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Complex and hypercomplex discrete Fourier transforms based on matrix exponential form of Euler's formula

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ABSTRACT

We show that the discrete complex, and numerous hypercomplex, Fourier transforms defined and used so far by a number of researchers can be unified into a single framework based on a matrix exponential version of Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, and a matrix root of -1 isomorphic to the imaginary root j. The transforms thus defined can be computed using standard matrix multiplications and additions with no hypercomplex code, the complex or hypercomplex algebra being represented by the form of the matrix root of -1, so that the matrix multiplications are equivalent to multiplications in the appropriate algebra. We present examples from the complex, quaternion and biquaternion algebras, and from Clifford algebras $C\ell_{1,1}$ and $C\ell_{2,0}$. The significance of this result is both in the theoretical unification, which permits comparisons between transforms in different hypercomplex algebras, and also in the scope it affords for insight into the structure of the various transforms, since the formulation is such a simple generalization of the classic complex case. It also shows that hypercomplex discrete Fourier transforms may be computed using standard matrix arithmetic packages without the need for a hypercomplex library, which is of importance in providing a reference implementation for verifying faster implementations based on hypercomplex code.

1. Introduction

The discrete Fourier transform is widely known and used in signal and image processing, and in many other fields where data is analyzed for frequency content [1]. In order to define our notation, the discrete Fourier transform in one dimension is classically formulated as:

$$\begin{aligned} F[u] &= S\sum_{m=0}^{M-1} f[m]\exp\left(-j2\pi\frac{mu}{M}\right) \\ f[m] &= T\sum_{u=0}^{M-1} F[u]\exp\left(j2\pi\frac{mu}{M}\right) \end{aligned} \tag{1}$$

where j is the imaginary root of -1, $f[m]$ is a real or complex valued discrete-time signal with M samples, $F[u]$ is complex valued, also with M samples, and the two scale factors S and T must multiply to $1/M$. If the transforms are to be unitary then S must equal T also. In this paper we discuss the formulation of the transform using a matrix exponential form of Euler's

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formula in which the imaginary square root of -1 is replaced by an isomorphic matrix root. This change means that the exponential has a matrix value, and therefore the samples of the signal must be either vectors or matrices conformable with the dimensions of the exponential matrix. This formulation works for the complex DFT, but more importantly, it works for hypercomplex DFTs (reviewed in Section 2). The matrix exponential formulation is equivalent to a large number of the known hypercomplex generalizations of the DFT known to the authors, based on quaternion, biquaternion or Clifford algebras, through a suitable choice of matrix root of -1, isomorphic to a root of -1 in the corresponding hypercomplex algebra. All associative hypercomplex algebras (and indeed the complex algebra) are known to be isomorphic to matrix algebras over the reals or the complex numbers. For example, Ward [2, Section 2.8], discusses isomorphism between the quaternions and 4×4 real or 2×2 complex matrices so that quaternions can be replaced by matrices, the rules of matrix multiplication then being equivalent to the rules of quaternion multiplication *by virtue of the pattern of the elements of the quaternion within the matrix*. Also in the quaternion case, Ickes [3] wrote an important paper showing how multiplication of quaternions could be accomplished using a matrix–vector or vector–matrix product that could accommodate reversal of the product ordering by a partial transposition within the matrix.

The fact that a hypercomplex DFT may be formulated using a matrix exponential may not be surprising. Nevertheless, to our knowledge, those who have worked on hypercomplex DFTs have not so far noted or exploited the observations made in this paper, which is surprising, given the ramifications discussed later.

2. Hypercomplex transforms

The first published descriptions of hypercomplex Fourier transforms that we are aware of date from the 1980s. Sommen, in two papers [4,5], presented hypercomplex generalisations of the classical Fourier and Laplace transforms using Clifford algebras. Later in the same decade (but apparently without knowledge of Sommen's work), Ernst and Delsuc separately published material on quaternion transforms defined for two-dimensional signals (that is functions of two independent variables), with a specific application to nuclear magnetic resonance. The difference in approach between the algebraic work of Sommen and the applied numerical work of Ernst and Delsuc persists to this day, with applied research lagging behind the algebraic theory. The present paper is concerned mainly with the applied numerical aspect of hypercomplex Fourier transforms, but in the discussion in Section 8 we consider some recent work on generalised geometric Fourier transforms [6] and its relation to the ideas presented in this paper.

The formulations of Ernst [7, Section 6.4.2], and Delsuc [8, Eq. 20], are almost equivalent (they differ only in the placing of the exponentials and the signal and the signs inside the exponentials)[1]:

$$F(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2) e^{\mathbf{i}\omega_1 t_1} e^{\mathbf{j}\omega_2 t_2} \, \mathrm{d}t_1 \, \mathrm{d}t_2$$

In a non-commutative algebra the ordering of exponentials within an integral is significant, and of course, two exponentials with different roots of -1 cannot be combined trivially.[2] Therefore there are other possible transforms that can be defined by positioning the exponentials differently. The first transform in which the exponentials were placed either side of the signal function was that of Ell [9,10]:

$$F(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\mathbf{i}\omega_1 t_1} f(t_1, t_2) e^{\mathbf{j}\omega_2 t_2} \, \mathrm{d}t_1 \, \mathrm{d}t_2$$

This style of transform was followed by Chernov [11], Bülow [12], Bülow and Sommer [13] and others since. In 1998 the present authors described a single-sided hypercomplex transform for the first time [14] exactly as in (1) except that f and F were quaternion-valued and j was replaced by a general quaternion root of -1. Expressed in the same form as the transforms above, this would be:

$$F(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\boldsymbol{\mu}(\omega_1 t_1 + \omega_2 t_2)} f(t_1, t_2) \, \mathrm{d}t_1 \, \mathrm{d}t_2$$

where $\boldsymbol{\mu}$ is now an arbitrary root of -1, not necessarily a basis element of the algebra. The realisation that an arbitrary root of -1 could be used meant that it was possible to define a hypercomplex transform applicable to one dimension:

$$F(\omega) = \int_{-\infty}^{\infty} e^{\boldsymbol{\mu}\omega t} f(t) \, \mathrm{d}t$$

Pei et al. have studied efficient implementation of quaternion FFTs and presented a transform based on commutative reduced biquaternions [15,16]. Ebling and Scheuermann defined a Clifford Fourier transform [17, Section 5.2], but their transform used the pseudoscalar (one of the basis elements of the algebra) as the square root of -1.

Apart from works by the present authors [14,18,19], the idea of using a root of -1 different to the basis elements of a hypercomplex algebra was not developed further until 2006, with the publication of a paper setting out the roots of -1

[1] In comparing the various formulations of hypercomplex transforms, we have changed the symbols used by the original authors in order to make the comparisons clearer. We have also made trivial changes such as the choice of basis elements used in the exponentials.

[2] The Baker-Campbell-Hausdorff formula applies to the product of two exponentials with non-commuting exponents.

in biquaternions (a quaternion algebra with complex numbers as the components of the quaternions) [20]. This work prepared the ground for a biquaternion Fourier transform [21] based on the present authors' one-sided quaternion transform [14]. More recently, the idea of finding roots of -1 in other algebras has been advanced in Clifford algebras by Hitzer and Abłamowicz [22,23] with the express intent of using them in Clifford Fourier transforms, perhaps generalising the ideas of Ebling and Scheuermann [17]. Finally, in this very brief summary of prior work we mention that the idea of applying hypercomplex algebras in signal processing has been studied by other authors apart from those referenced above. For an overview see [24].

In what follows we concentrate on DFTs in one dimension for simplicity, returning to the two dimensional case in Section 7. Matrix quantities are represented by sans serif font, for example: J. Vectors are represented by bold font, for example: $\mathbf{v}$.

3. Matrix formulation of the discrete Fourier transform

3.1. Matrix form of Euler's formula

The transforms presented in this paper depend on a generalization of Euler's formula: $\exp i\theta = \cos\theta + i\sin\theta$, in which the imaginary root of -1 is replaced by a matrix root, that is, a matrix that squares to give a negated identity matrix: $\exp \mathsf{J}\theta = \mathsf{I}\cos\theta + \mathsf{J}\sin\theta$. We give a proof of this result for completeness in Appendix A, Lemma 1.

Even among 2×2 matrices there is an infinite number of possible roots J [25, p16]. In the matrix generalization, the exponential must, of course, be a matrix exponential [26, Section 11.3]. Note that matrix versions of the trigonometric functions are not needed to compute the matrix exponential, because θ is a scalar. In fact, if the exponential is evaluated numerically using a matrix exponential algorithm or function, the trigonometric functions are not even explicitly evaluated [26, Section 11.3]. In practice, given that this is a special case of the matrix exponential, (because $\mathsf{J}^2 = -\mathsf{I}$), it is likely to be numerically preferable to evaluate the trigonometric functions and to sum scaled versions of I and J.

Notice that the matrix $e^{\mathsf{J}\theta}$ has a structure with the cosine of θ on the diagonal and the (scaled) sine of θ where there are non-zero elements of J.

3.2. Matrix form of DFT

The classic discrete Fourier transform of (1) may be generalized to a matrix–vector form in which the signals are vector-valued with N components per sample[3] and the root of -1 is replaced by an $N\times N$ matrix root J such that $\mathsf{J}^2 = -\mathsf{I}$. In this form, subject to choosing the correct representation for the matrix root of -1, we may represent a wide variety of complex and hypercomplex Fourier transforms.

Theorem 1. *The following are a discrete Fourier transform pair:*

$$F[u] = S\sum_{m=0}^{M-1}\exp\left(-\mathsf{J}2\pi\frac{mu}{M}\right)f[m] \tag{2}$$

$$f[m] = T\sum_{u=0}^{M-1}\exp\left(\mathsf{J}2\pi\frac{mu}{M}\right)F[u] \tag{3}$$

where J is a $N\times N$ matrix root of -1, $f[m]$ and $F[u]$ are vector-valued discrete-time signals with M samples; each sample, indexed by m or u respectively, being a column vector of N elements; and the two scale factors S and T multiply to give $1/M$.

Proof. The proof is based on substitution of the forward transform (2) into the inverse (3) followed by algebraic reduction to a result equal to the original signal f. We start by substituting (2) into (3), replacing m by $\mathcal{M}$ to keep the two indices distinct, and at the same time replacing the two scale factors by their product $1/M$:

$$f[m] = \frac{1}{M}\sum_{u=0}^{M-1}\left[e^{\mathsf{J}2\pi\frac{mu}{M}}\sum_{\mathcal{M}=0}^{M-1}e^{-\mathsf{J}2\pi\frac{\mathcal{M}u}{M}}f[\mathcal{M}]\right]$$

The exponential of the outer summation can be moved inside the inner, because it is constant with respect to the summation index $\mathcal{M}$:

$$f[m] = \frac{1}{M}\sum_{u=0}^{M-1}\sum_{\mathcal{M}=0}^{M-1}e^{\mathsf{J}2\pi\frac{mu}{M}}e^{-\mathsf{J}2\pi\frac{\mathcal{M}u}{M}}f[\mathcal{M}]$$

The two exponentials have the same root of -1, namely J, and therefore they can be combined:

[3] In this paper the N components of each vector signal sample are assumed to be either real or complex numbers. However, this is not a limitation that we are forced to impose, and therefore the possibility of values from other fields is not excluded.

$$f[m] = \frac{1}{M}\sum_{u=0}^{M-1}\sum_{\mathcal{M}=0}^{M-1} e^{\mathsf{J}\,2\pi\frac{(m-\mathcal{M})u}{M}} f[\mathcal{M}]$$

We now isolate out from the inner summation the case where $m = \mathcal{M}$. In this case the exponential reduces to an identity matrix,[4] and we have:

$$f[m] = \frac{1}{M}\sum_{u=0}^{M-1} \mathsf{I} f[m] + \frac{1}{M}\sum_{u=0}^{M-1}\left[\sum_{\mathcal{M}=0}^{M-1}\Bigg|_{\mathcal{M}\neq m} e^{\mathsf{J}\,2\pi\frac{(m-\mathcal{M})u}{M}} f[\mathcal{M}]\right]$$

The first summation on the right sums to $f[m]$, which is the original signal, as required. To complete the proof, we have to show that the second summation on the right reduces to zero. Taking the second summation alone, and changing the order of summation, we obtain:

$$\sum_{\mathcal{M}=0}^{M-1}\Bigg|_{\mathcal{M}\neq m}\left[\sum_{u=0}^{M-1} e^{\mathsf{J}\,2\pi\frac{(m-\mathcal{M})u}{M}}\right] f[\mathcal{M}]$$

Using Lemma 1 we now write the matrix exponential as the sum of a cosine and sine term.

$$\sum_{\mathcal{M}=0}^{M-1}\Bigg|_{m\neq\mathcal{M}}\left[\begin{array}{ll} \mathsf{I} & \sum_{u=0}^{M-1}\cos\left(2\pi\frac{(m-\mathcal{M})u}{M}\right) \\ +\mathsf{J} & \sum_{u=0}^{M-1}\sin\left(2\pi\frac{(m-\mathcal{M})u}{M}\right)\end{array}\right] f[\mathcal{M}]$$

Since both of the inner summations are sinusoids summed over an integral number of cycles, they vanish, and this completes the proof. □

Notice that the requirement for $\mathsf{J}^2 = -\mathsf{I}$ is the only constraint on J. It is not necessary to constrain elements of J to be real. Note that $\mathsf{J}^2 = -I$ implies that $\mathsf{J}^{-1} = -\mathsf{J}$, hence the inverse transform is obtained by negating or inverting the matrix root of -1 (the two operations are equivalent).

It is important to realize that (2) is totally different to the classical matrix formulation of the discrete Fourier transform, as given for example by Golub and Van Loan [26, Section 4.6.4]. The classic DFT given in (1) can be formulated as a matrix equation in which a large $M \times M$ Vandermonde matrix containing nth roots of unity multiplies the signal f *expressed as a vector of real or complex values*. The entire summation is represented by a single matrix–vector product, yielding a vector result of length M. The multiplication of elements of the matrix and elements of the vector is a standard real or complex multiplication. In contrast, the matrix–vector nature of (2) is at the level of individual multiplications *within* the summation. We have replaced a standard real or complex multiplication in (1) by a matrix–vector multiplication *representing* a complex or hypercomplex multiplication and producing equivalent numerical results. The dimensions of the matrix exponential and the vectors representing signal samples are set by the dimensionality of the algebra (2 for complex, 4 for quaternions *etc.*).

Readers who are already familiar with hypercomplex Fourier transforms should note that the ordering of the exponential within the summation (2) is not related to the ordering within the hypercomplex formulation of the transform (which is significant because of non-commutative multiplication). The formulation of (2) may be readily changed to place the exponential on the right by changing f and F to have row vectors as elements, and by transposing the matrix J (and hence the result of the matrix exponential). Irrespective of the placement of the exponential within (2), the hypercomplex ordering can be accommodated within the framework presented here by changing the representation of the matrix root of -1, in a non-trivial way, shown for the quaternion case by Ickes [3, Eq. 10] and called *transmutation*. The basis of Ickes' method is that, given a matrix P representing a hypercomplex number p, and a vector $\mathbf{q}$ representing a hypercomplex number q, the product $\mathsf{P}\mathbf{q}$ represents the hypercomplex product pq. By transmutation, we can obtain a matrix P' such that the product $\mathsf{P}'\mathbf{q}$ now represents the hypercomplex product qp. We have studied the generalisation of Ickes' method to the case of Clifford algebras, and it appears that there is a more general operation. In the cases we have studied this can be described as negation of the off-diagonal elements of the lower-right sub-matrix, excluding the first row and column.[5] We believe a more general result is known in Clifford algebra but we have not been able to locate a clear statement that we could cite. We therefore leave this for later work, as a full generalisation to Clifford algebras of arbitrary dimension requires further work.

4. Examples in specific algebras

In this section we present the information necessary for (2) and (3) to be verified numerically. In each of the cases below, we present an example root of -1 and a matrix representation.[6] We include in Appendix B a short MATLAB® function for

[4] The matrix exponential of a zero matrix is an identity matrix.

[5] This gives the same result as transmutation in the quaternion case.

[6] The matrix representations of roots of -1 are not unique – a transpose of the matrix, for example, is equally valid. The operations that leave the square of the matrix invariant probably correspond to fundamental operations in the hypercomplex algebra, for example negation, conjugation, reversion.

computing the transform in (2). The same code will compute the inverse by negating J. This may be used to verify the results in the next section and to compare the results obtained with the classic complex FFT. In order to verify the quaternion or biquaternion results, the QTFM toolbox [27] may be used.

4.1. Complex algebra

The 2×2 real matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ can be easily verified by eye to be a square root of the negated identity matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and it is easy to verify numerically that Euler's formula gives the same numerical results in the classic complex case and in the matrix case for an arbitrary θ. This root of -1 is based on the well-known isomorphism between a complex number $a + jb$ and the matrix representation $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ [2, Theorem 1.6].[7] The structure of a matrix exponential $e^{J\theta}$ using the above matrix for J is $\begin{pmatrix} C & -S \\ S & C \end{pmatrix}$ where $C = \cos\theta$ and $S = \sin\theta$.

4.2. Quaternion algebra

The quaternion roots of -1 were discovered by Hamilton [28, pp 203, 209], and consist of all unit pure quaternions, that is quaternions of the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ subject to the constraint $x^2 + y^2 + z^2 = 1$. A simple example is the quaternion $\boldsymbol{\mu} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$, which can be verified by hand to be a square root of -1 using the usual rules for multiplying the quaternion basis elements ($i^2 = j^2 = k^2 = ijk = -1$). Using the isomorphism with 4×4 matrices given by Ward [2, Section 2.8], between the quaternion $w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the matrix:

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix}$$

we have the following matrix representation:

$$\boldsymbol{\mu} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

Notice the structure that is apparent in this matrix: the 2×2 blocks on the leading diagonal at the top left and bottom right can be recognised as roots of -1 in the complex algebra as shown in Section 4.1

Proposition 1. *Any matrix of the form:*

$$\begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & x & 0 \end{pmatrix}$$

with $x^2 + y^2 + z^2 = 1$ is the square root of a negated 4×4 identity matrix. Thus the matrix representations of the quaternion roots of -1 are all roots of the negated 4×4 identity matrix.

Proof. The matrix is anti-symmetric, and the inner product of the ith row and ith column is $-x^2 - y^2 - z^2$, which is -1 because of the stated constraint. Therefore the diagonal elements of the square of the matrix are -1. Note that the rows of the matrix have one or three negative values, whereas the columns have zero or two. The product of the ith row with the jth column, $i \neq j$, is the sum of two values of opposite sign and equal magnitude. Therefore all off-diagonal elements of the square of the matrix are zero. □

The structure of a matrix exponential $e^{J\theta}$ using a matrix as in Proposition 1 for J is:

$$\begin{pmatrix} C & -xS & -yS & -zS \\ xS & C & -zS & yS \\ yS & zS & C & -xS \\ zS & -yS & xS & C \end{pmatrix}$$

[7] We have used the transpose of Ward's representation for consistency with the quaternion and biquaternion representations in the two following sections.

where, as before, $C = \cos\theta$ and $S = \sin\theta$.

4.3. Biquaternion algebra

The biquaternion algebra [2, Chapter 3] (quaternions with complex elements) can be handled exactly as in the previous section, except that the 4×4 matrix representing the root of -1 must be complex (and the signal matrix must have four complex elements per column). The set of square roots of -1 in the biquaternion algebra is given in [20]. A simple example is $\mathbf{i} + \mathbf{j} + \mathbf{k} + I(j - k)$ (where I denotes the classical complex root of -1, that is the biquaternion has real part $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and imaginary part $\mathbf{j} - \mathbf{k}$). Again, this can be verified by hand to be a root of -1 and its matrix representation is:

$$\begin{pmatrix} 0 & -1 & -1-I & -1+I \\ 1 & 0 & -1+I & 1+I \\ 1+I & 1-I & 0 & -1 \\ 1-I & -1-I & 1 & 0 \end{pmatrix}$$

Again, sub-blocks of the matrix have recognizable structure. The upper left and lower right diagonal 2×2 blocks are roots of -1, while the lower left and upper right off-diagonal 2×2 blocks are nilpotent – that is their square vanishes.

4.4. Clifford algebras

Recent work by Hitzer and Abłamowicz has explored the roots of -1 in Clifford algebras $C\ell_{p,q}$.[8] In their first paper on this topic [22] they explored algebras up to those with $p + q = 4$, which are 16-dimensional algebras, using explicit algebraic solutions for the coefficients of a multivector whose square is -1. The derivations of the roots of -1 for the 16-dimensional algebras are long and difficult. Therefore, for the moment, we confine the discussion here to lower-order algebras, noting that, since all Clifford algebras are isomorphic to a matrix algebra, we can be assured that if roots of -1 exist, they must have a matrix representation. Indeed, in a more recent paper [23], Hitzer, Helmstetter and Abłamowicz adopted a different approach to the study of roots of -1 in Clifford algebras based on matrix isomorphisms. Noting that all Clifford algebras are isomorphic to one of five possible matrix algebras, they study the roots of -1 in each of the matrix algebras. This is a more general method, because it is not specific to algebras of specific dimensions, but it does not yield the very specific information on multivector coefficients that we have used below to construct matrix roots of -1.

Using the results obtained by Hitzer and Abłamowicz in [22], and by finding from first principles the layout of a real matrix isomorphic to a Clifford multivector in a given algebra, it has been possible to verify that the transform formulation presented in this paper is applicable to at least the lower order Clifford algebras. The quaternions and biquaternions are isomorphic to the Clifford algebras $C\ell_{0,2}$ and $C\ell_{3,0}$ respectively so this is not surprising. Nevertheless, it is an important finding, because until now quaternion and Clifford Fourier transforms were defined in different ways, using different terminology, and it was difficult to make comparisons between the two. Now, with the matrix exponential formulation, it is possible to handle discrete quaternion and Clifford transforms (and indeed transforms in different Clifford algebras) within the same algebraic and/or numerical framework.

We present examples here from two of the 4-dimensional Clifford algebras, namely $C\ell_{1,1}$ and $C\ell_{2,0}$. These results have been verified against the CLICAL package [29] to ensure that the multiplication rules have been followed correctly and that the roots of -1 found by Hitzer and Abłamowicz are correct.

Following the notation in [22], we write a multivector in $C\ell_{1,1}$ as $\alpha + b_1 e_1 + b_2 e_2 + \beta e_{12}$, where $e_1^2 = +1,\ e_2^2 = -1,\ e_{12}^2 = +1$ and $e_1 e_2 = e_{12}$. A possible real matrix representation is as follows:

$$\begin{pmatrix} \alpha & b_1 & -b_2 & \beta \\ b_1 & \alpha & -\beta & b_2 \\ b_2 & -\beta & \alpha & b_1 \\ \beta & -b_2 & b_1 & \alpha \end{pmatrix}$$

In this algebra, the constraints on the coefficients of a multivector for it to be a root of -1 are as follows: $\alpha = 0$ and $b_1^2 - b_2^2 + \beta^2 = -1$ [22, Table 1].[9] Choosing $b_1 = \beta = 1$ gives $b_2 = \sqrt{3}$ and thus $e_1 + \sqrt{3}e_2 + e_{12}$ which can be verified algebraically or in CLICAL to be a root of -1. The corresponding matrix is then:

$$\begin{pmatrix} 0 & 1 & -\sqrt{3} & 1 \\ 1 & 0 & -1 & \sqrt{3} \\ \sqrt{3} & -1 & 0 & 1 \\ 1 & -\sqrt{3} & 1 & 0 \end{pmatrix}$$

[8] p and q are non-negative integers such that $p + q = n$ and $n \geqslant 1$. The dimension of the algebra (strictly the dimension of the space spanned by the basis elements of the algebra) is 2^n.

[9] We have re-arranged the constraint compared to [22, Table 1] to make the comparison with the quaternion case easier: we see that the signs of the squares of the coefficients match the signs of the squared basis elements.

Following the same notation in algebra $C\ell_{2,0}$, in which $e_1^2 = e_2^2 = +1,\ e_{12}^2 = -1$, a possible matrix representation is:

$$\begin{pmatrix} \alpha & b_1 & b_2 & -\beta \\ b_1 & \alpha & \beta & -b_2 \\ b_2 & -\beta & \alpha & b_1 \\ \beta & -b_2 & b_1 & \alpha \end{pmatrix}$$

The constraints on the coefficients are $\alpha = 0$ and $b_1^2 + b_2^2 - \beta^2 = -1$, and choosing $b_1 = b_2 = 1$ gives $\beta = \sqrt{3}$ and thus $e_1 + e_2 + \sqrt{3}e_{12}$ is a root of -1. The corresponding matrix is then:

$$\begin{pmatrix} 0 & 1 & 1 & -\sqrt{3} \\ 1 & 0 & \sqrt{3} & -1 \\ 1 & -\sqrt{3} & 0 & 1 \\ \sqrt{3} & -1 & 1 & 0 \end{pmatrix}$$

Notice that in both of these algebras the matrix representation of a root of -1 is very similar to that given for the quaternion case in Proposition 1, with zeros on the leading diagonal, an odd number of negative values in each row and an even number in each column. It is therefore simple to see that minor modifications to Proposition 1 would cover these algebras and the matrices presented above.

5. An example not based on a specific algebra

We show here using an arbitrary 2×2 matrix root of -1, that it is possible to define a Fourier transform without a specific algebra. Let an arbitrary real matrix be given as $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then by brute force expansion of $J^2 = -I$ we find the original four equations reduce to but two independent equations. Picking (a, b) and solving for the remaining coefficients we find that any matrix of the form:

$$\begin{pmatrix} a & b \\ -(1 + a^2)/b & -a \end{pmatrix}$$

with finite a and b, and $b \neq 0$, is a root of -1. Choosing instead (a, c) we get the transpose form:

$$\begin{pmatrix} a & -(1 + a^2)/c \\ c & -a \end{pmatrix}$$

where $c \neq 0$. Choosing the cross-diagonal terms (b, c) yields:

$$\begin{pmatrix} \pm\kappa & b \\ c & \mp\kappa \end{pmatrix} \tag{4}$$

where $\kappa = \sqrt{-1 - bc}$ and $bc \leqslant -1$.

In all cases the resulting matrix has eigenvalues of $\lambda = \pm i$. (This is a direct consequence of the fact that this matrix squares to -1.) Each form, however, has different eigenvectors. The standard matrix representation for the complex operator i is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with eigenvectors $v = [1, \pm i]$. In the matrix with (a, b) parameters the eigenvectors are $v = [1, -b/(a \pm i)]$ whereas the cross-diagonal form with (b, c) parameters has eigenvectors $v = [1, (\kappa \pm i)/c]$.

These forms suggest the interesting question: which algebra, if any, applies here[10]; and how can the Fourier coefficients (the 'spectrum') be interpreted? We are not able to answer the first question in this paper. The 'interpretation' of the spectrum is relatively simple. Consider a spectrum F containing only one non-zero value at index u_0 with value $\begin{pmatrix} x \\ y \end{pmatrix}$ and invert this spectrum using (3). Ignoring the scale factor, the result will be the signal:

$$f[m] = \exp\left(\mathrm{J}2\pi\frac{mu_0}{M}\right)\begin{pmatrix} x \\ y \end{pmatrix}$$

The form of the matrix exponential depends on J. In the classic complex case, as given in Section 4.1, the matrix exponential, as already seen, takes the form:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

[10] It is possible that there is no corresponding 'algebra' in the usual sense. Note that there are only two Clifford algebras of dimension 2, one of which is the algebra of complex numbers. The other has no multivector roots of -1 [22, Section 4], and therefore the roots of -1 given above cannot be a root of -1 in any Clifford algebra.

where $\theta = 2\pi\frac{mu_0}{M}$. This is a rotation matrix and it maps a real unit vector $\begin{pmatrix}1\\0\end{pmatrix}$ to a point on a circle in the complex plane. It embodies the standard *phasor* concept associated with sinusoidal functions. Using the same analysis, this time using the matrix in (4) above, one obtains for the matrix exponential the 'phasor' operator:

$$\begin{pmatrix}\cos\theta + \kappa\sin\theta & b\sin\theta \\ c\sin\theta & \cos\theta - \kappa\sin\theta\end{pmatrix}$$

Instead of mapping a real unit vector $\begin{pmatrix}1\\0\end{pmatrix}$ to a point on a circle, this matrix maps to an ellipse. Thus, we see that a transform based on a matrix such as that in (4) has basis functions that are projections of an elliptical, rather than a circular path in the complex plane, as in the classical complex Fourier transform. We refer the reader to a discussion on a similar point for the one-sided quaternion discrete Fourier transform in our own 2007 paper [19, Section IV], in which we showed that the quaternion coefficients of the Fourier spectrum also represent elliptical paths through the space of the signal samples.

It is possible that the matrices discussed in this section could be transformed by similarity transformations into matrices representing elements of a Clifford algebra.[11] Note that in the quaternion case, any root of -1 lies on the unit sphere in 3-space, and can therefore be transformed into another root of -1 by a rotation. It is possible that the same applies in other algebras, the transformation needed being dependent on the geometry. Clearly there are interesting issues to be studied here, and further work to be done.

6. Non-existence of transforms in algebras with odd dimension

In this section we show that there are no real matrix roots of -1 with odd dimension. This is not unexpected, since the existence of such roots would suggest the existence of a hypercomplex algebra of odd dimension. The significance of this result is to show that there is no discrete Fourier transform as formulated in Theorem 1 for an algebra of dimension 3, which is of importance for the processing of signals representing physical 3-space quantities, or the values of colour image pixels. We thus conclude that the choice of quaternion Fourier transforms or a Clifford Fourier transform of dimension 4 is inevitable in these cases. This is not an unexpected conclusion, nevertheless, in the experience of the authors, some researchers in signal and image processing hesitate to accept the idea of using four dimensions to handle three-dimensional samples or pixels. (This is despite the rather obvious parallel of needing two dimensions – complex numbers – to represent the Fourier coefficients of a real-valued signal or image.)

Theorem 2. *There are no $N \times N$ matrices J with real elements such that $\mathsf{J}^2 = -\mathsf{I}$ for odd values of N.*

Proof. The determinant of a diagonal matrix is the product of its diagonal entries. Therefore $|-\mathsf{I}| = -1$ for odd N. Since the product of two determinants is the determinant of the product, $|\mathsf{J}^2| = -1$ requires $|\mathsf{J}|^2 = -1$, which cannot be satisfied if J has real elements. $\square$

7. Extension to two-sided DFTs

There have been various definitions of two-sided hypercomplex Fourier transforms and DFTs. We consider here only one case to demonstrate that the approach presented in this paper is applicable to two-sided as well as one-sided transforms: this is a matrix exponential Fourier transform based on Ell's original two-sided two-dimensional quaternion transform [9, Theorem 4.1] and [10,30]. A more general formulation is:

$$F[u,v] = S\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-\mathsf{J}2\pi\frac{mu}{M}} f[m,n] e^{-\mathsf{K}2\pi\frac{nv}{N}} \tag{5}$$

$$f[m,n] = T\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} e^{+\mathsf{J}2\pi\frac{mu}{M}} F[u,v] e^{+\mathsf{K}2\pi\frac{nv}{N}} \tag{6}$$

in which *each element* of the two-dimensional arrays F and f is a square matrix representing a complex or hypercomplex number using a matrix isomorphism for the algebra in use, for example the representations already given in Section 4.2 in the case of the quaternion algebra; the two scale factors multiply to give $1/MN$, and J and K are matrix representations of two *arbitrary* roots of -1 in the chosen algebra. (In Ell's original formulation, the roots of -1 were **j** and **k**, that is two of the *orthogonal* quaternion basis elements. The theorem below shows that there is no requirement for the two roots to be orthogonal in order for the transform to invert.)

[11] We are grateful to Dr. Eckhard Hitzer for pointing this out.

Notice that here we are formulating the transform using matrices throughout because we have matrix representations of the exponential on both the left and the right of the signal. It is tempting to regard the signal/image $f[m,n]$ as a matrix, but this is neither necessary, nor useful to a mathematical description. As in the one-dimensional case, it is possible to represent a two-dimensional DFT in matrix form, so that the double summation is itself represented by matrix operations. This requires block matrices (matrices with matrix elements), but the operations within the matrix products would be real or complex. Again, in contrast, in the two-dimensional transform above, the products of triples of hypercomplex numbers *inside the double summation* are represented using matrix products, which is quite different. In what follows, we regard the signals as just two-dimensional *arrays* indexed by the pairs of values $[m,n]$ and $[u,v]$. The elements of these arrays are, however, matrices, with dimensionality set by the algebra in use (2 for complex, 4 for quaternions *etc.*).

Theorem 3. *The transforms in (5) and (6) are a two-dimensional discrete Fourier transform pair, provided that* $\mathsf{J}^2 = \mathsf{K}^2 = -\mathsf{I}$.

Proof. The proof follows the same scheme as the proof of Theorem 1. We start by substituting (2) into (3), replacing m and n by $\mathcal{M}$ and $\mathcal{N}$ respectively to keep the indices distinct:

$$f[m,n] = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} e^{\mathsf{J}2\pi\frac{mu}{M}} \times \left[\sum_{\mathcal{M}=0}^{M-1}\sum_{\mathcal{N}=0}^{N-1} e^{-\mathsf{J}2\pi\frac{\mathcal{M}u}{M}} f[\mathcal{M},\mathcal{N}] e^{-\mathsf{K}2\pi\frac{\mathcal{N}v}{N}}\right] \times e^{\mathsf{K}2\pi\frac{nv}{N}}$$

The scale factors can be moved outside both summations, and replaced with their product $1/MN$; and the exponentials of the outer summations can be moved inside the inner, because they are constant with respect to the summation indices $\mathcal{M}$ and $\mathcal{N}$. At the same time, adjacent exponentials with the same root of -1 can be merged. With these changes, and omitting the scale factor, the right-hand side of the equation becomes:

$$\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\sum_{\mathcal{M}=0}^{M-1}\sum_{\mathcal{N}=0}^{N-1} e^{\mathsf{J}2\pi\frac{(m-\mathcal{M})u}{M}} f[\mathcal{M},\mathcal{N}] e^{\mathsf{K}2\pi\frac{(n-\mathcal{N})v}{N}}$$

We now isolate out from the inner pair of summations the case where $\mathcal{M} = m$ and $\mathcal{N} = n$. In this case the exponentials reduce to identity matrices, and we have:

$$\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \mathsf{I} f[m,n]\,\mathsf{I}$$

This sums to $f[m,n]$, which is the original two-dimensional signal, as required. To complete the proof we have to show that the rest of the summation, excluding the case $\mathcal{M} = m$ and $\mathcal{N} = n$, reduces to zero. Dropping the scale factor, and changing the order of summation, we have the following inner double summation:

$$\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} e^{\mathsf{J}2\pi\frac{(m-\mathcal{M})u}{M}} f[\mathcal{M},\mathcal{N}] e^{\mathsf{K}2\pi\frac{(n-\mathcal{N})v}{N}}$$

Noting that the first exponential and f are independent of the second summation index v, we can move them outside the second summation (we could do similarly with the exponential on the right and the first summation):

$$\sum_{u=0}^{M-1} e^{\mathsf{J}2\pi\frac{(m-\mathcal{M})u}{M}} f[\mathcal{M},\mathcal{N}] \sum_{v=0}^{N-1} e^{\mathsf{K}2\pi\frac{(n-\mathcal{N})v}{N}}$$

and, as in Theorem 1, the summation on the right is over an integral number of cycles of cosine and sine, and therefore vanishes. □

Notice that it was not necessary to make assumptions about J and K equivalent to orthogonality of their quaternion equivalents **j** and **k**: it is sufficient that each be a root of -1. This has been verified numerically using the two-dimensional code given in Appendix B.

8. Discussion

We have shown that many discrete Fourier transforms in algebras that have a matrix representation, can be formulated in the way shown here. This includes the complex, quaternion, biquaternion, and Clifford algebras (although we have demonstrated only certain cases of Clifford algebras, we believe the result holds in general). This observation provides a theoretical unification of diverse hypercomplex DFTs which should permit comparisons between approaches using differing algebras.

Although we have shown that many existing one-sided complex and hypercomplex transforms can be represented using summations over matrix–vector products or triple matrix products, there are other possibilities. The two-sided transform of Ell cited in Section 7 is not the most general case. It is possible to have multiple exponentials on each side, indexed by the m and/or n indices, in various combinations. Recent work by Bujack et al. [6] has considered a most general formulation of

Fourier transforms over geometric algebras, with multiple exponential products to the left and right of the signal function. While we see no immediate reason why our method should not be applicable in these generalized cases, further work is needed to verify that the matrix formulation works in these cases, ideally including a numerical verification against an implementation in geometric algebra, which is not a trivial task.

Several other immediate possibilities for further work, as well as ramifications, now suggest themselves. Firstly, the study of roots of -1 is accessible from the matrix representation as well as direct representation in whatever algebra is employed for the transform. Most of the results obtained so far in hypercomplex algebras, and known to the authors [28, pp 203, 209], [20,22], were achieved by working *in the algebra* in question, that is by algebraic manipulation of quaternion, biquaternion or Clifford multivector values. An alternative approach would be to work in the equivalent matrix algebra, as demonstrated by Hitzer, Helmstetter and Abłamowicz in [23]. Following the reasoning in Section 5, it is possible to define matrix roots of -1 that appear not to be isomorphic to any Clifford or quaternion algebra, and these merit further study.

Secondly, the matrix formulation presented here lends itself to analysis of the structure of the transform, including possible factorizations for fast algorithms, as well as parallel or vectorized implementations for single-instruction, multiple-data (SIMD) processors, and of course, factorizations into multiple complex FFTs as has been done for quaternion FFTs (see for example [18]). In the case of matrix roots of -1 which do not correspond to Clifford or quaternion algebras, analysis of the structure of the transform may give insight into possible applications of transforms based on such roots.

Finally, at a practical level, hypercomplex transforms implemented directly in hypercomplex arithmetic are likely to be much faster than any implementation based on matrices, but the simplicity of the matrix exponential formulation discussed in this paper, and the fact that it can be computed using standard real or complex matrix arithmetic, *without using a hypercomplex library*, means that the matrix exponential formulation provides a very simple reference implementation which can be used for verification of the correctness of hypercomplex code. This is an important point, because verification of the correctness of hypercomplex FFT code is otherwise non-trivial. Verification of inversion is simple enough, but establishing that the spectral coefficients have the correct values is much less so.

Appendix A. Matrix exponential form of Euler's formula

The following Lemma is not claimed to be original but we have not been able to locate any published source that we could cite here. Since the result is essential to Theorem 1, we set it out here for completeness.

Lemma 1. *Euler's formula may be generalized as follows:*

$$e^{\mathsf{J}\theta} = \mathsf{I}\cos\theta + \mathsf{J}\sin\theta$$

where I *is an identity matrix, and* J *is a matrix such that* $\mathsf{J}^2 = -\mathsf{I}$.

Proof. The result follows from the series expansions of the matrix exponential and the trigonometric functions. From the definition of the matrix exponential [26, Section 11.3]:

$$e^{\mathsf{J}\theta} = \sum_{k=0}^{\infty} \frac{\mathsf{J}^k\theta^k}{k!} = \mathsf{J}^0 + \mathsf{J}\theta + \frac{\mathsf{J}^2\theta^2}{2!} + \frac{\mathsf{J}^3\theta^3}{3!} + \cdots$$

Noting that $\mathsf{J}^0 = \mathsf{I}$ (see [[31], Index Laws]), and separating the series into even and odd terms:

$$\begin{aligned} &= \mathsf{I} - \frac{\mathsf{I}\theta^2}{2!} + \frac{\mathsf{I}\theta^4}{4!} - \frac{\mathsf{I}\theta^6}{6!} + \cdots \\ &+ \mathsf{J}\theta - \frac{\mathsf{J}\theta^3}{3!} + \frac{\mathsf{J}\theta^5}{5!} - \frac{\mathsf{J}\theta^7}{7!} + \cdots \\ &= \mathsf{I}\cos\theta + \mathsf{J}\sin\theta \qquad \square \end{aligned}$$

Appendix B. MATLAB® code

We include here two short MATLAB® functions for computing the forward transform given in (2), and (5), *apart from the scale factors*. The inverses can be computed simply by interchanging the input and output and negating the matrix roots of -1. Neither function is coded for speed, on the contrary the coding is intended to be simple and easily verified against the equations. Since standard MATLAB® handles only flat matrices or vectors (not matrices or vectors with matrix or vector elements *etc.*), it is necessary to explain how the signals are represented. In the first function, *f* is an array with *M* columns, and the same number of rows as *J*, representing a signal with *M* samples, each with the same number of components as the number of rows in *J*.

```
function F = matdft (f, J)
M = size (f, 2);
F = zeros (size (f));
for m = 0:M-1
  for u = 0:M-1
    F (:, u + 1) = F (:, u + 1) ...
    + expm (-J .* 2 .* pi .* m .* u ./M) ...
    * f (:, m + 1);
  end
end
```

In the second function, f is a two-dimensional block matrix with M samples vertically and N horizontally. Each block in the matrix is the same size as J and represents one sample (pixel) using a matrix representation. Therefore the number of real/complex numbers in f will be MD vertically and ND horizontally, where J and K are of dimension $D \times D$.

```
function F = matdft2(f, J, K)
A = size (J, 1);
M = size (f, 1) ./ A;
N = size (f, 2) ./ A;
F = zeros (size (f));
for u = 0:M-1
  for v = 0:N-1
    for m = 0:M-1
      for n = 0:N-1
        F (A*u + 1:A*u + A, A*v + 1:A*v + A) = ...
        F (A*u + 1:A*u + A, A*v + 1:A*v + A) + ...
        expm (-J .* 2*pi .* m .* u ./M) ...
        * f (A*m + 1:A*m + A, A*n + 1:A*n + A) ...
        * expm (-K .* 2*pi .* n .* v ./N);
      end
    end
  end
end
```

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