MATH 529 — Lecture 26 (04/11/2024)

Today: * torsion in homology groups
* More on OHCP: LP/IP, total unimodularity ...

Recall: $H_1(\mathbb{K}^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $H_1(\mathbb{RP}^2, \mathbb{Z}) \cong \mathbb{Z}_2$, $H_1(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Torsion subgroup: If G is an Abelian group, an element $g \in G$ has **finite order** if $ng = 0$ for some $n \in \mathbb{Z}$, $n > 0$. ($ng \rightarrow g*g*\cdots*g$ (n times))

The set of all elements of finite order in G is a subgroup T of G, called the **torsion** subgroup.
If T does not contain any element other than the identity of G, we say G is **torsion-free**.

Fundamental Theorem of Finitely Generated Abelian Groups

Every finitely generated Abelian group G has a decomposition $G \cong F \oplus T$, where F is free abelian and T is the torsion subgroup of G, such that

$F \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\beta \text{ copies}}$ and $T \cong \mathbb{Z}/t_1 \oplus \mathbb{Z}/t_2 \oplus \cdots \oplus \mathbb{Z}/t_k$

where $t_i \in \mathbb{Z}$, $t_i > 1$, and $t_1 | t_2 | \cdots | t_k$.

$t_1 | t_2$ means t_1 divides t_2

This form $F \oplus T$, with the structure of F and T as shown, is a canonical form of G.

β is the **Betti number**, and t_i's are the **torsion coefficients** of G.

e.g., $H_1(\mathbb{K}^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \Rightarrow \beta = 1,\ t_1 = 2$.
$H_1(\mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}_2 \Rightarrow \beta = 0,\ t_1 = 2$.

Back to OHCP

(OHCP → cycle)

Given $\bar{c}$ in $[\bar{c}] \in H_p(K)$, find $\bar{x} \in [\bar{c}]$ with $\|\bar{x}\| = \sum_{i=1}^{m} w_i |x_i|$, where $w_i \geq 0$ is the weight for σ_i the i^{th} p-simplex, is smallest.

Q. Does an optimal homologous cycle always exist?

Yes, as long as K is finite and all $w_i \geq 0$.

We can prove that an optimal chain does indeed exist, mainly because of the finiteness of the complex. The challenge is in finding it efficiently.

Let's consider the optimization model for OHCP:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} w_i |x_i| \\
\text{s.t.} \quad & \bar{x} = \bar{c} + [\partial_{p+1}] \bar{y} \\
& \bar{x} \in \mathbb{Z}^m, \ \bar{y} \in \mathbb{Z}^n
\end{aligned}
$$

min ← minimize

s.t. ← "subject to"

$\sum w_i|x_i|$ → piecewise linear → can linearize by replacing x_i with $x_i^+ - x_i^-$ in constraints, and $x_i^+ + x_i^-$ in the objective function as shown below, along with $x_i^+, x_i^- \geq 0$.

K has m p-simplices and n $(p+1)$-simplices

$$
\left.
\begin{aligned}
\min \quad & \sum_{i=1}^{m} w_i (x_i^+ + x_i^-) \\
\text{s.t.} \quad & \bar{x}^+ - \bar{x}^- = \bar{c} + [\partial_{p+1}] \bar{y} \\
& \bar{x}^+, \bar{x}^- \in \mathbb{Z}^m_{\geq 0}, \ \bar{y} \in \mathbb{Z}^n
\end{aligned}
\right\}
$$

linear optimization problem with integrality constraints.

$\mathbb{Z}_{\geq 0}$ ← nonnegative

Intuition:

$|-3| = 3$. We write

$-3 = 2-5$, or

$-3 = 2024-2027$, or

$-3 = 0 - 3$.

In general, we want $-3 = x^+ - x^-$ (positive part: x^+; negative part: x^-)

$x^+, x^- \geq 0$,

and choose x^+, x^- such that $x^+ + x^-$ is minimum.

Similarly, $5 = 5 - 0$ (rather than $11-6$, for instance).

We can show that only one of x^+ and x^- will be > 0 in each such representation.

$\Rightarrow$ The OHCP can be modeled as an integer program (IP) (an integer linear program, or ILP, to be exact).

But IPs are hard to solve in general (NP-hard), while LPs can be solved in polynomial time.

$$\min\{\bar{c}^T\bar{x} \mid A\bar{x}=\bar{b}, \bar{x} \geq \bar{0}, \underline{\bar{x} \in \mathbb{Z}}\} \quad \text{—— (IP)}$$

$$\min\{\bar{c}^T\bar{x} \mid A\bar{x}=\bar{b}, \bar{x} \geq \bar{0}\} \quad \text{—— (LP)}$$

drop the integrality restriction to get the (LP) relaxation

The OHCP IP can be written in the above standard form of (IP).

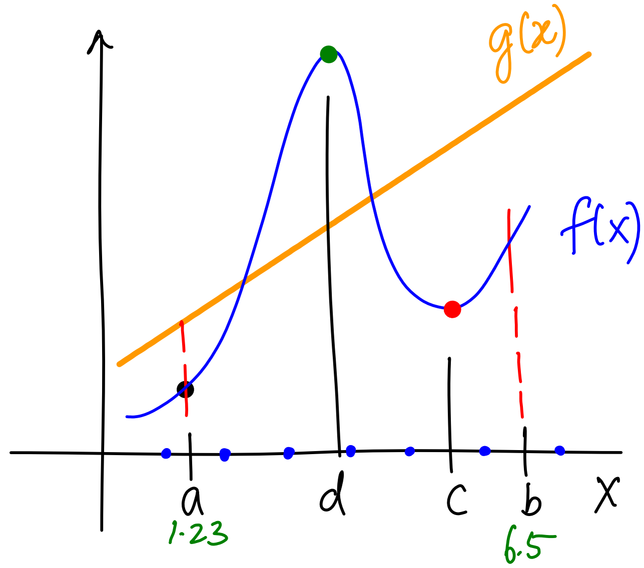
Let's review LP and IP from a 10,000-ft level...

Optimization for dummies

In calculus, we learned how to min $f(x)$ over $x \in [a,b]$.

We need:

$f'(x)=0$, and $f''(x)>0$ for x to be a minimum



$X=C$ is a (local) minimum,
but $X=a$ is the minimum of $f(x)$ over $x \in [a,b]$.

If $g(x)$ is linear, we need to check only the end points $x=a$ and $x=b$ to find the minimum.

Linear programming (LP) extends this linear case (of $g(x)$) to higher dimensions. The feasible set is the intersection of half spaces, and we need to still look only at vertices (or corner points). We can solve LPs "efficiently", i.e., in polynomial time.

Integer (linear) programming (IP) aims to find solutions to (LP) that have integer values. (IP) cannot be solved in polynomial time in the worst case.

Recall OHCP LP/IP

$$\min \sum_{i=1}^{m} w_i (x_i^+ + x_i^-)$$
$$\text{s.t.} \quad \bar{x}^+ - \bar{x}^- = \bar{c} + [\partial_{p+1}]\bar{y}$$
$$\bar{x}^+, \bar{x}^- \in \mathbb{Z}^m_{\geq 0}, \quad \bar{y} \in \mathbb{Z}^n$$

(IP) — (LP) with $\bar{x}^+, \bar{x}^- \geq \bar{0}$

Let's look at a 2D example that highlights the differences between (IP) and (LP).

A 2D LP/IP Example

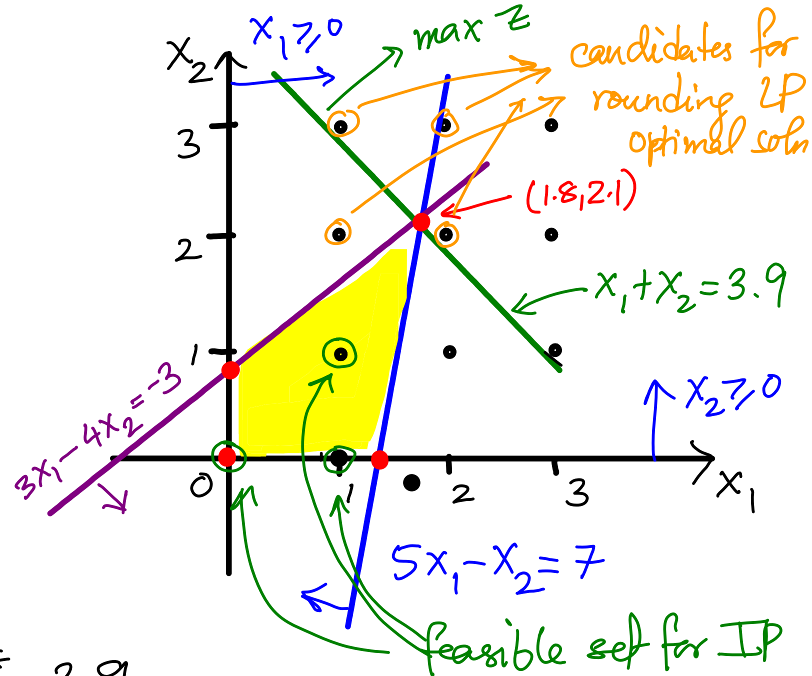
max $z = x_1 + x_2$

s.t. $5x_1 - x_2 \leq 7$

$3x_1 - 4x_2 \geq -3$

$x_1, x_2 \geq 0$

$x_1, x_2 \in \mathbb{Z}$



We slide the z-line in the direction of increase of z, while staying feasible.

Optimal solution is at (1.8, 2.9), $z^* = 3.9$.

For solving the IP, rounding (1.8, 2.9) will not work, as none of the options are even feasible here! Hence ignoring the integrality restriction could be quite bad.

But there are some special cases where we solve just the LP, and get integrality "for free"! One such special case involves the total unimodularity of the constraint matrix, and we will explore this concept.

Q. Why not solve OHCP over $\mathbb{Z}_2$?

Result: Chen and Freedman (2010)

OHCP over $\mathbb{Z}_2$ is NP-hard (even to approximate).

We could simulate addition over $\mathbb{Z}_2$ in the OHCP LP by setting $\bar{x}^+ - \bar{x}^- = \bar{c} + [\partial_{p+1}]\bar{y} + 2\bar{u}$, with $\bar{u} \in \mathbb{Z}^m$.

But the $2\bar{u}$ term destroys "nice structure"!

OHCP LP in standard form:

$$\left.\left.\begin{aligned} \min \ & \sum_{i=1}^{m} w_i(x_i^+ + x_i^-) \\ \text{s.t.} \ & \bar{x}^+ - \bar{x}^- = \bar{c} + [\partial_{p+1}](\bar{y}^+ - \bar{y}^-) \\ & \bar{x}^+, \bar{x}^- \geq \bar{0}, \ \bar{y}^+, \bar{y}^- \geq \bar{0} \end{aligned}\right\} (LP) \\ \bar{x}^+, \bar{x}^- \in \mathbb{Z}^m, \ \bar{y}^+, \bar{y}^- \in \mathbb{Z}^n \right\} (IP)$$

Writing it in standard form, we get

$$\begin{aligned} \min \ & \underbrace{[\bar{w}^T \ \bar{w}^T \ \bar{0} \ \bar{0}]}_{\bar{c}^T} \bar{x} \\ \text{s.t.} \ & \underbrace{[I \ -I \ -B \ B]}_{A} \underbrace{\begin{bmatrix} \bar{x}^+ \\ \bar{x}^- \\ \bar{y}^+ \\ \bar{y}^- \end{bmatrix}}_{\bar{x}} = \underbrace{\bar{c}}_{\bar{b}} \\ & \bar{x} \geq \bar{0} \\ & \bar{x} \in \mathbb{Z}^{2m+2n} \end{aligned}$$

standard form LP:

$$\begin{aligned} \min \ & \bar{c}^T\bar{x} \\ \text{s.t.} \ & A\bar{x} = \bar{b} \\ & \bar{x} \geq \bar{0} \end{aligned}$$

There is a well-studied special case where IP can be solved as an LP.

Total Unimodularity (TU)

$A \in \mathbb{Z}^{m\times n}, \bar{b} \in \mathbb{Z}^{m}$

Result Let $\min\{\bar{c}^T\bar{x} \mid A\bar{x}=\bar{b}, \bar{x}\geq\bar{0}, \bar{x}\in\mathbb{Z}^n\}$ — (IP)

and $\min\{\bar{c}^T\bar{x} \mid A\bar{x}=\bar{b}, \bar{x}\geq\bar{0}\}$. — (LP)

IP can **always** ($\forall\, \bar{c}\in\mathbb{R}^n$) be solved in polynomial time by solving (LP) iff A is totally unimodular (TU).
Equivalently, the vertices of (LP) have integer coordinates.

We consider whether we could use this result for the OHCP. We ask
When is $A = [I \; -I \; -B \; B]$ TU?

Def A matrix $B \in \mathbb{Z}^{m\times n}$ is **totally unimodular** (TU) if every subdeterminant of B is in $\{-1,0,1\}$.

B is **unimodular** if every nonsingular $m\times m$ submatrix of B has determinant ± 1.

In particular, $B_{ij} \in \{0,1,-1\}$.