EXERCISE 1 b)

FOR THE FUNCTION:

$$p(x, y) = \lambda \eta e^{-\lambda x - \eta y} = B \cdot \lambda e^{-\lambda x}$$

WITH UNKNOWN λ, WE SOLVE THE FOLLOWING PROBLEM:

GIVEN D AND $\eta \in (0, \infty)$

$$\min_{\lambda} l(\lambda; D)$$

SUBJECT TO ~~[illegible]~~ $\lambda \in (0, \infty)$

WHERE

$$l(\lambda; D) = -\sum_{n=1}^{N} \log(B \cdot \lambda e^{-\lambda x_n})$$

$$= -N \log \lambda + \sum_{n=1}^{N} \lambda x_n + \text{CONST.}$$

SOLVE BY DIFFERENTIATION:

$$0 = \nabla l(\lambda; D) = -\frac{N}{\lambda} + \sum_{n=1}^{N} x_n$$

$$\Rightarrow \lambda = \frac{N}{\sum_{n=1}^{N} x_n}$$

I.E. THE ML ESTIMATOR FOR λ IS THE INVERSE OF THE SAMPLE MEAN

EXERCISE 1c)

FOR THE FUNCTION:

$$p(x, y) = \lambda \eta e^{-\lambda x - \eta y} \quad \text{WITH} \quad \eta = \frac{1}{\lambda}$$

$$\Rightarrow p(x, y) = e^{-\lambda x - \frac{y}{\lambda}}$$

WE PROCEED ANALOGOUSLY TO 1b), BUT NOW:

$$l(\lambda; D) = -\sum_{n=1}^{N} \log e^{-\lambda x_n - \frac{y_n}{\lambda}} = -\sum_{n=1}^{N}\left(-\lambda x_n - \frac{y_n}{\lambda}\right)$$

$$= \lambda X + \frac{Y}{\lambda} \quad \text{WHERE} \quad X \equiv \sum_{n=1}^{N} x_n \quad \text{AND} \quad Y \equiv \sum_{n=1}^{N} y_n$$

SOLVE BY DIFFERENTIATION:

$$0 = \nabla l(\lambda; D) = X - \frac{Y}{\lambda^2}$$

$$\Rightarrow \lambda^2 = \frac{Y}{X} \quad \text{OR} \quad \lambda = \pm\sqrt{\frac{\sum_{n=1}^{N} y_n}{\sum_{n=1}^{N} x_n}}$$

BUT $\lambda > 0$, SO:

$$\lambda = \sqrt{\frac{\sum_{n=1}^{N} y_n}{\sum_{n=1}^{N} x_n}}$$

EXERCISE 1d)

SUBSTITUTING FOR $\eta = (1-\lambda)$, OUR FUNCTION BECOMES:

$$p(x,y) = \lambda(1-\lambda)e^{-\lambda x-(1-\lambda)y}$$

$$\Rightarrow l(\lambda; D) = -\sum_{n=1}^{N} \log\left(\lambda(1-\lambda)e^{-\lambda x_n-(1-\lambda)y_n}\right)$$

$$= -\sum_{n=1}^{N}\left[\log(\lambda(1-\lambda)) - \lambda x_n - (1-\lambda)y_n\right]$$

$$= -N\log(\lambda(1-\lambda)) + \lambda X + (1-\lambda)Y$$

WHERE X, Y AS IN 1c)

SOLVE BY DIFFERENTIATION:

$$0 = \nabla l(\lambda; D) = \frac{-N(1-2\lambda)}{\lambda(1-\lambda)} + X - Y$$

$$\Rightarrow \lambda^2(X-Y) + \lambda(Y-X-2N) + N = 0$$

$$\Rightarrow \lambda = \frac{-Y+X+2N \pm \sqrt{(Y-X-2N)^2 - 4N(X-Y)}}{2(X-Y)}$$

$$= 1 \text{ OR } \frac{2N}{Y-X}$$

$\lambda = 1$ IS UNPERMISSIBLE SINCE IT IMPLIES $\eta = 0$

HENCE $\lambda = \dfrac{2N}{\sum_{n=1}^{N} y_n - \sum_{n=1}^{N} x_n}$ FOR ALL $\sum_{n=1}^{N} y_n > \sum_{n=1}^{N} x_n$