denotes possible forms of behavior, where a_3 and a_4 are arbitrary strengths. Let a_3 be equal to a_4 in magnitude, but opposite in sign. Let $\lambda_4 = \lambda_3 + \delta\lambda$. Then the expression above becomes

$$a_3[e^{\lambda_3 t} - e^{(\lambda_3+\delta\lambda)t}],$$

which describes a humped waveform, rising from zero and than falling back; but as $\delta\lambda$ approaches zero the expression also goes to zero. Since, however, the coefficients a_k are arbitrary, and the modes may be evoked in any strength, we are quite entitled to amplify the faint behavior as $\delta\lambda$ diminishes. Suppose that as we move from system to system we make the coefficients a_3 and a_4 increase in inverse proportion to $\delta\lambda$. Then a sequence of possible behaviors is

$$\frac{e^{(\lambda_3+\delta\lambda)t} - e^{\lambda_3 t}}{\delta\lambda},$$

an expression which in the limit is the derivative of $\exp \lambda_3 t$ with respect to λ_3. Therefore, in the limiting system, where λ_4 is equal to λ_3,

$$te^{\lambda_3 t}$$

is a possible time variation and any multiple thereof. It might seem that the merging modes would have to be hit extremely hard in order to produce this result, but in fact only a finite amount of energy is required. Hence, if repeated roots occur, derivatives of $\exp \lambda_k t$ with respect to λ_k must be included in the expression for the most general natural behavior.

IMPULSE RESPONSE AND TRANSFER FUNCTION

Let $V_2(t) = I(t)$ when $V_1(t) = \delta(t)$, irrespective of the choice of the origin of t. Then $I(t)$ may be called the impulse response, and by the superposition property,

$$V_2(t) = \int_{-\infty}^{\infty} I(u)V_1(t-u)\,du.$$

Now if $V_1(t)$ is an exponential function,

$$V_1(t) = e^{pt},$$

where p is some complex constant, it follows that

$$V_2(t) = \int_{-\infty}^{\infty} I(u)e^{p(t-u)}\,du$$

$$= \left[\int_{-\infty}^{\infty} I(u)e^{-pu}\,du\right]e^{pt}.$$

The result shows that the response to exponential stimulus is also exponential, with the same constant p, but with an amplitude given by the quantity in brackets. We call this quantity the *transfer function*, and it is seen to be the Laplace trans-

form of the impulse response. From the subsidiary equation given earlier we see that the transfer function is

$$\frac{a_0 + a_1 p + a_2 p^2 + \ldots}{b_0 + b_1 p + b_2 p^2 + \ldots}.$$

For lumped electrical networks, one can write the transfer function readily from experience with calculating a-c impedance. Hence, through the Laplace transformation, the impulse response also becomes readily accessible. Thus in the circuit of Fig. 14.4, the ratio of $V_2(t)$ to $V_1(t)$ when $V_1(t)$ varies as $\exp pt$ is, by inspection,

$$R\frac{\dfrac{\dfrac{1}{R+L_2p}}{\dfrac{1}{R+L_2p}+Cp}}{L_1p+\dfrac{(R+L_2p)\dfrac{1}{Cp}}{R+L_2p+\dfrac{1}{Cp}}} = R\frac{1}{R+(L_1+L_2)p+L_1RCp^2+L_1L_2Cp^3}.$$

By the method of partial fractions, this ratio of polynomials can be expressed in the form

$$\frac{A}{p+a} + \frac{B}{p+b} + \frac{C}{p+c}$$

and thus the impulse response is

$$(Ae^{-at} + Be^{-bt} + Ce^{-ct})H(t).$$

In problems where $V_2(t)$ must be found when $V_1(t)$ is given, two distinct courses must always be borne in mind: one is to proceed by multiplication with the transfer function in the transform domain; the other method is to proceed, by use of the impulse response, directly through the superposition integral.

A strange difference between the two procedures may be noted. The superposition integral

$$V_2(t_1) = \int_{-\infty}^{\infty} I(u)V_1(t_1-u)\,du$$

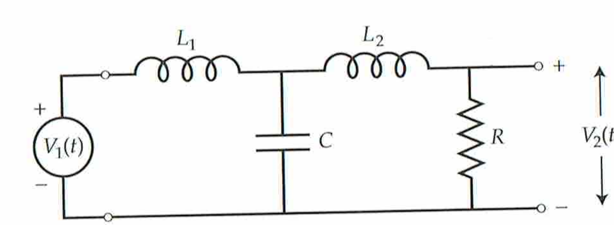

Fig. 14.4 **Find $V_2(t)$ when $V_1(t)$ is given.**

does not make use of values of $V_1(t)$ for $t > t_1$ during the course of calculating $V_2(t_1)$. (This is clear because of the fact that the integrand vanishes for negative values of u.) In other words, to calculate what is happening now, we need only know the history of excitation up to the present moment.

But the procedure in the transform domain normally makes use of information about the future excitation. Indeed, to calculate the Laplace transform of the excitation $V_1(t)$ we have to know its behavior indefinitely into the future.

As a rule, one obtains the same result for the response at $t = t_1$ irrespective of what $V_1(t)$ does after t_1 passes, but occasional trouble can be expected. For example, it is perfectly reasonable to ask what is the response of a system to $V_1(t) = \exp t^2[H(t)]$, and the superposition integral gives it readily, but the Laplace integral diverges because of troubles connected with the infinitely remote future. One way out of this dilemma is to assume that the excitation is perpetually on the point of ceasing; in other words, assume an excitation $\exp t^2[H(t) - H(t - t_1)]$.

INITIAL-VALUE PROBLEMS

Various types of initial-value problems may be distinguished. One type represents the classical problem of a differential equation plus boundary conditions. Thus the driving function is given for all[1] $t > 0$, but no information is provided about the prior excitation of the system. Consequently, some continuing response to prior excitation may still be present, and therefore extra facts must be furnished. The number of extra facts is the same as, or closely connected with, the number of modes of natural behavior, for the continuing response may be regarded as a mixture of natural modes excited by a stimulus that ceased at $t = 0$. Thus sufficient facts must be supplied to allow the strength of each natural mode to be found. These data often consist of the "initial" response and a sufficient number of "initial" derivatives. The word "initial" refers to $t = 0+$; the value at $t = 0$, which may seem to have more claim to the title, is frequently different. To deal with this problem, calculate the response due to a stimulus $V_1(t)$ that is equal to zero for $-\infty < t < 0$,[2] and equal to the given driving function for $t > 0$.[3] Of the many possible particular solutions of the differential equation, this one corresponds to the condition of no stored energy prior to $t = 0$. At $t = 0+$, this response function and its derivatives possess values that may be compared with the initial data. If they are the same, then there was no stored energy prior to $t = 0$, and the solution is complete. But if there are discrepancies, the differences must

[1] The exciting function $V_1(t)$ is given sometimes for $t \geq 0$ and sometimes for $t > 0$, but the statement of the problem, as in the example that follows, often does not say which.

[2] In this statement $-\infty$ means a past epoch subsequent to the assembly of a physical system which has since remained in the same linear, time-invariant condition.

[3] At $t = 0$, take a value equal to $\frac{1}{2}V(0+)$; or take any other *finite* value, since, as previously seen, there is no response to a null function.

be used to fix the strengths a_k of the natural modes by solving a small number of simultaneous equations of the form

$$V_2(0+) = \sum_k a_k e^{\lambda_k t}$$

$$V_2'(0+) = \sum_k a_k \lambda_k e^{\lambda_k t},$$

$$\ldots$$

A much better method, which injects the boundary conditions in advance, depends on assuming that the *response*, not the stimulus, is zero for $t < 0$. Let it be required to solve the differential equation

$$b_0\hat{V}_2(t) + b_1\hat{V}_2'(t) + b_2\hat{V}_2''(t) + \ldots = \hat{V}_1(t)$$

for $\hat{V}_2(t)$, given $\hat{V}_1(t)$ for $t > 0$, and the initial values $\hat{V}_2(0+), \hat{V}_2'(0+), \ldots$. The circumflex accents indicate that the functions are undefined save for positive t. We replace the differential equation by a new one for $V_2(t)$ where

$$V_2(t) = \begin{cases} \hat{V}_2(t) & t > 0 \\ 0 & t < 0. \end{cases}$$

The function $V_2(t)$ is thus defined for all t, and agrees with $\hat{V}_2(t)$, where the latter is defined. Since

$$V_2'(t) = \hat{V}_2(0+)\delta(t) + \begin{cases} \hat{V}_2'(t) & t > 0 \\ 0 & t < 0 \end{cases}$$

and

$$V_2''(t) = \hat{V}_2'(0+)\delta(t) + \hat{V}_2(0+)\delta'(t) + \begin{cases} \hat{V}_2''(t) & t > 0 \\ 0 & t < 0, \end{cases}$$

the new differential equation is

$$b_0V_2(t) + b_2V_2'(t) + b_2V_2''(t) + \ldots = V_1(t),$$

where

$$V_1(t) = [b_1\hat{V}_2(0+)\delta(t)] + [b_2\hat{V}_2'(0+)\delta(t) + b_2\hat{V}_2(0+)\delta'(t)] + \ldots + \begin{cases} \hat{V}_1(t) & t > 0 \\ 0 & t < 0. \end{cases}$$

This differential equation differs from the original one by the inclusion in the driving function of impulses necessary to produce the initial discontinuities of $V_2(t)$ and its derivatives. Taking Laplace transforms throughout, we have the subsidiary equation

$$b_0\overline{V}_2(p) + b_1p\overline{V}_2(p) + b_2p^2\overline{V}_2(p) + \ldots = \overline{V}_1(p).$$

Thus

$$\overline{V}_2(p) = \frac{\overline{V}_1(p)}{b_0 + b_1p + b_2p^2 + \ldots}$$

$$= \frac{[b_1\hat{V}_2(0+)] + [b_2\hat{V}_2'(0+) + b_2\hat{V}_2(0+)p] + \ldots + \int_{0+}^{\infty} \hat{V}_1(t)e^{-pt}\,dt}{b_0 + b_1p + b_2p^2 + \ldots}.$$