3.8 (a) Letting $f_i(\mathbf{x}) = \partial f(\mathbf{x})/\partial x_i$, show that,

$$\sigma_{ij}(\mathbf{x}) \equiv -\frac{x_i f_i(\mathbf{x}) + x_j f_j(\mathbf{x})}{f_j^2(\mathbf{x}) f_{ii}(\mathbf{x}) + 2 f_i(\mathbf{x}) f_j(\mathbf{x}) f_{ij}(\mathbf{x}) + f_i^2(\mathbf{x}) f_{jj}(\mathbf{x})} \frac{f_i(\mathbf{x}) f_j(\mathbf{x})}{x_i x_j}.$$

S: First note that:

$$\sigma_{ij}(\mathbf{x}) \equiv \left(\frac{r}{MRTS_{ij}(\mathbf{x}(r))} \frac{\partial MRTS_{ij}(\mathbf{x}(r))}{\partial r} \right)^{-1} = \left(\frac{r}{f_i(\mathbf{x}(r))/f_j(\mathbf{x}(r))} \frac{\partial (f_i(\mathbf{x}(r))/f_j(\mathbf{x}(r)))}{\partial r} \right)^{-1},$$

where the derivatives are evaluated at $r = x_j^0/x_i^0$. To simplify the notation, lets write only f_i for $f_i(\mathbf{x}(r))$ and similarly for the other derivatives of f. Using chain rule, we obtain:

$$\begin{aligned}
\frac{\partial (f_i/f_j)}{\partial r} &= \frac{1}{f_j} \sum_{k=1}^{n} f_{ik} \frac{\partial x_k(r)}{\partial r} - \frac{f_i}{f_j^2} \sum_{k=1}^{n} f_{jk} \frac{\partial x_k(r)}{\partial r} \\
&= \frac{1}{f_j} \left(f_{ii} \frac{\partial x_i(r)}{\partial r} + f_{ij} \frac{\partial x_j(r)}{\partial r} \right) - \frac{f_i}{f_j^2} \left(f_{jj} \frac{\partial x_j(r)}{\partial r} + f_{ji} \frac{\partial x_i(r)}{\partial r} \right), \qquad (2)
\end{aligned}$$

since $\partial x_k(r)/\partial r = 0$ for all $k \neq i, j$. We need then to find $\partial x_i(r)/\partial r$ and $\partial x_j(r)/\partial r$. Observe that:

$$\frac{\partial x_j/x_i}{\partial r} = 1 \quad \Rightarrow \quad \frac{1}{x_i} \frac{\partial x_j(r)}{\partial r} - \frac{x_j}{x_i^2} \frac{\partial x_i(r)}{\partial r} = 1 \quad \Rightarrow \quad \frac{\partial x_i(r)}{\partial r} = \frac{x_i}{x_j} \frac{\partial x_j(r)}{\partial r} - \frac{x_i^2}{x_j}$$

Since $f(\mathbf{x}(r)) \equiv f(\mathbf{x}^0)$, we have:

$$f_i \frac{\partial x_i(r)}{\partial r} + f_j \frac{\partial x_j(r)}{\partial r} = 0 \quad \Rightarrow \quad f_i \left(\frac{x_i}{x_j} \frac{\partial x_j(r)}{\partial r} - \frac{x_i^2}{x_j} \right) + f_j \frac{\partial x_j(r)}{\partial r} = 0$$

Solving this last equation for $\partial x_j(r)/\partial r$, we find that:

$$\frac{\partial x_j(r)}{\partial r} = \frac{x_i^2 f_i}{x_j f_j + x_i f_i},$$

what gives:

$$\frac{\partial x_i(r)}{\partial r} = \frac{-x_i^2 f_j}{x_j f_j + x_i f_i}$$

Plugging these expressions for $\partial x_i(r)/\partial r$ and $\partial x_j(r)/\partial r$ in equation (2) yields:

$$\begin{aligned}\frac{\partial(f_i/f_j)}{\partial r} &= \frac{1}{f_j}\left(\frac{-x_i^2 f_j f_{ii} + x_i^2 f_i f_{ij}}{x_j f_j + x_i f_i}\right) - \frac{f_i}{f_j^2}\left(\frac{x_i^2 f_i f_{jj} - x_i^2 f_j f_{ij}}{x_j f_j + x_i f_i}\right)\\ &= \frac{-x_i^2(f_j^2 f_{ii} - 2 f_j f_i f_{ij} + f_i^2 f_{jj})}{f_j^2(x_j f_j + x_i f_i)}\end{aligned}$$

Now plugging everything together, we find:

$$\begin{aligned}\sigma_{ij} &= \left(\frac{x_j f_j}{x_i f_i}\frac{\partial(f_i/f_j)}{\partial r}\right)^{-1} = \frac{x_i f_i}{x_j f_j}\left(\frac{f_j^2(x_j f_j + x_i f_i)}{-x_i^2(f_j^2 f_{ii} - 2 f_j f_i f_{ij} + f_i^2 f_{jj})}\right)\\ &= -\frac{x_i f_i + x_j f_j}{f_j^2 f_{ii} - 2 f_i f_j f_{ij} + f_i^2 f_{jj}}\,\frac{f_i f_j}{x_i x_j},\end{aligned}$$

where all the functions above are evaluated at $\mathbf{x}$.

(b) Using the formula in (a), show that $\sigma_{ij}(\mathbf{x}) \geq 0$ whenever f is increasing and concave. (The elasticity of substitution is non-negative when f is merely quasiconcave but you need not show this.)

S: If f is increasing, then all first-derivatives are non-negative. And if f is concave, then $f_j^2 f_{ii} - 2 f_i f_j f_{ij} + f_i^2 f_{jj}$ is non-positive (this expression is equal to the $\mathbf{e}_{ij}^T \cdot H(\mathbf{x}) \cdot \mathbf{e}_{ij}$, where $H(\mathbf{x})$ is the Hessian of f and $\mathbf{e}_{ij}$ is the vector of zeros in all coordinates, but i and j, where it is equal to f_j at coordinate i and $-f_i$ at coordinate j. Theorem A2.4 implies that f concave means that $H(\mathbf{x})$ is negative semidefinite, that is, $\mathbf{z}^T \cdot H(\mathbf{x}) \cdot \mathbf{z} \leq 0$, for any vector $\mathbf{z}$). Thus, the expression found for $\sigma_{ij}(\mathbf{x})$ in part (a) implies that $\sigma_{ij}(\mathbf{x}) \geq 0$ whenever f is increasing and concave.