sheet05

November 19, 2018

1 Expectation-Maximization

In this assignment we will be using the Expectation Maximization method to estimate the parameters of the same three coin experiment as in the theoretical part. We will examine the behavior of the algorithm for various combinations of parameters.

1.1 Description of the Experiment

The following procedure generates the data for the three coin experiment.

The parameters are:

- λ := The probability of heads on the hidden coin H.
- p_1 := The probability of heads on coin A.
- p_2 := The probability of heads on coin B.

Each of the N samples is collected the following way:

- The secret coin (H) is tossed.
- If the result is heads, coin A is tossed M times and the results are recorded.
- If the result is tails, coin B is tossed M times and the results are recorded.

Heads are recorded as 1.
Tails are recorded as 0.

The data is returned as an $N \times M$ matrix, where each of the N rows correspond to the trials and contains the results of the corresponding sample (generated either by coin A or by coin B).

1.2 Description of Provided Functions

Three functions are provided for your convenience:

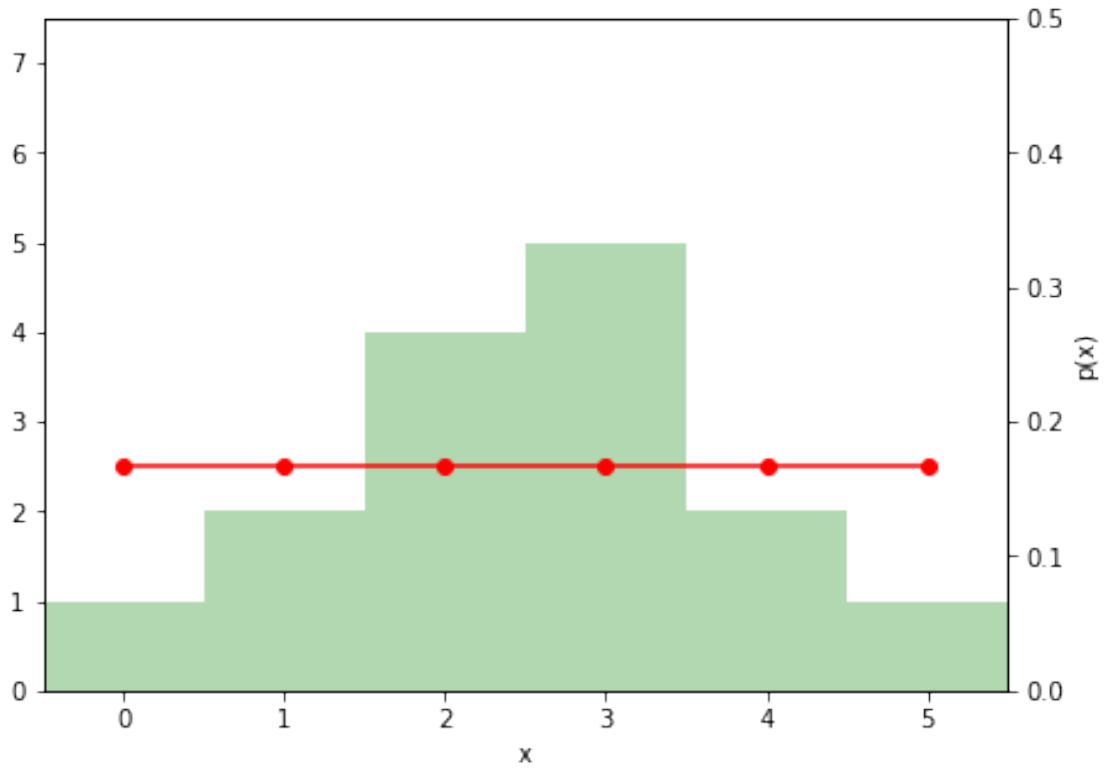
- `utils.generateData(lambda,p1,p2,N,M)`: Performs the experiment N times with coin parameters specified as argument and returns the results in a $N \times M$ matrix.
- `utils.unknownData()` Returns a dataset of size $N \times M$ where generation parameters are unknown.

- `utils.plot(data,distribution)`: Plot a histogram of the number of heads per trial along with the probability distribution. This function will be used to visualize the progress of the EM algorithm at every iteration.

An example of use of these two functions is given below:

```
In [17]: %matplotlib inline
         import numpy,utils

         # Print the data matrix as a result of the three coins experiment with parameter 0.5, 0
         data = utils.generateData(0.5,0.8,0.2,15,5)
         print(data)

         # Print the data histogram along with a uniform probability distribution.
         utils.plot(data,numpy.ones([data.shape[1]+1])/(data.shape[1]+1))
```

```
[[0 0 0 0 0]
 [1 0 1 1 1]
 [0 1 0 1 1]
 [0 1 1 0 0]
 [1 1 1 0 1]
 [1 0 1 0 1]
 [1 0 1 1 0]
 [0 0 1 0 0]
 [1 1 1 1 1]
 [0 1 0 1 1]
 [0 1 0 0 0]
 [0 0 1 0 1]
 [1 1 0 0 1]
 [0 1 0 1 0]
 [0 0 1 1 0]]
```



1.3 Calculate the Log-Likelihood (10 P)

Implement a function which calculates the log likelihood for a given dataset and parameters. The log-likelihood is given by:

$$LL = \frac{1}{N}\sum_{i=1}^{N}\log \sum_{z\in\{\text{heads,tails}\}} P(X = x_i, Z = z \mid \theta) = \frac{1}{N}\sum_{i=1}^{N}\log\left[\lambda \cdot p_1^{h(x_i)} \cdot (1-p_1)^{t(x_i)} + (1-\lambda)\cdot p_2^{h(x_i)} \cdot (1-p_2)^{t(x_i)}\right]$$

where $h(x_i)$ and $t(x_i)$ denote the number of heads and tails in sample i, respectively. Note that we take the averaged log-likelihood over all trials, hence the multiplicative term $\frac{1}{N}$ in front.

```
In [18]: t = lambda x: x.shape[1]-h(x)
         h = lambda x: x.sum(axis=1)

         def loglikelihood(x,lam,p1,p2):
             N=data.shape[0]
             return (1/N)*numpy.log(lam*p1**h(x)*(1-p1)**t(x)+
                                    (1-lam)*p2**h(x)*(1-p2)**t(x)).sum()

         loglikelihood(data,.5,.5,.5)

Out[18]: -3.4657359027997274
```

1.4 Implementing and Running the EM Algorithm (30 P)

Implement a function which iteratively determines the values of λ, p_1 and p_2. The function starts with some initial estimates for the parameters and returns the results of the method for those parameters.

In each iteration, the following update rules are used for the parameters:

$$\lambda^{new} = \frac{E(\#heads(coin\_H))}{\#throws(coin\_H)} = \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda p_1^{h(x_i)}(1-p_1)^{t(x_i)}}{\lambda p_1^{h(x_i)}(1-p_1)^{t(x_i)} + (1-\lambda)p_2^{h(x_i)}(1-p_2)^{t(x_i)}}$$

$$p_1^{new} = \frac{E(\#heads(coin\_A))}{E(\#throws(coin\_A))} = \frac{\sum_{i=1}^{N} R_1(i)h(x_i)}{M\sum_{i=1}^{N} R_1(i)}$$

$$p_2^{new} = \frac{E(\#heads(coin\_B))}{E(\#throws(coin\_B))} = \frac{\sum_{i=1}^{N} R_2(i)h(x_i)}{M\sum_{i=1}^{N} R_2(i)}$$

where $h(x_i)$ and $t(x_i)$ denote the number of heads and tails in sample i, respectively, and

$$R_1(i) = \frac{\lambda p_1^{h(x_i)}(1-p_1)^{t(x_i)}}{\lambda p_1^{h(x_i)}(1-p_1)^{t(x_i)} + (1-\lambda)p_2^{h(x_i)}(1-p_2)^{t(x_i)}}$$

$$R_2(i) = \frac{(1-\lambda) p_2^{h(x_i)}(1-p_2)^{t(x_i)}}{\lambda p_1^{h(x_i)}(1-p_1)^{t(x_i)} + (1-\lambda)p_2^{h(x_i)}(1-p_2)^{t(x_i)}}$$

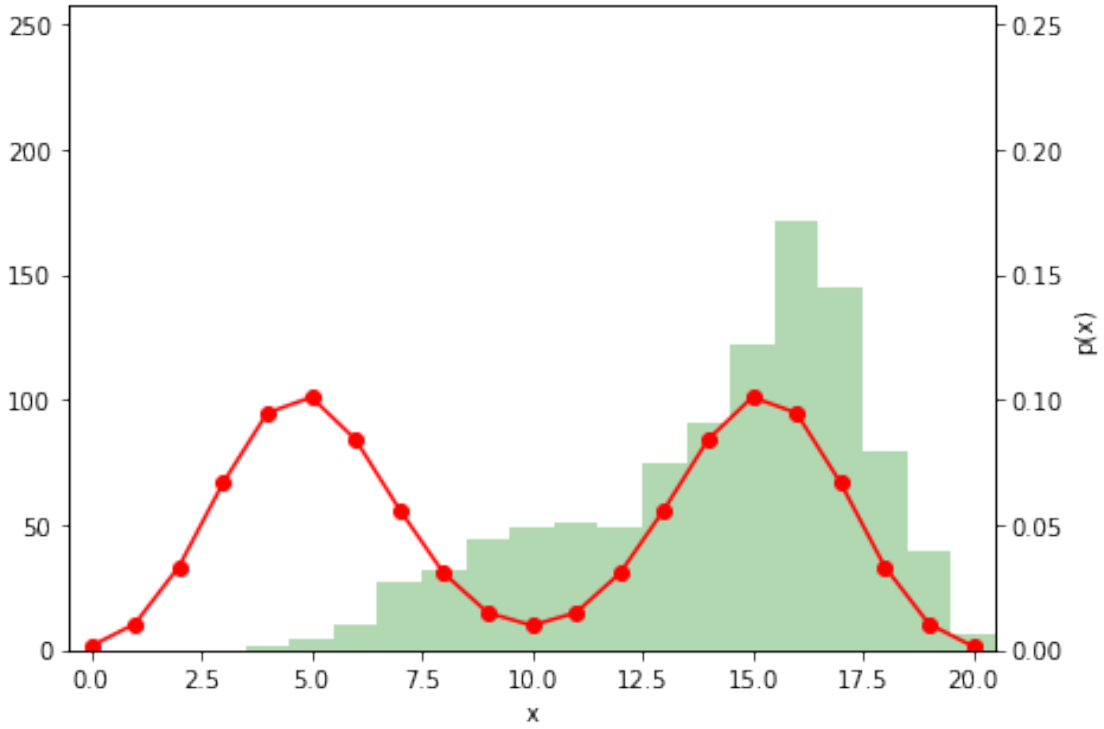
TODO:

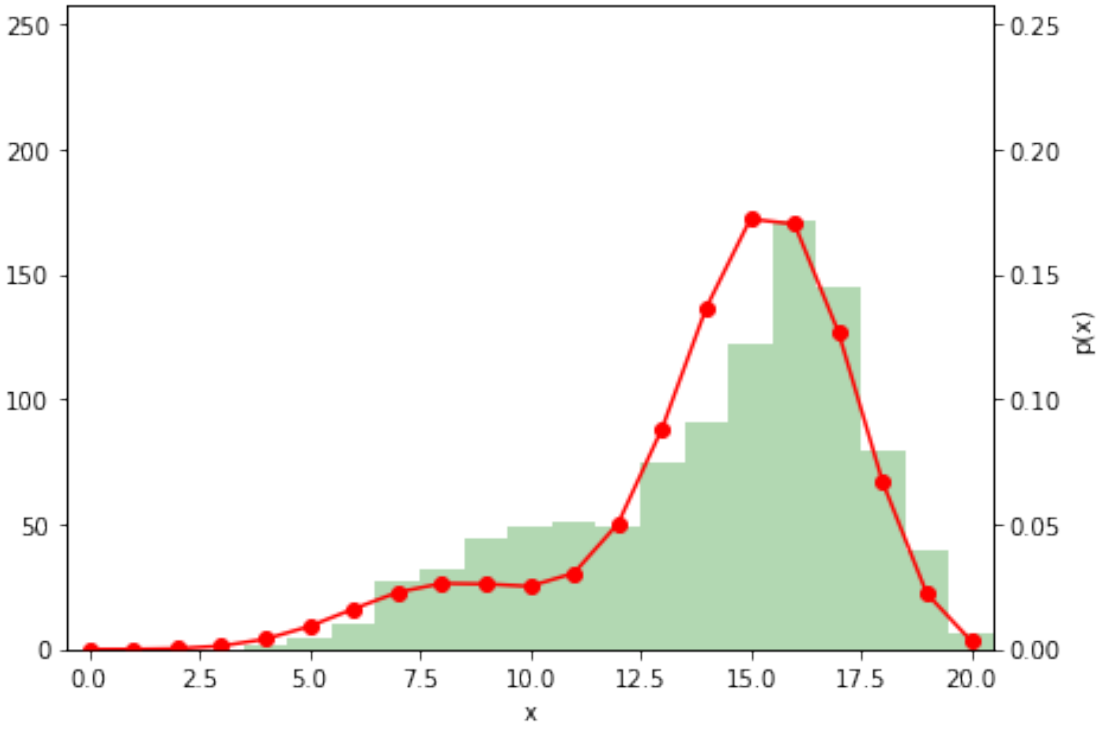
- **Implement the EM learning procedure.**
- **Use as stopping criterion the improvement of log-likelihood between two iterations to be smaller than** 0.001**.**
- **Run the EM procedure on the data returned by function** `utils.unknownData()`**. Use as an initial solution for your model the parameters** $\lambda = 0.5$**,** $p_1 = 0.25$**,** $p_2 = 0.75$ **.**
- **At each iteration of the EM procedure, print the log-likelihood and the value of your model parameters, and plot the learned probability distribution using the function utils.plot().**

```
In [19]: import utils
         %matplotlib inline
         from scipy.stats import binom

         data = utils.unknownData()
         N, M = data.shape

         def F(lam, p1, p2):
             x = numpy.arange(M+1).reshape(-1,1)
             return binom.pmf(x, M, p1)*lam + binom.pmf(x, M, p2)*(1-lam)

         def Lambda(x, lam, p1, p2):
             return (1/N)*((lam*p1**h(x)*(1-p1)**t(x))/
```

```
                    (lam*p1**h(x)*(1-p1)**t(x)+(1-lam)*p2**h(x)*(1-p2)**t(x))).sum()

def R1(x, lam, p1, p2):
    return ((lam*p1**h(x)*(1-p1)**t(x))/
            (lam*p1**h(x)*(1-p1)**t(x)+(1-lam)*p2**h(x)*(1-p2)**t(x)))

def R2(x, lam, p1, p2):
    return (((1-lam)*p2**h(x)*(1-p2)**t(x))/
            (lam*p1**h(x)*(1-p1)**t(x)+(1-lam)*p2**h(x)*(1-p2)**t(x)))

def P(R, x, lam, p1, p2):
    R_res = R(x, lam, p1, p2)
    return (R_res*h(data)).sum()/(M*R_res).sum()


# ----------------------------------------
# Template for your code
# ----------------------------------------
lam = .5
p1  = .25
p2  = .75
it = 0
l_old = 0
criterion = False

# Iterate until the stopping criterion is satisfied
while not criterion:
    it += 1

    # - Plot data histogram and the learned probability distribution
    utils.plot(data, F(lam,p1,p2))

    # - Perform one step of EM
    lam_n = Lambda(data,lam,p1,p2)
    p1_n = P(R1, data, lam, p1, p2)
    p2_n = P(R2, data, lam, p1, p2)
    l = loglikelihood(data, lam, p1, p2)
    lam = lam_n
    p1 = p1_n
    p2 = p2_n

    # - Print the log-likelihood and the model parameters
    print('it:%2d  lambda: %.2f  p1: %.2f  p2: %.2f  log-likelihood: %.3f'
          %(it, lam, p1, p2, l))

    # - Determine if stopping criterion is satisfied
    if abs(l_old-l) < 0.001:
        criterion = True
```

```
            l_old = l

it: 1  lambda: 0.15  p1: 0.41  p2: 0.76  log-likelihood: -12.201
it: 2  lambda: 0.18  p1: 0.44  p2: 0.77  log-likelihood: -11.709
it: 3  lambda: 0.21  p1: 0.46  p2: 0.78  log-likelihood: -11.684
it: 4  lambda: 0.23  p1: 0.47  p2: 0.78  log-likelihood: -11.672
it: 5  lambda: 0.24  p1: 0.48  p2: 0.78  log-likelihood: -11.666
it: 6  lambda: 0.25  p1: 0.48  p2: 0.79  log-likelihood: -11.664
it: 7  lambda: 0.26  p1: 0.48  p2: 0.79  log-likelihood: -11.662
it: 8  lambda: 0.26  p1: 0.49  p2: 0.79  log-likelihood: -11.662
```

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p(x)
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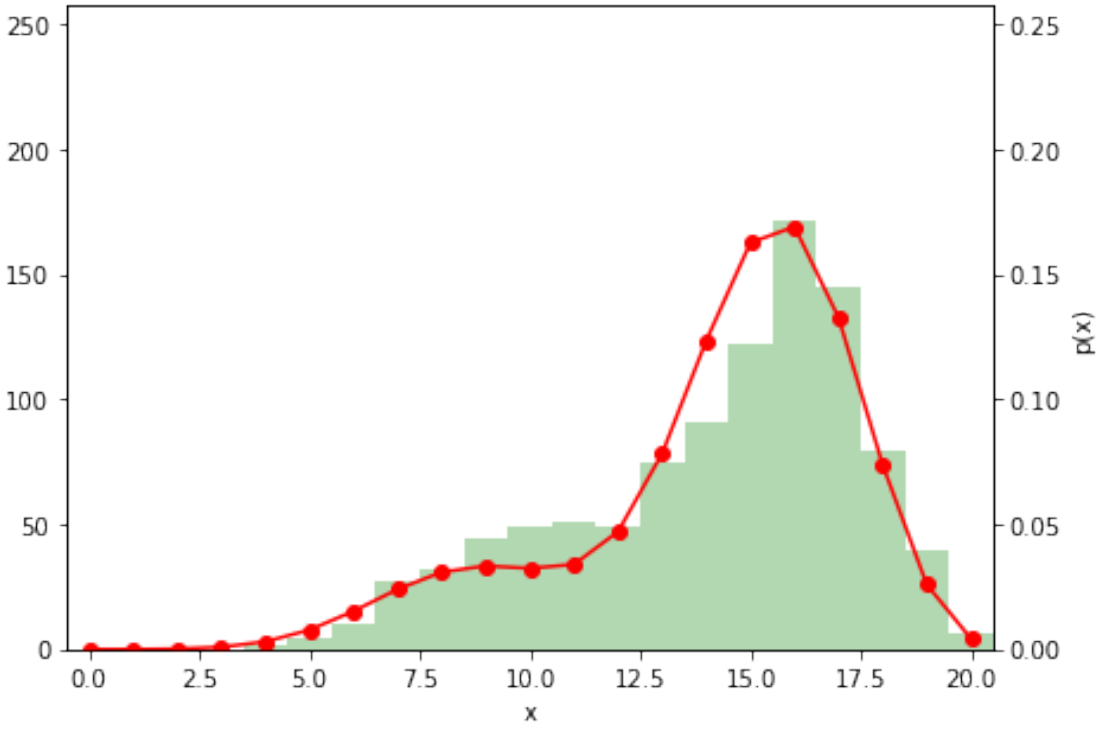
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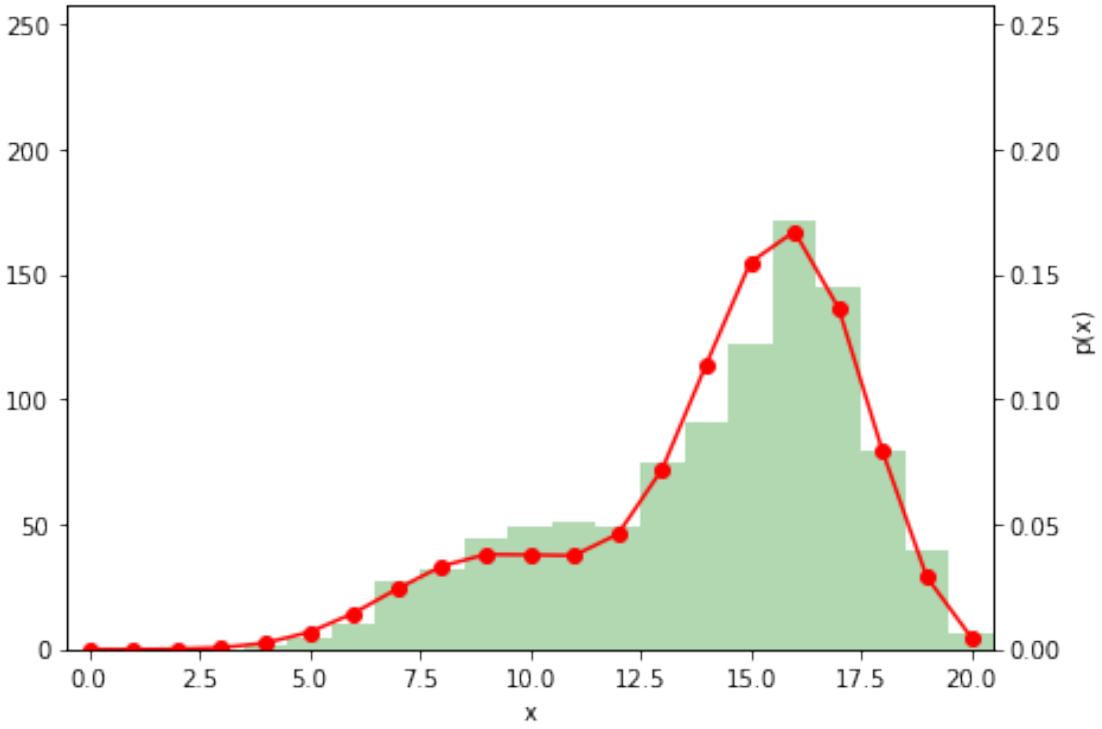
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7.5
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15.0
17.5
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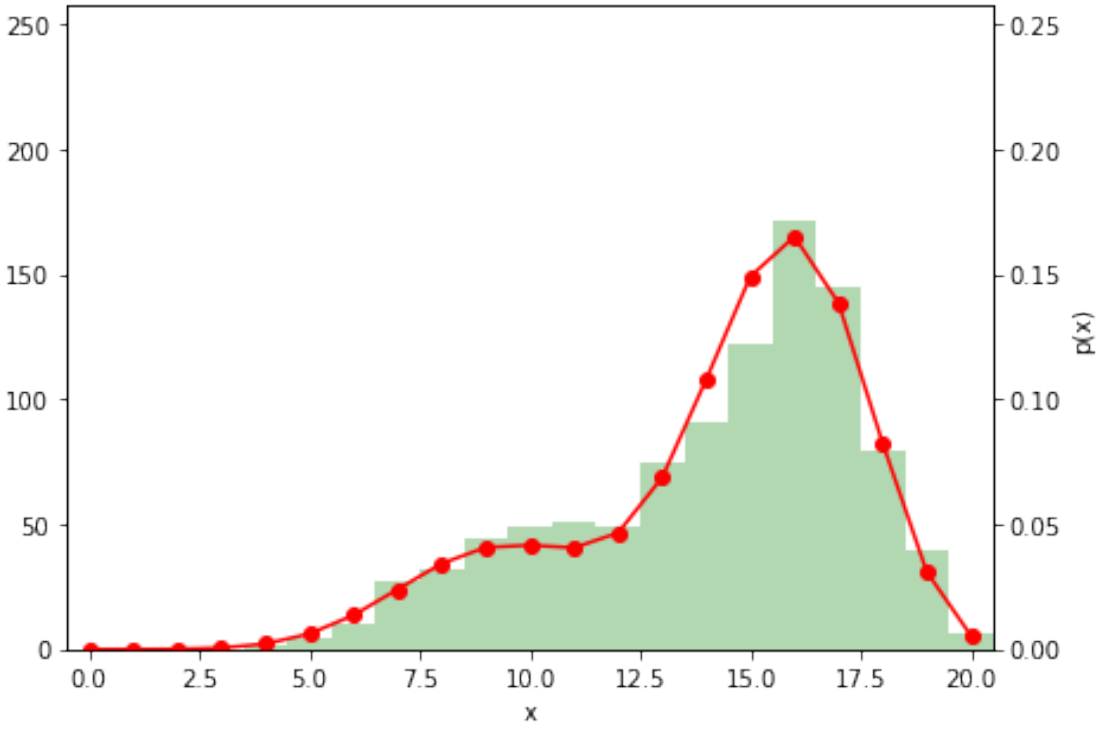
x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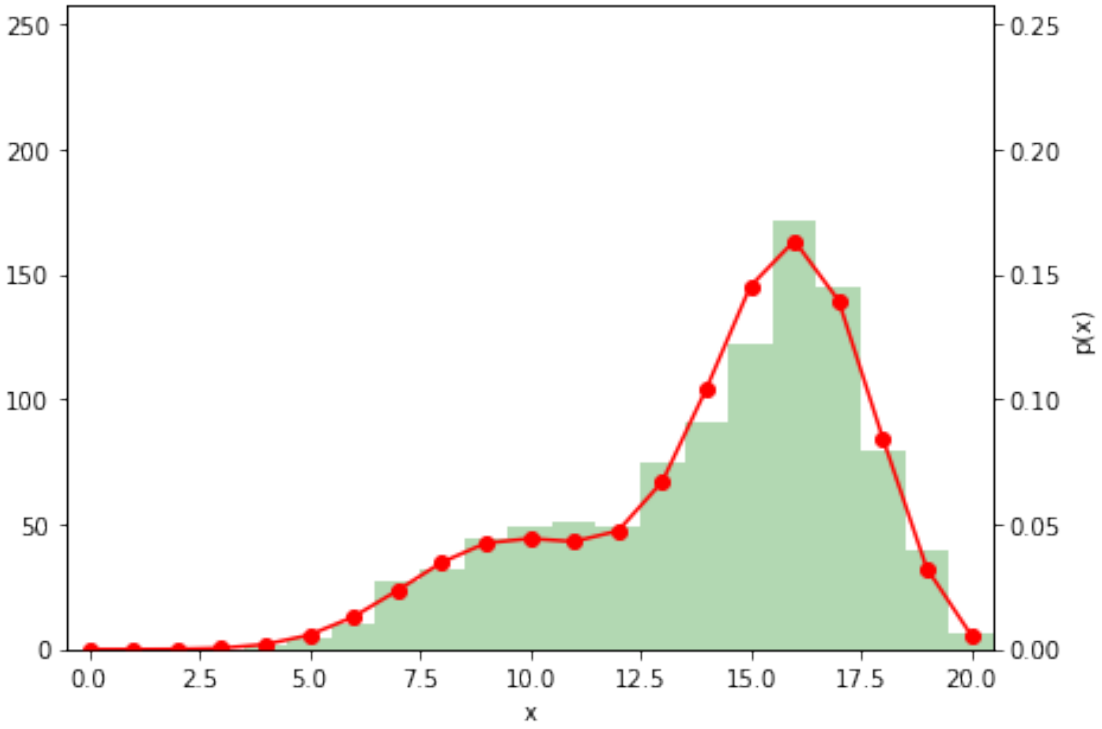
x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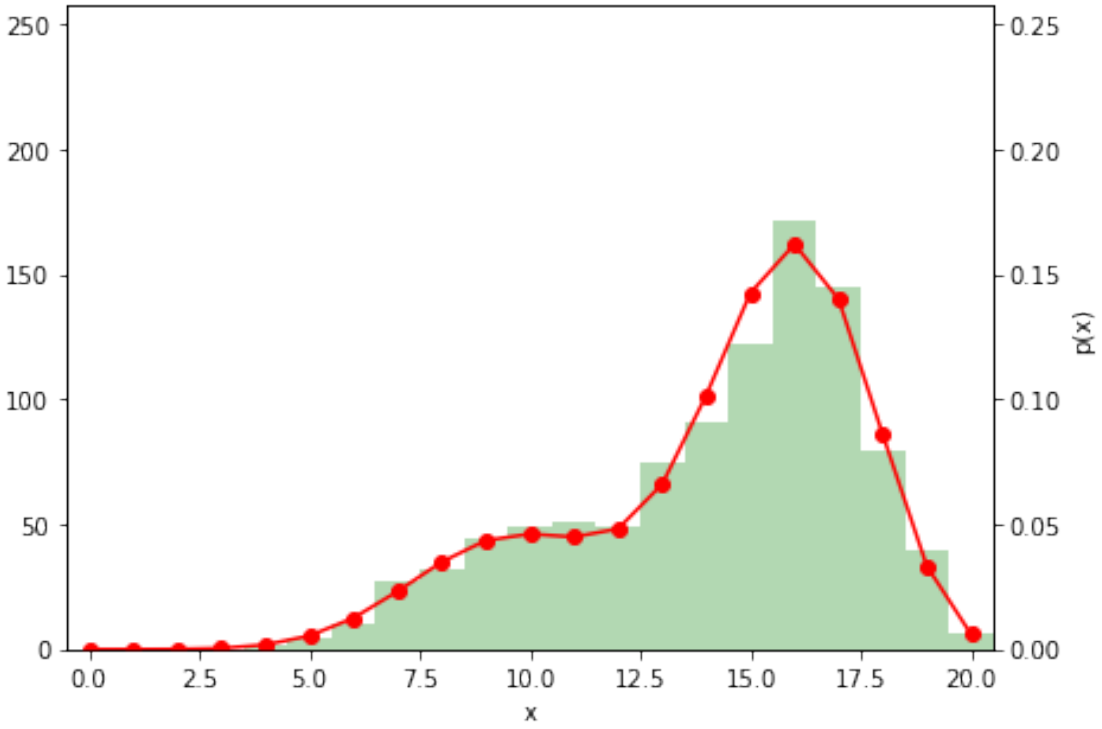
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p(x)
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15.0
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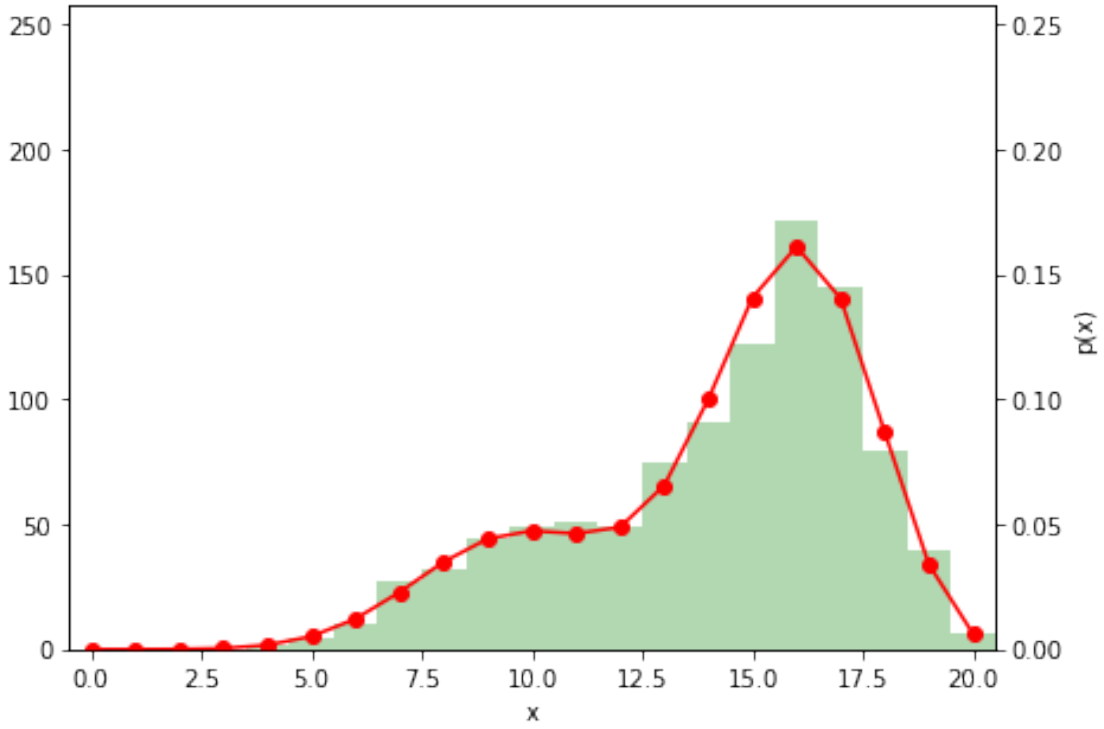
x



1.5 More Experiments (10 P)

Examine the behaviour of the EM algorithm for various combinations of data generation parameters and initializations (for generating various distributions, use the method `utils.generateData(...)`). In particular, find settings for which:

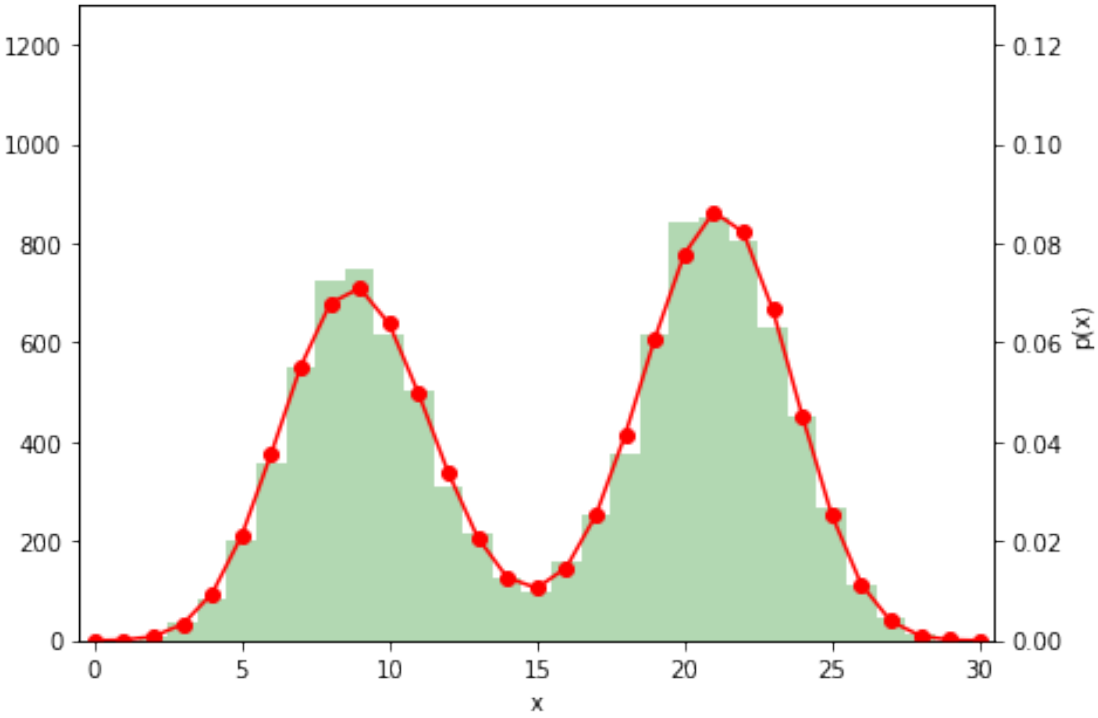
- The role of coins A and B are permuted between the data generating model and the learned model (i.e. $\hat{p}_1 \approx p_2$, $\hat{p}_2 \approx p_1$ and $\hat{\lambda} \approx 1 - \lambda$).
- The EM procedure takes a long time to converge.

Print the parameters and log-likelihood objective at each iteration. Only display the plot for the converged model.

```
In [20]: N, M = 10000, 30
         lam = .45
         p1  = .3
         p2  = .7
         data = utils.generateData(lam,p1,p2,N, M)
         p1  = p2*numpy.random.rand()*0.1
         p2  = p1*numpy.random.rand()*0.1
         lam = (1-lam)*numpy.random.rand()*0.1
         it = 0
         l_old = 0
```

```
    criterion = False

    # Iterate until the stopping criterion is satisfied
    while not criterion:
        it += 1

        # - Perform one step of EM
        lam_n = Lambda(data,lam,p1,p2)
        p1_n = P(R1, data, lam, p1, p2)
        p2_n = P(R2, data, lam, p1, p2)
        l = loglikelihood(data, lam, p1, p2)
        lam = lam_n
        p1 = p1_n
        p2 = p2_n

        # - Print the log-likelihood and the model parameters
        print('it:%2d  lambda: %.2f  p1: %.2f  p2: %.2f  log-likelihood: %.3f'
              %(it, lam, p1, p2, l))

        # - Determine if stopping criterion is satisfied
        if abs(l_old-l) < 0.001:
            criterion = True
        l_old = l

    # - Plot data histogram and the learned probability distribution
    utils.plot(data, F(lam,p1,p2))
```

```
it: 1  lambda: 1.00  p1: 0.52  p2: 0.09  log-likelihood: -51.784
it: 2  lambda: 0.97  p1: 0.53  p2: 0.15  log-likelihood: -20.707
it: 3  lambda: 0.86  p1: 0.57  p2: 0.21  log-likelihood: -20.385
it: 4  lambda: 0.69  p1: 0.64  p2: 0.26  log-likelihood: -19.753
it: 5  lambda: 0.59  p1: 0.68  p2: 0.29  log-likelihood: -19.174
it: 6  lambda: 0.56  p1: 0.70  p2: 0.30  log-likelihood: -19.000
it: 7  lambda: 0.55  p1: 0.70  p2: 0.30  log-likelihood: -18.986
it: 8  lambda: 0.55  p1: 0.70  p2: 0.30  log-likelihood: -18.986
```


1200
1000
800
600
400
200
0
0.12
0.10
0.08
0.06
0.04
0.02
0.00
p(x)
0
5
10
15
20
25
30
x