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Numerical Methods for Differential Equations

Assignment 11

Upload solutions until 18 July 2022, 3pm

Exercise 11.1 (Transport equation) **(2+4+4+6+4 = 20 points)**

Consider the (linear) transport equation, which is given in one space dimension for a constant parameter $a > 0$ as

$$a\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = 0. \tag{1}$$

Let $\Omega \subseteq \mathbb{R}$ denote the space domain and let $(0,T) \subset \mathbb{R}$ denote the time interval. One can formulate the following initial value problem as

$$\begin{cases} \forall (x,t) \in \Omega \times (0,T): & a\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = 0 \\ \forall x \in \Omega: & u(x,0) = g(x) \\ \forall t \in (0,T): & u(0,t) = 1 \end{cases}.$$

This exercise aims to compute a numerical solution of the differential equation. In order to do so, one starts with replacing the differential quotients in (1) by difference quotients. Then, a discrete version of (1) is then given as

$$aD_x u(t,x) + D_t u(t,x) = 0 \tag{2}$$

with the difference quotients

$$D_x u = \frac{u_k^n - u_{k-1}^n}{\Delta x}, \quad D_t u = \frac{u_k^{n+1} - u_k^n}{\Delta t}.$$

Here, Δx denotes a given stepsize in space and Δt a stepsize in time respectively. Note that u_k^n is the numerical approximation of u at the n-th step in time and k-th step in space.

(a) Rearrange (2) such that the term u_k^{n+1} is on the left hand side of the equality sign and everything else on the right hand side.

(b) Use this equation in order to set up an equation (i.e. compute a matrix A) of the form

$$u^{n+1} = Au^n, \tag{3}$$

where $u^n = (u_1^n, ..., u_N^n)^T \in \mathbb{R}^N$ denotes the vector with the values which the numerical approximation $u(t,x)$ takes during the n-th time step on the grid points $\{x_1, ..., x_N\}$. Pay special attention on how to compute the values on the left spacial boundary, i.e. u_1^n.

Step (b) opens the possibility to compute the solution $u(t,x)$ on the discrete grid time step by time step. If u^n is given, we may compute the solution for the next time step u^{n+1} by solving the system (3) with the data u^n from the previous step. For the first time step, the right hand side of the equation system is given by $u_0(x)$, evaluated on the grid points $\{x_1, ..., x_N\}$.

Hint: This method to solve a PDE is called 'finite differences'. If you have problems computing the matrix A, have a look in the upcoming chapter in the lecture notes.

(c) Write a `Matlab` function file `generate_matrix(N, dx, dt, a)` which returns the matrix A from (3). Hereby, the input parameter `N` denotes the number of space grid point $\{x_1, ..., x_N\}$.

(d) Write a `Matlab` script `solve_transport_eqn.m` which computes a numerical solution of the transport equation (1) with

$$a = \frac{\pi}{4}, \quad g(x) = \exp\left(-\frac{x^2}{2}\right),$$
$$\Omega = [0,5], \quad T = 4, \quad \Delta x = \Delta t = \frac{1}{10}.$$

Mind that the first and last node of the equidistant grid should equal the interval boundaries, i.e. $0 = x_1 < ... < x_N = 5$.
Store the vectors u^n from each time step as columns in a matrix, i.e.

$$U = \begin{bmatrix} | & | & \\ u^1, & u^2, & ... \\ | & | & \end{bmatrix}.$$

Hereby, the first vector u^1 corresponds to the evaluation of $g(x)$ on the grid $\{x_1, ..., x_N\}$. The last vector should represent the numerical solution at $T = 4$.

(e) Create a plot which animates the numerical solution for increasing time $0 \leq t \leq 4$, i.e. each frame of the animation is supposed to depict the solution of one time step.

Notice: Hand in your solutions for (a) and (b) in a separate file in .pdf/.jpg/.jpeg/.png/.txt format (i.e. .pdf / image files if you take a photo/scan of your handwritten solutions; .txt files if you want to type you solution.)