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Numerical Methods for Differential Equations

Assignment 7

Upload solutions until 20 June 2022, 3pm

Exercise 7.1 (Newton's method) (6+6+6+2=20 points)

(a) Write a `Matlab` function file `newton2d` which takes as input a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, its Jacobian ∇F, an initial guess $\mathbf{x}_0 \in \mathbb{R}^2$ and an optional string `output`. If `output` equals `"full"`, `newton2d` should return a matrix `X`. The columns of `X` are supposed to be the vectors which occur after each application of the Newton iteration rule. In this case the last column of `X` is the computed solution of the equation $F(\mathbf{x}) = \mathbf{0}$. If `output` does not equal `"full"` or is left away, `newton2d` should return the numerical solution of $F(\mathbf{x}) = \mathbf{0}$ only.
In both cases use the following convergence criteria:

$$\frac{||\mathbf{x}_{k+1} - \mathbf{x}_k||_2}{||\mathbf{x}_k||_2} \le 10^{-6}$$

(b) Now, use `newton2d` to solve the equation

$$F(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} = \mathbf{0} \tag{1}$$

with

$$\begin{aligned} f_1(\mathbf{x}) = f_1(x_1, x_2) &= 6x_1 - \cos x_1 - 2x_2, \\ f_2(\mathbf{x}) = f_2(x_1, x_2) &= 8x_2 - x_1 x_2^2 - \sin x_1 \end{aligned}$$

and the initial guess $\mathbf{x}_0 = (0, 0)$. (For this part of the exercise, you may leave the `output` parameter away). Verify whether the solution obtained by your implementation of the Newton method is a solution for the equation.

(c) Use `newton2d` again to solve a system of equations as (1), this time for

$$\begin{aligned} f_1(\mathbf{x}) = f_1(x_1, x_2) &= \exp(-\exp(-(x_1 + x_2))) - x_2(1 + x_1^2), \\ f_2(\mathbf{x}) = f_2(x_1, x_2) &= x_1 \cos(x_2) + x_2 \sin(x_1) - \frac{1}{2} \end{aligned}$$

and the initial guess $\mathbf{x}_0 = (0, 0)$. Set `output` to full in order to get the matrix whose columns represent the waypoints on the path which the Newton method takes in order to get closer to a solution of (1). Plot the described path on a suitable clipping of $\mathbb{R}^2$. Connect the single waypoints (vectors) by straight lines.

(d) Can you find an initial guess $\mathbf{x}_0$ for which the Newton method for the problem in (c) diverges?