

# TTIC 31230 Fundamentals of Deep Learning

## Problems for Graphical Models.

**Problem 1. Pseudolikelihood of a one dimensional spin glass.** We let  $\hat{x}$  be an assignment of a value to every node. We define the score of  $\hat{x}$  by

$$f(\hat{x}) = \sum_{i=1}^{N-1} \mathbf{1}[\hat{x}[i] = \hat{x}[i+1]]$$

The probability distribution over assignments is defined by a softmax.

$$Q_f(\hat{x}) = \underset{\hat{x}}{\text{softmax}} f(\hat{x})$$

What is the **Pseudolikelihood** of the all ones assignment?

**Solution:**

$$\tilde{P}_f(\hat{x}) = \prod_i P_f(\hat{x}[i] \mid \hat{x}/i)$$

where  $\hat{x}/i$  consists of all components of  $\hat{x}$  other than  $i$ . In a graphical model  $P_f(\hat{x}[i] \mid \hat{x}/i)$  is determined by the neighbors of  $i$  and we can consider only how a value is scored against its neighbors. For  $\hat{x}$  equal to all ones we have

$$f(\hat{x}) = N - 1$$

$$f(\hat{x}[i=0]) = \begin{cases} N-3 & \text{for } 1 < i < N \\ N-2 & \text{for } i=1 \text{ or } i=N \end{cases}$$

For  $1 < i < N$  we get

$$\begin{aligned} Q_f(\hat{x}[i=1] \mid \hat{x}/i) &= \frac{e^{N-1}}{e^{N-1} + e^{N-3}} \\ &= \frac{1}{1 + e^{-2}} \end{aligned}$$

and for  $i=1$  or  $i=N$  we get

$$Q_f(\hat{x}[i=1] \mid \hat{x}/i) = \frac{1}{1 + e^{-1}}$$

This gives

$$\tilde{Q}(\hat{x}) = (1 + e^{-1})^{-2} (1 + e^{-2})^{-(N-2)}$$

**Problem 2. Pseudolikelihood for images.** Consider a semantic segmentation  $\hat{y}[i]$  on pixels  $i$  with  $\hat{y}[i]$  a semantic class label in  $\{C_1, \dots, C_K\}$ . We also assume a scoring function  $s_\Phi$  on semantic segmentations defining

$$P_\Phi(\hat{y}) = \underset{\hat{y}}{\text{softmax}} \ s_\Phi(\hat{y})$$

Pseudolikelihood is defined by

$$\tilde{P}_\Phi(\hat{y}) = \prod_i P_\Phi(\hat{y}[i] \mid \hat{y} \setminus i)$$

where  $\hat{y} \setminus i$  assigns a class to every pixel other than  $i$ , and  $\hat{y}[i := c]$  is the semantic segmentation identical to  $\hat{y}$  except that pixel  $i$  is labeled with semantic class  $c$ . In a typical graphical model for images we have

$$P_\Phi(\hat{y}[i] \mid \hat{y} \setminus i) = P_\Phi(\hat{y}[i] \mid \hat{y}[N(i)])$$

where  $\hat{y}[N(i)]$  is  $\hat{y}$  restricted to those pixels which are neighbors of pixel  $i$ .

(a) show

$$\frac{P_\Phi(\hat{y})}{\sum_c P_\Phi(\hat{y}[i := c])} = \underset{c}{\text{softmax}} \ s_\Phi(\hat{y}[i := c]) \quad \text{evaluated at } c = y[i]$$

**Solution:**

$$\begin{aligned} \frac{P_\Phi(\hat{y})}{\sum_c P_\Phi(\hat{y}[i := c])} &= \frac{\frac{1}{Z} e^{s_\Phi(\hat{y})}}{\sum_c \frac{1}{Z} e^{s_\Phi(\hat{y}[i := c])}} \\ &= \frac{e^{s_\Phi(\hat{y})}}{\sum_c e^{s_\Phi(\hat{y}[i := c])}} \\ &= \underset{c}{\text{softmax}} \ s_\Phi(\hat{y}[i := c]) \quad \text{evaluated at } c = y[i] \end{aligned}$$

(b) How many scores need to be computed in the worst case for computing  $P_\Phi(\hat{y})$ . Given the result of part (a), how many for computing  $\tilde{P}_\Phi(\hat{y})$ ?

**Solution:**  $K^N$  for  $P_\Phi$  and  $KN$  for  $\tilde{P}_\Phi$ .

(c) Consider a distribution on semantic segmentations where for each pixel the class assigned to that pixel is uniquely determined by the classes of its neighbors. Can this distribution be defined by a softmax over scores? Explain your answer.

**Solution:** No. Since  $e^s > 0$  for any finite  $s$ , all semantic segmentations must have nonzero probability.

(d) If  $P_\Phi$  is a distribution defined in some other way such that the class of each pixel is completely determined by the other pixels, given a simple expression for  $\tilde{P}_\Phi(\hat{y})$  in the case where  $\hat{y}$  has nonzero probability under  $P_\Phi$ .

**Solution:** We have  $P_\Phi(\hat{y}|\hat{y}\setminus i) = 1$  which implies  $\tilde{P}(\hat{y}) = 1$ .