

Sketch Proof of the Erdős-Straus Conjecture

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The Erdős-Straus conjecture states that, for every positive integer $n \geq 2$, there exists a solution in positive integers a, b, c to the equation

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

We provide a simple proof. First, we show that proving the general case of n is equivalent to proving the case of prime n . We rewrite the initial equation above and use this to derive the basis of our proof. We show that, in general, for each prime n there is an a from which the values of b, c can be recovered, provided that either:

1. one of the divisors a^2 is congruent to 2 modulo 3, or
2. the sum of a and one of the divisors of a^2 is divisible by 3, which is true in general.

To begin, each positive non-prime integer $n > 1$ can be written as $n = pq$ where p is prime and $q > 1$ is a positive integer. Therefore, if we possess a solution in a, b, c for p , then we have an equivalent solution for n , since

$$\begin{aligned} \frac{4}{p} &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ \frac{4}{n} \Rightarrow \frac{4}{pq} &= \frac{1}{aq} + \frac{1}{bq} + \frac{1}{cq}. \end{aligned}$$

We consider only the case of prime n . It is clear that each of a, b, c is either divisible by n or not, and at least one must be, as opposed to collectively sharing divisors of which the product is n . We rewrite the initial equation as

$$\begin{aligned} \frac{4}{p} &= \frac{1}{a} + \frac{1}{bp} + \frac{1}{c} \\ pb \cdot (a + c) &= ac \cdot (4b - 1) \\ \frac{bp}{4b - 1} &= \frac{ac}{a + c}. \end{aligned} \tag{1}$$

It is important to know when $\frac{pb}{4b-1}$ will be in its lowest terms. We note that b is necessarily coprime to $4b-1$, so it is only when p divides $4b-1$ that the fraction may be simplified any further.

Thus,

$$k \cdot b = ac, \quad k \cdot \frac{4b-1}{p} = a+c, \quad \text{if } p \mid (4b-1) \quad (2)$$

$$k \cdot bp = ac, \quad k \cdot (4b-1) = a+c, \quad \text{if } p \nmid (4b-1). \quad (3)$$

We introduce the k because the expansion of the left hand-sides into ac and $a+c$ may require multiplication by a constant (i.e. multiplying the fraction by k/k).

We rearrange the equations to solve for c , and equate them to one another such that in (2)

$$a^2 = k \cdot \frac{(a \cdot (4b-1) - bp)}{p}, \quad (4)$$

and in (3)

$$a^2 = k \cdot (a \cdot (4b-1) - bp). \quad (5)$$

We consider the above in terms of the function $f(a)$, a line of the form $f(x) = Ax + B$, such that $a^2 = k \cdot f(a)$. It is clear that for a given p , the line will have a “fix-point” at $f\left(\frac{p}{4}\right) = -\frac{1}{4}$ for (4) and at $f\left(\frac{p}{4}\right) = -\frac{p}{4}$ for (5), irrespective of the parameter b , which determines its slope.

Since we wish to know whether for a given p there exists some positive integer a such that either of (4) or (5) is satisfied, it is sufficient to know whether the line to $f(a)$ from the fix-point has a positive integer slope. We use this fix-point to show in the case of (2) that

$$\frac{\frac{a^2}{k} + \frac{1}{4}}{a - \frac{p}{4}} = \frac{4a^2 + k}{k \cdot (4a - p)},$$

and in the case of (3) that

$$\frac{\frac{a^2}{k} + \frac{p}{4}}{a - \frac{p}{4}} = \frac{4a^2 + kp}{k \cdot (4a - p)}.$$

We set $a^2 = k \cdot d$, which leaves $\frac{4d+1}{4a-p}$ and $\frac{4d+p}{4a-p}$, respectively.

Given the above, we consider the general case for positive integer Q such that $4a - Q$ divides either $4d + 1$ or $4d + Q$, where $Q < 4a$.

We call the result of either fraction m and solve for Q , then

$$\begin{aligned}\frac{4d+1}{4a-Q} &= m \\ Q &= \frac{4(am-d)-1}{m}\end{aligned}\tag{6}$$

and

$$\begin{aligned}\frac{4d+Q}{4a-Q} &= m \\ Q &= \frac{4(am-d)}{m+1}.\end{aligned}\tag{7}$$

In the case of (6) we see that $4(am-d)-1$ is odd, so m must be odd to divide it; i.e. for Q to be an integer, which also is odd. In the case of (7), however, m might be odd or even depending on whether $m+1$ is a divisor of the even number $4(am-d)$.

We restrict m such that $am > d$ for the appropriate values of a, d .

Suppose that a is fixed. We enumerate the values of d , and the corresponding m and Q for both (6) and (7), where $a = 1, 2, \dots, 5$. The largest values of m and Q for each d are emphasized.

For $a = 1$, in the case of (6) and (7), respectively:

d	1
m	5
Q	3

d	1	
m	3	7
Q	2	3

For $a = 2$:

d	1	2	4		
m	1	5	3	9	17
Q	3	7	5	7	7

d	1	2	4			
m	2, 3	11	7	15	3, 5, 7, 11	23
Q	4, 5	7	6	7	2, 4, 5, 6	7

For $a = 3$:

d	1	3	9	
m	1	5	13	37
Q	7	11	11	11

d	1	3	9			
m	3, 7	15	2, 3, 5, 7, 11	23	5, 7, 11, 15, 23	47
Q	8, 10	11	4, 6, 8, 9, 10	11	4, 6, 8, 9, 10	11

For $a = 4$:

d	1		2		4	8		16	
m	1	5	1, 3	9	17	3, 11	33	5, 13	65
Q	11	15	7, 13	15	15	5, 13	15	3, 11	15

d	1		2		4		8		16		
m	3, 4, 9		19	2, 5, 7, 11	23	3, 7, 15	31	5, 7, 15, 23	47	7, 9, 15, 19, 39	79
Q	11, 12, 14		15	8, 12, 13, 14	15	8, 12, 14	15	8, 10, 13, 14	15	6, 8, 11, 12, 14	15

For $a = 5$:

d	1		5		25
m	1	5	3, 7	21	101
Q	15	19	13, 17	19	19

d	1		5		25		
m	2, 3, 7, 11		23	4, 9, 19	39	7, 9, 11, 14, 19, 23, 29, 39, 59	119
Q	12, 14, 17, 18		19	12, 16, 18	19	5, 8, 10, 12, 14, 15, 16, 17, 18	19

In the case of (6), for each a and $d = 1$ we see that the largest $m = 5$. The difference between consecutive largest m is $4(d_{i+1} - d_i)$.

In the case of (7), for each a and $d = 1$ we see that the largest $m = 3 + 4a$. The difference between consecutive largest m is also $4(d_{i+1} - d_i)$.

The largest $Q = 4a - 1$, in either case.

We write the largest m for each d_i as $M_{1,i}$ in the case of (6), and, as $M_{2,i}$ in the case of (7). We wish to characterize the smaller values of m for each d_i in terms of these “maxima”, such that

$$Q - R = \frac{4(a \cdot (M_{1,i} - r) - d_i) - 1}{M_{1,i} - r}$$

in the case of (6), and

$$Q - R = \frac{4(a \cdot (M_{2,i} - r) - d_i)}{M_{2,i} - r + 1}$$

in the case of (7) where the smaller values of m are $M_{1/2,i} - r$.

These maxima can be computed with our observations above as

$$\begin{aligned} M_{1,i} &= 1 + 4 \left(1 + \sum_{1 \leq j \leq i-1}^{i-1} (d_{j+1} - d_j) \right) \\ &= 1 + 4d_i \end{aligned} \tag{8}$$

and

$$\begin{aligned} M_{2,i} &= 3 + 4 \left(a + \sum_{1 \leq j \leq i-1}^{i-1} (d_{j+1} - d_j) \right) \\ &= 3 + 4(a + d_i - 1). \end{aligned} \tag{9}$$

We show that this is indeed the case by substituting (8) into (6), such that

$$\begin{aligned} Q &= \frac{4(a \cdot (1 + 4d_i) - d_i) - 1}{1 + 4d_i} \\ &= 4a - 1 \end{aligned}$$

and by substituting (9) into (7), such that

$$\begin{aligned} Q &= \frac{4(a \cdot (3 + 4(a + d_i - 1)) - d_i)}{3 + 4(a + d_i - 1) + 1} \\ &= 4a - 1 \end{aligned}$$

which is what we expect.

We ignore, for the moment, the maximal case of m , and consider the general case:

For the first case, we substitute (6) for Q and solve for r , which yields

$$\begin{aligned} Q - R &= \frac{4(a \cdot (m - r) - d) - 1}{m - r} \\ r &= \frac{m^2 R}{4d + mR + 1}. \end{aligned} \tag{10}$$

For the second case, we substitute (7) for Q and solve for r , which yields

$$\begin{aligned} Q - R &= \frac{4(a \cdot (m - r) - d)}{m - r + 1} \\ r &= \frac{(m + 1)^2 \cdot R}{4(a + d) + (m + 1) \cdot R}. \end{aligned} \tag{11}$$

We recall that the largest Q for each a , irrespective of the divisor d , is $4a - 1$. Thus, we have a clear case in which there is a solution to (1) for each $a \geq 1$, which accounts for the positive integers $Q \equiv 3 \pmod{4}$ and therefore the primes $p \equiv 3 \pmod{4}$.

In general, the corresponding values in (1) for b, c can be recovered by considering (4) and (5), which we restate here:

$$a^2 = k \cdot \frac{(a \cdot (4b - 1) - pb)}{p},$$

$$a^2 = k \cdot (a \cdot (4b - 1) - pb).$$

This can be rewritten to facilitate recovery, such that

$$\begin{aligned} a \cdot (4b - 1) - pb &= 4ab - a - pb \\ &= b \cdot (4a - p) - a. \end{aligned}$$

In the context of (4) and (6), we see that $d = \frac{b \cdot (4a - Q) - a}{Q}$.

In the context of (5) and (7), we see that $d = b \cdot (4a - Q) - a$.

Thus, given some $a, d, Q = p$, we can recover b by

$$b = \frac{dp + a}{4a - p}$$

in the case of (6), and

$$b = \frac{d + a}{4a - p}$$

in the case of (7). We recover $k = \frac{a^2}{dp}$ or $k = \frac{a^2}{d}$, from the restatements above, and $c = \frac{k \cdot bp}{a}$, from (2) and (3).

We wish to show that there exists also at least one $Q \equiv 1 \pmod{4}$ for each a and therefore for each prime $p \equiv 1 \pmod{4}$. In fact, we wish to show that at least one of either

$$4a - 3 = \frac{4(a \cdot (M_{1,i} - r) - d_i) - 1}{M_{1,i} - r}$$

or

$$4a - 3 = \frac{4(a \cdot (M_{2,i} - r) - d_i)}{M_{2,i} - r + 1}$$

is satisfied for a positive integer r . We know how to express r in terms of a, d, m, R , and therefore, want to find a solution to either (10) or (11) for some $a, d = d_i, m = M_{1/2,i}, R = 2$.

In the case of (10), then

$$r = \frac{2M_{1,i}^2}{4d_i + 2M_{1,i} + 1}.$$

We substitute (8) for $M_{1,i}$ such that

$$\begin{aligned} r &= \frac{2(1 + 4d_i)^2}{4d_i + 2(1 + 4d_i) + 1} \\ &= \frac{2(1 + 4d_i)}{3}. \end{aligned} \tag{12}$$

In the case of (11), then

$$r = \frac{2(M_{2,i} + 1)^2}{4(a + d_i) + 2(M_{2,i} + 1)}.$$

We substitute (9) for $M_{2,i}$ such that

$$\begin{aligned} r &= \frac{2(3 + 4(a + d_i - 1) + 1)^2}{4(a + d_i) + 2(3 + 4(a + d_i - 1) + 1)} \\ &= \frac{8(a + d_i)}{3}. \end{aligned} \tag{13}$$

We consider the two possible cases:

1. For (12) we have that $Q = 4a - 3$ when at least one of the divisors of a^2 is congruent to 2 modulo 3, since $4(3x - 1) + 1 = 3(4x - 1)$ for any x .
2. For (13) the sum of a and one of the divisors of a^2 is divisible by 3.
 - If $a \equiv 1 \pmod{3}$, then $a^2 \equiv 1 \pmod{3}$, since $(3x - 2)^2 = 3(3x^2 - 4x - 1) - 1$ for any x .
 - If $a \equiv 2 \pmod{3}$, then $a^2 \equiv 1 \pmod{3}$, since $(3x - 1)^2 = 3(3x^2 - 2x + 1) - 2$ for any x .
 - If $a \equiv 0 \pmod{3}$, then $a^2 \equiv 0 \pmod{3}$, for any a .

With these two cases, the second in particular, we can now account for each $Q \equiv 1 \pmod{4}$, and therefore we prove the Erdős-Straus conjecture.

QED.