Hatch Mathematics & Physics

Tutorial Examples
Mathematics

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Chapter 1

Mathematics

1.1 Overview

I wanted to write some brief notes to capture the key points covered in some of our weekly tutorials. Becoming a good mathematition is possible only by applying our recently acquired mathematical knowledge. For this reason these notes include some exercises to provide the reader a chance to build some skill in applying these new mathematical facts.

In all cases getting mathematics wrong is a part of this learning process; we expect and need to fail to efficiently learn. The only thing that separates folks whom consider themselves capable in mathematics, is the level of persistence in the presence of this inevitable and expected failure.

We should therefore always expect to get a little bit stuck; but try, try and try again!

1.2 Fractions

The value of a fraction is unaltered if the *numerator* and *denominator* are multiplied or divided by the same amount. It is important that we understand the basic operations of fractions, since not only are these operations used when we are performing simplifications of numeric only examples but these also constitute the same basic operations required when tackling symbolic or algebraic manipulation.

1.2.1 Addition and Subtraction

$$\frac{x}{a} + \frac{y}{b} = \frac{x \times b + y \times a}{a \times b} = \frac{xb + ya}{ab} \tag{1.1}$$

$$\frac{x}{a} - \frac{y}{b} = \frac{x \times b - y \times a}{a \times b} = \frac{xb - ya}{ab} \tag{1.2}$$

1.2.2 Multiplication and Division

$$\frac{x}{a} \times \frac{y}{b} = \frac{x \times y}{a \times b} = \frac{xy}{ab} \tag{1.3}$$

$$\frac{x}{a} \div \frac{y}{b} = \frac{x}{a} \times \frac{b}{y} = \frac{x \times b}{a \times y} = \frac{xb}{ay} \tag{1.4}$$

1.2.3 Exercise fractions

Simplify ...

$$3\frac{1}{7} + \frac{2}{3} \tag{1.5}$$

$$3\frac{1}{7} - \frac{2}{3} \tag{1.6}$$

$$\frac{3}{7} \times \frac{1}{3} \tag{1.7}$$

$$\frac{3}{7} \div \frac{7}{3} \tag{1.8}$$

$$\frac{4x}{y} \times \frac{x}{6y} \tag{1.9}$$

$$2st \times \frac{3t}{s^2} \tag{1.10}$$

$$\frac{4\pi r^2}{3} \div 2\pi r \tag{1.11}$$

$$\frac{4uv}{3} \div \frac{u}{2v} \tag{1.12}$$

$$\frac{\pi x^3}{3} \div 8\pi x \tag{1.13}$$

$$\frac{3x^2}{2y} \times \frac{y}{y-2} \tag{1.14}$$

1.3 Surds and Indices

When we express a number as the product of two equal factors, that factor is called the *square root* of the number, for example $4 = 2 \times 2$ thus the square root of 4 is 2. This is written as $2 = \sqrt{4}$. Now -2 is also a square root of 4, as $4 = -2 \times -2$. We can write that $\pm\sqrt{4} = \pm 2$.

1.3.1 Simplifying Surds

Consider $\sqrt{18}$ since one of the factors of 18 is 9 and 9 has an exact square root,

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} \tag{1.15}$$

However since $\sqrt{9} = 3$ therefore $3 \times \sqrt{2}$ or $3\sqrt{2}$. Thus $3\sqrt{2}$ is the simplist possible form for the surd $\sqrt{18}$. Similarly

$$\sqrt{\frac{2}{25}} = \frac{\sqrt{2}}{\sqrt{25}} = \frac{\sqrt{2}}{5} \tag{1.16}$$

Rationalising a Surd denominator

In the example $\frac{2}{\sqrt{3}}$ the square root in the denominator can be removed if we multiply it by another $\sqrt{3}$. Thus we can apply as below:

$$\frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3} \tag{1.17}$$

1.3.2 Exercise surds and indices

Simplify ...

$$\frac{\sqrt{80}}{\sqrt{16}} \tag{1.18}$$

$$\left(\frac{\sqrt{80}}{3} + \frac{1}{3}\right) \tag{1.19}$$

$$\left(\frac{\sqrt{80}}{3} + \frac{1}{6}\right) \tag{1.20}$$

$$\left(\frac{\sqrt{80}}{3} + 2\frac{1}{6}\right) \tag{1.21}$$

$$\left(\frac{\sqrt{100}}{20} + \frac{1}{8} + \frac{1}{8}\right) \tag{1.22}$$

$$\left(\frac{\sqrt{150}}{8} + 7\frac{1}{16}\right) \tag{1.23}$$

$$\frac{1}{\sqrt{7}} \tag{1.24}$$

$$\frac{2}{\sqrt{11}} \tag{1.25}$$

$$\frac{3\sqrt{2}}{\sqrt{5}} \tag{1.26}$$

$$\frac{\sqrt{5}}{\sqrt{10}} \tag{1.27}$$

$$\frac{\sqrt{1}}{\sqrt{27}} \tag{1.28}$$

1.4 Base and Index

In an expression such as 3^4 the *base* is 3 and the 4 is called the *power* or *index*, working with indices involves using some properties which apply to any base, we we express these rules in terms of a general base a, which stands for any number.

1.4.1 Rule 1

Because a^3 means $a \times a \times a$ and a^2 means $a \times a$ it follows that

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5 \tag{1.29}$$

More generally ...

$$a^p \times a^q = a^{p+q} \tag{1.30}$$

1.4.2 Rule 2

Dealing with division

$$a^7 \div a^4 = \frac{\cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a} \times a \times a \times a}{\cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a}} = a^3 \tag{1.31}$$

this can be also be read as $a^7 \div a^4 = a^{7-4}$ or more generally ...

$$a^p \div a^q = a^{p-q} \tag{1.32}$$

Thus from rule 2 say:

$$a^3 \div a^5 = a^{3-5} = a^{-2} = \frac{1}{a^2} \tag{1.33}$$

This tells us that a^{-2} means $\frac{1}{a^2}$ which more generally

$$a^{-p} = \frac{1}{a^p} \tag{1.34}$$

This a^{-p} means the reciprocal of a^p.

Finally for rule 2 any base to the power zero is equal to 1. In fact any number at all raised to the power zero is always 1.

$$a^0 = 1 \tag{1.35}$$

1.4.3 Rule 3

$$(a^2)^3 = (a \times a)^3 \tag{1.36}$$

$$(a \times a)^3 = (a \times a) \times (a \times a) \times (a \times a) = a^6 \tag{1.37}$$

More generally ...

$$(a^p)^q = a^{p \times q} \tag{1.38}$$

Not that this is different to rule 2, since in rule 3 case we have $(a^p)^q$ where (a^p) is raised to the power of q relative to previous rule 2 case where $a^p \times a^q$.

1.4.4 Rule 4

This rule explains the meaning of a fractional index

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a \tag{1.39}$$

Thus

$$a = a^{\frac{1}{2}} \times a^{\frac{1}{2}} \tag{1.40}$$

Therefore $a^{\frac{1}{2}}$ means $\sqrt{a}$ the positive square root of a

$$a = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = \sqrt{a} \times \sqrt{a} = a^1 = a \tag{1.41}$$

More generally ...

$$a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} \tag{1.42}$$

or

$$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p \tag{1.43}$$

1.4.5 Exercise base and index

Simplify ...

$$\frac{2^3 \times 2^7}{4^3} \tag{1.44}$$

$$(x^2)^7 \times x^{-3} \tag{1.45}$$

1.5 Venn Diagrams and Sets

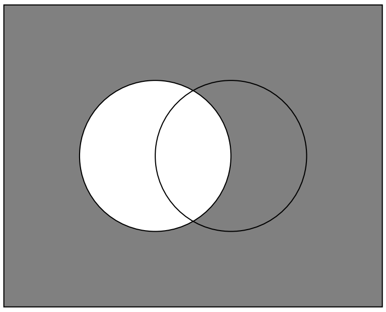
A Venn Diagram is a pictorial representation of the relationships between sets.

$\overline{A} \cap B$ - The intersection $\cap$ represents the overlap or overlay between two sets. It is often useful to consider these two sets separately and then try and mentally overlap these two separate diagrams.

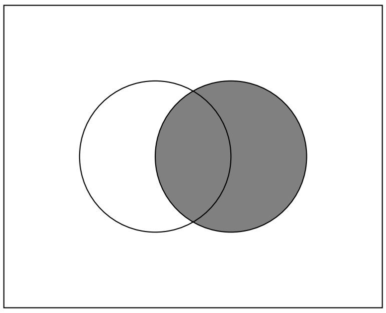
1.5.1 $\overline{A} \cap B$

In the example below $\overline{A} \cap B$ is split into two separate diagrams $\overline{A}$ and B before we finally complete the overlap of $\cap$ for $\overline{A} \cap B$. Many students get confused thinking that the "overlap in the middle" represents $\cap$. This is not the case; we need to first construct both sides of the expression; in this case $\overline{A}$ and B before we attempt to visualise where these two separate sets overlap. As you can see from the third diagram below; this third set is more then the "bit in the middle" between the two sets.

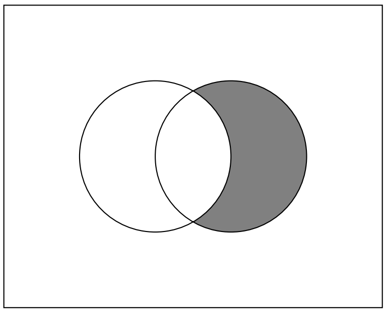
$\overline{A}$



B



$\overline{A} \cap B$



Union $\cap$

The union of $\cap$ between two sets is simply another way of saying that we need to add the the the two separate sets together. In the middle example shown in figure 1.5 we can see $a \cap B$ as everything in A added or $\cap$ or Union with everything in B. Whilst this example is trivial; in any case where we have a $\cap$ we just need to think about shading the region from both sides of the union.

Exercises

The following figure 1.5 show how to shade regions of Venn Diagrams for two sets: A intersect B, A union B, A', A intersect B', A' intersect B, A union B', A' union B, A' union B' = (A intersect B)', A' intersect B' = (A union B)'.

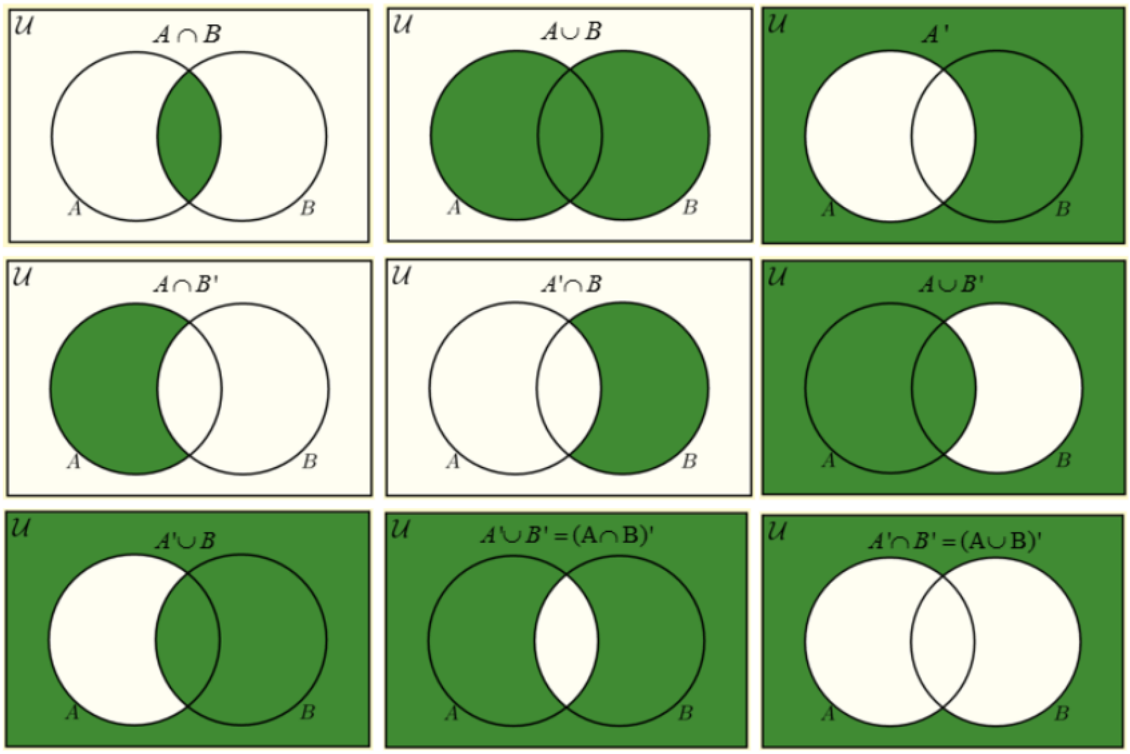


Figure 1.1: Two set examples

Figure 1.2 includes more examples of how to shade Venn Diagrams to represent the required regions of two sets and three sets where we are interested in the complement of a set.

1.6 Conversion of Units

Often in examinations we are given measurements in units that are not fungible with each other. For example we want to convert $18.45mm$ into meters or even kilometers? We need to be able to confidently move between units of length. The notes below develop some shortcuts of heuristics based on our everyday understanding of SI units.

1.6.1 Length

We can start from the intuitive understanding from elementary maths and experience as below:

$10mm \equiv 1cm$ $100cm \equiv 1m$

So how does knowing these help us say convert a length of $18.45mm$ into say meters or kilometers? We can start by making the observation that we can manipulate any ratio in a similar way to an equation or fraction; as long as we apply the same factor to both sides of the ratio, then the same ratio remains

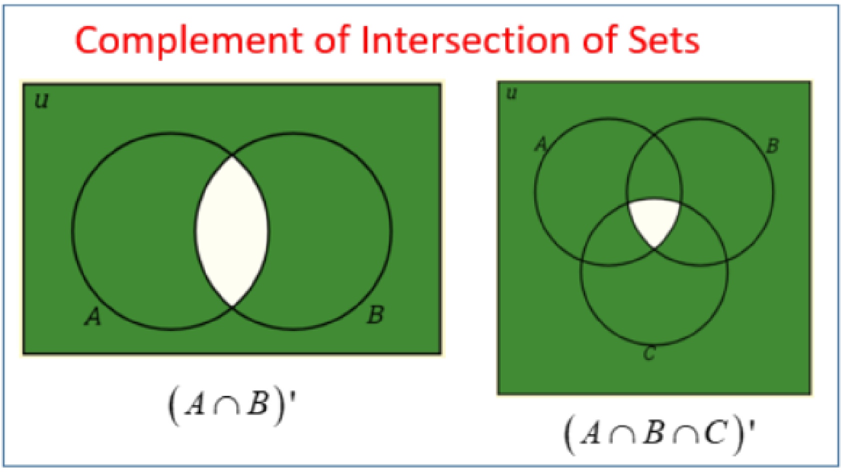


Figure 1.2: Set complement $\overline{A}$

valid.

Using $10mm \equiv 1cm$ we can safely divide both sides by 10 this giving $\frac{10}{10}mm \equiv \frac{1}{10}cm$ which can be simplified as $1mm \equiv 0.1cm$. Since we also know that $100cm \equiv 1m$ we can apply a similar trick as $\frac{100}{100}cm \equiv \frac{1}{100}m$ which results in $1cm \equiv \frac{1}{100}m$.

Appying one final scaling to we can again $\div 10$ on both sides; giving us $0.1cm \equiv \frac{1}{1000}m$. Thus we have

$1mm \equiv 0.1cm \equiv 0.1cm \equiv \frac{1}{1000}m$. This tells us that to move from mm into m we just need to multiple by $\frac{1}{1000}$ or $\times 10^{-3}$.

Going back to the original example converting $18.45mm$ into meters; we just need to multiply 18.45×10^{-3} which gives us $0.01845m$. We can extend this more broadly using the conversion rules shown in figure 1.3.

Figure 1.3 shows us the multipliers that we need to apply; for example we can use $\times 10^{-3}$ when moving between mm and m. However moving between mm and km we need $\times 10^{-6}$. In the opposite sense if we have km we can move into m by $\times 10^{+6}$. You will notice that the rules involve a simple addition of the base ten power. Moving between mm to cm and then m and finallly km involved some $\times 10^{-6}$ separate multiplications which separately can be applied or grouped into a single $\times 10^{-6}$ operator.

The simplicity and beatuy of these simple unit conversions, together with the human sized $10mm \equiv 1cm$ and $100cm \equiv 1m$ are some of the reasons why as a society we chose to make the rather painful decision to migrate to these bases.

1.6.2 Conversion Examples

- *e.g.* 18.45 mm to meters we $(\times 10^{-3})$ $18.45 \times 10^{-3} = 0.01845$ m
- *e.g.* 18.45 km to mm we $(\times 10^{6})$ $18.45 \times 10^{6} = 18,450,000$ mm
- *e.g.* 18.45 m to mm we $(\times 10^{3})$ $18.45 \times 10^{3} = 18,450$ mm

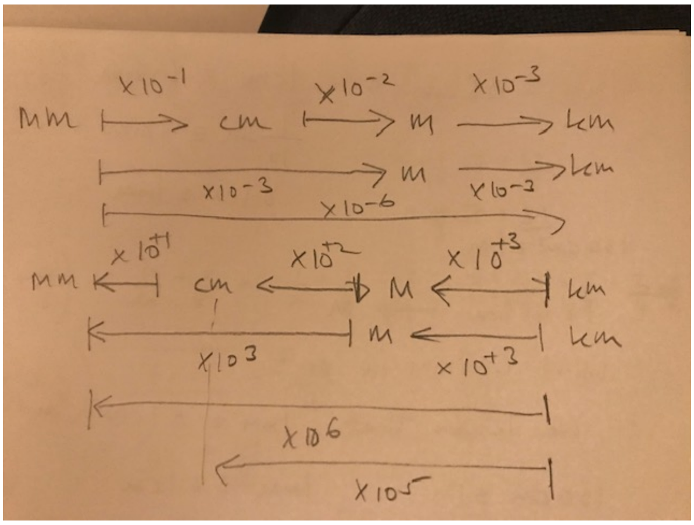


Figure 1.3: General rules to move between length units.

- *e.g.* 18.45 mm to km we ($\times 10^{-6}$) $18.45 \times 10^{-6} = 0.00001845$ km

1.7 Solutions

1.7.1 Problems section 1.2.3

$$3\frac{1}{7} + \frac{2}{3} = \frac{22}{7} + \frac{2}{3} = \frac{66+14}{21} = \frac{80}{21} \tag{1.46}$$

$$3\frac{1}{7} - \frac{2}{3} = \frac{66-14}{21} = \frac{52}{21} \tag{1.47}$$

$$\frac{3}{7} \times \frac{1}{3} = \frac{3}{21} = \frac{1}{7} \tag{1.48}$$

$$\frac{3}{7} \div \frac{7}{3} = \frac{3}{7} \times \frac{3}{7} = \frac{9}{49} = \frac{1}{9} \tag{1.49}$$

$$\frac{4x}{y} \times \frac{x}{6y} = \frac{4x2}{6y^2} = \frac{2x2}{3y^2} = \frac{2}{3}\left(\frac{x}{y}\right)^2 \tag{1.50}$$

$$2st \times \frac{3t}{s^2} = \frac{6st^2}{s^2} = \frac{6t^2}{s} \tag{1.51}$$

$$\frac{4\pi r^2}{3} \div 2\pi r = \frac{4\pi r^2}{3} \times \frac{1}{2\pi r} = \frac{2r}{3} \tag{1.52}$$

$$\frac{4uv}{3} \div \frac{u}{2v} = \frac{4uv}{3} \times \frac{2v}{u} = \frac{8}{3}v^2 \tag{1.53}$$

$$\frac{\pi x^3}{3} \div 8\pi x = \frac{\pi x^3}{3} \times \frac{1}{8\pi x} = \frac{x^2}{24} \tag{1.54}$$

$$\frac{3x^2}{2y} \times \frac{y}{y-2} = \frac{3}{2}\frac{x^2}{(y-2)} \tag{1.55}$$

1.7.2 Problems section 1.3.2

$$\frac{\sqrt{80}}{\sqrt{16}} = \frac{\sqrt{5 \times 16}}{\sqrt{16}} = 4\frac{\sqrt{5}}{4} = \sqrt{5} \tag{1.56}$$

$$\left(\frac{\sqrt{80}}{3} + \frac{1}{3}\right) = \left(\frac{\sqrt{5 \times 16}}{3} + \frac{1}{3}\right) = \left(4\frac{\sqrt{5}}{3} + \frac{1}{3}\right) = \left(\frac{1 + 4\sqrt{5}}{3}\right) = \frac{1}{3} + \frac{4\sqrt{5}}{3} \tag{1.57}$$

$$\left(\frac{\sqrt{80}}{3} + \frac{1}{6}\right) = \left(\frac{\sqrt{5 \times 16}}{3} + \frac{1}{6}\right) = \frac{25\sqrt{5} + 3}{18} = \frac{1}{6} + \frac{4}{3}\sqrt{5} \tag{1.58}$$

$$\left(\frac{\sqrt{80}}{3} + 2\frac{1}{6}\right) = \left(\frac{\sqrt{5 \times 16}}{3} + \frac{13}{6}\right) = \frac{24\sqrt{5} + 39}{18} = \frac{39}{18} + \frac{4}{3}\sqrt{5} \tag{1.59}$$

$$\left(\frac{\sqrt{100}}{20} + \frac{1}{8} + \frac{1}{8}\right) \tag{1.60}$$

$$\left(\frac{\sqrt{150}}{8} + 7\frac{1}{16}\right) = \frac{10}{20} + \frac{2}{8} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \tag{1.61}$$

$$\frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{\sqrt{7}}{7} \tag{1.62}$$

$$\frac{2}{\sqrt{11}} = \frac{2\sqrt{11}}{11} \tag{1.63}$$

$$\frac{3\sqrt{2}}{\sqrt{5}} = \frac{3\sqrt{2} \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{3\sqrt{2} \times \sqrt{5}}{5} = \frac{3\sqrt{10}}{5} = \frac{3}{5}\sqrt{10} \tag{1.64}$$

$$\frac{\sqrt{5}}{\sqrt{10}} = \frac{\sqrt{5}\sqrt{10}}{\sqrt{10}\sqrt{10}} = \frac{\sqrt{50}}{10} = \frac{1}{10}\sqrt{2 \times 25} = \frac{\sqrt{2}}{2} \tag{1.65}$$

$$\frac{\sqrt{1}}{\sqrt{27}} = \frac{\sqrt{\sqrt{27}}}{27} \tag{1.66}$$

1.7.3 Problems section 1.4.5

$$\frac{2^3 \times 2^7}{4^3} = \frac{2^{10}}{4^3} = \frac{2^{10}}{2^6} = 2^{10} \times 2^{-6} = 2^4 \tag{1.67}$$

$$(x^2)^7 \times x^{-3} = \frac{x^{14}}{x^3} = x^{14-3} = x^{11} \tag{1.68}$$

1.8 Quadratic Equations

A number of separate techniques exist for solving equations that are of the form $ax^2+bx+c=0$ some methods are quite simple and others whilst less simple are more beautiful. None less beautiful than completing the square coupled with a geometic explanation as shown in the next section.

1.8.1 Completing the Square

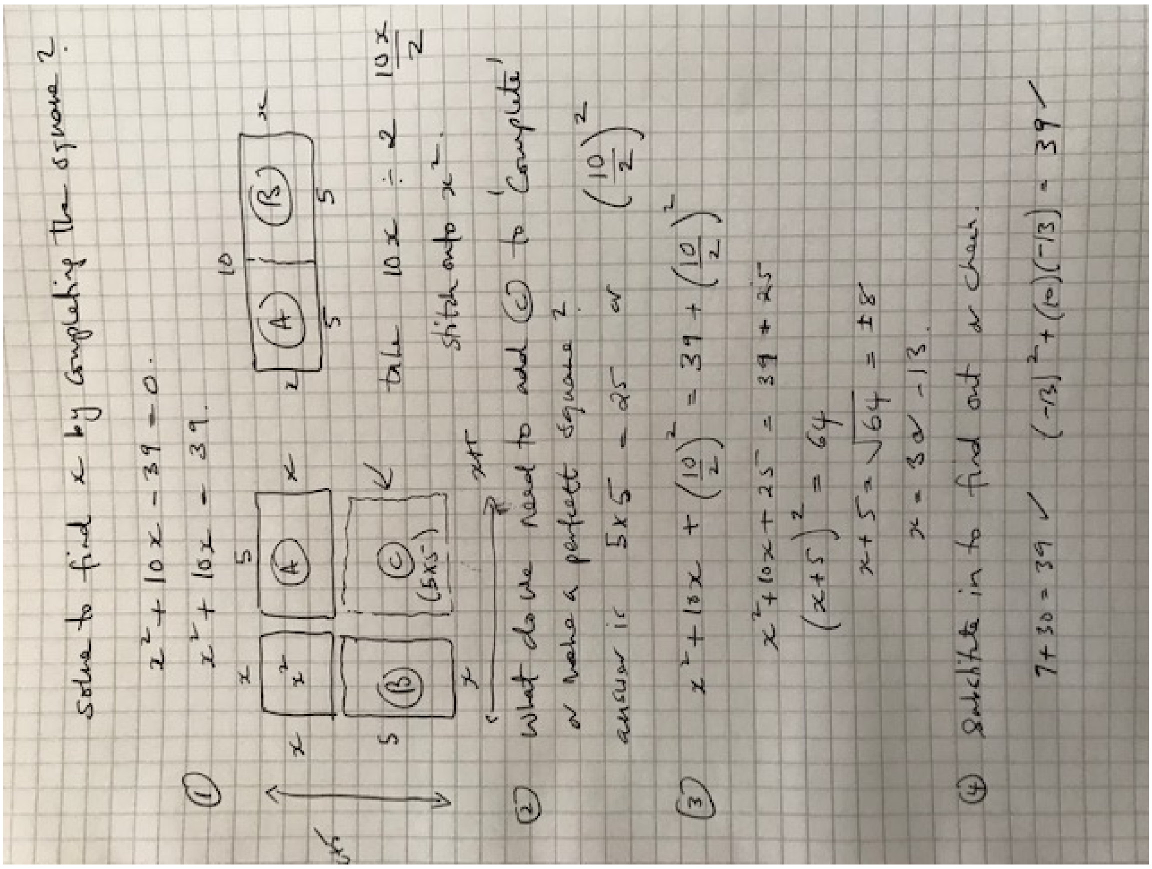


Figure 1.4: Complete Example - Completing the Square

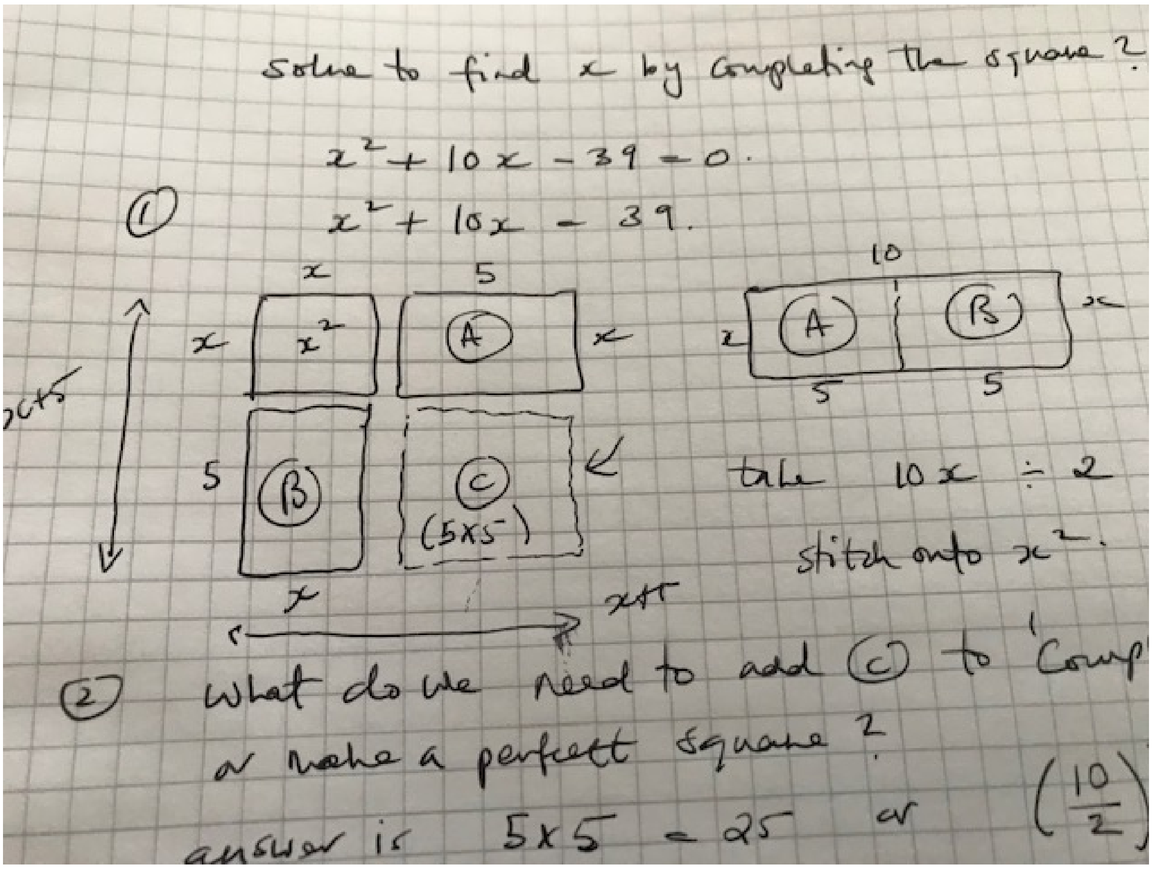


Figure 1.5: Two set examples

Chapter 2

Physics

2.1 Overview

Looking at and thinking about circuits, we touched on some important conceptual differences between what we label as charge Q, current I and resistance R.

We now know that charge is often assembled into nice packets; typically labelled or called electrons e^{-1} (other types of similar packets also exists). These are special bundles of energy that move around the place at different speeds; sometimes slow and sometimes much faster. We shouldn't get confused and think about electromagnetic radiation or photons when we think about electrons these are two different things. Electrons as we know are much more typically found in the stable bound shells around the outsides of atomic nuclei (why don't they fall into the nucleus and or why don't all the electrons in an atom fall into the lowest shell?) than freely moving. But in some special materials electrons are less well bound and can be separated from outer shells quite easily by applying what we call a potential difference across the material.

2.1.1 Potential difference & Current

So how should we think most basically about potential difference, current and charge; how are these related to each other. Before we jump in and begin to look at any equations, let's first consider or think about what we mean as by potential difference or voltage. Creating or applying a potential difference or voltage, is a little bit like tilting or raising a snooker table from one end; we introduce a bias or tendancy in the underlying field in this case the electromagnetic field for charges to be displaced. Since we can't observe this induced field displacement we struggle to understand what this manifests; however we can feel a potential difference, normally as a result of an electric shock. (Why would we experience this has some simple and more complex answers?)

With a raised or tilted snooker table any snooker balls at the raised end of the table will gradually begin to roll away towards the lower end, flowing towards the lower potential difference. As with all stuff in the universe, we can more generally say that everything is trying ultimately in similar ways to find the lowest energy state available. People sit, but would rather sleep and snooker balls want to similarly be in a lower more stable energy state. That is just the way the entire universe works, everything likes being in ground state energy.

So how and more importantly what does this mean for our snooker table, now we have a potential difference, we have raised one end and now we have our snooker balls or electrons in our conducting metal material similarly moving. Moving charge is then labelled as a current, or put another way the rate of change of charge moving we say is a measure of current. This is governed by the simple equation (2.1) where we measure current I in Ampheres as the change in charge Q measured in Coulombs that have passed a point on our snooker table in a certain time T in seconds.

$$I = \frac{\delta Q}{\delta T} = \frac{Q}{T} \tag{2.1}$$

When our snooker balls roll across the green baise whilst relatively fricitionless, there is still some microscopic frictional forces that retard or hold back the snooker balls. This propensity of the medium to hold up the charge is known more generally as resistance and we measure this tendancy using the units of σ or ohms.

Ohms law (2.2) allows us to finally then link together voltage V, current I and resistance R. This simple equation allows us to say many important things about circuits.

$$V = IR \tag{2.2}$$

2.1.2 Circuits

We can plug different components together in loops also known as electrical circuits. These circuits can then be designed to exhibit a variety of complex behaviours including ultimately the computer circuit that I am using to write this document. These components have a variety of different symbols that allow us to see quickly the intended behaviour two important examples at the `Resistor` and `Variable Resistor` these are also shown below.

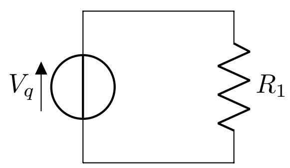
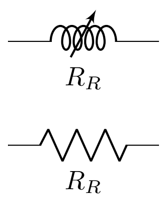


Figure 2.1: Example circuit with potential difference and resistor.



Rearranging Ohms law (2.2) to (2.3) allows us to calculate the current I flowing in circuit 2.1 by substituing known values from measurement.

$$I = \frac{V_q}{R_1} \tag{2.3}$$