Green's Identities and Green's Functions

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Abstract

This project delves into the fundamental concepts of Green's Identities and Green's Functions, which are powerful tools in the analysis and solution of partial differential equations (PDEs). We will explore the theoretical underpinnings of Green's Identities, derived from the divergence theorem, and subsequently introduce Green's Functions as particular solutions to non-homogeneous boundary value problems. Emphasis will be placed on their application in solving various types of PDEs, including Laplace, Poisson, and wave equations, with a focus on understanding their physical interpretations and mathematical constructions. Numerous solved examples will illustrate the practical utility of these concepts in diverse physical and engineering contexts.

Partial Differential Equations (PDEs) are indispensable mathematical tools for describing a vast array of physical phenomena, from heat conduction and fluid dynamics to electromagnetism and quantum mechanics. Their omnipresence in science and engineering stems from their ability to capture continuous changes in systems governed by rates of change in space and time. However, finding analytical solutions to PDEs, especially for complex geometries or with non-homogeneous terms and boundary conditions, often presents significant challenges.

This graduation project aims to explore two exceptionally powerful concepts that greatly simplify the analysis and solution of many PDE problems: **Green's Identities** and **Green's Functions**. Green's Identities are fundamental integral theorems derived from the Divergence Theorem, providing crucial relationships between volume and surface integrals of functions and their derivatives. These identities lay the groundwork for understanding the behavior of solutions to PDEs and for deriving uniqueness theorems.

Building upon Green's Identities, **Green's Functions** emerge as a particular type of solution to non-homogeneous PDEs with a point source (represented by the Dirac delta function). They encapsulate the fundamental response of a system to a localized disturbance and, by superposition, allow us to construct solutions for arbitrary source distributions and boundary conditions. This project will systematically develop these concepts, starting from their theoretical foundations, moving through their construction for various operators, and finally demonstrating their practical utility through a series of illustrative solved examples. Our exploration will primarily draw upon the principles outlined in "Partial Differential Equations for Scientists and Engineers" by Stanley J. Farlow and "Partial Differential Equations: An Introduction" by Walter A. Strauss.

Before diving into Green's Identities, it is essential to recall some fundamental concepts from vector calculus, particularly the **Divergence Theorem**, which forms the bedrock for their derivation.

1. Divergence Theorem (Gauss's Theorem)

The Divergence Theorem, also known as Gauss's Theorem, relates the flux of a vector field through a closed surface to the divergence of the field within the volume enclosed by that surface.

Theorem 1.1. (Divergence Theorem) *Let V be a bounded region in $\mathbb{R}^3$ with a piecewise smooth boundary surface ∂V. Let $\mathbf{F}(\mathbf{x})$ be a continuously differentiable vector field defined on $V \cup \partial V$. Then,*

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS \tag{1}$$

where $\mathbf{n}$ is the outward unit normal vector to the surface ∂V, and dS is the surface area element.

This theorem states that the total outward flux of a vector field from a closed surface is equal to the integral of the divergence of the field over the volume enclosed by that surface. It can be seen as a higher-dimensional generalization of the Fundamental Theorem of Calculus.

2. Gradient, Divergence, and Curl Operators

For a scalar function $\phi(\mathbf{x})$ and a vector field $\mathbf{F}(\mathbf{x}) = (F_1, F_2, F_3)$, the key differential operators are:

Green's Identities are a set of three fundamental integral identities that are derived from the Divergence Theorem. They are particularly useful in the study of partial differential equations, especially those involving the Laplace operator.

3. Green's First Identity

Green's First Identity is derived by applying the Divergence Theorem to a specific vector field.

Theorem 3.1. (Green's First Identity) *Let $u(\mathbf{x})$ and $v(\mathbf{x})$ be two scalar functions that are twice continuously differentiable in a bounded region $V \subset \mathbb{R}^3$ with a smooth boundary ∂V. Then,*

$$\iiint_V (\nabla u \cdot \nabla v + u\nabla^2 v)\, dV = \iint_{\partial V} u\frac{\partial v}{\partial n}\, dS \tag{2}$$

where $\frac{\partial v}{\partial n} = \nabla v \cdot \mathbf{n}$ is the normal derivative of v on the boundary ∂V.

Proof. Consider the vector field $\mathbf{F} = u\nabla v$. We can compute its divergence using the product rule for divergence:

$$\nabla \cdot (u\nabla v) = (\nabla u) \cdot (\nabla v) + u(\nabla \cdot \nabla v)$$

Since $\nabla \cdot \nabla v = \nabla^2 v$ (the Laplacian of v), we have:

$$\nabla \cdot (u\nabla v) = \nabla u \cdot \nabla v + u\nabla^2 v$$

Now, apply the Divergence Theorem to the vector field $\mathbf{F} = u\nabla v$:

$$\iiint_V \nabla \cdot (u\nabla v)\, dV = \iint_{\partial V} (u\nabla v) \cdot \mathbf{n}\, dS$$

Substituting the expression for $\nabla \cdot (u\nabla v)$ on the left side:

$$\iiint_V (\nabla u \cdot \nabla v + u\nabla^2 v)\, dV = \iint_{\partial V} u(\nabla v \cdot \mathbf{n})\, dS$$

Recognizing that $\nabla v \cdot \mathbf{n} = \frac{\partial v}{\partial n}$, we obtain Green's First Identity:

$$\iiint_V (\nabla u \cdot \nabla v + u\nabla^2 v)\, dV = \iint_{\partial V} u\frac{\partial v}{\partial n}\, dS$$ □

4. Green's Second Identity (Symmetry Property)

Green's Second Identity is obtained by subtracting Green's First Identity with the roles of u and v swapped. This identity highlights a crucial symmetry property.

Theorem 4.1. (Green's Second Identity) *Let $u(\mathbf{x})$ and $v(\mathbf{x})$ be two scalar functions that are twice continuously differentiable in a bounded region $V \subset \mathbb{R}^3$ with a smooth boundary ∂V. Then,*

$$\iiint_V (u\nabla^2 v - v\nabla^2 u)\, dV = \iint_{\partial V} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{3}$$

Proof. From Green's First Identity, we have:

$$\iiint_V (\nabla u \cdot \nabla v + u\nabla^2 v)\, dV = \iint_{\partial V} u \frac{\partial v}{\partial n}\, dS \tag{4}$$

Now, swap the roles of u and v in equation (4):

$$\iiint_V (\nabla v \cdot \nabla u + v\nabla^2 u)\, dV = \iint_{\partial V} v \frac{\partial u}{\partial n}\, dS \tag{5}$$

Since $\nabla u \cdot \nabla v = \nabla v \cdot \nabla u$, subtract equation (5) from equation (4):

$$\iiint_V [(u\nabla^2 v + \nabla u \cdot \nabla v) - (v\nabla^2 u + \nabla v \cdot \nabla u)]\, dV = \iint_{\partial V} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

This simplifies to Green's Second Identity:

$\iiint_V (u\nabla^2 v - v\nabla^2 u)\, dV = \iint_{\partial V} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$ ▯

5. Green's Third Identity

Green's Third Identity is a special case of Green's Second Identity where one of the functions, usually v, is chosen to be the **free-space Green's function** for the Laplacian. This identity provides a formula to express the value of a function inside a domain based on its boundary values and source terms.

Definition 5.2. (Free-Space Green's Function for Laplace's Equation) *The free-space Green's function $G_0(\mathbf{x}, \mathbf{x}')$ for the Laplace operator in $\mathbb{R}^3$ satisfies*

$$\nabla^2 G_0(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function, representing a point source at $\mathbf{x}'$. In $\mathbb{R}^3$, $G_0(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$.

If we let $v = G_0(\mathbf{x}, \mathbf{x}')$ in Green's Second Identity and assume $u(\mathbf{x})$ is a function satisfying $\nabla^2 u = f(\mathbf{x})$ in V, we need to carefully handle the singularity of G_0 at $\mathbf{x} = \mathbf{x}'$. This is done by excluding a small sphere V_ϵ of radius ϵ around $\mathbf{x}'$ from the domain V. The result is Green's Third Identity:

Theorem 5.3. (Green's Third Identity) *Let $u(\mathbf{x})$ be a twice continuously differentiable function in V satisfying $\nabla^2 u = f(\mathbf{x})$, and let $G_0(\mathbf{x}, \mathbf{x}')$ be the free-space Green's function. Then for any point $\mathbf{x}' \in V$,*

$$u(\mathbf{x}') = \iiint_V G_0(\mathbf{x}, \mathbf{x}') f(\mathbf{x})\, dV + \iint_{\partial V} \left(u(\mathbf{x})\frac{\partial G_0(\mathbf{x}, \mathbf{x}')}{\partial n} - G_0(\mathbf{x}, \mathbf{x}')\frac{\partial u(\mathbf{x})}{\partial n}\right) dS \tag{6}$$

If $u(\mathbf{x})$ is a harmonic function (i.e., $\nabla^2 u = 0$ in V), the volume integral term vanishes:

$$u(\mathbf{x}') = \iint_{\partial V} \left(u(\mathbf{x})\frac{\partial G_0(\mathbf{x}, \mathbf{x}')}{\partial n} - G_0(\mathbf{x}, \mathbf{x}')\frac{\partial u(\mathbf{x})}{\partial n}\right) dS \tag{7}$$

This identity is a representation formula, stating that the value of u at an interior point $\mathbf{x}'$ is determined by the source term f and the values of u and its normal derivative $\frac{\partial u}{\partial n}$ on the boundary.

6. Physical Interpretation and Applications

Green's Identities provide fundamental insights into the behavior of solutions to PDEs.

Example 6.4. (Using Green's First Identity to prove uniqueness for Laplace's equation) ***Problem:*** *Prove that the solution to Laplace's equation, $\nabla^2 u = 0$, in a bounded region V with Dirichlet boundary condition $u = g$ on ∂V is unique.*

Solution:

Assume there are two solutions, $u_1(\mathbf{x})$ and $u_2(\mathbf{x})$, that both satisfy:

$$\nabla^2 u_1 = 0 \text{ in } V, u_1 = g \text{ on } \partial V$$

$$\nabla^2 u_2 = 0 \text{ in } V, u_2 = g \text{ on } \partial V$$

Let $w(\mathbf{x}) = u_1(\mathbf{x}) - u_2(\mathbf{x})$. Then, $w(\mathbf{x})$ satisfies:

$$\nabla^2 w = \nabla^2(u_1 - u_2) = \nabla^2 u_1 - \nabla^2 u_2 = 0 - 0 = 0 \text{ in } V$$

And on the boundary:

$$w = u_1 - u_2 = g - g = 0 \text{ on } \partial V$$

Now, apply Green's First Identity with both u and v replaced by w:

$$\iiint_V (\nabla w \cdot \nabla w + w\nabla^2 w)\, dV = \iint_{\partial V} w \frac{\partial w}{\partial n}\, dS$$

Substituting $\nabla^2 w = 0$ and $w = 0$ on ∂V:

$$\iiint_V (|\nabla w|^2 + w \cdot 0)\, dV = \iint_{\partial V} 0 \cdot \frac{\partial w}{\partial n}\, dS$$

This simplifies to:

$$\iiint_V |\nabla w|^2\, dV = 0$$

Since $|\nabla w|^2$ is a non-negative quantity, its integral over a volume can only be zero if $|\nabla w|^2 = 0$ everywhere in V.
This implies $\nabla w = \mathbf{0}$ in V.
If the gradient of w is zero, then w must be a constant throughout the region V.
Since $w = 0$ on the boundary ∂V, this constant must be zero. Therefore, $w(\mathbf{x}) = 0$ for all $\mathbf{x} \in V$.
Thus, $u_1(\mathbf{x}) - u_2(\mathbf{x}) = 0$, which means $u_1(\mathbf{x}) = u_2(\mathbf{x})$. This proves that the solution to Laplace's equation with Dirichlet boundary conditions is unique.

Green's Functions are central to solving linear non-homogeneous differential equations, particularly partial differential equations. They represent the response of a system to an impulse or point source.

7. Definition of Green's Function

For a linear differential operator L, the **Green's Function** $G(\mathbf{x}, \mathbf{x}')$ is defined as the solution to the equation:

$$LG(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{8}$$

subject to specific homogeneous boundary conditions relevant to the problem. Here, $\delta(\mathbf{x} - \mathbf{x}')$ is the **Dirac delta function**, representing an idealized point source at $\mathbf{x}'$.

8. Dirac Delta Function and its Properties

The Dirac delta function $\delta(x)$ is a generalized function (or distribution) characterized by:

Its most important property is the "sifting property":

$$\int_{-\infty}^{\infty} f(x)\delta(x - a)\, dx = f(a) \tag{9}$$

For multiple dimensions, $\delta(\mathbf{x} - \mathbf{x}') = \delta(x - x')\delta(y - y')\delta(z - z')$.

9. Motivation for Green's Functions

Green's functions allow us to construct the solution to a non-homogeneous PDE $Lu = f$ using the principle of superposition. If we know the response to a point source, we can sum (integrate) over all point sources that make up the distributed source $f(\mathbf{x})$.

Recall Green's Second Identity:

$$\iiint_V (u\nabla^2 v - v\nabla^2 u)\, dV = \iint_{\partial V} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\, dS$$

Let $L = \nabla^2$, so we want to solve $\nabla^2 u = f(\mathbf{x})$. Let $v = G(\mathbf{x}, \mathbf{x}')$ be the Green's function for ∇^2, satisfying $\nabla^2 G = \delta(\mathbf{x} - \mathbf{x}')$.
Substituting these into Green's Second Identity:

$$\iiint_V \left(u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}') - G(\mathbf{x},\mathbf{x}')f(\mathbf{x})\right) dV = \iint_{\partial V} \left(u(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n} - G(\mathbf{x},\mathbf{x}')\frac{\partial u(\mathbf{x})}{\partial n}\right) dS$$

Using the sifting property of the Dirac delta function, the first term on the left side becomes $u(\mathbf{x}')$:

$$u(\mathbf{x}') - \iiint_V G(\mathbf{x},\mathbf{x}')f(\mathbf{x})\,dV = \iint_{\partial V} \left(u(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n} - G(\mathbf{x},\mathbf{x}')\frac{\partial u(\mathbf{x})}{\partial n}\right) dS$$

Rearranging, we get the fundamental solution formula:

$$u(\mathbf{x}') = \iiint_V G(\mathbf{x},\mathbf{x}')f(\mathbf{x})\,dV + \iint_{\partial V} \left(G(\mathbf{x},\mathbf{x}')\frac{\partial u(\mathbf{x})}{\partial n} - u(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n}\right) dS \qquad (10)$$

This formula expresses the solution $u(\mathbf{x}')$ at any point $\mathbf{x}'$ inside the domain as a superposition of the effects of the source term $f(\mathbf{x})$ and the boundary conditions.

10. Properties of Green's Functions

Green's functions can be thought of as the inverse of linear differential operators, often describing the effect of a point source (a Dirac delta function) on a system. They play a critical role in solving numerous problems in mathematical physics and engineering.

Fundamental Properties of Green's Functions

A Green's function $G(\mathbf{x},\xi)$ for a linear differential operator L is a function that satisfies the following differential equation:

$$LG(\mathbf{x},\xi) = \delta(\mathbf{x}-\xi)$$

Here, $\delta(\mathbf{x}-\xi)$ is the **Dirac delta function**, indicating a **point source** at position ξ.

To solve a linear, non-homogeneous differential equation of the form $Lu(\mathbf{x}) = f(\mathbf{x})$ for a function $u(\mathbf{x})$, the function $G(\mathbf{x},\xi)$ can be used to express the solution in the following integral form:

$$u(\mathbf{x}) = \int G(\mathbf{x},\xi)f(\xi)\,d\xi$$

This integral is based on the **principle of superposition**: the function $f(\mathbf{x})$ can be thought of as a sum of a series of point sources, and the response to each point source (the Green's function) is summed to obtain the total response.

Here are some important properties of Green's functions:

1. Defining Equation (Source Term)

The Green's function is defined by the differential operator applied to it resulting in the **Dirac delta function**. This shows that the Green's function represents the system's response to a **point impulse**.

$$LG(\mathbf{x}, \xi) = \delta(\mathbf{x} - \xi)$$

2. Satisfies the Homogeneous Equation

The Green's function satisfies the **homogeneous differential equation** in regions where $\mathbf{x} \neq \xi$ (i.e., everywhere except at the source):

$$LG(\mathbf{x}, \xi) = 0 \text{ for } \mathbf{x} \neq \xi$$

3. Boundary Conditions

The Green's function must satisfy the **homogeneous boundary conditions** to which the differential equation being solved is subject. For instance, for Dirichlet boundary conditions, $G(\mathbf{x}, \xi) = 0$ is satisfied on the boundary. This ensures that the boundary conditions are naturally met in the general solution.

4. Continuity and Jump in Derivative

For a second-order differential operator (e.g., Sturm-Liouville operators):

5. Symmetry (Reciprocity Principle)

If the differential operator L is **self-adjoint**, the Green's function exhibits the property of **symmetry**:

$$G(\mathbf{x}, \xi) = G(\xi, \mathbf{x})$$

Physically, this means that the response at point $\mathbf{x}$ due to a point source at ξ is equal to the response at point ξ due to a point source at $\mathbf{x}$. This property is known as the **reciprocity principle**.

6. Uniqueness

In general, a Green's function may **not be unique** if the associated homogeneous equation has non-trivial solutions. However, when appropriate additional conditions (such as radiation conditions or causality) are applied, the Green's function usually becomes unique.

7. Eigenfunction Expansion

If the operator L possesses a **complete set of eigenfunctions** (i.e., the solutions $\psi_n(\mathbf{x})$ of $L\psi_n = \lambda_n \psi_n$ form a basis), the Green's function can be expressed as a series expansion in terms of these eigenfunctions:

$$G(\mathbf{x}, \xi) = \sum_n \frac{\psi_n(\mathbf{x})\psi_n^*(\xi)}{\lambda_n}$$

Here, λ_n are the eigenvalues, and ψ_n^* is the complex conjugate of ψ_n. This expression requires careful consideration for cases where $\lambda_n = 0$.

11. Construction of Green's Functions: Free Space

The simplest type of Green's function to construct is the free-space Green's function, which applies to an infinite domain without boundaries.

Example 11.5. (Finding the free-space Green's function for Laplace's equation in 3D) ***Problem:*** *Find the free-space Green's function $G_0(\mathbf{r}, \mathbf{r}')$ for Laplace's equation in $\mathbb{R}^3$, satisfying $\nabla^2 G_0 = \delta(\mathbf{r} - \mathbf{r}')$.*

Solution:

Let the source point $\mathbf{r}'$ be at the origin for simplicity. Then we are looking for a solution to $\nabla^2 G_0 = \delta(\mathbf{r})$.

Due to the spherical symmetry of the delta function at the origin, we expect G_0 to depend only on the radial distance $r = |\mathbf{r}|$. So, $G_0 = G_0(r)$.

For $r \neq 0$, $\nabla^2 G_0 = 0$. In spherical coordinates, for a spherically symmetric function, the Laplacian simplifies to:

$$\nabla^2 G_0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG_0}{dr}\right) = 0 \text{ for } r \neq 0$$

Integrating once with respect to r:

$$r^2 \frac{dG_0}{dr} = C_1$$

$$\frac{dG_0}{dr} = \frac{C_1}{r^2}$$

Integrating again:

$$G_0(r) = -\frac{C_1}{r} + C_2$$

Now we need to determine C_1 using the singular behavior at the origin. We integrate $\nabla^2 G_0 = \delta(\mathbf{r})$ over a small sphere V_ϵ of radius ϵ centered at the origin:

$$\iiint_{V_\epsilon} \nabla^2 G_0 \, dV = \iiint_{V_\epsilon} \delta(\mathbf{r}) \, dV = 1$$

By the Divergence Theorem:

$$\iiint_{V_\epsilon} \nabla \cdot (\nabla G_0) \, dV = \iint_{\partial V_\epsilon} \nabla G_0 \cdot \mathbf{n} \, dS$$

On the surface of the sphere ∂V_ϵ, $\mathbf{n}$ is in the radial direction, so $\nabla G_0 \cdot \mathbf{n} = \frac{dG_0}{dr}$ evaluated at $r = \epsilon$.

$$\frac{dG_0}{dr} = -\frac{C_1}{r^2}$$

So, at $r = \epsilon$: $\frac{dG_0}{dr} = -\frac{C_1}{\epsilon^2}$.
The surface area of the sphere is $4\pi\epsilon^2$.

$$\iint_{\partial V_\epsilon} \left(-\frac{C_1}{\epsilon^2}\right) dS = \left(-\frac{C_1}{\epsilon^2}\right)(4\pi\epsilon^2) = -4\pi C_1$$

Equating this to the integral of the delta function:

$$-4\pi C_1 = 1 \implies C_1 = -\frac{1}{4\pi}$$

We typically choose $C_2 = 0$ for the free-space Green's function, so that $G_0 \to 0$ as $r \to \infty$.
Therefore, the free-space Green's function in 3D is:

$$G_0(r) = \frac{1}{4\pi r}$$

More generally, with the source at $\mathbf{r}'$:

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{11}$$

In 2D, the free-space Green's function for Laplace's equation is :

$G_0(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}'|$.

When dealing with finite domains, the Green's function must not only satisfy the source condition but also the specified homogeneous boundary conditions. The general form of the Green's function for a region V with boundary ∂V is given by:

$$G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') + H(\mathbf{x}, \mathbf{x}')$$

where $G_0(\mathbf{x}, \mathbf{x}')$ is the free-space Green's function (satisfying $LG_0 = \delta(\mathbf{x} - \mathbf{x}')$), and $H(\mathbf{x}, \mathbf{x}')$ is a harmonic function (i.e., $LH = 0$) chosen such that $G(\mathbf{x}, \mathbf{x}')$ satisfies the desired homogeneous boundary conditions on ∂V.

12. Method of Images

The method of images is a powerful technique for constructing Green's functions for certain simple geometries (e.g., half-spaces, quarter-spaces, spheres) by placing fictitious "image" sources outside the domain. These image sources are chosen such that their combined potential with the original source satisfies the homogeneous boundary conditions.

Example 12.6. (Method of Images for Dirichlet Boundary Conditions in a Half-Space) ***Problem:*** *Find the Green's function for Laplace's equation in the half-space* $z > 0$ *with the Dirichlet boundary condition* $G(\mathbf{x}, \mathbf{x}') = 0$ *on the plane* $z = 0$*. The source point is* $\mathbf{x}' = (x', y', z')$*, where* $z' > 0$.

Solution:

The free-space Green's function in 3D is $G_0(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$.

We want $G(\mathbf{x}, \mathbf{x}') = 0$ *for all* $\mathbf{x}$ *on the plane* $z = 0$.

To satisfy this, we introduce an image source of opposite sign at the reflection of $\mathbf{x}'$ *across the* $z = 0$ *plane. Let the image source be at* $\mathbf{x}'' = (x', y', -z')$.

The Green's function is then proposed as:

$$G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') - G_0(\mathbf{x}, \mathbf{x}'')$$

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} - \frac{1}{4\pi|\mathbf{x}-\mathbf{x}''|}$$

Let's check if this satisfies the conditions:
1. ***Source condition:*** *For* $\mathbf{x} \in V$ *(*$z > 0$*),* $\mathbf{x}''$ *is outside* V*. So,* $\nabla^2 G_0(\mathbf{x}, \mathbf{x}'') = 0$ *for* $\mathbf{x} \in V$.

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \nabla^2 G_0(\mathbf{x}, \mathbf{x}') - \nabla^2 G_0(\mathbf{x}, \mathbf{x}'') = \delta(\mathbf{x}-\mathbf{x}') - 0 = \delta(\mathbf{x}-\mathbf{x}')$$

This is satisfied for $\mathbf{x} \in V$.
2. ***Boundary condition:*** *On the plane* $z = 0$, $\mathbf{x} = (x, y, 0)$.

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{(x-x')^2 + (y-y')^2 + (0-z')^2} = \sqrt{(x-x')^2 + (y-y')^2 + (z')^2}$$

$$|\mathbf{x}-\mathbf{x}''| = \sqrt{(x-x')^2 + (y-y')^2 + (0-(-z'))^2} = \sqrt{(x-x')^2 + (y-y')^2 + (z')^2}$$

Since $|\mathbf{x}-\mathbf{x}'| = |\mathbf{x}-\mathbf{x}''|$ *when* $z = 0$*, we have:*

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} - \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} = 0 \text{ on } z = 0$$

The Dirichlet boundary condition is satisfied.
Thus, the Green's function for the half-space $z > 0$ *with Dirichlet boundary conditions is:*

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}''|}\right)$$

13. Solving Poisson's Equation with Green's Functions

Once the Green's function for a specific domain and boundary conditions is known, the solution to Poisson's equation ($\nabla^2 u = f$) with homogeneous boundary conditions (e.g., $u = 0$ on ∂V) can be directly found using the integral representation.

If $G(\mathbf{x}, \mathbf{x}')$ is the Green's function for ∇^2 with $G = 0$ on ∂V, then the boundary integral terms in the general solution formula vanish (since $u = 0$ on boundary, and $G = 0$ on boundary).

$$u(\mathbf{x}') = \iiint_V G(\mathbf{x}, \mathbf{x}') f(\mathbf{x})\, dV$$

This provides a powerful method to solve Poisson's equation for arbitrary source terms.

Example 13.7. (Solving Poisson's equation with Green's function in a half-space) ***Problem:*** *Solve Poisson's equation $\nabla^2 u = f(\mathbf{x})$ in the half-space $z > 0$ with the boundary condition $u(\mathbf{x}) = 0$ on $z = 0$.*

Solution:

We use the Green's function derived in the previous example:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x} - \mathbf{x}''|}\right)$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{x}' = (x', y', z')$ (the integration variables), and $\mathbf{x}'' = (x', y', -z')$. Note that in the formula, we switch $\mathbf{x}$ and $\mathbf{x}'$ as the integration variables usually take the 'prime' notation, and the solution is evaluated at the 'unprimed' coordinates.
So, the solution $u(\mathbf{x})$ is:

$$u(\mathbf{x}) = \iiint_{V'} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, dV'$$

$$u(\mathbf{x}) = \iiint_{z'>0} \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x} - \mathbf{x}''|}\right) f(x', y', z')\, dx'dy'dz'$$

This integral provides the explicit solution for $u(\mathbf{x})$ for any given source term $f(\mathbf{x}')$.

Green's functions can also be extended to solve time-dependent partial differential equations, such as the heat equation and the wave equation. These are often called **propagators**. For time-dependent problems, the Green's function is typically zero for times $t < t'$ (where t' is the time of the impulse), reflecting causality. Such Green's functions are known as **retarded Green's functions**.

14. Green's Function for the Heat Equation

The one-dimensional heat equation with a point source is:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = \delta(x - x_0)\delta(t - t_0)$$

The free-space Green's function for the 1D heat equation (or fundamental solution) is:

$$G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}} \exp\left(-\frac{(x - x_0)^2}{4k(t - t_0)}\right) H(t - t_0)$$

where $H(t - t_0)$ is the Heaviside step function, ensuring that the effect only occurs after the source is applied. This solution represents the temperature distribution at time t due to an instantaneous point heat source at x_0 at time t_0.

15. Green's Function for the Wave Equation

The one-dimensional wave equation with a point source is:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \delta(x - x_0)\delta(t - t_0)$$

The free-space Green's function for the 1D wave equation is:

$$G(x, t; x_0, t_0) = \frac{1}{2c} H\left(t - t_0 - \frac{|x - x_0|}{c}\right)$$

This Green's function shows that the disturbance propagates at speed c, forming a cone in the $x - t$ plane. The Heaviside function ensures that the effect is felt only when the "light cone" from the source point reaches the observation point.

Green's functions are ubiquitous across various scientific and engineering disciplines due to their ability to provide solutions to problems involving sources and boundaries.

16. Electrostatics

In electrostatics, the electric potential $\phi(\mathbf{x})$ due to a charge density $\rho(\mathbf{x})$ is governed by Poisson's equation:

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

The Green's function for this problem represents the potential due to a point charge. Once the Green's function $G(\mathbf{x}, \mathbf{x}')$ satisfying appropriate boundary conditions is known, the potential can be found as:

$$\phi(\mathbf{x}) = \iiint_V G(\mathbf{x}, \mathbf{x}')\left(-\frac{\rho(\mathbf{x}')}{\epsilon_0}\right) dV' + \text{Boundary Terms}$$

17. Fluid Dynamics

In incompressible potential flow, the velocity potential Φ satisfies Laplace's equation. Green's functions can be used to model flow fields generated by point sources, sinks, or dipoles, which are fundamental building blocks for more complex flow patterns around objects.

18. Quantum Mechanics

In quantum mechanics, Green's functions (often called propagators) are crucial in scattering theory, where they describe the propagation of quantum particles. For example, the Lippmann-Schwinger equation uses Green's functions to describe the scattering of a particle by a potential.

19. Acoustics and Heat Transfer

Similar to electrostatics, Green's functions are used to model sound propagation from point sources in acoustics and temperature distributions from point heat sources in heat transfer problems. The Green's functions for the wave equation and heat equation, respectively, are direct applications.

While powerful, Green's functions are not always easy to construct for arbitrary domains and operators.

20. Numerical Methods for Green's Functions

For problems where analytical Green's functions are intractable, numerical methods are often employed. Techniques like the **Boundary Element Method (BEM)** explicitly utilize the Green's function (often the free-space one) to convert a PDE into an integral equation over the boundary, which can then be discretized and solved numerically. Other methods, like Finite Element Method (FEM), do not directly use Green's functions but solve the PDE directly.

This project has provided a comprehensive overview of Green's Identities and Green's Functions, highlighting their theoretical foundations and practical applications in solving partial differential equations. We began by establishing the groundwork with the

Divergence Theorem and subsequently derived Green's First, Second, and Third Identities, demonstrating their utility in proving uniqueness for solutions to PDEs.

The core of this project centered on Green's Functions, defined as the response of a system to a point source. We showed how the Dirac delta function facilitates their definition and how they can be systematically constructed, starting with the free-space Green's function for the Laplacian. The powerful Method of Images was presented as an elegant technique for handling simple boundary conditions, and the general solution formula for non-homogeneous PDEs was derived. We briefly touched upon their extension to time-dependent problems, showcasing their role as propagators for the heat and wave equations.

The widespread applicability of Green's functions across diverse fields such as electrostatics, fluid dynamics, quantum mechanics, acoustics, and heat transfer underscores their immense significance in mathematical physics and engineering. While analytical construction for complex geometries remains a challenge, the conceptual framework provided by Green's functions often guides the development of numerical solution techniques.

In conclusion, Green's Identities and Green's Functions offer an elegant and systematic approach to understanding and solving linear partial differential equations. They provide a profound physical intuition for the influence of sources and boundaries on a system's behavior, making them indispensable tools for scientists and engineers alike.