

Aabir_AK_HW03

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```
[1]: import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from sklearn.model_selection import train_test_split
from sklearn.linear_model import LinearRegression
from sklearn.metrics import mean_squared_error as MSE
import tabulate as tb

import seaborn as sns

sns.set_style("whitegrid")
```

```
[2]: np.random.seed(42)
```

0.1 Conceptual exercises

0.1.1 Generating data (X) and computing Y with noise

```
[3]: X = np.random.normal(loc=np.random.randint(0, 100, 20), scale=np.random.
    ↪ randint(0, 25, 20), size=(1000, 20))
```

```
[4]: beta = np.random.normal(0, 30, 20)
for i in range(beta.shape[0]):
    if np.random.random() >= 0.70:
        beta[i] = 0

beta = np.array(beta)

assert any([i==0 for i in beta.T]), "beta contains no zeros"
zero_pos = [i for i in range(len(beta)) if beta[i]==0]
print(f"zero values for beta at positions: {zero_pos}")

epsilon = np.random.randn(1000,)

print(X.shape, beta.shape, epsilon.shape)
```

zero values for beta at positions: [6, 10, 17, 19]
(1000, 20) (20,) (1000,)

```
[5]: Y = np.matmul(X, beta) + epsilon
      print(Y.shape)
```

(1000,)

0.1.2 Splitting the data

```
[6]: X_train, X_test, Y_train, Y_test = train_test_split(X, Y, train_size=100)
      print(X_train.shape, X_test.shape, Y_train.shape, Y_test.shape)
```

(100, 20) (900, 20) (100,) (900,)

0.1.3 Performing best subset selection and compiling results on *training* set

```
[7]: # I analyze results of the forward selection algorithm. For backward selection,
      ↪ use forward = False

      forward = True

      best_featureset = [] if forward else list(range(20))
      best_MSE = []
      best_model = []
      model_size = []
      for select_step in range(1, 21):
          s = select_step if forward else 21-select_step
          model_size.append(s)
          #print(f'best_featureset : {best_featureset}')
          new_mse = []
          all_models = []
          for feature_no in range(20):
              if forward:
                  if feature_no in best_featureset:
                      new_mse.append(np.inf)
                      all_models.append(None)
                  else:
                      #print(f"{feature_no} not in {best_featureset}")
                      f_list = best_featureset+[feature_no]
                      mdl = LinearRegression(fit_intercept=True).fit(X_train[:,
                      ↪ f_list], Y_train)
                      Y_train_pred = mdl.predict(X_train[:, f_list])
                      new_mse.append(MSE(Y_train, Y_train_pred))
                      all_models.append(mdl)
              else:
                  if feature_no in best_featureset:
                      f_list = best_featureset.copy()
```

```

        f_list.remove(feature_no)
        mdl = LinearRegression(fit_intercept=True).fit(X_train[:,
→f_list], Y_train)
        Y_train_pred = mdl.predict(X_train[:, f_list])
        new_mse.append(MSE(Y_train, Y_train_pred))
        all_models.append(mdl)
    else:
        new_mse.append(np.inf)
        all_models.append(None)
    #print(new_mse)
    if forward:
        best_feat = np.argmin(new_mse)
        #print(f'adding new best feature # {best_feat}')
        best_featureset.append(best_feat)
        best_MSE.append(np.min(new_mse))
        best_model.append(all_models[best_feat])
    else:
        worst_feat = np.argmax(new_mse)
        best_featureset.pop(worst_feat)
        best_MSE.append(np.min(new_mse))
        #print(f"best subset of size {model_size}: {best_featureset}\n\tMSE =
→{best_MSE[-1]}")

#assert all([MSE(Y_train, best_model[i].predict(X_train))== best_MSE[i] for i
→in range(20)])

```

```

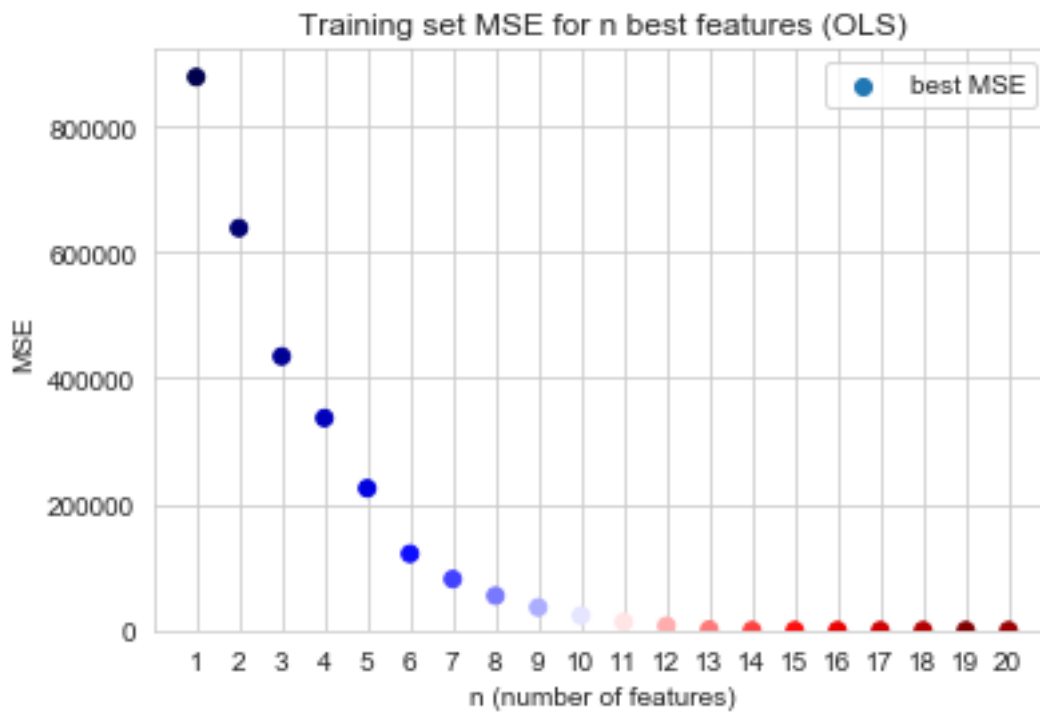
[8]: plt.clf()
ranks = np.argsort(best_MSE)
colnums = list(reversed(np.linspace(0.3, 0.99, 20)))
cols = np.zeros(20)
for i, r in enumerate(ranks):
    cols[r] = colnums[i]
plt.scatter(range(1, 21), best_MSE, label='best MSE', c=cols, cmap=plt.cm.
→seismic)
plt.title('Training set MSE for n best features (OLS)')
plt.xticks(range(1, 21))
plt.xlabel('n (number of features)')
plt.ylabel('MSE')
plt.ylim(0)
plt.legend()
plt.show()
print(f'Train MSE is lowest for the best {ranks[0]+1} features.\n\n')
print(tb.tabulate(dict(zip(['rank', 'feature #', 'beta weight', 'MSE',
    'L2 of coefs', '#abs-sum of betas',
    'L2 b/w coefs and betas'
    ],
    ranks,
    best_featureset,
    best_MSE,
    [sum(coef**2 for coef in mdl.coef_) for mdl in best_model],
    [sum(abs(coef) for coef in mdl.coef_) for mdl in best_model],
    [sum((mdl.coef_ - best_model[0].coef_)**2 for mdl in best_model)]
    ),
    headers=[0, 1, 2, 3, 4, 5, 6],
    floatfmt='%.3f',
    intfmt='%d',
    maxcolwidth=100,
    maxrowwidth=100,
    tablinestyle='plain'))

```

```

                                [list(range(1, 21)), [f"f{i+1}" for i in
↪best_featureset],
                                beta[best_featureset], best_MSE,
                                [np.linalg.norm mdl.coef_ for mdl in
↪best_model],
                                #[np.abs(beta[best_featureset][:i+1]).sum() for
↪i in range(len(beta))]
                                [np.linalg.norm(beta[:i+1] - mdl.coef_) for i,
↪mdl in enumerate(best_model)]
                                ])), headers="keys"))

```



Train MSE is lowest for the best 19 features.

rank	feature #	beta weight	MSE	L2 of coefs	L2 b/w coefs
and betas					
1	f5	24.4879	878367	26.3608	
17.9478					
2	f16	27.3752	638503	40.9984	
24.4321					
3	f4	-38.9013	434847	54.946	
28.2849					

4	f12	-25.2876	336982	58.0339
31.1381				
5	f2	17.5056	225363	63.318
30.9881				
6	f6	20.2686	121443	70.5721
30.8221				
7	f13	11.5036	81299.5	69.2825
31.2721				
8	f8	11.2583	54823	70.4401
31.6916				
9	f19	-35.6605	36339.9	77.4138
39.2698				
10	f9	-11.9184	23392.2	76.9538
38.3716				
11	f17	-3.0314	13116.9	77.584
39.3817				
12	f15	-4.06156	7129.68	77.3908
46.029				
13	f10	-5.3283	1275.6	76.4245
48.296				
14	f14	3.57638	62.4206	77.3582
48.8478				
15	f1	8.41296	0.622241	77.6478
50.3792				
16	f18	0	0.576682	77.6476
57.2873				
17	f11	0	0.573091	77.6495
57.3696				
18	f20	0	0.569609	77.6504
57.3727				
19	f7	0	0.569525	77.6504
67.5517				
20	f3	-21.3005	0.569525	77.6504
67.5517				

As the variance of each feature is quite large, and the beta values are large too, the MSE decreases rapidly until the 16th feature is added. The f16 through f19 are those for which $\beta = 0$, which intuitively makes sense.

It is a little unusual that f3 (with a relatively large magnitude β is the ‘worst’ performing feature by this method.

0.1.4 Compiling results on *test* set

```
[9]: plt.clf()
test_MSEs = [MSE(Y_test, mdl.predict(X_test[:, best_featureset[:i+1]])) for i,
             ↪mdl in enumerate(best_model)]
```

```

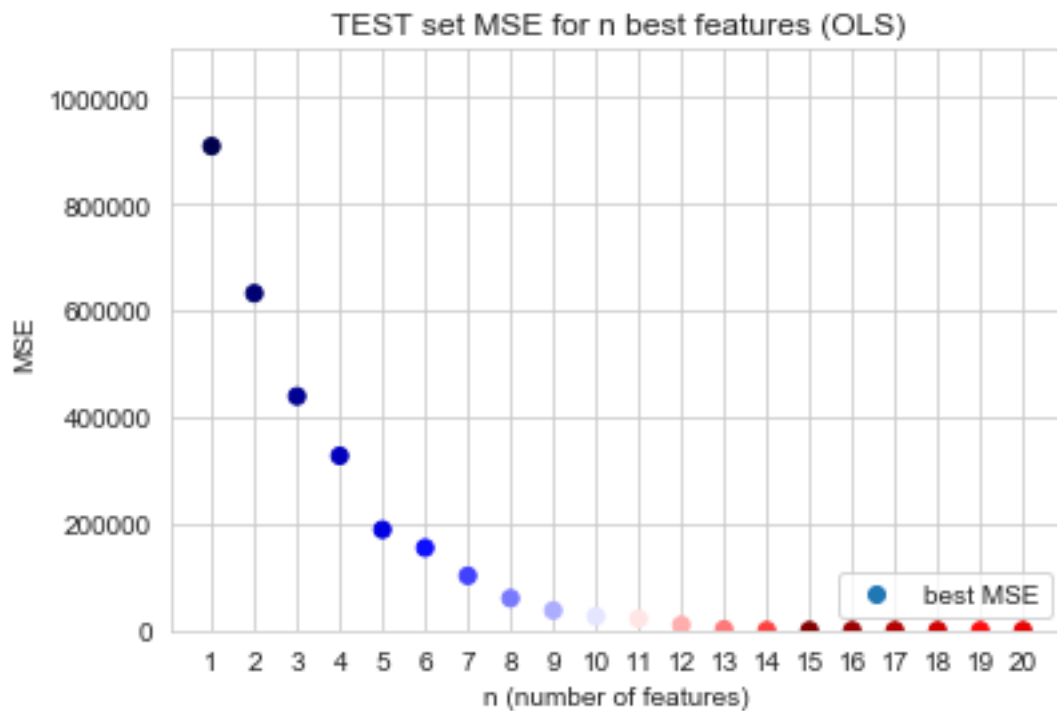
ranks = np.argsort(test_MSEs)
colnums = list(reversed(np.linspace(0.3, 0.99, 20)))
cols = np.zeros(20)
for i, r in enumerate(ranks):
    cols[r] = colnums[i]

plt.scatter(range(1, 21), test_MSEs, label='best MSE', c=cols, cmap=plt.cm.
    ↳seismic)
plt.title('TEST set MSE for n best features (OLS)')
plt.xlabel('n (number of features)')
plt.ylabel('MSE')
plt.xticks(range(1, 21))
plt.ylim(0, 1.2*max(test_MSEs))
plt.legend(loc='lower right')
plt.show()

print(f'Test MSE is lowest for the best {ranks[0]+1} features.\n\n')

print(tb.tabulate(dict(zip(['rank', 'feature #', 'train MSE', 'test MSE'
    ],
    [list(range(1, 21)), [f"f{i+1}" for i in_
    ↳best_featureset],
    best_MSE, test_MSEs
    ])), headers="keys"))

```



Test MSE is lowest for the best 15 features.

rank	feature #	train MSE	test MSE
1	f5	878367	907799
2	f16	638503	631939
3	f4	434847	438833
4	f12	336982	327119
5	f2	225363	188702
6	f6	121443	154619
7	f13	81299.5	102197
8	f8	54823	60055.1
9	f19	36339.9	37190.1
10	f9	23392.2	26241.2
11	f17	13116.9	21473.9
12	f15	7129.68	10034
13	f10	1275.6	1779.84
14	f14	62.4206	81.0563
15	f1	0.622241	1.2726
16	f18	0.576682	1.31484
17	f11	0.573091	1.31802
18	f20	0.569609	1.32218
19	f7	0.569525	1.32324
20	f3	0.569525	1.32324

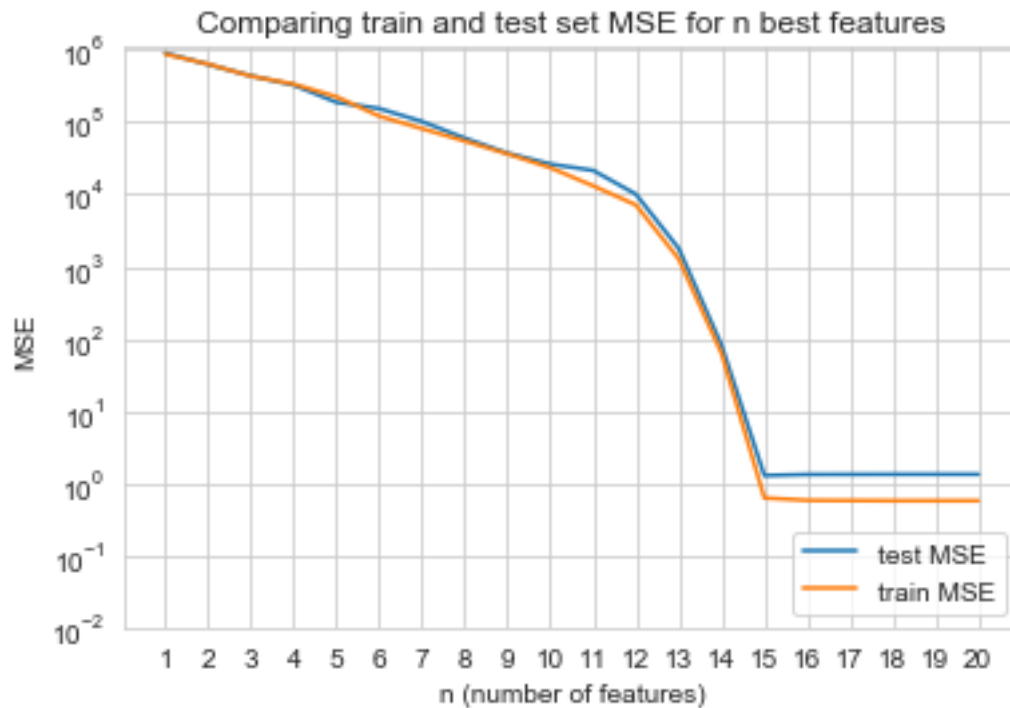
```
[10]: # plotting train and test set MSEs

plt.plot(range(1, 21), test_MSEs, label='test MSE')
plt.plot(range(1, 21), best_MSE, label='train MSE')
plt.title('Comparing train and test set MSE for n best features')
plt.xlabel('n (number of features)')
plt.ylabel('MSE')
plt.xticks(range(1, 21))
plt.ylim(0.01, 1.2*max(test_MSEs))
plt.legend(loc='lower right')
plt.yscale('log')
plt.show()

print('Based on test MSE, it would appear that the best featureset is the top_
→15 features')
f_string15 = 'f'+', '.join([str(x) for x in best_featureset[:15]])
print(f'{f_string15}')

print("\n\n\nThe 5 worst performing features are excluded:")
f_string5 = 'f'+', '.join([str(x) for x in best_featureset[15:]])
```

```
print(f'{f_string5}')
```



Based on test MSE, it would appear that the best featureset is the top 15 features
 f4, f15, f3, f11, f1, f5, f12, f7, f18, f8, f16, f14, f9, f13, f0

The 5 worst performing features are excluded:
 f17, f10, f19, f6, f2

```
[11]: print('Comparing betas and trained weights for top 15\n\n')

sgn = np.sign(np.multiply(beta[:15], best_model[14].coef_))[:15]
print(tb.tabulate({'feature': [f'f{i}' for i in best_featureset[:15]],
                  'beta': beta[best_featureset[:15]], 'coef': best_model[14].
    ↪ coef_,
                  'absolute difference': np.abs(beta[best_featureset[:
    ↪ 15]] - best_model[14].coef_)
                  #'signum(beta x coef)': sgn
                  },
                  headers='keys'))

print(f'\n\nmodel intercept: {best_model[-1].intercept_}')
```



```
print(f'\n\nY_train mean: {np.mean(Y_train)}\nY_train std: {np.std(Y_train)}')
print(f'\n\nY_test mean: {np.mean(Y_test)}\nY_test std: {np.std(Y_test)}')
```

Comparing betas and trained weights for top 15

feature	beta	coef	absolute difference
f4	24.4879	24.4866	0.00131481
f15	27.3752	27.3841	0.00885102
f3	-38.9013	-38.9028	0.00152833
f11	-25.2876	-25.2945	0.00696448
f1	17.5056	17.4989	0.00673293
f5	20.2686	20.2472	0.0213728
f12	11.5036	11.5112	0.00759373
f7	11.2583	11.2474	0.0108286
f18	-35.6605	-35.664	0.00343784
f8	-11.9184	-11.9061	0.0122746
f16	-3.0314	-3.03784	0.00644101
f14	-4.06156	-4.05922	0.00233925
f9	-5.3283	-5.3328	0.00449614
f13	3.57638	3.58911	0.0127271
f0	8.41296	8.40132	0.0116457

model intercept: -305.3161570324494

Y_train mean: -1422.4022211681897

Y_train std: 1099.459363803579

Y_test mean: -1358.61291988624

Y_test std: 1078.7174916080571

Clearly the coefficients are very similar between the true generating process and the best fit model. This is expected as we are fitting a linear model to a process with a linear generating process.

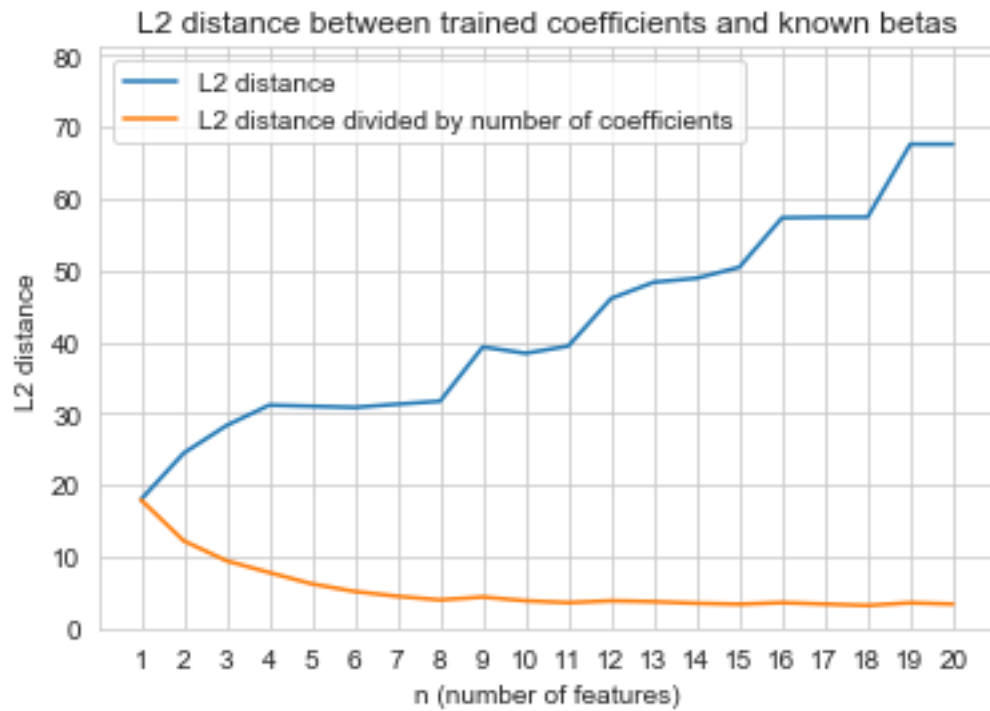
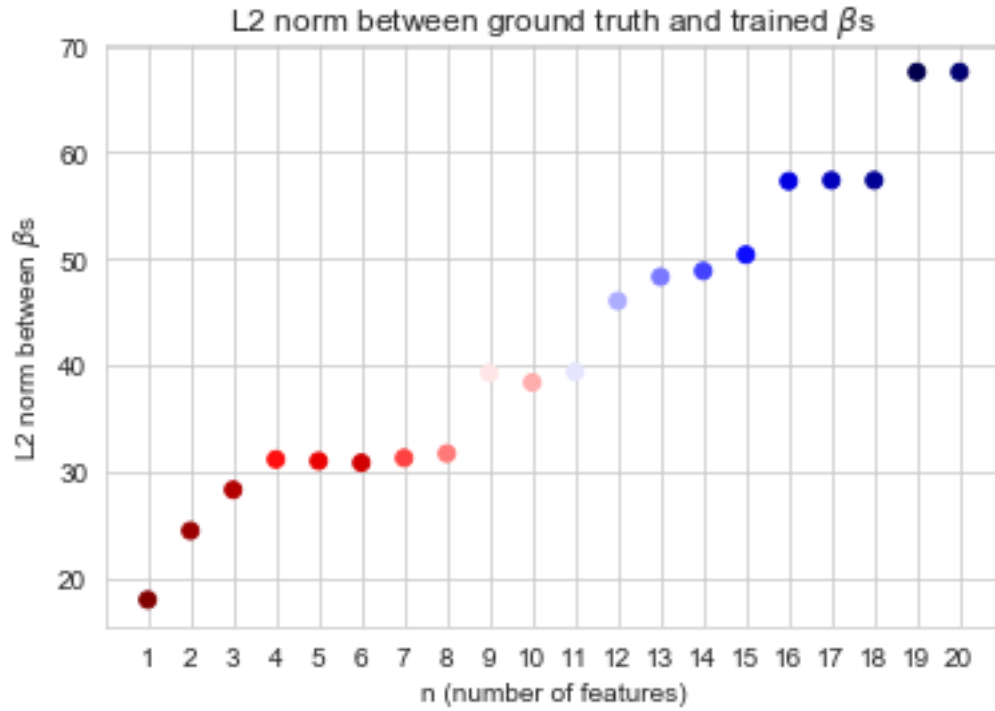
It is worth noting some of the factors that may be leading to the minor mismatch: - There is some noise added to the Y values (in the form of epsilon) - 4 the betas are set to zero. Of the 5 features excluded from the model, 4 are the ones with $\beta = 0$. However, one of the 'predictive' features is excluded. - The generating process has no intercept, but the linear regression does. If the generated data clusters heavily, the intercept could explain a significant degree of the variance, allowing other coefficients. As we can see, the Ys have a mean around -1400 and standard deviation of around 1000. This means an intercept of -300 would do a decent job of beginning to model the data.

0.2 Plot of $\sqrt{\sum_{j=1}^p (\beta_j - \hat{\beta}_j^r)^2}$

```
[12]: plt.clf()
beta_L2 = [np.linalg.norm(beta[:i+1] - mdl.coef_) for i, mdl in
    ↪ enumerate(best_model)]
ranks = np.argsort(beta_L2)
colnums = list(reversed(np.linspace(0.3, 0.99, 20)))
cols = np.zeros(20)
for i, r in enumerate(ranks):
    cols[r] = colnums[i]
plt.scatter(range(1, 21), beta_L2, c=cols, cmap=plt.cm.seismic)
plt.title(r'L2 norm between ground truth and trained $\beta$s')
plt.xticks(range(1, 21))
plt.xlabel('n (number of features)')
plt.ylabel(r'L2 norm between $\beta$s')
#plt.ylim(0)
plt.show()

# plotting L2 norm between trained weights and known betas

l2_diff = [(np.linalg.norm(beta[:i+1] - mdl.coef_)) for i, mdl in
    ↪ enumerate(best_model)]
l2_div = [(np.linalg.norm(beta[:i+1] - mdl.coef_)/(i+1) for i, mdl in
    ↪ enumerate(best_model)]
plt.plot(range(1, 21), l2_diff, label='L2 distance')
plt.plot(range(1, 21), l2_div, label='L2 distance divided by number of
    ↪ coefficients')
plt.title('L2 distance between trained coefficients and known betas')
plt.xlabel('n (number of features)')
plt.ylabel('L2 distance')
plt.xticks(range(1, 21))
plt.ylim(0, 1.2*max(l2_diff))
plt.legend(loc='upper left')
#plt.yscale('log')
plt.show()
```



As we can see, the L2 distance between trained coefficients and known betas for each best model

of size r increases steadily, though it does flatten out in a few zones. We observe flattening from 4-8, 9-11 and 13-15.

Comparing this with the plot on test MSE error, we observe **no real correlations**. This can be attributed to the gradual accumulation of difference in the trained coefficients. Typically, we would hope that we would reach a sweet spot between over-fitting and under-fitting as the learned parameters begin to approximate the known β s. In this case, the L2 distance between learned and known coefficients would hopefully also show some kind of local optima.

However, for reasons discussed above, this does not happen in this example.

0.3 Application exercises with GSS data

Note that as per scikit-learn convention, I treat `alpha` (in sklearn) as λ and `l1_ratio` (in sklearn) as α .

```
[13]: train_d = pd.read_csv('./data/gss_train.csv')
      test_d = pd.read_csv('./data/gss_test.csv')
```

```
[14]: print(train_d.shape, test_d.shape)
```

```
(1481, 78) (493, 78)
```

```
[15]: train_d.head()
```

```
[15]:
```

	age	attend	authoritarianism	black	born	childs	colath	colrac	colcom	\
0	21	0	4	0	0	0	1	1	0	
1	42	0	4	0	0	2	0	1	1	
2	70	1	1	1	0	3	0	1	1	
3	35	3	2	0	0	2	0	1	0	
4	24	3	6	0	1	3	1	1	0	

	colmil	...	zodiac_GEMINI	zodiac_CANCER	zodiac_LEO	zodiac_VIRGO	\
0	1	...	0	0	0	0	
1	0	...	0	0	0	0	
2	0	...	0	0	0	0	
3	1	...	0	0	0	0	
4	0	...	0	0	0	0	

	zodiac_LIBRA	zodiac_SCORPIO	zodiac_SAGITTARIUS	zodiac_CAPRICORN	\
0	0	0	0	0	
1	0	0	0	0	
2	0	0	0	0	
3	0	1	0	0	
4	0	1	0	0	

	zodiac_AQUARIUS	zodiac_PISCES
0	0	0
1	0	0

2	0	0
3	0	0
4	0	0

[5 rows x 78 columns]

```
[16]: Y_train, X_train = train_d['egalit_scale'], train_d[train_d.columns.
      ↪difference(['egalit_scale'])]
      Y_test, X_test = test_d['egalit_scale'], test_d[test_d.columns.
      ↪difference(['egalit_scale'])]

      print(Y_train.shape, X_train.shape, Y_test.shape, X_test.shape)
```

(1481,) (1481, 77) (493,) (493, 77)

```
[17]: from sklearn.linear_model import RidgeCV, LassoCV, ElasticNetCV
      from sklearn.model_selection import cross_validate

      iters = 10000

      ols, rdg = LinearRegression().fit(X_train, Y_train), RidgeCV(alphas=np.
      ↪linspace(0.1,2,10))
      las, ela = LassoCV(n_alphas=10, max_iter=iters), ElasticNetCV(l1_ratio=np.
      ↪linspace(0.1, 1, 10), max_iter=iters)

      ols_err1, ols_err2 = MSE(Y_train, ols.predict(X_train)), MSE(Y_test, ols.
      ↪predict(X_test))

      print(f'For OLS Linear Regression:\nTrain error = {ols_err1}\n\nTest error =_
      ↪{ols_err2}')
```

For OLS Linear Regression:

Train error = 55.12263854924573

Test error = 63.213629623015

```
[18]: def get_alpha_and_MSE(modelCV):
      modelCV.fit(X_train, Y_train)
      al = modelCV.alpha_
      tr_mse = MSE(Y_train, modelCV.predict(X_train))
      te_mse = MSE(Y_test, modelCV.predict(X_test))
      return al, tr_mse, te_mse, modelCV

      n = X_train.shape[1]

      print(f"Ordinary Linear Regression: {sum(np.abs(ols.coef_)<0.1)}/{n}_
      ↪coefficients are in (-0.1, 0.1)\n")
```

```

print('Cross validation results\n')
rdg_alpha, rdg_mse1, rdg_mse2, rdgmod = get_alpha_and_MSE(rdg)
print(f"Ridge Regression: {sum(np.abs(rdgmod.coef_)<0.1)}/{n} coefficients are in (-0.1, 0.1)")

las_alpha, las_mse1, las_mse2, lasmod = get_alpha_and_MSE(las)
print(f"Lasso Regression: {sum(np.abs(lasmod.coef_)<0.1)}/{n} coefficients are in (-0.1, 0.1)")

ela_alpha, ela_mse1, ela_mse1, elamod = get_alpha_and_MSE(ela)
print(f"ElasticNet1: {sum(np.abs(elamod.coef_)<0.1)}/{n} coefficients are in (-0.1, 0.1)")

ela2 = ElasticNetCV(alphas=[.1, *list(np.linspace(0.12, 0.45, 5)),
                                .5, .7, .9, .95, .99, 1],
                    l1_ratio = [elamod.l1_ratio_], max_iter=iters)
ela2_alpha, ela2_mse1, ela2_mse2, elamod2 = get_alpha_and_MSE(ela2)
print(f"ElasticNet2: {sum(np.abs(elamod.coef_)<0.1)}/{n} coefficients are in (-0.1, 0.1)\n\n")

```

Ordinary Linear Regression: 8/77 coefficients are in (-0.1, 0.1)

Cross validation results

Ridge Regression: 9/77 coefficients are in (-0.1, 0.1)

Lasso Regression: 59/77 coefficients are in (-0.1, 0.1)

ElasticNet1: 59/77 coefficients are in (-0.1, 0.1)

ElasticNet2: 59/77 coefficients are in (-0.1, 0.1)

```

[19]: train_errs = [ols_err1, rdg_mse1, las_mse1, ela_mse1, ela2_mse1]
test_errs = [ols_err2, rdg_mse2, las_mse2, ela_mse2, ela2_mse2]

rdg_alpha, las_alpha, ela_alpha, ela2_alpha = [round(i, 2) for i in [rdg_alpha,
las_alpha, ela_alpha, ela2_alpha]]

print(tb.tabulate({'CV model': ['Linear Regression', f'Ridge Regression (alpha={rdg_alpha})',
f'Lasso Regression (alpha={las_alpha})',
f'ElasticNet (alpha={ela_alpha}, l1_ratio={elamod.l1_ratio_})',
f'ElasticNet (alpha={ela2_alpha}, l1_ratio={elamod2.l1_ratio_})'],
'Train Error': train_errs, 'Test Error': test_errs,

```

```

        'Notes': ['', '', '', 'optimizing alpha w/ default l1_ratio',
                  'optimizing l1_ratio w/ optimal alpha']],
        headers='keys'))

print(f"\n\nMean egalit_scale value of {Y_test.shape[0]} samples = {Y_test.
    ↪mean()}")
print(f"Standard deviation = {Y_test.std()}")

```

```

    ↪-----

NameError                                Traceback (most recent call
↪last)

<ipython-input-19-d6827bba18de> in <module>
      1 train_errs = [ols_err1, rdg_mse1, las_mse1, ela_mse1, ela2_mse1]
----> 2 test_errs = [ols_err2, rdg_mse2, las_mse2, ela_mse2, ela2_mse2]
      3
      4 rdg_alpha, las_alpha, ela_alpha, ela2_alpha = [round(i, 2) for i in
↪[rdg_alpha, las_alpha, ela_alpha, ela2_alpha]]
      5

NameError: name 'ela_mse2' is not defined

```

In the ordinary regression model, 8/77 coefficients are within the threshold to be considered small i.e. within $(-0.1, 0.1)$. In these terms, ridge regression only has one more coefficient that drops into this threshold. As expected, we see a gradual increase in the training error with more regularization. However, as the model is able to generalize better, the test error decreases.

Lasso regression seems to be overzealous with the regularization (with nearly 80% of weights being set to very small values), but still performs slightly better than ridge regression. This means that it is likely reducing the effect of most variables on the dataset on the outcome of interest. Given the complexity of social behavior, it is quite likely that there may not be any meaningful linear relationship between egalitarian outlooks and arbitrary GSS variables like `tvhours` (number of hours of TV consumed per week), or the 12 fields `zodaic_*` representing zodiac sign. This could therefore potentially be ignoring confounders in the dataset.

This is worth noting especially because the subsequent regularization methods of Lasso and ElasticNet reduce the test MSE by only around 0.13 and 0.26 respectively. In fact, searching `alpha` for ElasticNet while keeping `l1_ratio` at the default value produces the best performance. Somehow, searching `l1_ratio` over a range of values with this optimal `alpha` actually performs worse on the test set. This could be because of overfitting on the training data, has to balance the two parameters to effectively regularize.

The best performer on the test set is ElasticNet. Note that the marginal improvement is quite small from regularization.

All in all, we do a pretty good job of predicting an individual's self-reported inclination towards egalitarian behavior. The standard deviation of the test set is around 10, and our test MSE on nearly 500 samples is just about 63. This is a fair level of accuracy.

This can be justified by the fact that the dataset also contains many variables that are known to be predictors of egalitarian attitudes - such as political orientation, confidence in science, opinions on personal liberty (such as homosexuality, marijuana consumption, etc.).