MATH 524 – Lecture 7 (09/12/2023)

Today: * two results on abelian groups
* orientation of simplices,
* chains, chain groups

Structure of finitely generated abelian groups

Two main results that we will use in characterizing the structure of homology groups on simplicial complexes.

Theorem 4.2 [M] Let F be a free abelian group. If R is a subgroup of F, then R is a free abelian group. If $\operatorname{rank}(F) = n$, then $\operatorname{rank}(R) = r \leq n$. Furthermore, there is a basis $e_1, \dots, e_n$ of F and numbers $t_1, \dots, t_k$ ($t_i \in \mathbb{Z}_{>0}$) such that

(1) $t_1 e_1, \dots, t_k e_k, e_{k+1}, \dots, e_r$ is a basis for R, and

(2) $t_1 | t_2 | \cdots | t_k$, i.e., t_i divides t_{i+1} $\forall\, i \geq 1$. ($i \leq k-1$).

The t_i's are uniquely determined by F and R.

Intuitively, the subgroup inherits the structure of the original group...

Theorem 4.3 [M] (Fundamental theorem of finitely generated abelian groups).
Let G be a finitely generated abelian group, and let T be its torsion subgroup. The following results hold.

(a) There is a free abelian subgroup H of G such that $G = H \oplus T$. The rank of H $rk(H) = \beta$, a finite number.

(b) There exist finite cyclic groups $T_1, \dots, T_k$ with $|T_i| = t_i > 1$, and $t_1 | t_2 | \cdots | t_k$ such that

$$T = T_1 \oplus \cdots \oplus T_k.$$

(c) The numbers β and $t_1, \dots, t_k$ are uniquely determined by G.

β is the **betti number** of G, and $t_1, \dots, t_k$ are the **torsion coefficients** of G.

"torsion" meaning "twistedness" or "cyclic nature"; as opposed to the free part.

A quick example on torsion...

Example What is the torsion subgroup of the multiplicative group $\mathbb{R}^*$ of all nonzero real numbers?
$G = \mathbb{R}/\{0\}$, operation is $*$ (multiplication), identity is 1, $g^{-1} = \frac{1}{g}$ $\forall g \in G$.
The answer is $\{1, -1\}$.

Here is the main consequence of the previous theorem:

Any finitely generated abelian group G can be written as a direct sum of cyclic groups, i.e.,
↳ is isomorphic to

$$G \simeq \underbrace{(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})}_{\beta} \oplus \mathbb{Z}/t_1 \oplus \mathbb{Z}/t_2 \oplus \cdots \oplus \mathbb{Z}/t_k$$

where $\beta \geq 1$, $t_i > 1$, and $t_i \mid t_{i+1}$ $\forall i$. This is a canonical form, called the **invariant factor decomposition** of G.

We can also get the **primary decomposition**, which is another canonical form:

$$G \simeq \underbrace{(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})}_{\beta} \oplus \mathbb{Z}/(p_1)^{n_1} \oplus \cdots \oplus \mathbb{Z}/(p_r)^{n_r}$$

for primes $p_1, \ldots, p_r$.

Examples

1. What are the betti number and torsion coefficients of

$$G = \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z} \ ?$$

$$G \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3,$$ so $\beta = 2$.

Notice that $4 \times 3 = (2)^2 \times 3 = 2 \times 6$, and $2 \mid 6$. Hence $t_1 = 2, t_2 = 6$ are the torsion coefficients.

2. Find the primary and invariant factor decompositions of $\mathbb{Z}/_4 \times \mathbb{Z}/_{12} \times \mathbb{Z}/_{18}$.

$$\cancel{(2)^2} \qquad \mathbb{Z}/_3 \times \mathbb{Z}/_4 \qquad \mathbb{Z}/_2 \times \mathbb{Z}/_9$$

We do not get $\mathbb{Z}/_4 \simeq \mathbb{Z}/_2 \times \mathbb{Z}/_2$, as 2 and 2 are not coprime.

Primary decomposition: $\mathbb{Z}/_2 \times \mathbb{Z}/_4 \times \mathbb{Z}/_4 \times \mathbb{Z}/_3 \times \mathbb{Z}/_9$.

invariant factor decomposition:

$$\begin{array}{ccc} 2 & 4 & 4 \\ & \times & \times \\ & 3 & 9 \\ \hline 2 & 12 & 36 \end{array}$$

$$\mathbb{Z}/_2 \times \mathbb{Z}/_{12} \times \mathbb{Z}/_{36}$$

Notice that $2 \mid 12 \mid 36$.

A standard "trick" is to write the factors for each prime in a line in a right justified fashion. Then multiply the numbers in each column to get the torsion coefficients.

Homology Groups

We now study groups and homomorphisms defined on simplicial complexes. Questions about topological similarity are posed as equivalent questions on corresponding groups' structure.

We need a few foundational concepts.

Orientation of a simplex

Let σ be a simplex (geometric or abstract). We define two orderings of its vertex set to be equivalent if they differ by an even permutation, i.e., you can go from one ordering to the other using an even number of pairwise swaps.

If $\dim(\sigma) > 0$, the orderings fall into two equivalence classes. Each class is an **orientation** of σ.

If $\dim(\sigma) = 0$, it has only one orientation.

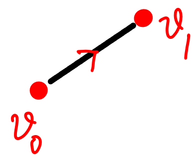
An **oriented simplex** is a simplex σ together with an orientation of σ.

Notation: Let $v_0, \ldots, v_p$ be independent. Then $\sigma = v_0 v_1 \cdots v_p$ is the simplex spanned by $v_0, \ldots, v_p$, and $[v_0, \ldots, v_p]$ denotes the oriented simplex σ with the orientation $(v_0, \ldots, v_p)$.

GI if $\bar{v}_0, \ldots, \bar{v}_p \in \mathbb{R}^d$ and distinct if $v_0, \ldots, v_p$ are (just) 'labels' in the abstract setting.

When it is clear from the context, we will use σ to denote both the simplex as well as its orientation (or the oriented simplex).

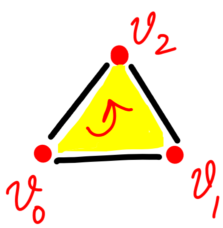
1-simplex $[v_0, v_1]$, $[v_1, v_0] \rightarrow$ opposite orientation



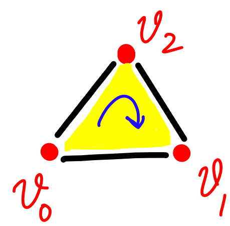
$[v_0, v_1]$ equivalent to orienting the edge from v_0 to v_1.
$[v_1, v_0] \rightarrow$ draw the arrow the other way.

2-simplex Notice that $[v_0, v_1, v_2]$ is the same as $[v_1, v_2, v_0]$.

$(v_0 v_1 v_2) \rightarrow (v_1 v_0 v_2) \rightarrow (v_1 v_2 v_0)$ two pairwise swaps

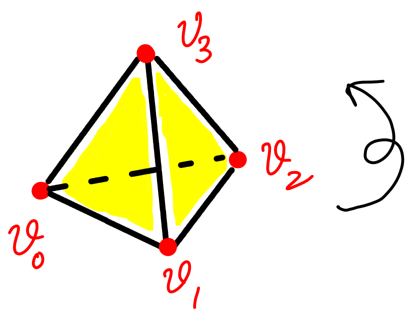


$[v_0 v_1 v_2] \rightarrow$ can be the counterclockwise orientation



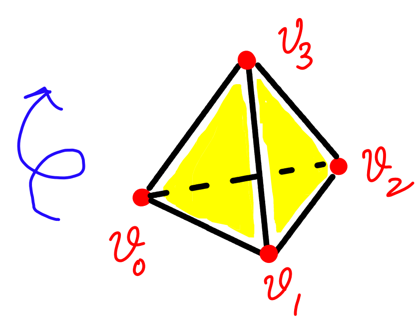
$[v_0 v_2 v_1] \rightarrow$ is the clockwise orientation

3-simplex $[v_0 v_1 v_2 v_3]$



We could imagine orienting the tetrahedron as per the right-hand thumb rule — $v_0 \rightarrow v_1 \rightarrow v_2$ as the fingers of your right hand curl around, and $v_2 \rightarrow v_3$ points up along your thumb.

Notice that $[v_0 v_2 v_1 v_3]$, the opposite orientation, then corresponds to the left-hand thumb rule.



Starting with oriented simplices, we define collections of them as functions, and then consider adding these functions — to define groups.

Def Let K be a simplicial complex. A **p-chain** on K is a function c from the set of oriented p-simplices of K to $\mathbb{Z}$ such that

(1) $c(\sigma) = -c(\sigma')$ if σ, σ' are opposite orientations of the same simplex; and

(2) $c(\sigma) = 0$ for all but finitely many p-simplices σ.

Thus, even on infinite simplicial complexes, each p-chain has nonzero values on only finitely many p-simplices.

We can add two p-chains by adding their values. The resulting group is the group of oriented p-chains of K, $\mathbf{C_p(K)}$. If $p<0$ or $p>\dim(K)$, $C_p(K)$ is trivial.

One can indeed check that $C_p(K)$ is a group - identity (0), inverse $(c(\sigma'))$, and associativity all hold. In fact $C_p(K)$ are abelian groups, as adding the functions is commutative.

Are there really only two orientations of higher dimensional simplices?

YES! Consider a 3-simplex $\sigma = [v_0 v_1 v_2 v_3]$ and $-\sigma = [v_1 v_0 v_2 v_3]$, its reverse orientation. What about $[v_2 v_0 v_3 v_1]$, for instance?

$$[v_2 v_0 v_3 v_1] \xrightarrow{1} [v_0 v_2 v_3 v_1] \xrightarrow{2} [v_0 v_2 v_1 v_3] \xrightarrow{3} [v_0 v_1 v_2 v_3]$$ 3 swaps, i.e., odd.

Hence $[v_2 v_0 v_3 v_1]$ should be the opposite orientation to $[v_0 v_1 v_2 v_3]$. But then it should be the same orientation as $[v_1 v_0 v_2 v_3]$.

Let's check: $$[v_2 v_0 v_3 v_1] \xrightarrow{1} [v_1 v_0 v_3 v_2] \xrightarrow{2} [v_1 v_0 v_2 v_3]$$ 2 swaps, i.e., even!

For oriented simplex σ, the **elementary chain** c corresponding to σ is the function defined as follows:

$c(\sigma) = 1$,

$c(\sigma') = -1$, where σ' is the opposite orientation of σ,

$c(\tau) = 0$, $\forall\, \tau \neq \sigma$.

The correspondence to unit vectors in a Euclidean space is indeed direct here. The elementary chains have value $+1$ for exactly one p-simplex. Later on, we will see that these elementary chains correspond to unit vectors representing each p-simplex — at least in the case when K is finite.

Notation: σ denotes the simplex, oriented simplex, or the elementary chain corresponding to the simplex. Then we can write $\sigma' = -\sigma$ (where σ' is the simplex with orientation opposite to that of σ).

Lemma 5.1 [M] $C_p(K)$ is free abelian, and a basis for $C_p(K)$ can be obtained by orienting each p-simplex, and using the corresponding elementary chains.

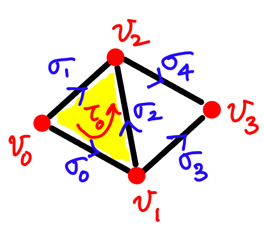
Notice that $C_0(K)$ has a "natural" basis, since each 0-simplex has only one orientation. But we do need to choose an orientation for each p-simplex to get a basis when $p > 0$. And there exist many bases when $p > 0$.

Corollary [M] Any function f from oriented p-simplices of K to abelian group G extends naturally to a homomorphism from $C_p(K)$ to G provided $f(-\sigma) = -f(\sigma)$ for all oriented p-simplices σ in K.

reverse orientation of σ.

Let's consider a small example.

K:



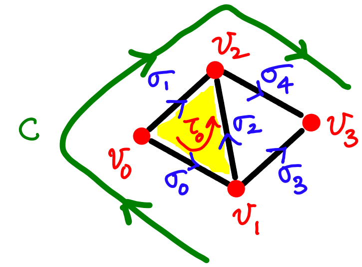
Let K be a shown here, with 4 vertices (v_0, v_1, v_2, v_3), 5 edges $(\sigma_0 - \sigma_4)$, and 1 triangle (τ_0). We orient the edges lexicographically, i.e., $[v_i v_j]$ with $i < j$. The triangle τ_0 is oriented as $[v_0 v_1 v_2]$, or CCW as shown here.

A 1-chain c can be specified as follows:

$c(\sigma_0) = -1$

$c(\sigma_1) = 1 \quad c(\sigma_2) = 0,$

$c(\sigma_4) = 1 \quad c(\sigma_3) = 0$

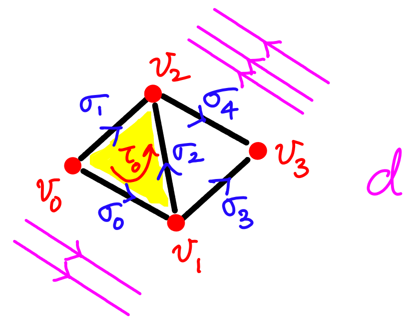


Notice that a value of -1 on σ_0 could be interpreted as "traversing" σ_0 once in its reverse orientation, i.e., going from v_1 to v_0 once. Following the same logic, we see that c here represents the piecewise linear "curve" going $v_1 \to v_0 \to v_2 \to v_3$ (once).

Here is another 1-chain d:

$d(\sigma_0) = 2, \; d(\sigma_4) = -3$

$d(\sigma_j) = 0, \; j = 1, 2, 3.$



In particular, notice that the chains need not represent single connected pieces all the time.

A 2-chain can be $g(\tau_0) = 2$, which represents two copies of the single triangle in K.

Now that we have defined the chain groups $C_p(K)$ for each p, we now talk about how to connect/relate the $C_p(K)$ for various p. In particular, how are $C_p(K)$ and $C_{p-1}(K)$ related?