What is the radius of the little circle?

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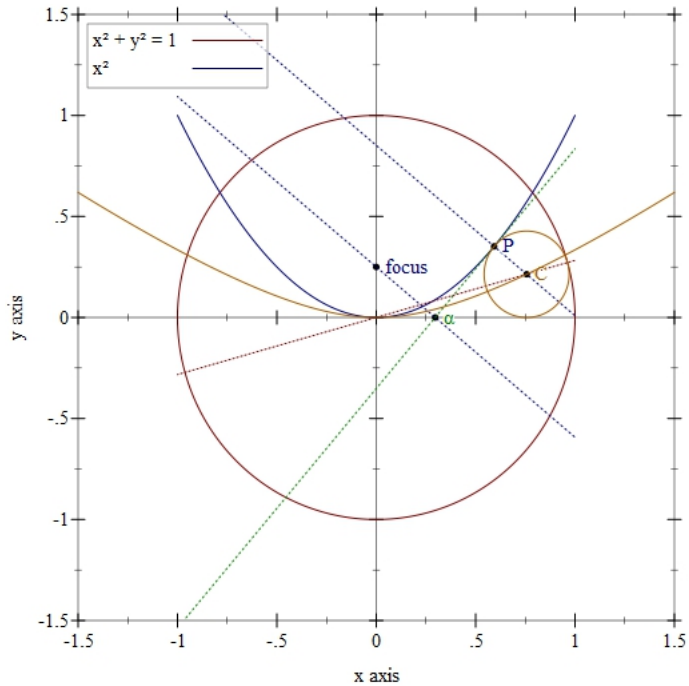
We are presented with a unit circle and the parabola x^2, and are required to find the radius of the largest circle which fits below the parabola inside of the unit circle while touching the x-axis, which we'll refer to as the "little circle".



The line going through the focus, $\left(0, \frac{1}{4}\right)$, of the parabola x^2 and a point $(\alpha, 0)$ on the axis is given by

$$f(x) = (\alpha - x)(4\alpha)^{-1}.$$

The line perpendicular to $f(x)$ which is tangent to the parabola is given by

$$g(x) = (x - \alpha)(4\alpha)^{+1}.$$

We calculate the point of intersection between x^2 and $g(x)$, which point we call P. We solve for its x coordinate,

$$\begin{aligned} x^2 &= g(x) \\ &= (x-\alpha)(4\alpha) \\ x^2 - 4\alpha x + 4\alpha^2 &= 0, \end{aligned}$$

where

$$\begin{aligned} x &= \frac{4\alpha \pm \sqrt{16\alpha^2 - 16\alpha^2}}{2} \\ &= 2\alpha. \end{aligned}$$

Thus, we have $P = \left(2\alpha, 4\alpha^2\right)$.

The line perpendicular to $g(x)$ which intersects the center of the little circle is given by

$$h(x) = f(x-\alpha) + 4\alpha^2,$$

because we know that $f(x)$ and $h(x)$ are parallel, and the point which intersects the parabola and the little circle is $P = \left(2\alpha, 4\alpha^2\right)$, essentially moving $f(x)$ right by α and up by $4\alpha^2$.

Now, we know that the center of the little circle will lie on $C = (2\alpha + \Delta, r)$ where $\Delta > 0$ and r is the radius of the little circle.

From the above, we know that the change in x from P to C is Δ and the change in y is $s = h(2\alpha) - h(2\alpha + \Delta)$. Furthermore, we know that $r^2 = \Delta^2 + s^2$, and $r = 4\alpha^2 - s$.

$$\begin{aligned} r^2 &= \Delta^2 + s^2 \\ \left(4\alpha^2 - s\right)^2 &= \Delta^2 + s^2 \\ 16\alpha^4 - 8\alpha^2 s + s^2 &= \Delta^2 + s^2 \\ 16\alpha^4 - 8\alpha^2 s &= \Delta^2. \end{aligned}$$

We see that $s = f(\alpha) - f(\alpha + \Delta) = \dfrac{\Delta}{4\alpha}$ which we substitute into the above such that

$$\begin{aligned} 16\alpha^4 - 8\alpha^2 \frac{\Delta}{4\alpha} &= \Delta^2 \\ 16\alpha^4 - 2\alpha\Delta - \Delta^2 &= 0. \end{aligned}$$

We solve for Δ using the formula for quadratic roots,

$$\begin{aligned} \Delta &= \frac{-2\alpha \pm \sqrt{4\alpha^2 + 64\alpha^4}}{2} \\ &= \frac{-2\alpha \pm 2\alpha\sqrt{16\alpha^2 + 1}}{2} \\ &= \pm\alpha\sqrt{16\alpha^2 + 1} - \alpha. \end{aligned}$$

We know that $\Delta > 0$, so we choose the root where $\Delta = \alpha\left(\sqrt{16\alpha^2+1}-1\right)$.

We know that the line which intersects C and the perimeter of the unit circle also intersects the origin, because the little circle must be tangent to the unit circle. This line is given by

$$k(x) = \frac{r}{2\alpha+\Delta}x.$$

We wish to know where $k(x)$ intersects the unit circle, the point (u, v).

$$\begin{aligned}
k(u)^2 &= 1-u^2 \\
\frac{r^2}{(2\alpha+\Delta)^2}u^2 &= 1-u^2 \\
u^2 &= \frac{(2\alpha+\Delta)^2}{r^2+(2\alpha+\Delta)^2} \\
u &= \frac{2\alpha+\Delta}{\sqrt{r^2+(2\alpha+\Delta)^2}}.
\end{aligned}$$

It follows that v can be computed as

$$\begin{aligned}
v^2 + \frac{(2\alpha+\Delta)^2}{r^2+(2\alpha+\Delta)^2} &= 1 \\
v &= \frac{r}{\sqrt{r^2+(2\alpha+\Delta)^2}}.
\end{aligned}$$

Using the trigonometric identities for sin and cos, we have

$$\begin{aligned}
\sin &= \frac{\text{opposite}}{\text{hypotenuse}} \\
&= \frac{r}{\sqrt{r^2+(2\alpha+\Delta)^2}}, \\
\cos &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
&= \frac{2\alpha+\Delta}{\sqrt{r^2+(2\alpha+\Delta)^2}},
\end{aligned}$$

because the radius of the unit circle is 1, so the hypotenuse must be 1.

The distance of the line from the origin to C, is given by $d = \sqrt{r^2 + (2\alpha + \Delta)^2}$, and we know that the total distance of the line from the origin to the perimeter of the unit circle is 1, therefore $d = 1 - r$.

$$\begin{aligned} \sin &= \frac{r}{1-r}, \\ \cos &= \frac{2\alpha + \Delta}{1-r}. \end{aligned}$$

We employ the trigonometric identity, $\sin^2 + \cos^2 = 1$, and write

$$\begin{aligned} \left(\frac{r}{1-r}\right)^2 + \left(\frac{2\alpha+\Delta}{1-r}\right)^2 &= 1 \\ \frac{r^2 + (2\alpha+\Delta)^2}{(1-r)^2} &= 1 \\ r^2 + (2\alpha+\Delta)^2 &= (1-r)^2 \\ (2\alpha+\Delta)^2 &= 1 - 2r \\ 2r &= 1 - (2\alpha+\Delta)^2. \end{aligned}$$

Next, we substitute $r = 4\alpha^2 - \dfrac{\Delta}{4\alpha}$ such that

$$\begin{aligned} 8\alpha^2 - \frac{\Delta}{2\alpha} &= 1 - 4\alpha^2 - 4\alpha\Delta - \Delta^2 \\ 0 &= 1 - 12\alpha^2 - 4\alpha\Delta + \frac{\Delta}{2\alpha} - \Delta^2 \\ &= 2\alpha - 24\alpha^3 - 8\alpha^2\Delta + \Delta - 2\alpha\Delta^2. \end{aligned}$$

Likewise we substitute $\Delta = \alpha\left(\sqrt{16\alpha^2+1} - 1\right)$, and simplify such that

$$\begin{aligned} 2\alpha - 24\alpha^3 - 8\alpha^3\left(\sqrt{16\alpha^2+1}-1\right) + \alpha\left(\sqrt{16\alpha^2+1}-1\right) - 2\alpha^3\left(\sqrt{16\alpha^2+1}-1\right)^2 &= 0 \\ 1 - 20\alpha^2 - 4\alpha^2\sqrt{16\alpha^2+1} + \sqrt{16\alpha^2+1} - 32\alpha^4 &= 0 \\ 32\alpha^4 + 20\alpha^2 - 1 &= \sqrt{16\alpha^2+1}\left(1 - 4\alpha^2\right) \\ \left(32\alpha^4 + 20\alpha^2 - 1\right)^2 &= \left(16\alpha^2+1\right)\left(1-4\alpha^2\right)^2 \\ 1024\alpha^8 + 1280\alpha^6 + 336\alpha^4 - 40\alpha^2 + 1 &= 256\alpha^6 - 112\alpha^4 + 8\alpha^2 + 1 \\ 64\alpha^6 + 64\alpha^4 + 28\alpha^2 - 3 &= 0. \end{aligned}$$

We find that the positive real root of the polynomial yields $\alpha \approx 0.29651428340842...$.

From this we may calculate that the radius of the little circle is $r \approx 0.2138417792353557...$.