond to a point in the s-plane where multiple roots of the characteristic equation occur

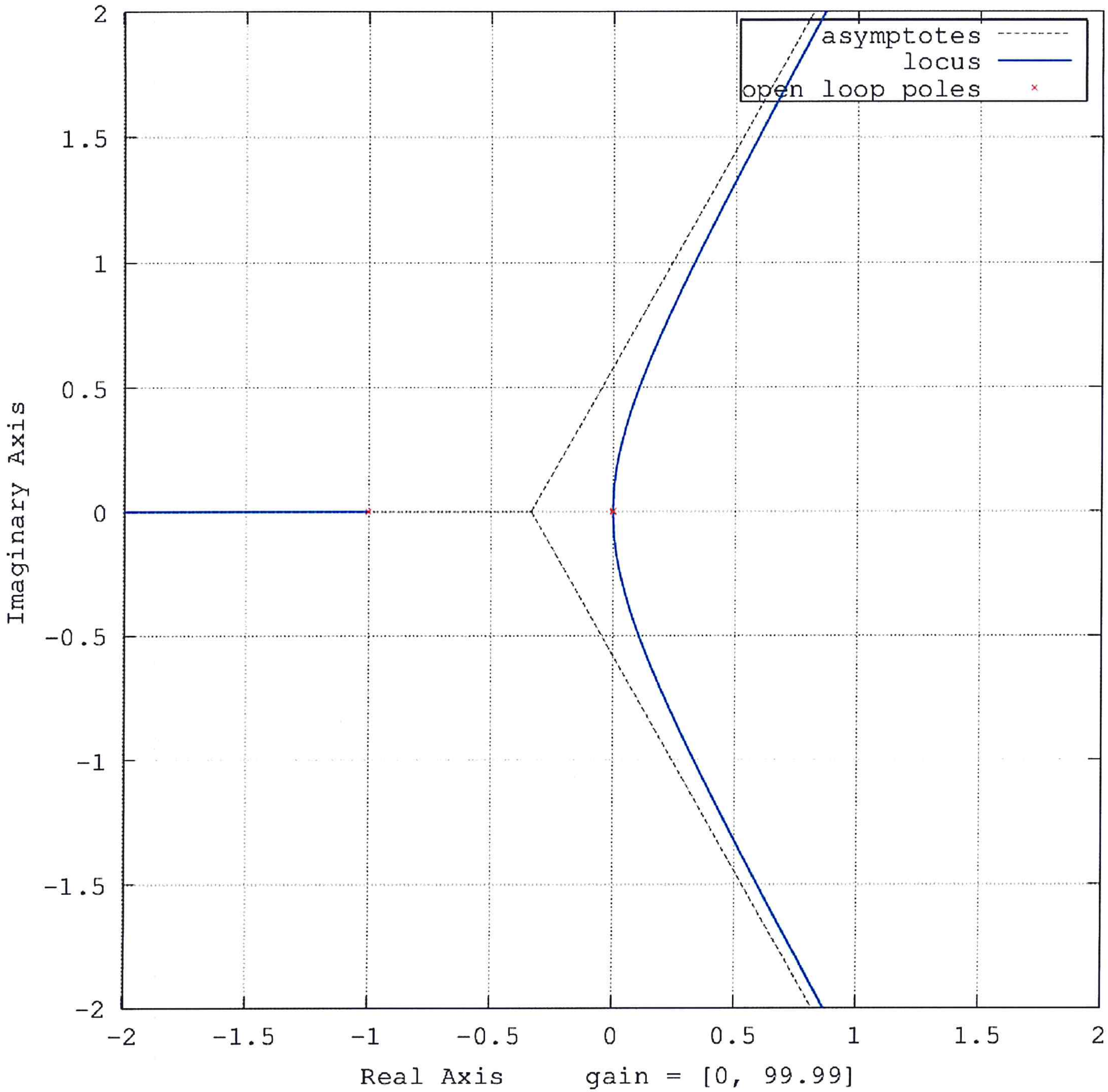
    ↳ thus, for this problem, it occurs at $s=0$

- determine angle of departure from poles

  since the poles are on the real axis, the root loci for p_1 and p_2 will migrate toward the asymptotes

∴ The above analysis provides sufficient information to plot the root loci for this transfer function. Compare the hand-drawn plot on the previous page to that of the one generated by Octave on the following page using rlocus(). Both compare favorably.

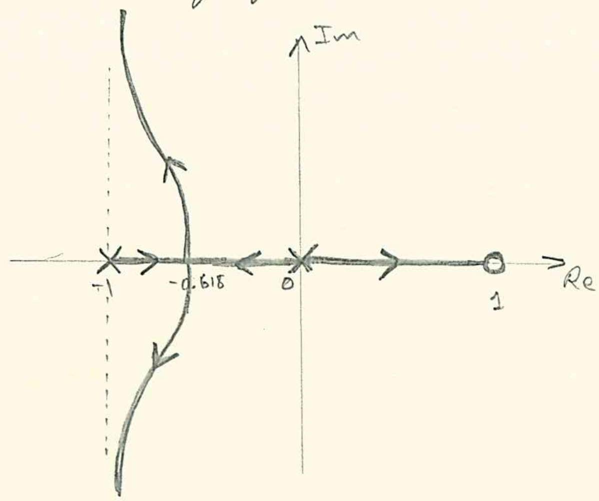
Root-Locus Plot of G1(s) = 1/[s^2(s+1)]



(b) $G_2(s) = \dfrac{s-1}{s^2(s+1)}$

poles: $p_1 = 0$, $p_2 = 0$, $p_3 = -1$

zeros: $z_1 = 1$



• root loci angle and magnitude cond's

$$\angle G_2(s) = \pm 180°(2k+1) \quad k = 0, 1, 2, \ldots$$

$$|G_2(s)| = 1$$

• determine root loci on real axis

- test pt b/w 1 and ∞

$\angle s = \theta_1 = \theta_2 = 0°$ $\quad \angle s-1 = \phi_1 = 0°$

$\angle s+1 = \theta_3 = 0°$

$$\angle G_2(s) = \phi_1 - \theta_1 - \theta_2 - \theta_3 = 0° \neq \pm 180°(2k+1)$$

- test pt b/w -1 and 0

$\angle s = \theta_1 = \theta_2 = 180°$ $\quad \angle s-1 = \phi_1 = 180°$

$\angle s+1 = \theta_3 = 0°$

$$180° - 180° - 180° - 0° = \pm 180°(2k+1) \checkmark$$

↳ see shaded line b/w -1 and 1

- test pt b/w 0 and 1

$\angle s = \theta_1 = \theta_2 = 0°$ $\quad \angle s-1 = \phi_1 = 180$

$\angle s+1 = \theta_3 = 0°$

$$-3(0°) + 180° = \pm 180°(2k+1) \checkmark$$

↳ see shaded line b/w 0 and 1 above

- test pt b/w -∞ & -1

$\angle s = \theta_1 = \theta_2 = 180°$

$\angle s+1 = \theta_3 = 180°$ $\quad \angle s-1 = \phi_1 =$

$$180° - 3(180°) \neq \pm 180°(2k+1)$$

• determine asymptotes of root loci

$$\text{angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} \qquad n = 3, \quad m = 1$$

$$= \pm 90°(2k+1) \Rightarrow 90°, -90° \rightarrow \text{see above}$$

- find pt of intersection w/ real axis using eq. 6-13

$$s = -\frac{(p_1 + p_2 + p_3) - z_1}{n-m}$$

$$= -\frac{(0 + 0 - 1) - 1}{3 - 1} = \frac{2}{2} \Rightarrow s = -1 \rightarrow \text{see above}$$

- find the breakaway and/or breakin pts

  - there will be a breakaway pt at the pole of multiplicity 2, i.e. p_1 and p_2. There will also be a breakaway pt somewhere between 0 and -1 on the real axis.

breakaway pts occur at $\frac{dK}{ds} = 0$

$$\frac{K(s-1)}{s^2(s+1)} + 1 = 0 \Rightarrow K = -\frac{s^2(s+1)}{s-1} = -\frac{s^3+s^2}{s-1}$$

$$K = -\left(s^2 + 2s + 2 + \frac{2}{s-1}\right)$$

$$\frac{dK}{ds} = -\left(2s + 2 - \frac{2}{(s-1)^2}\right) = 0$$

$$2s(s-1)^2 + 2(s-1)^2 - 2 = 0$$

$$(2s+2)(s-1)^2 - 2 = 0$$

$$2s^3 - 2s^2 - 2s = 0$$

$$s^3 - s^2 - s = 0 \Rightarrow \quad s_1 = 0 \quad s_{2,3} = \frac{1 \pm \sqrt{5}}{2}$$

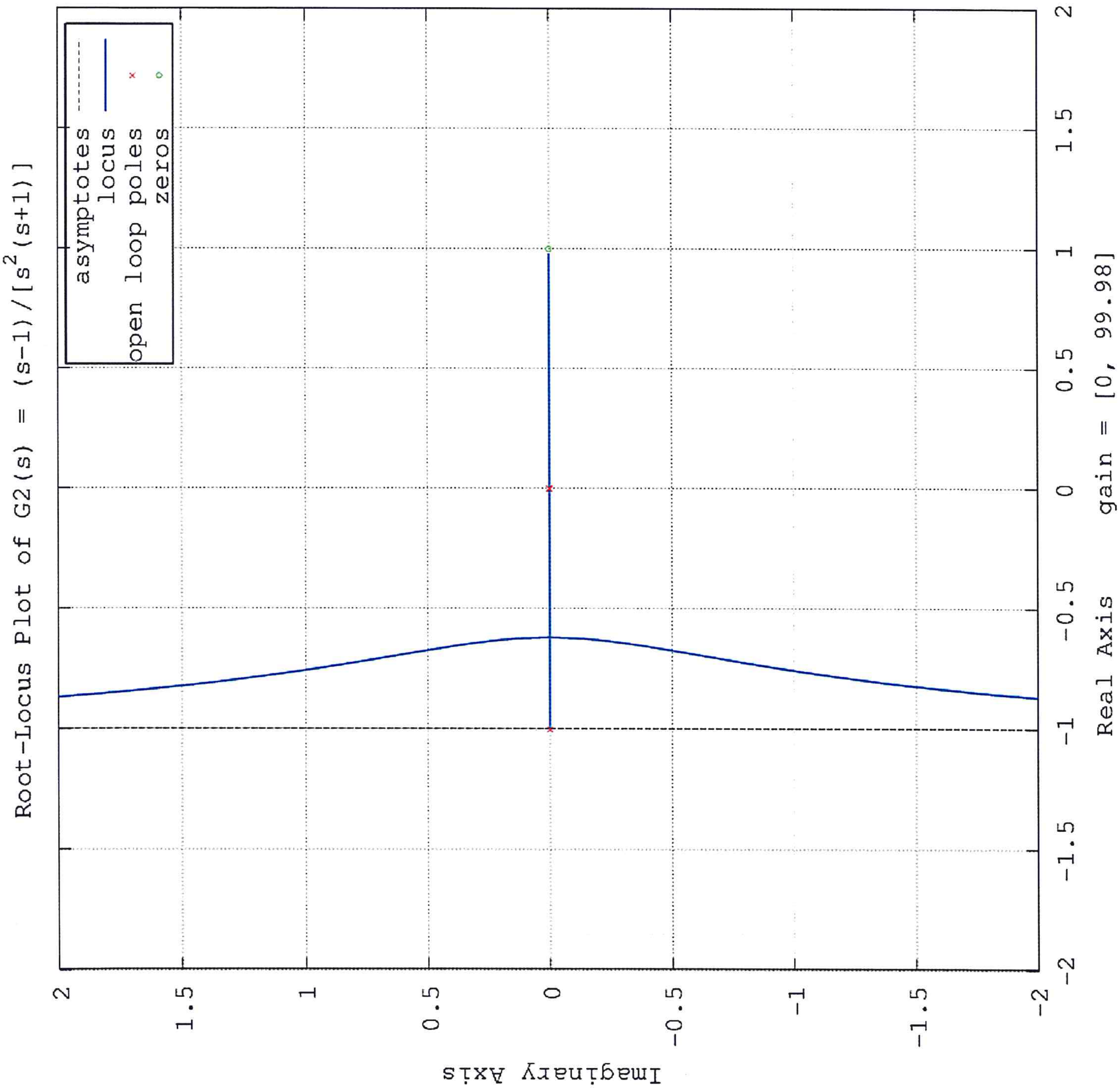
$$s_2 = 1.618 \qquad s_3 = -0.618$$

↑ not possible

- determine angles of departure/arrival

Since the poles/zeros are on the real axis, the root loci will depart/arrive on this axis. The root loci will break away from this axis at $-0.618 = s$.

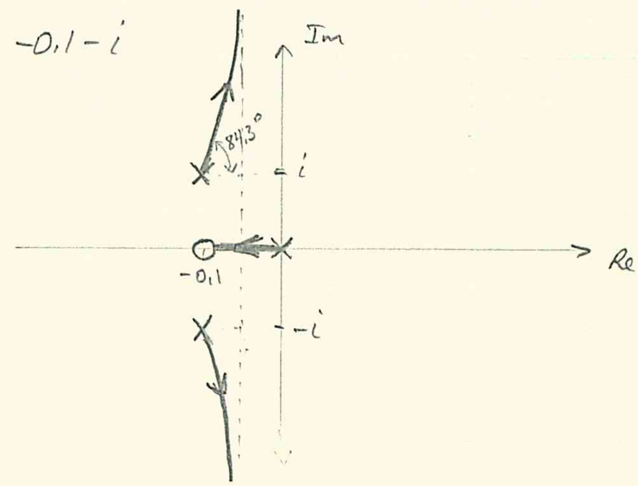
∴ The above analysis provides sufficient information to plot the root loci for this transfer function. Compare the hand-drawn plot on the previous page to that of the one generated by Octave on the next page. Both compare favorably.


Root-Locus Plot of G2(s) = (s-1)/[s^2(s+1)]
asymptotes
locus
open loop poles
zeros
Real Axis
gain = [0, 99.98]
Imaginary Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2

(c) $G_3(s) = \dfrac{(s+0.1)}{s(s^2+0.2s+1.01)} = \dfrac{(s+0.1)}{s(s+0.1-i)(s+0.1+i)}$

poles: $p_1 = 0$, $p_2 = -0.1+i$, $p_3 = -0.1-i$

zeros: $z_1 = -0.1$



- determine root loci on real axis
  - test pt on positive real axis

$\angle s = \theta_1 = 0°$

$\angle s-p_2 + \angle s-p_3 = \theta_2 + \theta_3 = 360°$

$\angle s+0.1 = 0° = \phi_1$

$\phi_1 - \theta_1 - (\theta_2+\theta_3) = -360° \neq \pm 180°(2k+1)$

- test pt b/w $-\infty$ and -0.1

$\angle s = \theta_1 = 180°$

$\theta_2 + \theta_3 = 360°$ $\quad \angle s+0.1 = 180°$

$180° - 180° - 360° \neq \pm 180°(2k+1)$

- test pt b/w -0.1 and 0

$\angle s = \theta_1 = 180°$ $\quad \angle s+0.1 = 0° = \phi_1$

$\theta_2 + \theta_3 = 360°$

$\phi_1 - \theta_1 - (\theta_2+\theta_3) = -540° = \pm 180°(2k$ ✓

↳ see shaded line b/w -0.1 and

- determine asymptotes of root loci

angles of asymptotes $= \dfrac{\pm 180°(2k+1)}{n-m}$ $\quad n = 3$, $n = 1$

$= \pm 90°(2k+1) \Rightarrow 90°, -90°$

- find pt of intersection w/ real axis using eq. 6-13 from Ogata

$$s = -\frac{(p_1+p_2+p_3) - z_1}{n-m}$$

$$= -\frac{0 + (-0.1+i) + (-0.1-i) + 0.1}{3-1} = -\frac{0.1}{2} = -0.05$$ → see dashed asymptotes lines above

- find the breakaway and/or breakin pts

  - since there are no points in the s-plane where multiple roots can occur, there are no breakaway or breakin pts.

- determine angle of departure from poles

$$\theta_{2,dep} = 180 - (\theta_1 + \theta_3) + \phi_1$$

$$\tan \psi = \frac{0.1}{1.0}$$

$$= 180 - (95.7^\circ + 90^\circ) + 90^\circ$$

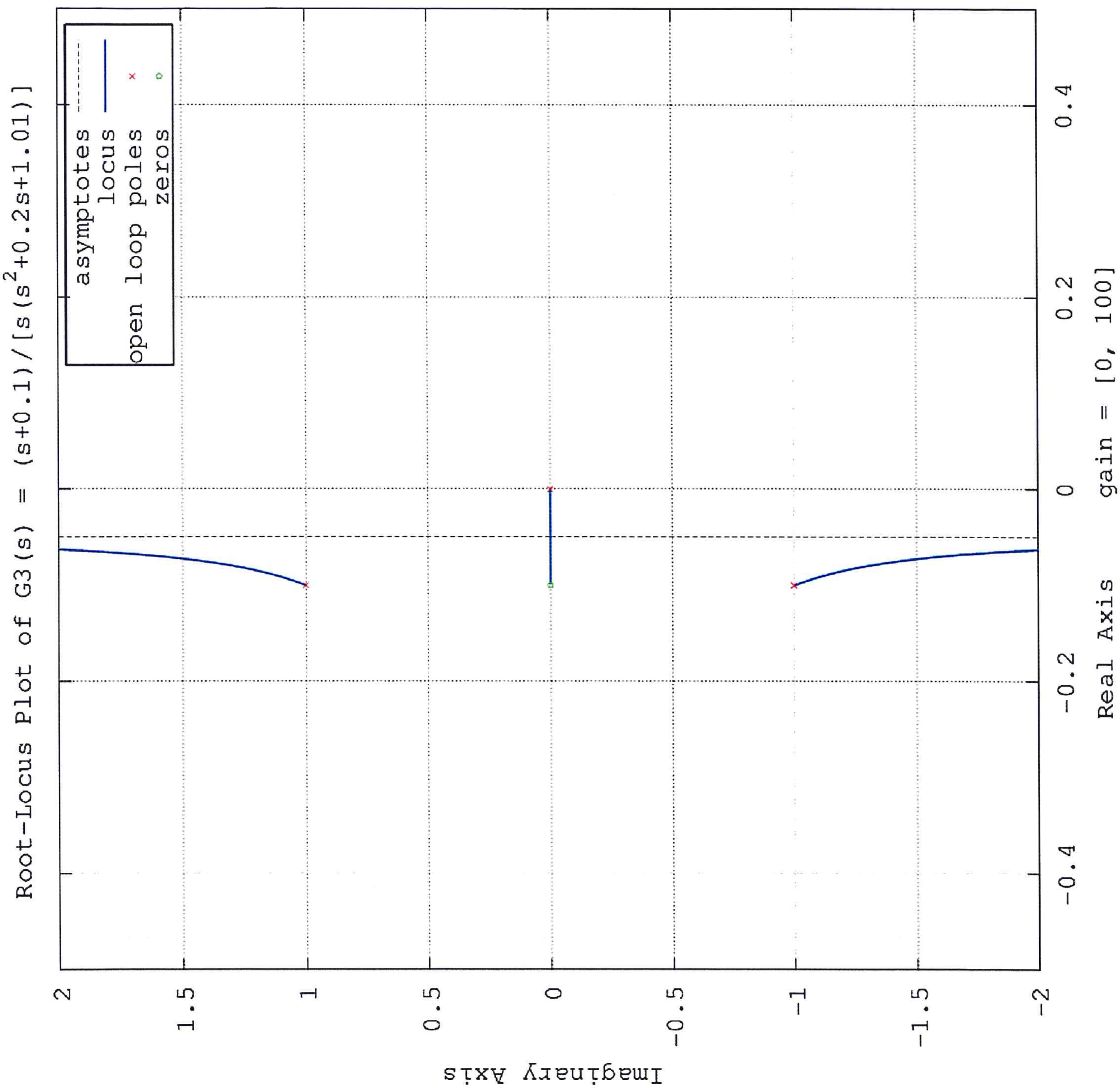
$$\psi = 5.7^\circ$$

$$= 84.3^\circ$$

with symmetry about real axis

$$\theta_{3,dep} = -84.3^\circ$$

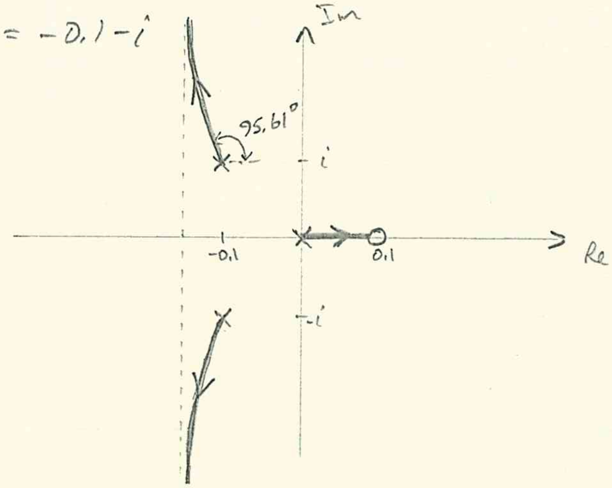
∴ The above analysis provides sufficient information to plot the root loci for this transfer function. Compare this hand-drawn plot to that generated by Octave on the next page. Both compare favorably.


Root-Locus Plot of G3(s) = (s+0.1)/[s(s^2+0.2s+1.01)]
asymptotes
locus
open loop poles
zeros
Real Axis gain = [0, 100]
Imaginary Axis
-0.4
-0.2
0
0.2
0.4
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2

(d) $G_3(s) = \frac{(s-0.1)}{s(s^2+0.2s+1.01)} = \frac{(s-0.1)}{s(s+0.1-i)(s+0.1+i)}$

poles: $p_1 = 0$, $p_2 = -0.1+i$, $p_3 = -0.1-i$

zeros: $z_1 = 0.1$



- determine root loci on real axis
  - test pt b/w 0.1 and ∞

$\angle s = \theta_1 = 0°$ $\quad \angle s-0.1 = 0°$

$\theta_2 + \theta_3 = 360°$

$0° - 0° - 360° \neq \pm 180°(2k+1)$

- test pt b/w −∞ and 0

$\angle s = \theta_1 = 180°$ $\quad \angle s-0.1 = 180°$

$\theta_2 + \theta_3 = 360°$

$180° - 180° - 360° \neq \pm 180°(2k+1)$

- test pt b/w 0 and 0.1

$\angle s = 0° = \theta_1$ $\quad \angle s-0.1 = 180° = \phi_1$

$\theta_2 + \theta_3 = 360°$

$180° - 0° - 360° = \pm 180(2k+1)$ ✓

↳ see shaded line b/w 0 and 0.1

- determine asymptotes of root loci

angles of asymptotes $= \frac{\pm 180°(2k+1)}{n-m}$ $\quad n = 3$, $m = 1$

$= \pm 90°(2k+1) \Rightarrow 90°, -90°$

- find pt of intersection w/ real axis using eq. 6-13 from Ogata

$$s = -\frac{(p_1+p_2+p_3) - z_1}{n-m}$$

$$= -\frac{0-0.1+i-0.1-i-0.1}{3-1} = 0.15 \Rightarrow s = -0.15$$

↳ see dashed line for asymptotes above

- find the breakaway and/or breakin pts
  - since there are no points in the s-plane where multiple roots can occur, there are no breakaway or breakin pts.
- determine angle of departure from complex poles

$$\theta_{2,dep} = 180^\circ - (\theta_1 + \theta_3) + \phi_1$$

$$\tan\psi = \frac{0.2}{1}$$

$$= 180^\circ - (95.7^\circ + 90^\circ) + 101.31^\circ$$

$$\psi = 11.31^\circ$$

$$= 95.61^\circ$$

w/ symmetry about real axis

$$\theta_{3,dep} = -95.61^\circ$$

∴ The above analysis provides sufficient information to plot the root loci for this transfer function. Compare this hand-drawn plot to that generated by Octave on the next page. Both compare favorably.

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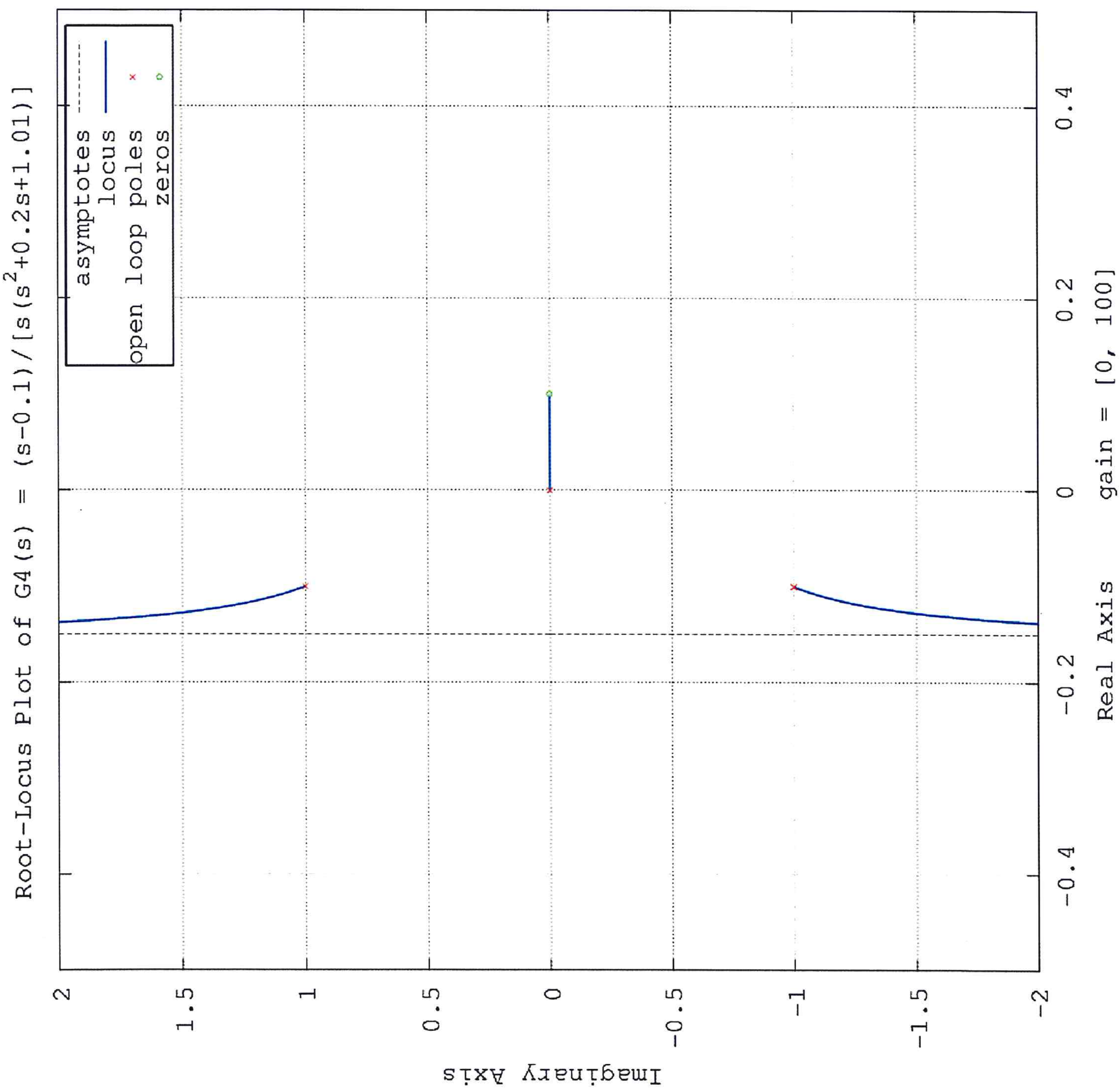

Root-Locus Plot of G4(s) = (s-0.1)/[s(s^2+0.2s+1.01)]
asymptotes
locus
open loop poles
zeros
Real Axis
gain = [0, 100]
Imaginary Axis
-0.4
-0.2
0
0.2
0.4
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2

(2) (i)

$$\underline{\underline{G}}(s) = \frac{\underline{\underline{Y}}(s)}{\underline{\underline{U}}(s)} = \underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}} + \cancel{\underline{\underline{D}}}^{=0}$$

$$\underline{\underline{G}}(s) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} s-1 & 0 & 0 & 0 \\ 0 & s-2 & 0 & 0 \\ 0 & 0 & s+1 & 0 \\ 0 & 0 & 0 & s+2 \end{bmatrix}^{-1}}_{\underline{\underline{E}}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{\underline{E}} = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 & 0 & 0 \\ 0 & \frac{1}{s-2} & 0 & 0 \\ 0 & 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\underline{\underline{G}}(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-2} & \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{s-1} + \frac{1}{s-2} + \frac{1}{s+1} + \frac{1}{s+2}$$

$$\boxed{\underline{\underline{G}}(s) = \frac{2s(2s^2-5)}{(s-1)(s-2)(s+1)(s+2)}} \leftarrow \text{transfer function matrix}$$

$\boxed{\text{system poles: } p_1 = 1, \; p_2 = 2, \; p_3 = -1, \; p_4 = -2}$

system zeros: $z_1 = 0$, $z_2 = \sqrt{5/2}$, $z_3 = -\sqrt{5/2}$

The system is unstable since there are poles that lie in the right half-plane of the complex plane.

(ii) $$\underline{\underline{G}}(s) = \frac{2s(2s^2-5)}{(s-1)(s-2)(s+1)(s+2)} = \frac{4s^3-10s}{s^4-5s^2+4}$$

$$\underline{\underline{A}}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix}$$

s^4	s^3	s^2	s^1	s^0
b_0	b_1	b_2	b_3	b_4
0	4	0	-10	0
	a_1	a_2	a_3	a_4
	0	-5	0	4

$$\underline{\underline{B}}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{\underline{C}}_0 = \begin{bmatrix} b_4 - a_4 b_0 & b_3 - a_3 b_0 & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix}$$

$$\underline{\underline{C}}_0 = \begin{bmatrix} 0 & -10 & 0 & 4 \end{bmatrix}$$

(iii) $$\underline{\underline{F}} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}$$

desired closed-loop poles: $0, -1/3, -1, -2$

$$p_c(s) = s(s+1/3)(s+1)(s+2) = s^4 + \tfrac{10}{3}s^3 + 3s^2 + \tfrac{2}{3}s$$

$$\left| s-\underline{\underline{I}} - \underline{\underline{A}}_0 - \underline{\underline{B}}_0\underline{\underline{F}} \right| = \left| \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ 4 & 0 & -5 & s \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ 4-f_1 & -f_2 & -5-f_3 & s-f_4 \end{bmatrix} \right| = s^4 - f_4 s^3 + (-5-f_3)s^2 + f_2 s + 4 - f_1$$

Compare the coefficients here with those of $p_c(s)$ above

$$f_4 = -10/3 \quad f_3 = -8 \quad f_2 = -\tfrac{2}{3} \quad f_1 = 4$$

$$\boxed{\underline{\underline{F}} = \begin{bmatrix} 4 & -2/3 & -8 & -10/3 \end{bmatrix}}$$

(iv) $Y(s) = \underline{C}_0 (s\underline{I} - \underline{A}_0 - \underline{B}_0 \underline{E})^{-1} \underline{X}_0$

$$\underline{X}_0 = [0 \quad 0 \quad 0 \quad 1]^T$$

$$Y(s) = [0 \ -10 \ 0 \ 4]\left(\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ 4 & 0 & -5 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -2/3 & -8 & -10/3 \end{bmatrix}\right)^{-1} \underline{X}_0$$

$$Y(s) = [0 \ -10 \ 0 \ 4]\underbrace{\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ 0 & 2/3 & 3 & s+10/3 \end{bmatrix}^{-1}}_{\underline{K}} \underline{X}_0$$

$$\underline{K} = \frac{1}{3}\begin{bmatrix} \frac{3}{s} & \frac{9s^2+30s+27}{3s^4+10s^3+9s^2+2s} & \frac{9s+30}{3s^4+10s^3+9s^2+2s} & \frac{9}{3s^4+10s^3+9s^2+2s} \\ 0 & \frac{9s^2+30s+27}{3s^3+10s^2+9s+2} & \frac{9s+30}{3s^3+10s^2+9s+2} & \frac{9}{3s^3+10s^2+9s+2} \\ 0 & -\frac{6}{3s^3+10s^2+9s+2} & \frac{9s^2+30s}{3s^3+10s^2+9s+2} & \frac{9s}{3s^3+10s^2+9s+2} \\ 0 & -\frac{6s}{3s^3+10s^2+9s+2} & \frac{27s+6}{3s^3+10s^2+9s+2} & \frac{9s^2}{3s^3+10s^2+9s+2} \end{bmatrix}$$

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

$$Y(s) = [0 \quad -10 \quad 0 \quad 4] \underline{\underline{K}} \, \underline{x}_0$$

$$[1\times 4] \qquad [4\times 4] \; [4\times 1]$$

$$Y(s) = [-10k_{21} + 4k_{41} \quad -10k_{22} + 4k_{42} \quad -10k_{23} + 4k_{43} \quad -10k_{24} + 4k_{44}] \, \underline{x}_0$$

$$Y(s) = -10k_{24} + 4k_{44}$$

$$= -\frac{30}{3s^3+10s^2+9s+2} + \frac{12s^2}{3s^3+10s^2+9s+2} = \frac{12s^2-30}{3s^3+10s^2+9s+2}$$

$$Y(s) = \frac{4s^2-10}{s^3+\frac{10}{3}s^2+3s+2/3}$$

using residue:

$$= -\frac{8.6}{s+1/3} + \frac{9}{s+1} + \frac{3.6}{s+2}$$

$\mathcal{L}^{-1}$

$$\boxed{y(t) = -8.6e^{-t/3} + 9e^{-t} + 3.6e^{-2t}}$$

Is the system observable?

$$\text{let } \underline{\underline{O}} = \left[C_0^* \;\vdots\; A_0^* C_0^* \;\vdots\; (A_0^*)^2 C_0^* \;\vdots\; (A_0^*)^3 C_0^* \right]$$

this formula is from Ogata

$$= \begin{bmatrix} 0 & -16 & 0 & -40 \\ -10 & 0 & -16 & 0 \\ 0 & 10 & 0 & 34 \\ 4 & 0 & 10 & 0 \end{bmatrix}$$

rank ($\underline{\underline{O}}$) = 4, which is equal to $n = 4$, making it completely observable.