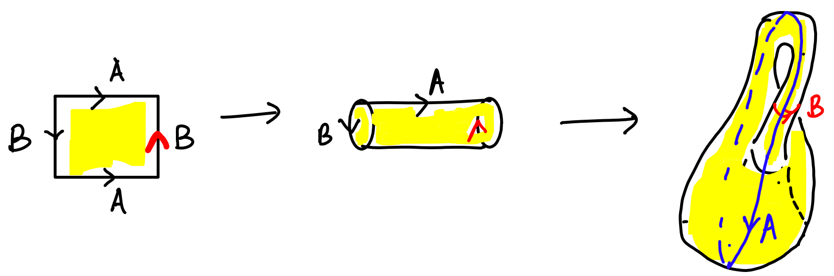
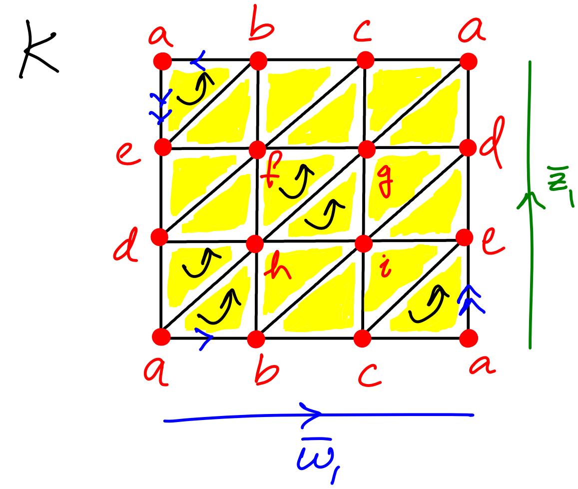
MATH 524 – Lecture 10 (09/21/2023)

Today: * Homology of $\mathbb{K}^2$, $\mathbb{R}P^2$
* Homology of connected sums

Theorem 6.3 [M] (Klein bottle)





Let K be the complex shown, and L the rectangle with the labels. $|K| \approx$ Klein bottle. Then

$$H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}/_2 \quad \text{and} \quad H_2(K) = 0.$$

Let $\bar{w}_1 = [a,b] + [b,c] + [c,a]$ and
$\bar{z}_1 = [a,e] + [e,d] + [d,a]$

Then the torsion subgroup of $H_1(K)$ is represented by $\bar{z}_1$, and the free part is generated by $\bar{w}_1$.

Proof

We follow the same technique as in the case of the torus. Indeed, we can push an input chain off of all the edges in the middle, as before. But notice that the edges in A (boundary of the square) do not all behave identically. For instance, $[a,b]$ does get opposite orientations from $\partial_2[abh]$ and $\partial_2[aeb]$. But $[a,e]$ gets $+1$ from both $\partial_2[aec]$ and $\partial_2[aeb]$. As such, $\partial_2 \bar{r} \neq 0$ here!

Similar to The previous case, we get the following results.

(1) Every 1-cycle of K is homologous to a 1-cycle carried by A.

(2) If $\bar{d}$ is a 2-chain of K and $\partial\bar{d}$ is carried by A, then $\bar{d}$ is a multiple of $\bar{r}$.

We also get (3):

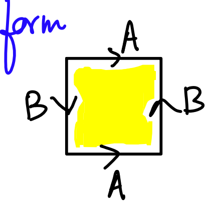
(3) If $\bar{c}$ is a 1-cycle carried by A, then $\bar{c} = m\bar{w}_1 + n\bar{z}_1$ for $m, n \in \mathbb{Z}$.

But instead of (4), we get

(4') $\partial_2 \bar{r} = 2\bar{z}_1$.

Like last time, we get that $\bar{c}$, a 1-cycle of K, is homologous to $m\bar{w}_1 + n\bar{z}_1$. If $\bar{c} = \partial_2\bar{d}$, then $\bar{c} = \partial_2\bar{d} = 2p\bar{z}_1$, $p \in \mathbb{Z}$. Hence, $\bar{c}$ is a boundary iff $m = 0$, n is even. Hence we get $H_1(K) \simeq \underbrace{\mathbb{Z}}_{\text{free}} \oplus \underbrace{\mathbb{Z}/2}_{\text{torsion.}}$, with $\bar{w}_1$ generating the free part and $\bar{z}_1$ generating the torsion part.

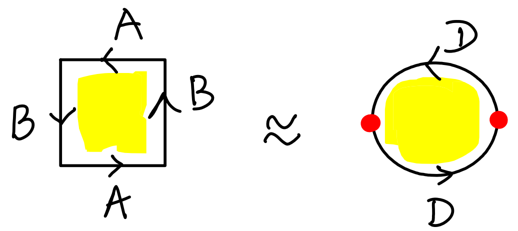
Intuitively, one can see from the "pasting" picture itself that the boundary of the square space is 2B. In the case of the torus, both A and B do not form boundaries, but here, 2B is the boundary.

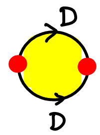


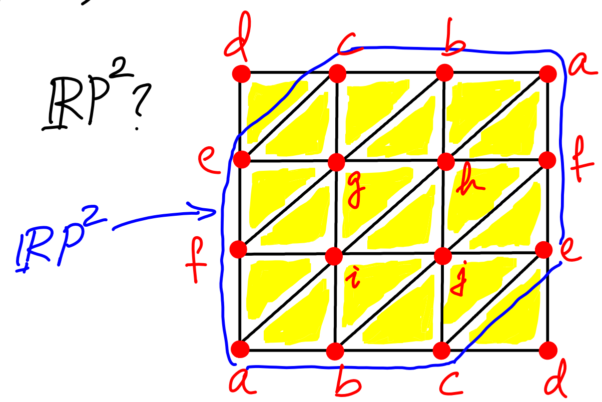
$H_2(K)$: Since $\partial_2\bar{r} = 2\bar{z}_1 \neq 0$, $Z_2(K) = 0$, and hence $H_2(K) = 0$.

Intuitively, the Klein bottle does not enclose a 3D space like the torus. Hence, its 2nd homology group is trivial.

Theorem 6.4 [M] $\mathbb{RP}^2$ (projective plane)

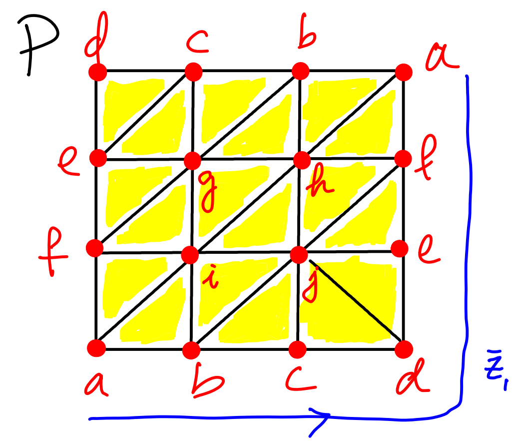


(Note :  $\approx S^2$!)



Is there a problem with the simplicial complex? Notice that edge $[c,e]$ is shared by three triangles $[cde]$, $[ceg]$, $[cej]$. If you work it out, this simplicial complex is in fact almost correct, i.e., its underlying space is $\mathbb{RP}^2$ with a "flap" which is triangle $[cde]$.

But we can fix the simplicial complex by flipping one copy of $\overline{ce}$ for an off-diagonal edge, e.g., $\overline{dj}$.
You can check that every edge in the simplicial complex P is shared by exactly two triangles.



Let P be the simplicial complex, and L is the underlying space (rectangle).
$|P| \approx \mathbb{RP}^2$. We get $H_1(P) \simeq \mathbb{Z}/2$, and $H_2(P) = 0$.

<u>Proof</u>

Let $g: |L| \to |P|$ be the pasting map, and let $A = g(|\mathrm{Bd}\,L|)$.
Here, A is a circle. Let

$$\bar{z}_1 = [a,b] + [b,c] + [c,d] + [d,e] + [e,f] + [f,a].$$

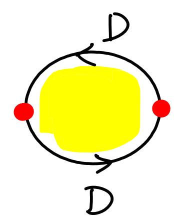
We get corresponding results to (1)-(4) as for torus & Klein bottle. (1) and (2) hold as before. Again, we push chains off of the edges in the middle, and can give criteria describing the homology in terms of structure of cycles carried by A, the boundary of $|L|$. In place of (3), (4), we get the following results.

(3') Every 1-cycle carried by A is a multiple of $\bar{z}_1$.

(4'') $\partial_2 \bar{r} = 2\bar{z}_1$.

Hence $H_1(P) \simeq \mathbb{Z}/2$, $H_2(P) = 0$. $\rightarrow$ as $\partial_2 \bar{r} \neq 0$

We could come to the same conclusion directly from this diagram – notice that ∂D is the boundary of the 2D space modeled by the disc.

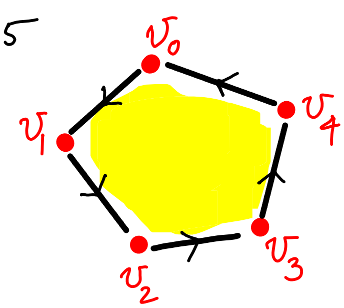


Hence, $H_1(P)$ has no free part, but only the torsion part.

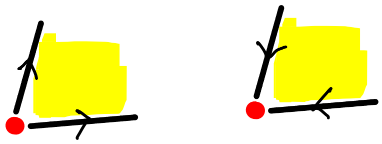
So, $\beta_1(P) = 0$, and $\bar{z}_1$ generates the torsion part.

The projective plane is a special case of the k-fold dunce hat for $k=2$. The general space is obtained by taking a k-sided polygon (k-gon) with vertices $v_0, \dots, v_{k-1}$ and edges $v_i v_{i+1}$ for $i = 0, \dots, k-2$ along with $v_{k-1} v_0$. We then identify consecutive pairs of edges $[(v_i v_{i+1})$ and $(v_{i+1} v_{i+2})]$.

$k=5$ D_k



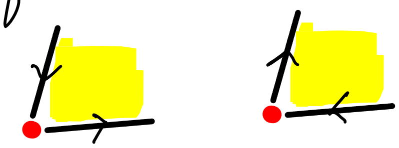
The arrows here indicate how you identify the edges, and not their orientations.



no twist

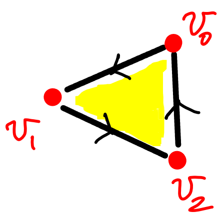
When "gluing" edges connected to a single node, with the arrows indicating the order of identification/gluing, if the arrows both come in or both go out, then we glue them without a twist.

But if one arrow comes in and the other goes out, we glue with a twist, as we do in the case of Möbius strip.



with twist

D_3
$(k=3)$

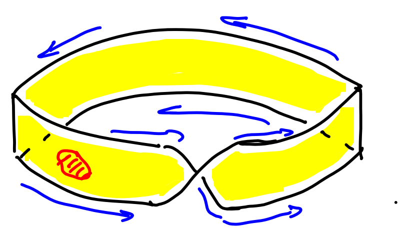


$$H_1(D_k) \simeq \mathbb{Z}/k.$$

Essentially, the boundary of the 2D space modeled by the k-gon is the cycle going k-times around.

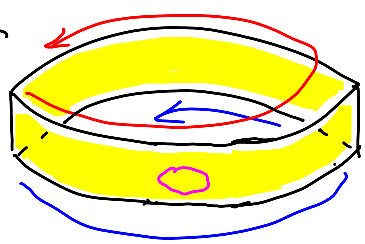
Here are two more examples — Möbius strip and cylinder.

M



$$H_1(M) \simeq \mathbb{Z}.$$

C



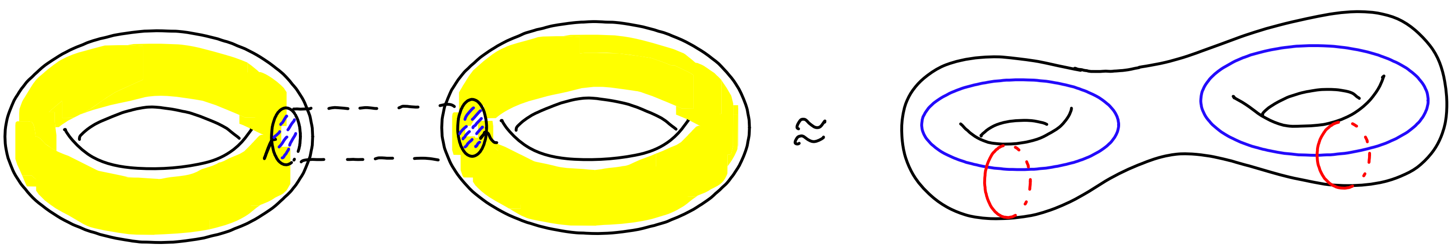
$$H_1(C) \simeq \mathbb{Z}$$

Indeed, both the cylinder and Möbius strip have the same homology groups (and not just H_1).

We now talk about how to join surfaces to get more general surfaces, and how to figure out their homology groups

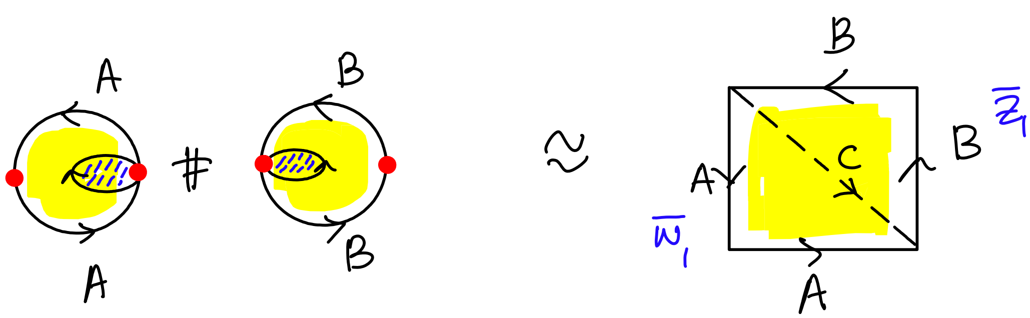
Def The **connected sum** of two surfaces S_1 and S_2 is the space obtained by deleting an open disc from each, and pasting the remaining pieces along the edge of the removed disc. We denote this connected sum as $S_1 \# S_2$.

e.g., $\mathbb{T}^2 \# \mathbb{T}^2$ double torus



$H_1(\mathbb{T}^2 \# \mathbb{T}^2) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

$\underline{\mathbb{RP}^2 \# \mathbb{RP}^2}$



Theorem 6.5 [M] $H_1(\mathbb{RP}^2 \# \mathbb{RP}^2) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$ and

$$H_2(\mathbb{RP}^2 \# \mathbb{RP}^2) = 0.$$

We can figure out the structure of these homology groups directly from the diagram above, rather than use a simplicial complex.

Let $\overline{w}_1$ be the 1-cycle represented by "A" (left and below), and $\overline{z}_1$ be the 1-cycle represented by "B" (right and above). We get (1) and (2) as before, and (3') & (4') as follows.

(3') Every 1-cycle carried by A is of the form $m\overline{w}_1 + n\overline{z}_1$, $m, n \in \mathbb{Z}$.

(4') $\partial_2 \overline{r} = 2\overline{w}_1 + 2\overline{z}_1$.

So $H_2(\mathbb{R}P^2 \# \mathbb{R}P^2) = 0$. What about $H_1(\mathbb{R}P^2 \# \mathbb{R}P^2)$?

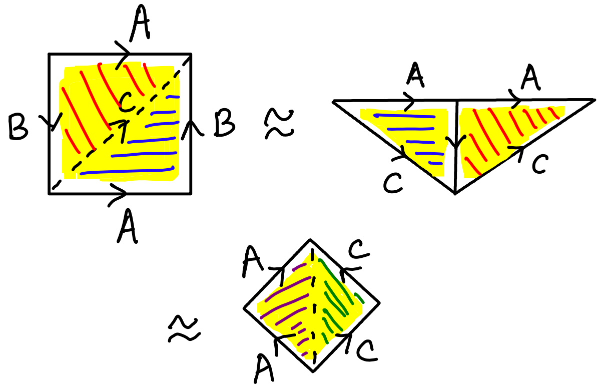
$\{\overline{w}_1, \overline{z}_1\}$ is a basis, but the torsion part is not "separated" in the basis. We can use $\{\overline{w}_1, \underbrace{\overline{w}_1 + \overline{z}_1}_{\overline{z}_1'}\}$ as another basis instead, as $\overline{z}_1 = -(\overline{w}_1) + (\overline{w}_1 + \overline{z}_1)$.

With $\{\overline{w}_1, \overline{z}_1'\}$ as the basis, we can directly see that $2\overline{z}_1'$ is a boundary, so $H_1(\mathbb{R}P^2 \# \mathbb{R}P^2) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

We could've used $\{3\overline{w}_1 + 4\overline{z}_1, \overline{w}_1 + \overline{z}_1\}$ also! Or $\{\overline{z}_1, \overline{w}_1 + \overline{z}_1\}$.

Notice that $H_1(\mathbb{R}P^2 \# \mathbb{R}P^2) \simeq H_1(\mathbb{K}^2)$. In fact, $\mathbb{R}P^2 \# \mathbb{R}P^2 \approx \mathbb{K}^2$!

Here's a proof by picture.



To glue the "B" edges with a twist, we first take the mirror image of the right triangle across the horizontal axis and then slide it over to the left side to glue. The resulting space is indeed the connected sum of two $\mathbb{R}P^2$'s, as shown below.