MATH 529: Lecture 1 (01/13/2026)

Today: * syllabus, logistics
* two motivating applications

Call me Bala!

Introduction to **Computational** topology → focus for this course

This course will be offered completely electronically:
- scribes will be posted as "course notes"; videos will also be posted.
- assignments to be turned in electronically
  (you could submit scanned versions of handwritten assignments).
- web page has all the docs/info.

Topology "Topo" → place or space, "logos" → study or talk } in Greek

Topology talks about how space is connected.

topology
- point set topology (open/closed, connected, ...)
- algebraic topology (groups, addition, basis, ...)

We will concentrate on algebraic topology.

Computational topology

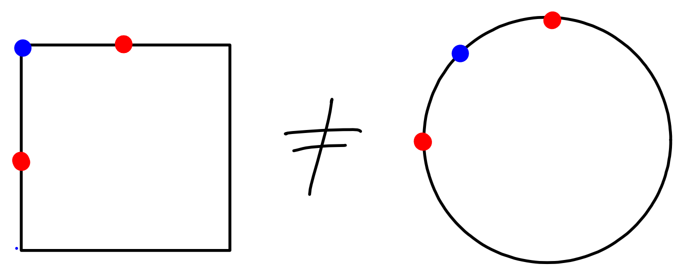
→ combine efficient algorithms and data structures with results from topology to analyze real-life data.

Let's start with an intuition for what we mean by connectivity of spaces.

An Example

According to geometry, the square and circle are not equal.

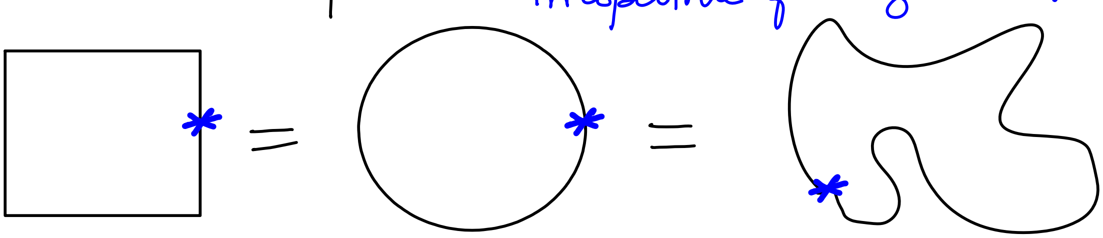
Size (length, here) is critical in geometry, but not so much in topology.



But topology says they are same as far as how they are connected!

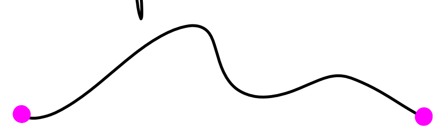
For instance, take a string, and tie a knot to make a loop.

We want to study connectivity irrespective of size here!



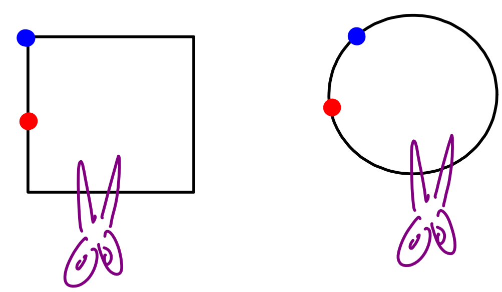
All these objects are same, i.e., they are connected the same way.

But, if you did not tie the knot, the loose string (open thread) differs from any of the above tied loops in connectivity.

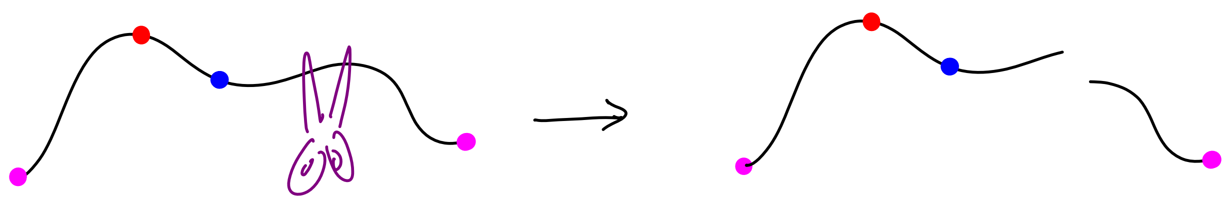


Note that the two end points have different "neighborhoods."

Here is another way to understand connectivity. Consider cutting the string (tied into a loop) once. Such a cut leaves the string in one piece, i.e., connected.

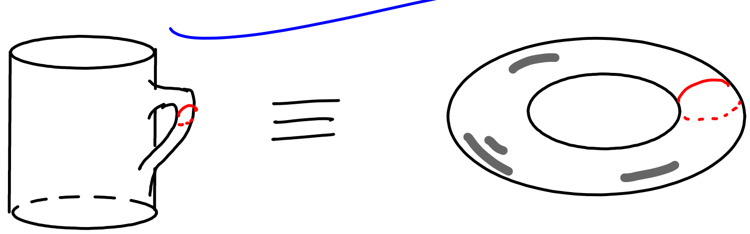


But cutting the open thread once leaves two pieces, i.e., it is disconnected.



two pieces

A popular quote: "A topologist cannot distinguish the coffee cup from a donut!"



the coffee cup, if made of clay, could be deformed to look like the donut, without changing its connectivity.

A more practical example: how we are able to read (recognize) letters of the alphabet in different fonts.

A **A** $\mathcal{A}$ $\neq$ B **B** $\mathcal{B}$ $\neq$ C C **C**

Two Illustrations of Computational Topology

1. Patient antibiotic trajectories

https://doi.org/10.1145/3307339.3342143

? How do agents and doses affect length of stay?

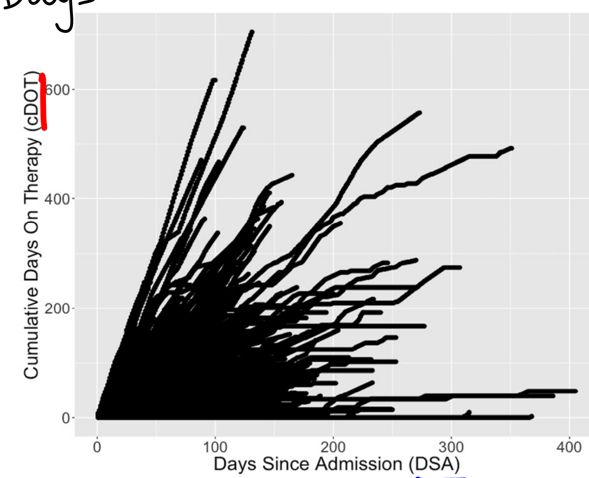
Number of hospitals	25
Number of hospital unit-categories	9
Number of distinct patient-admission records	349, 610
Number of adult patients	334, 207
Number of male patients	148, 540
Number of female patients	201, 052
Average LOS per admission	7 days
Longest LOS → length of stay	405 days
Number of antibacterials used	66
Most used antibacterial	Vancomycin
Average DOT per admission	6
Number of agent ranks	4
Most used agent rank	rank 3

→ Days On Therapy

If a patient gets one dose of an agent (antibiotic) that is counted as 1 Day On Therapy (DOT).

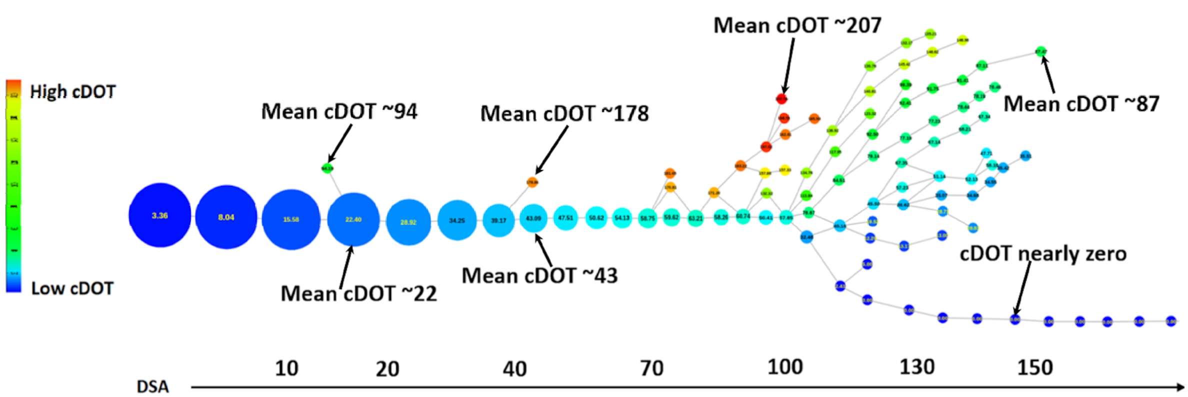
cDOT : cumulative Days On Therapy

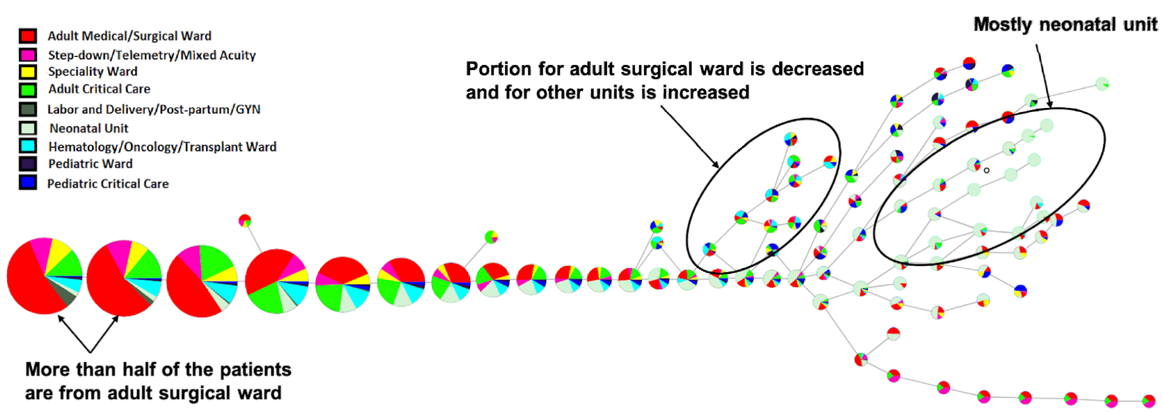
DSA : Days Since Admission.



This plot by itself is not very informative or insightful (even if we were to use color...)

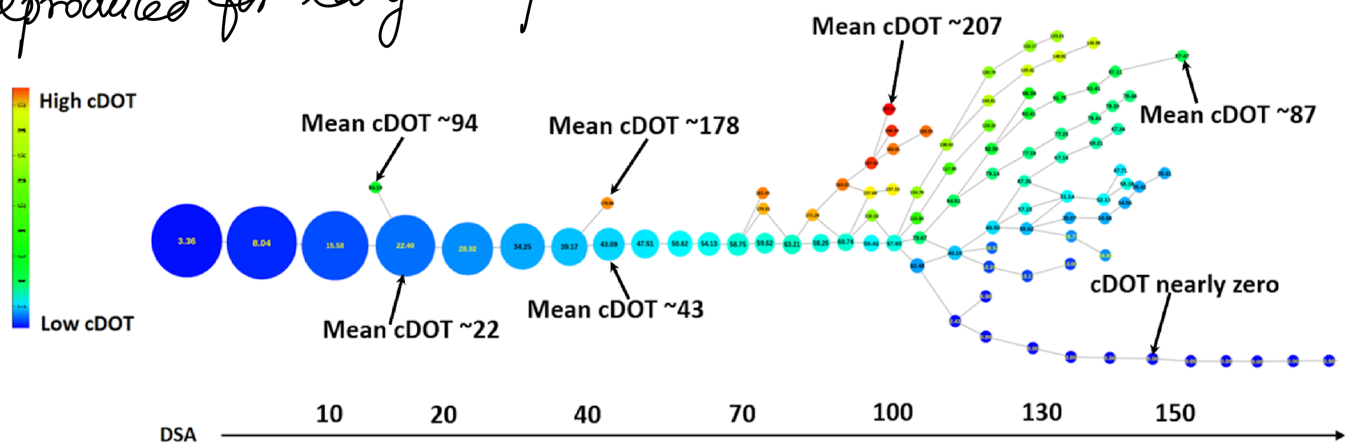
Here are two versions of a Mapper representation of the same data:

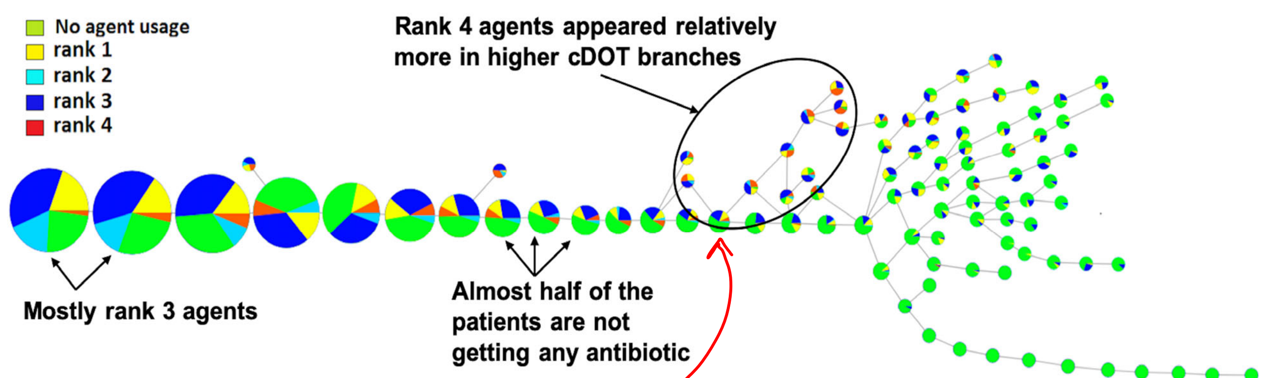




Each node represents a cluster of patient trajectories. A lot of the patients got low cDOTs and they had small(er) DSAs – as captured by the big clusters on the left. As the second Mapper shows, many of these patients were treated in the adult surgical ward — one of the most common types of admissions to hospitals.

Here is another version of the Mapper showing ranks (1–4, 4 is strongest) of the antibiotics. The first mapper (using cDOT) is reproduced for easy comparison.



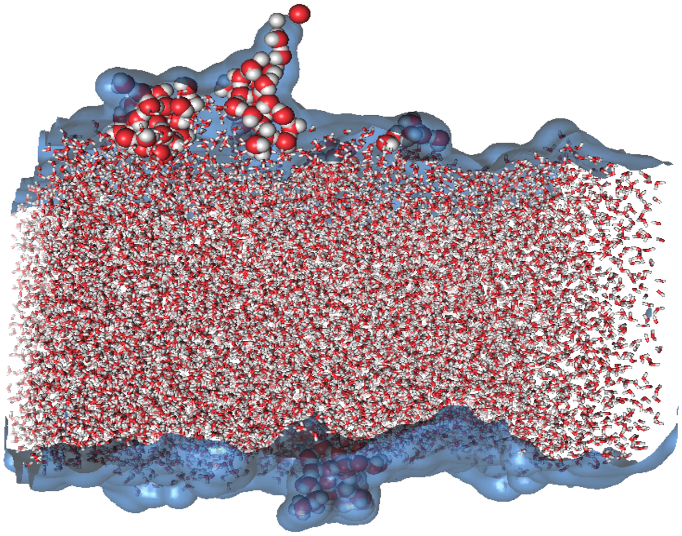


The high cDOT + high rank sub-branches had more patients in the other (higher risk) wards. Similarly, the much higher DSA group (120+) on the right end with relatively smaller cDOT values turned out to be patients in neonatal ICU's.

Note that these nontrivial subgroups are identified in an unsupervised manner — no learning is involved!

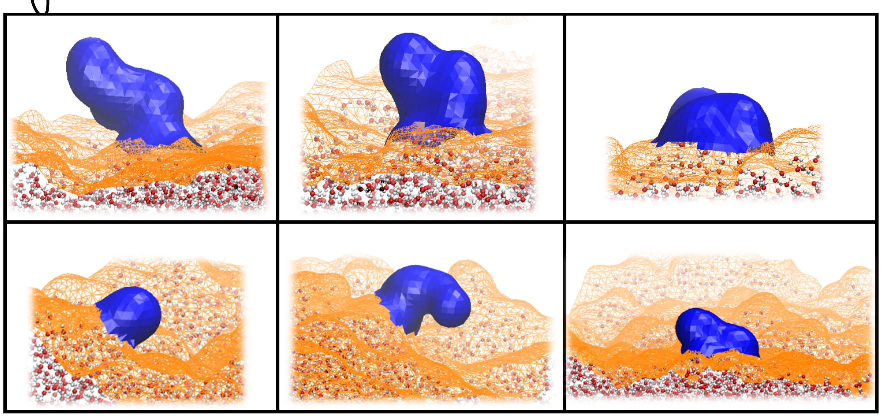
2. Interface features in Chemistry

https://doi.org/10.1021/acs.jctc.0c00260 (https://doi.org/10.26434/chemrxiv.11988048.v1) → preprint



An interface surface separates a water layer from a hexane (organic) layer. When a reagent is added, the reaction is initiated and the water molecules escape to the hexane layer through finger-like features in the interface called "protrusions". These features were identified manually (by observation!)

The goal was to identify and characterize protrusions using geometric measure theory and computational topology.



Which of these six features do you think are protrusions?

It is not easy to guess!

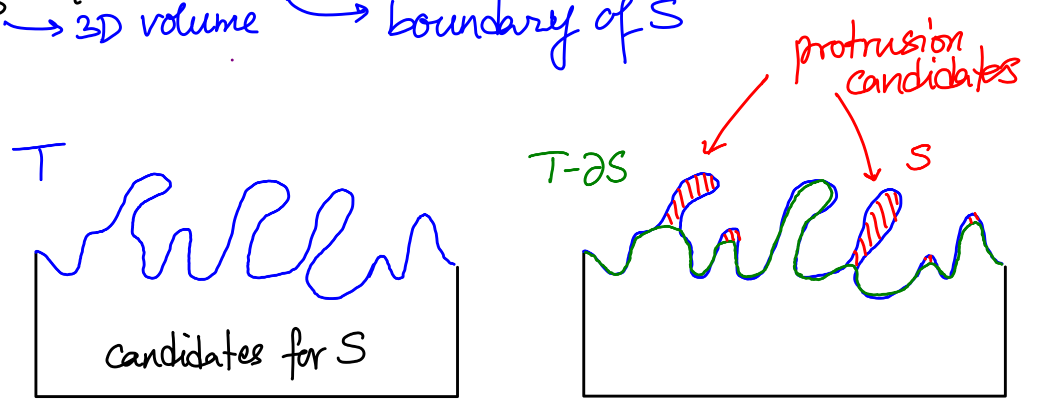
We use the notion of multiscale flat norm of surface T:

$$F_\lambda(T) = \min_{S} \{ \text{Area}(T - \partial S) + \lambda \text{Volume}(S) \}, \quad \lambda \geq 0 \rightarrow \text{scale parameter}$$

$S \rightarrow$ 3D volume

$\partial S \rightarrow$ boundary of S

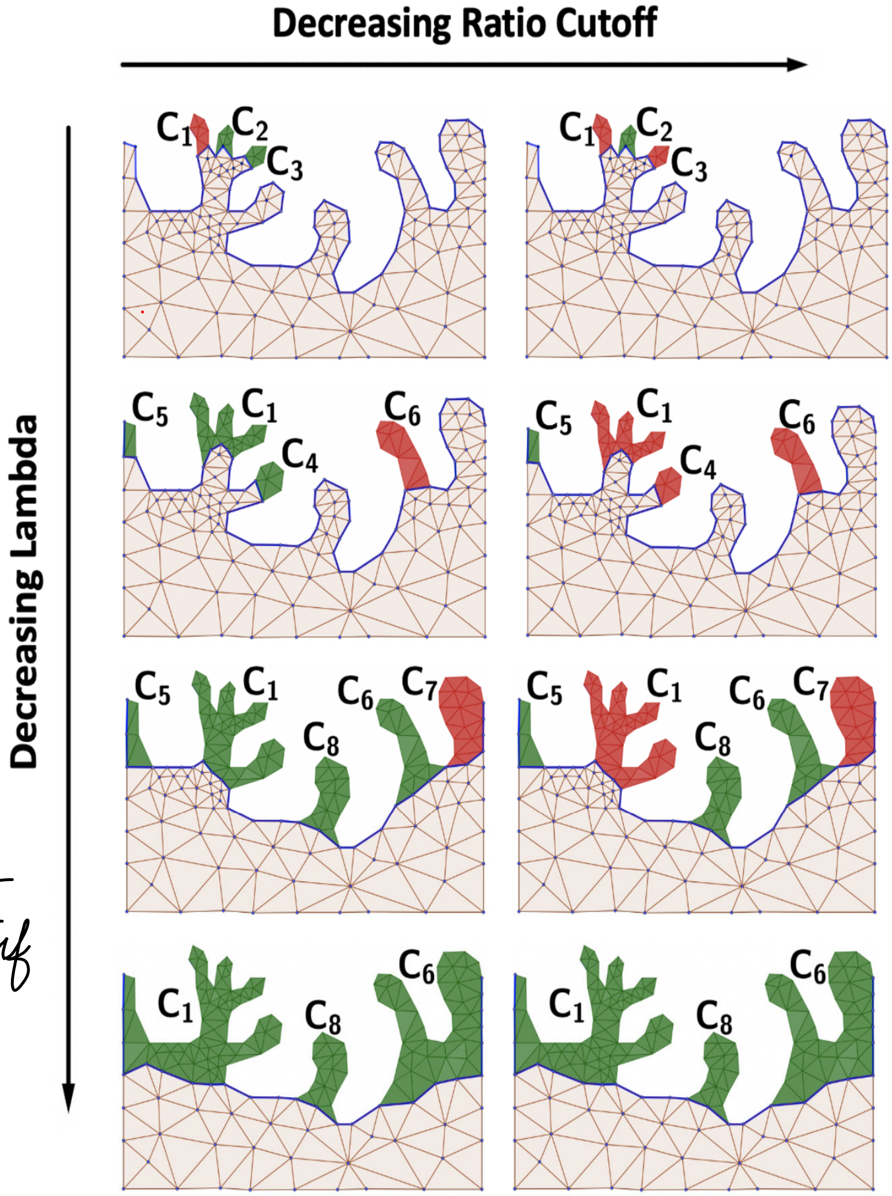
Illustration in 2D:



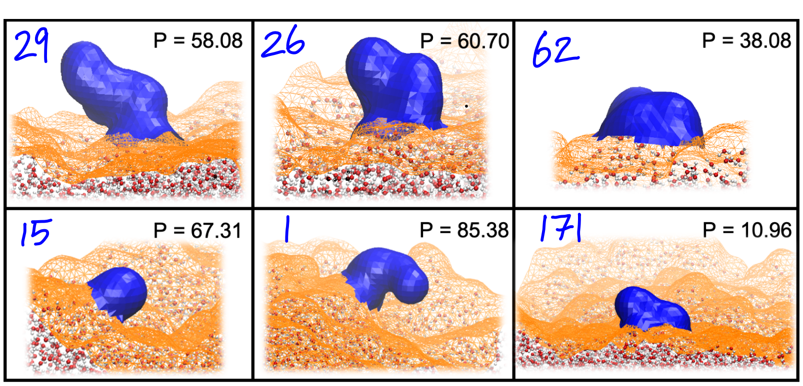
We keep track of connected components in S as $\lambda \downarrow$.
We relabel them and also keep track of merging behavior.

We also track the ratio $\dfrac{Vol(C)}{Vol(B(1/\lambda))}$ for each component C, where $B(1/\lambda)$ is the ball with radius $1/\lambda$ and $Vol(C)$ is the volume of component C.

A component C is "alive" at ratio cutoff r and scale λ if $\dfrac{Vol(C)}{Vol(B(1/\lambda))} > r$.



The longer a component is alive, the more likely it is to be a protrusion.



It turned out that all six of these features were protrusions!
The probabilities (as %'s) along with their ranks among 195 candidate features (lower rank $\Rightarrow$ more likely to be a protrusion) are shown here.

While this example is described in 3D, the underlying concepts are more general, and in fact generate certain key fundamental questions in geometric measure theory (GMT).

In fact, when we talk about applied algebraic topology, the application could be to pure mathematics! We will talk about this aspect toward the end of the semester.

Note that we are showing a discrete version of the surface—in the form of a triangular mesh. Indeed, we need to discretize continuous spaces to perform computations!

Here is a notion of connectivity in the "discrete setting":

The neighborhood, i.e., the set of nearby points, of the two • points are different — they each have only one neighbor, while the • points all have two neighbors each.

MATH 529: Lecture 2 (01/15/2026)

Today: * topology, open/closed sets
* homeomorphism, examples

We define topology as a mathematical method to define and study how a space is connected.

Notation For a set X, we denote by 2^X the power set of X, which is the set of all subsets of X.

Def A **topology** on a set X is a subset $\mathcal{T}$ of 2^X such that the following conditions hold.

1. $A_1, A_2 \in \mathcal{T} \Rightarrow A_1 \cap A_2 \in \mathcal{T}$ (finite intersections)

2. $\{A_j \mid j \in J\} \in \mathcal{T} \Rightarrow \bigcup_{j \in J} A_j \in \mathcal{T}$ (infinite unions or finite)

index set infinite or finite

3. $\phi, X \in \mathcal{T}$ empty set

$(X, \mathcal{T})$ is a topological space, denoted $\mathbb{X}$ ($\mathcal{T}$ is understood from context).

$A \in \mathcal{T}$ is an **open set** of $\mathbb{X}$.

The complement of A, i.e., $X - A$ (or $X \setminus A$) "set minus" is a **closed set**.

Some sets can be both open and closed at the same time, e.g., ϕ, X are both open and closed in any topology.

We typically specify a topology by specifying its open sets.

interior $\mathring{A}$ of $A \subseteq \mathbb{X}$: $\mathring{A} = \overset{\text{union}}{\bigcup}(\text{open sets contained in } A)$

closure $\bar{A}$ of $A \subseteq \mathbb{X}$: $\bar{A} = \overset{\text{intersection}}{\bigcap}(\text{closed sets containing } A)$.

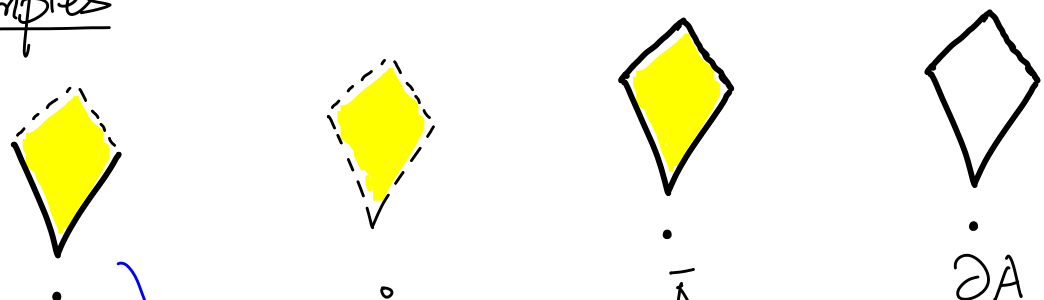
→ minimal closed set that contains A.

boundary ∂A of $A \subseteq \mathbb{X}$: $\partial A = \bar{A} - \mathring{A}$.

$\partial A = \{$points in A that intersect both $\bar{A}$ and $\overline{(X-A)}\}$.

Examples

1.



$A \subseteq \mathbb{X}$ $\qquad \mathring{A}$ $\qquad \bar{A}$ $\qquad \partial A$

→ A consists of the "half-open" rhombus and a separate point.

2. A discrete example. Let $X = \{a,b,c\}$.

We can define different topologies on X.

Let $T_1 = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ and

$T_2 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$.

Under T_1, $\{a,b\}$ is open, its complement $\{c\}$ is closed. With $A = \{a,b\}$, $\mathring{A} = \bigcup\{\emptyset, \{b\}, \{a,b\}\} = \{a,b\} = A$.

We can specify other topologies on X, e.g., $T_3 = 2^X$, where each set in T is both open and closed. But $T_4 = \{\emptyset, \{a\}, \{b\}, \{a,b,c\}\}$ is not a topology, as, e.g., $\{a\} \cup \{b\} = \{a,b\} \notin T_4$.

Neighborhood Let $\mathbb{X}=(X,T)$. A neighborhood of $x \in X$ is any $A \in T$ such that $x \in \mathring{A}$.

More generally, some books define a neighborhood as any set that includes, i.e., contains as a subset, an open set which contains X. Under this definition, the neighborhood could be a closed set (or neither open nor closed).

Now that we have defined topology, we consider the natural next question of comparing two spaces — how do we say two given spaces have the "same topology"? We introduce the notion of homeomorphism as a (strong) notion of topological similarity.

Homeomorphism

In geometry, we can study transformations that preserve "shape" of a rigid body, e.g., rotation and translation. These transformations "do not change" the geometry of the body.

In topology, we permit more types of transformations — e.g., stretch, shrink, expand, twist, etc., as long as you do not cut one piece into two or more, or join two pieces into one, or poke a hole in your object. All such permitted transformations "preserve topology".

A series of such permitted transformations that preserve topology constitute a homeomorphism. And two spaces are topologically "similar" if such "nice" functions exist from one space to the other and also back. We define what we mean by "nice" here.

We start with some background and definitions on functions.

Def Let A, B be sets. A function $f: A \to B$ is a rule that assigns exactly one $b \in B$ for every $a \in A$.

dom f: domain of $f = A$, cod f: codomain of $f = B$

im f: image of $f = \{b \in B \mid f(a) = b$ for some $a \in A\} = \{f(a) \mid a \in A\}$.

im f is also called the range of f. Note that im $f \subseteq$ cod f.

"for every"

$f: A \to B$ is 1-to-1 or **injective** if $\forall b \in B$, there exists **at most** one $a \in A$ with $f(a) = b$.

can be none

$f: A \to B$ is onto or **surjective** if $\forall b \in B$, there exists **at least** one $a \in A$ with $f(a) = b$.

can be more

If f is both injective and surjective, we say that f is **bijective**, or that f is a bijection.

Def A function $f: X \to Y$ is **continuous** if for every open set $B \subseteq Y$, $f^{-1}(B)$ is open in X. "takes" open sets to open sets.

A continuous function is also called a **map**.

Def A **homeomorphism** $f: X \to Y$ is a bijective function such that both f and f^{-1} are continuous.

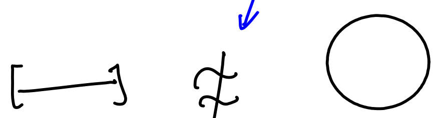
We say X is homeomorphic to Y, or $X \approx Y$.

We also say that X and Y have the same topological type.

Examples

It's often easier to argue why two spaces are not homeomorphic — we just identify one (or more) place(s) where things don't work.

"not homeomorphic"

1. 

$[0,1] \not\approx S^1$

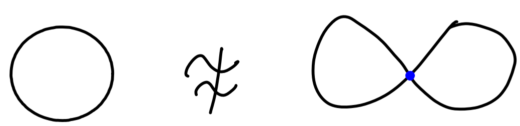
0 1

$[0,1]$ S^1

1-sphere, or circle

We would need a map that assigns both end points of $[0,1]$ to a single point in S^1.

But the inverse of any map that takes both end points of $[0,1]$ to one point in S^1 is not bijective.

2. 

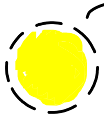
$S^1 \not\approx$ Figure 8

The crossing point in ∞ (x) cannot be mapped to a corresponding point in S^1.

Also, we could map S^1 to one of the two circles in figure-8, but not both.

On the other hand, two show that two spaces are homeomorphic, we need to specify the maps f and f^{-1}!

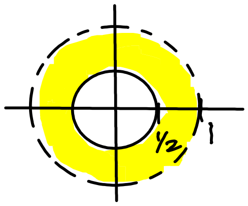
3. $D_1 = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1\} \rightarrow$ open unit disc



$$D_1 \underset{f^{-1}=g}{\overset{f}{\rightleftarrows}} \mathbb{R}^2 \qquad D_1 \approx \mathbb{R}^2$$

for ease of notation

Intuitively, we can shrink all of $\mathbb{R}^2$ into D_1. Similarly, we can stretch D_1 to fill all of $\mathbb{R}^2$.



$$g(\bar{x}) = \frac{\bar{x}}{1+\|\bar{x}\|_2} \qquad g: \mathbb{R}^2 \to D$$

Euclidean norm

g maps all of D_1 (in $\mathbb{R}^2$) to fit within $D_{1/2} = \{\bar{x} \in \mathbb{R} \mid \|\bar{x}\| < \frac{1}{2}\}$, and then fits all of $\mathbb{R}^2$ outside D_1 within the half open annulus with radii $\frac{1}{2}$ and 1.

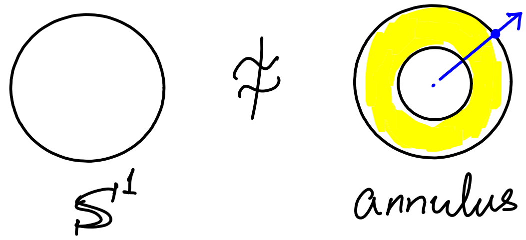
The continuous function going from D_1 to $\mathbb{R}^2$ can be similarly defined:

$f: D \to \mathbb{R}^2$ where $f(\bar{x}) = \dfrac{\bar{x}}{1-\|\bar{x}\|}$. f is an "infinite stretch".

Note that points $\bar{x}$ in D that are close to the edge, i.e., have $\|\bar{x}\|$ close to 1, are mapped so as to fill up the entire $\mathbb{R}^2$ outside $\bar{D}$. We stretch the open disc so as to fill the entire plane, and hence it is called an infinite stretch.

Usually, we try to define the continuous maps f and f^{-1} to show that two spaces $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic. At the same time, the intuition (geometric when possible) is also important to grasp. On the other hand, to show that $\mathbb{X} \not\cong \mathbb{Y}$, it is often sufficient to identify subset(s) that create the obstructions, e.g., the $\times$ in figure-8 v/s $\mathbb{S}^1$.

4.

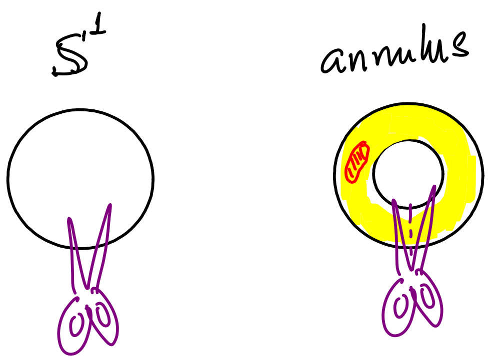


Both these spaces have the shape of a "hole".

We could shrink the annulus so that it reduces to the circle. The corresponding function maps every point on the annulus radially onto the outer circle, for instance. But we cannot uniquely map the circle back to the annulus — would need to "map" each point on the circle to (infinitely) many points on the thick strip of the annulus.

Another observation highlights the neighborhoods of points in the circle and the annulus. Every point on the circle has open neighborhoods that look like the number line ($\mathbb{R}^1$). On the other hand, points in the annulus have neighborhoods that look like the open disc ($\mathbb{R}^2$) or open half disc (the points on the boundary).

Intuitively, the annulus is 2-dimensional, while S^1 is one-dimensional.



Notice that the two spaces behave the same way under a "cut" as we had been talking about earlier with the string.

In particular, a straight cut along one "edge" of either space would leave them both connected. At the same time, one could "carve out" a 2D disc ( ) from the annulus, but not from the circle.

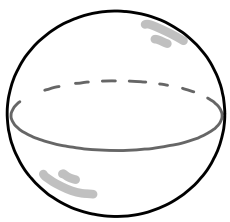
If we "relax" our definition of topological similarity, the two spaces would be considered the same — they both look like a hole, after all. Indeed we will see that checking for homeomorphism is difficult (both theoretically and computationally). We'll work with looser concepts of topological similarity later on — homology!

MATH 529: Lecture 3 (01/20/2026)

Today: * 1 more example of homeomorphism
* manifolds

Examples of homeomorphism (continued...)

5. sphere $\not\approx \mathbb{R}^2$
$\mathbb{S}^2$ (2-sphere)



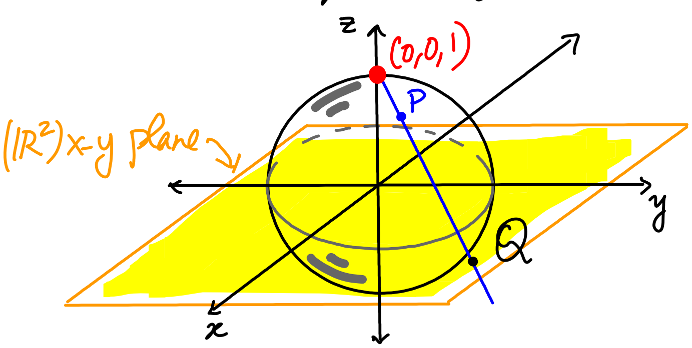
Here is an observation: the sphere encloses a 3D pocket (or void), while $\mathbb{R}^2$ does not do so.

Such enclosed voids are the 3D analogues of holes (which are 2D).

But $\mathbb{R}^2 \cup \{\infty\} \approx \mathbb{S}^2$ → By stereographic projection, which is used to map out the surface of the earth on to a (planar) map, for instance.

"point at infinity"

Recall the equation for $\mathbb{S}^2$: $x^2+y^2+z^2=1$.



If you poke a hole in the sphere you can spread it out on the 2D plane, like a pierced balloon.

The equation of the line connecting (0,0,1) and $P(x,y,z)$ is given by $\bar{x} = (0,0,1) + t\,(x-0, y-0, z-1)$, $t \in \mathbb{R}$.
This line intersects the x-y plane at Q, which has $z=0$. Hence we get $t(z-1)+1=0 \Rightarrow t = \frac{1}{1-z}$.
Thus, Q is $\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$.

$P(x,y,z)$ on $\mathbb{S}^2$ projected from $(0,0,1)$ to $\mathbb{R}^2$ is

North pole

$Q\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$.

This formula is valid for all points on $\mathbb{S}^2$, except the north pole $(0,0,1)$.

The lower hemisphere of $\mathbb{S}^2$ gets mapped to D (unit disc), and the upper hemisphere gets mapped to the rest of $\mathbb{R}^2$.

According to a topologist, "a sphere is nothing but the plane with a point added at infinity"!

Note that every point on $\mathbb{S}^2$ has a neighborhood that looks like $\mathbb{R}^2$, i.e., "it feels locally Euclidean". Such objects are called **manifolds** and are among the most commonly studied spaces in computational topology.

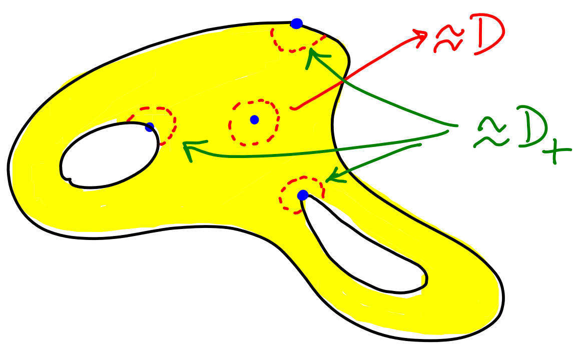
Def A topological space $\mathbb{M}$ is a **2-manifold** if all points in $\mathbb{M}$ lie in open discs, i.e., every point has a neighborhood $\approx D = \{\bar{x} \in \mathbb{R}^2 \mid \|x\| < 1\}$.

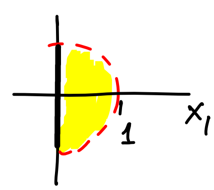
e.g., $\mathbb{S}^2$, $\mathbb{R}^2$.

these are 2-manifolds without boundary.

A **2-manifold with boundary** is a topological space $\mathbb{M}$ whose every point has a neighborhood homeomorphic to D or to $D_+ = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1, x_1 \geq 0\}$ (but not both), and there exist some points of the latter type.

1st entry



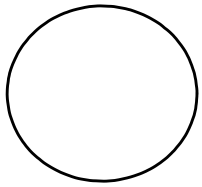
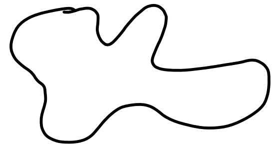
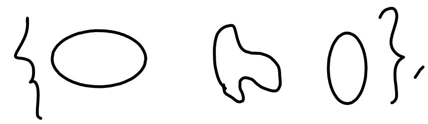


points on the boundary have neighborhoods that are half discs.

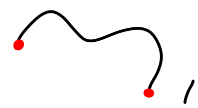
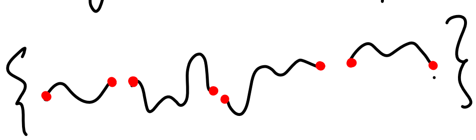
Def The **boundary** of a 2-manifold with boundary $\mathbb{M}$ is the set of points in $\mathbb{M}$ that have neighborhoods homeomorphic to the half disc.

(The definition of ∂A used for sets A is equivalent to this definition).

Notice that the 1-manifold is just the circle ($\mathbb{S}^1$),

 or  or ,

a collection of disjoint circles.

A 1-manifold with boundary: , or 

boundary is indeed the set of end points.

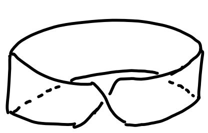
only one end point is included here!

0-manifold: Any collection of distinct points (discrete set of points).

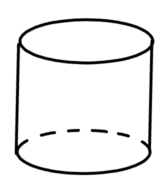
2-manifolds, also called surfaces, are a well studied class of spaces (objects), both from the theoretical as well as applied points of view. We will present several details of the properties of 2-manifolds first. To define and study d-manifolds for $d \geq 2$, we will need a few more definitions and concepts from analysis/point set topology.

By default, we assume a manifold (w/ or w/o boundary) is connected.

A 2-manifold (for that matter, d-manifold for $d \geq 2$) is orientable or non-orientable.



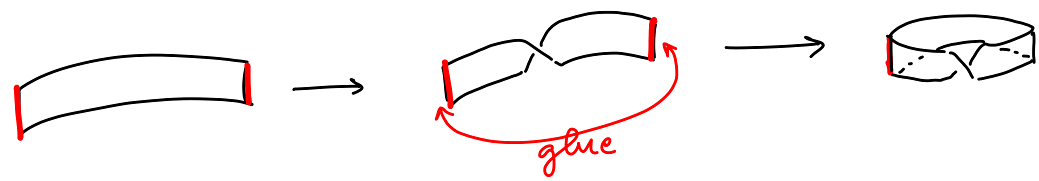
Möbius strip is non-orientable

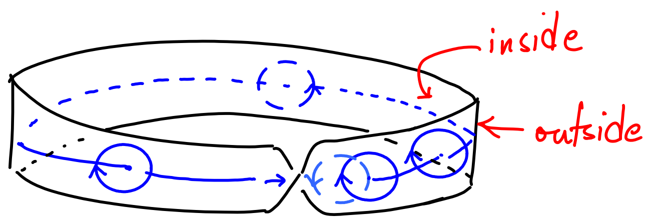


cylinder is orientable

The Möbius strip has only one "side," while the cylinder has two "sides" — inside and outside.

We obtain the Möbius strip by taking a rectangular strip of paper, and gluing its short edges together after twisting the strip once.





there are two "sides" at each point (locally) on the Möbius strip — front/back or inside/outside.

Consider sliding an oriented loop, or a clock ◯ along the surface of the Möbius strip. Look at the path followed by the center of the clock. Once the center comes back to where it started, its orientation is reversed (as it goes over the "twist").

The path traced by the center of the clock here is hence an **orientation reversing** closed curve. If the orientation is not reversed this way, the curve is said to be **orientation preserving**.

Def If all closed curves in a 2-manifold (with or without boundary) are orientation preserving, then the 2-manifold is **orientable**, else it is **nonorientable**.

orientable: $\mathbb{R}^2$, $\mathbb{S}^2$, torus, etc.

nonorientable: Möbius strip, Klein bottle, etc.

Classification of Manifolds

Enumerate all possible manifolds upto homeomorphisms for each dimension.

We mention the classification for 0- and 1-dimensional manifolds, but will come back to give some more definitions to finish the discussion for 2-manifolds.

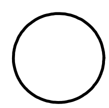
each point has to have a neighborhood $\approx \mathbb{R}^0$, i.e., a point.

0-manifolds: a discrete space, e.g., $\mathbb{Z}^2$ – all points with integer coordinates in $\mathbb{R}^2$.

Notice that $2\mathbb{Z}^2$, all points with even integer coordinates in $\mathbb{R}^2$ is homeomorphic to $\mathbb{Z}^2$.

We will define "compact" soon...

1-manifolds
connected

- compact: $\approx \mathbb{S}^1$, 
- compact with boundary: $\approx [0,1]$ closed unit interval
- non-compact: $\approx \mathbb{R}^1$ or $(0,1)$
- non-compact with boundary $\approx \mathbb{R}^1_+$ or $[0,1)$
  - $\mathbb{R}^1_+$: $x \geq 0$
  - $[0,1)$: half-open unit interval

if a 1-manifold is not a single connected space, each connected subspace has one of these structures.

MATH 529: Lecture 4 (01/22/2026)

Today: * d-manifolds in general
* Classification of 2-manifolds

We will now introduce some concepts which we will use to define d-manifolds in general (in particular, for $d \geq 2$).

Def A **cover** of $A \subseteq X$ is a family $\{C_j \mid j \in J\}$ in 2^X such that $A \subseteq \bigcup_{j \in J} C_j$.
J: index set

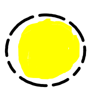
An **open cover** is a cover made of open sets.

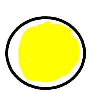
A **subcover** of A is a cover $\{C_k \mid k \in K\}$ such that $K \subseteq J$.
index set is a subset

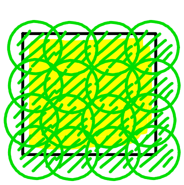
Def A set A is **compact** if every open cover of A has a finite subcover. Correspondingly, a topological space $\mathbb{A} \subseteq \mathbb{X}$ is compact if every open cover of $\mathbb{A}$ has a finite subcover.

Note: In $\mathbb{R}^d$, closed + bounded $\iff$ compact.

e.g., $\mathbb{S}^2$ is compact, but $\mathbb{R}^2$ is not.

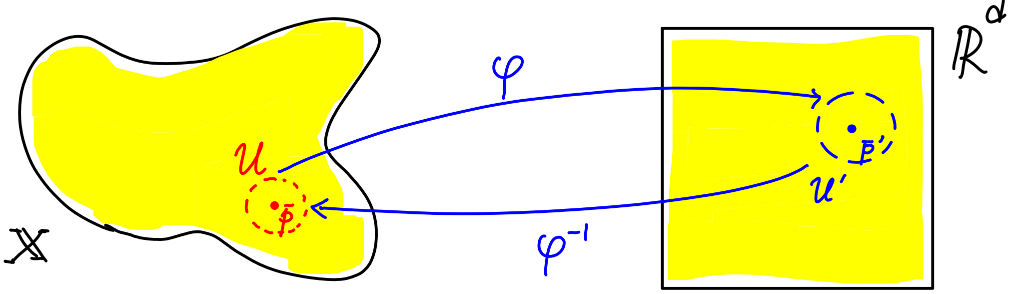
 D_1 is not compact.

 $\overline{D}_1$ (closed disc) is a compact 2-manifold with boundary



Example of finite subcover: consider a unit square, which is a subset of $\mathbb{R}^2$. Consider open discs of radius $1/4$ centered at each rational point within the square (denoted here as ◯). There are infinitely many such discs, which together cover the square. But a finite subset of those discs also covers the square.

Def A **chart** at $\bar{p} \in \mathbb{X}$ is a homeomorphism $\varphi : \mathcal{U} \to \mathbb{R}^d$ for $\mathcal{U} \in \mathbb{X}$ an open set containing $\bar{p}$. The **dimension** of the chart is d.



Def (Hausdorff) A topological space $\mathbb{X}$ is **Hausdorff** if $\forall x, y \in X$, $x \neq y$, there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of x, y, respectively, such that $\mathcal{U} \cap \mathcal{V} = \emptyset$.

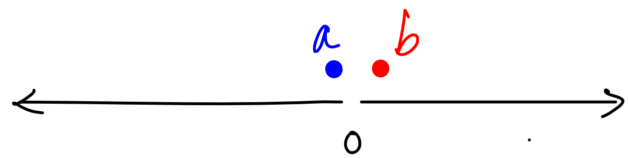
an open set containing the point in question

e.g., $\mathbb{R}^2$.

Example of a non-Hausdorff space: $X = L \cup \{a, b\}$ where

$L = \mathbb{R} \setminus \{0\}$

a and b are both used in place of the origin. Open sets are the usual open intervals in $\mathbb{R}$.



and such that

$L \cup \{a\} \approx \mathbb{R}$ and $L \cup \{b\} \approx \mathbb{R}$.

But every pair of open sets $\mathcal{U}$ and $\mathcal{V}$ containing a and b, respectively, intersect!

Def A topological space is **completely separable** or second countable if it has a countable basis, i.e., it has a countable collection of open sets such that every open set can be written as a union of open sets from this collection (basis).

think $\mathbb{Z}$, integers, as opposed to $\mathbb{R}$, which is uncountable

e.g., $\mathbb{R}$ is completely separable – it can be shown that open intervals with rational lengths centered at only rational points works as a countable basis.

A space that is not completely separable: take uncountably many copies of $[0,1)$, e.g., with the 0 of $[0,1)$ anchored at all irrational points – called the long line or Alexandroff line.

Def (Manifold) d-dimensional manifold A completely separable, Hausdorff space $\mathbb{X}$ is a **d-manifold** if there exists a d-dimensional chart at every $\bar{x} \in \mathbb{X}$, i.e., $\bar{x}$ has a neighborhood homeomorphic to $\mathbb{R}^d$.

$\mathbb{X}$ is a d-manifold with boundary if every $\bar{x} \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$ or $\mathbb{H}^d = \{\bar{x} \in \mathbb{R}^d \mid x_1 \geq 0\}$ (d-dimensional half space).

The boundary of $\mathbb{X}$, denoted by $\partial\mathbb{X}$, is the set of $\bar{x} \in \mathbb{X}$ with a neighborhood homeomorphic to $\mathbb{H}^d$.

The **dimension** of the manifold is d here.

Notice the correspondence between the definition of 2-manifolds introduced previously, and the general definition here. The main condition is the existence of neighborhoods $\approx \mathbb{R}^d$ around each point.

Def (Embedding) An **embedding** of $\mathbb{X}$ in $\mathbb{Y}$ is a map $g: \mathbb{X} \to \mathbb{Y}$ whose restriction to $g(\mathbb{X})$ is a homeomorphism.

Manifolds are manifolds irrespective of their embedding! $\mathbb{S}^2$ is a 2-manifold even if it is not sitting in $\mathbb{R}^3$.

We will introduce alternative representations of manifolds to highlight this point. In fact, in many cases, we can study the manifold easily using such representations.

Classification of Manifolds (continued)

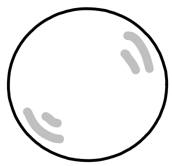
Enumerate all possible manifolds of a given dimension upto homeomorphism. We already listed the classifications for 0- and 1-dimensional manifolds.

We now consider the case of compact, connected, closed 2-manifolds. We first list the "basic building blocks", so to speak, which include the 2-sphere, torus, Möbius strip, and the real projective plane. We can build larger 2-manifolds by gluing these building blocks together.

2-Manifolds (We consider compact 2-manifolds)

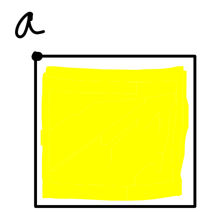
First, let us study several typical 2-manifolds, some of which we already saw previously.

1. $\mathbb{S}^2$



2-sphere

$\{\bar{x} \in \mathbb{R}^3 \mid \|\bar{x}\|_2 = 1\}$

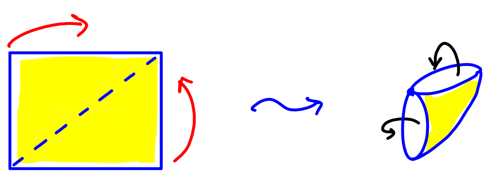


"identify" all points on boundary with the point a.

This is a "diagram" of $\mathbb{S}^2$.

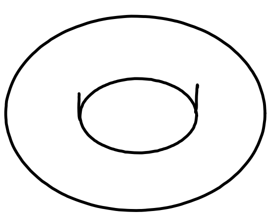
both are 2-spheres!

Start with a square sheet of paper and glue its all its edges together to make a sphere.



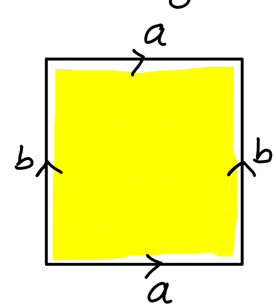
Arrows capture how edges are glued – with or without twist.

2.



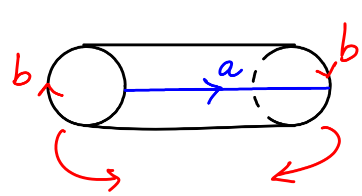
$\mathbb{T}^2$ torus

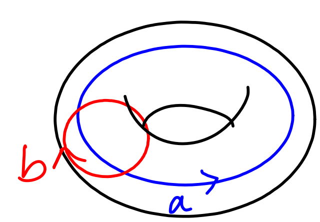
$\approx$



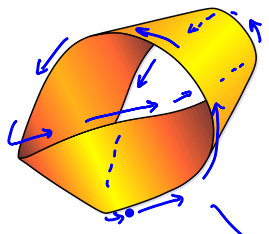
we could mathematically Study this representation

Imagine folding a rectangular sheet of paper first into an open cylinder, and then gluing its end circles to form a torus.



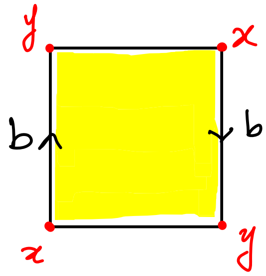


3. Möbius strip → 2-manifold with boundary



(image: www)

$\approx$

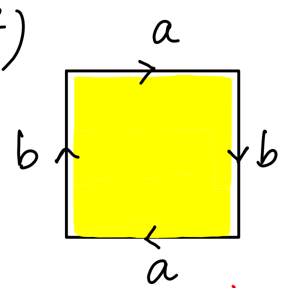


Notice that we can traverse the "edge" of the Möbius strip in one go — it's one big circle!

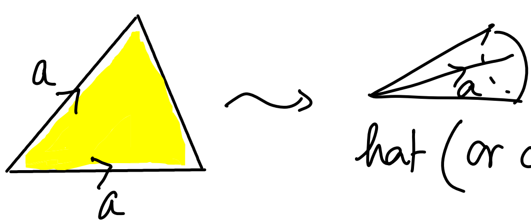
Notice that we are not identifying the horizontal edges. So they remain as boundaries. At the same time all four edges are identified pairwise in the case of the torus. Indeed, the Möbius strip is a manifold with boundary, while the torus is a manifold (without boundary).

cannot be embedded in $\mathbb{R}^3$!

4. (Real) Projective plane ($\mathbb{RP}^2$)
(also, Dunce hat)

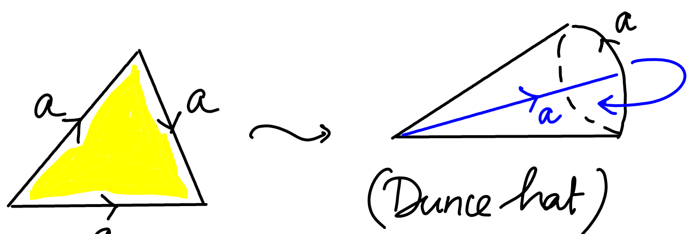


identify the free edges of Möbius strip in an opposing sense.



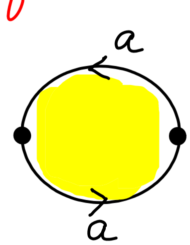
hat (or cone)

Same as gluing the boundary of a disc to the boundary of a Möbius strip.

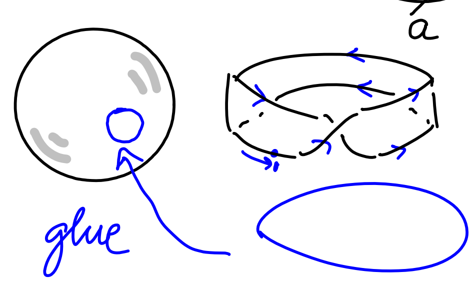


(Dunce hat)

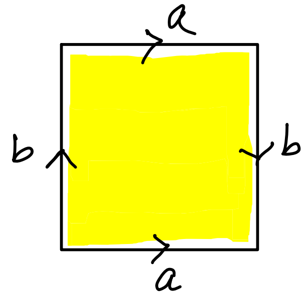
Here's another representation



Yet another way to make $\mathbb{RP}^2$: Cut an open disc out of S^2 and glue a Möbius strip along the edge left by the cut.

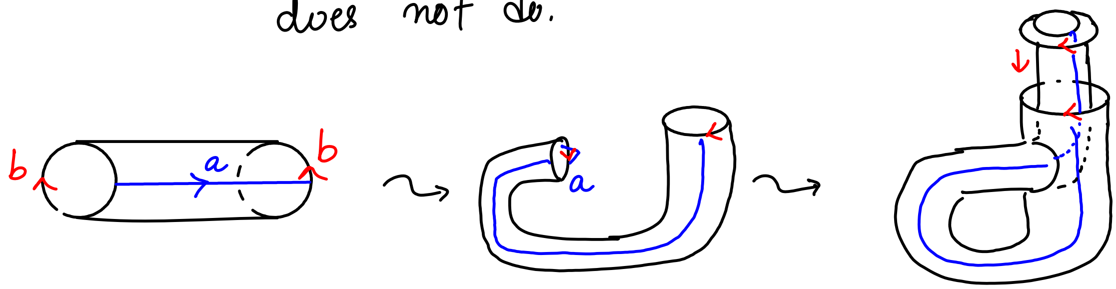


5. Klein bottle ($\mathbb{K}^2$)



Identify free edges of the Möbius strip in the same direction.

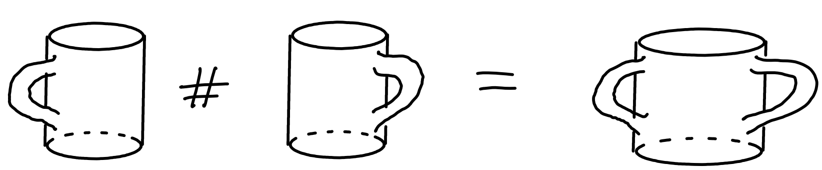
An "immersion" in 3D:
→ allows self intersection, which an embedding does not do.



We get $\mathbb{K}^2$ also by gluing together two Möbius strips along their boundary circles. Or, cut two discs out of S^2 and glue a Möbius strip each along the edges of both cuts.

Note that S^2 and $\mathbb{T}^2$ are orientable manifolds, while the Möbius strip, $\mathbb{RP}^2$, and $\mathbb{K}^2$ are non-orientable manifolds (with or without boundary).

We can obtain more general 2-manifolds by "gluing" these basic shapes together. For example, we can connect two coffee cups to get one coffee cup with two handles!



We can do this kind of "gluing" to join manifolds in any dimension (as long as the manifolds being joined have the same dimension). This ing is formally termed connected sum.

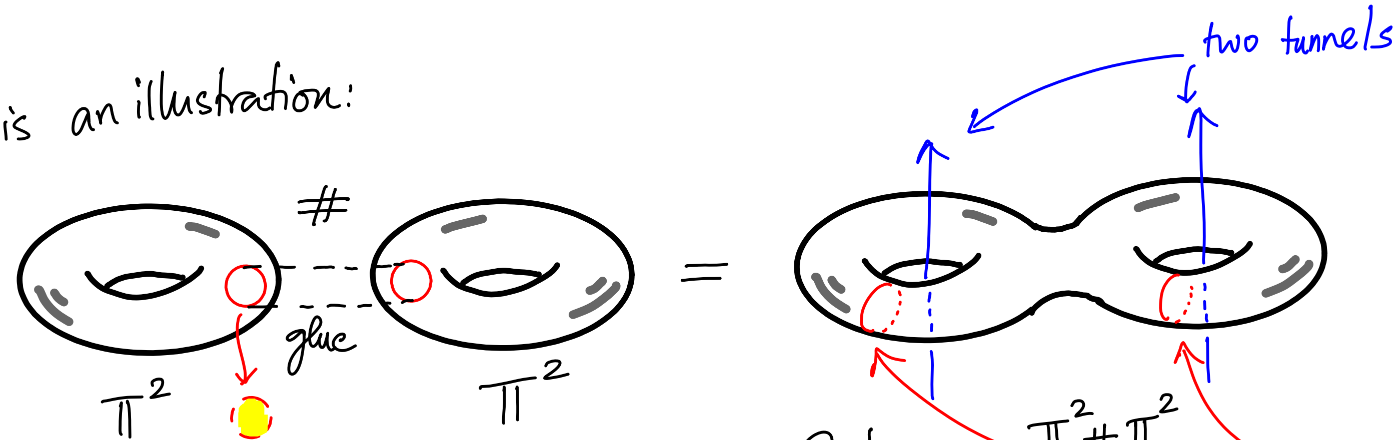
Def (Connected sum). Let $\mathbb{M}_1, \mathbb{M}_2$ be d-manifolds. The **connected sum** of these d-manifolds is another d-manifold defined as follows.

$$\mathbb{M}_1 \# \mathbb{M}_2 = (\mathbb{M}_1 - D_1^d) \bigcup_{\partial \bar{D}_1^d \cong \partial \bar{D}_2^d} (\mathbb{M}_2 - D_2^d)$$

↳ identified by homeomorphism

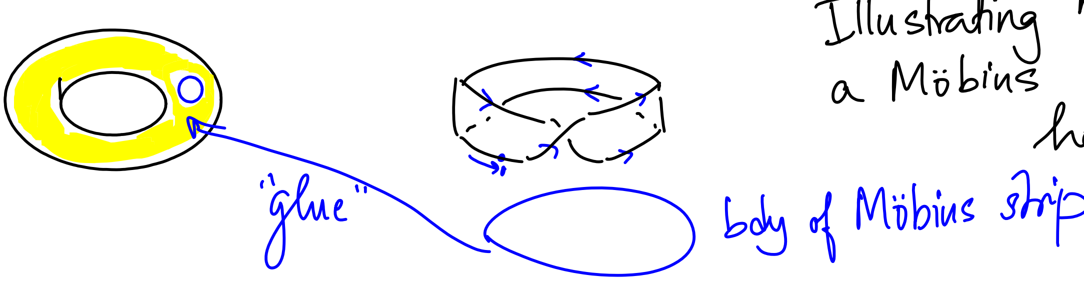
D_1^d, D_2^d are d-dimensional open discs in $\mathbb{M}_1, \mathbb{M}_2$, respectively.

Here is an illustration:



2-torus
(double torus)

Remove open dics from both tori, and "glue" along the boundaries of these circular holes.



Illustrating how to glue a Möbius strip to a hole in a torus.

MATH 529: Lecture 5 (01/27/2026)

Today:
* classification of 2-manifolds
* simplices and simplicial complexes
* ~~abstract simplicial complexes~~ didn't get to it ...

Classification of Compact, connected 2-manifolds

Result Every compact, connected 2-manifold is homeomorphic to $\mathbb{S}^2$, or a connected sum of copies of $\mathbb{T}^2$, or a connected sum of copies of $\mathbb{RP}^2$. If a 2-manifold is not connected, each component has this structure.

Examples

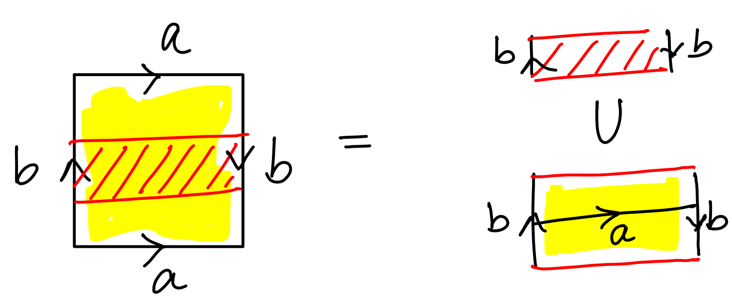
1. $\mathbb{S}^2 \# \mathbb{RP}^2 \approx \mathbb{RP}^2$

   you just close back the open disc cut out from $\mathbb{M}$!

   In fact, $\mathbb{S}^2 \# \mathbb{M} \approx \mathbb{M}$ for any 2-manifold $\mathbb{M}$.

2. $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx \mathbb{K}^2$ Here's a "proof" by pictures:

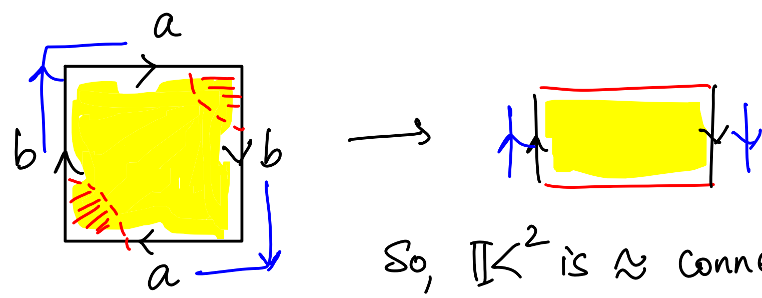
$\mathbb{K}^2$:



Connected sum of 2 Möbius strips!

And $\mathbb{RP}^2 - D^2$:

open 2-disc



Mobius strip

So, $\mathbb{K}^2$ is $\approx$ connected sum of 2 $\mathbb{RP}^2$'s.

Also, we get $\mathbb{RP}^2$ as $\mathbb{S}^2 \# M$, for a Möbius strip M.

3. $\mathbb{T}^2 \# \mathbb{RP}^2 \approx \#(\mathbb{RP}^2)^3$

   orientable nonorientable

   Once you join a non-orientable 2-manifold, the result stays non-orientable.

As one would expect, the corresponding result for 3-manifolds is much more complicated. Thurston's geometrization conjecture states that all compact 3-manifolds can be canonically decomposed into submanifolds that have geometric structure. This conjecture implies the famous Poincaré conjecture, which states that every compact, simply connected 3-manifold is homeomorphic to $\mathbb{S}^3$, the 3-sphere.

(Informally, an object is simply connected if there are no "holes" passing through the object).

Perelman presented a proof of the Poincaré conjecture, almost 100 years after it was originally proposed (in 1904; Perelman's proof appeared in 2003). The corresponding result for n-manifolds with $n \geq 4$ turns out to be easier to prove, informally because of the "increased geometric freedom" one can afford in higher dimensions.

The main concepts used in Perelman's proof can be used to provide a proof for Thurston's geometrization conjecture (Perelman presented such a proof in 2003, along with his proof of the Poincaré conjecture).

Simplices

While we can study sample 2-manifolds as is, we cannot do computations on them. For this purpose, we need a discretized version of the spaces in question, which could be stored and handled naturally by a computer. Can we, for instance, use some sort of "counting arguments" to distinguish S^2 from $\mathbb{T}^2$?

We introduce the concept of simplicial complexes in this context, and use concepts from combinatorial algebraic topology. The idea is that we can handle the combinatorics using efficient algorithms. Similarly, there are efficient data structures that can be used to work with simplicial complexes modeling the spaces. We also want to separate the topology from the geometry of the object. we use the geometry in many applications.

We first introduce some definitions.

Def (Combinations) Let $S = \{\bar{p}_0, \dots, \bar{p}_k\} \subseteq \mathbb{R}^d$. A **linear combination** of $\bar{p}_i$ is $\bar{x} = \sum_{i=0}^{k} \lambda_i \bar{p}_i$, $\lambda_i \in \mathbb{R}\ \forall i$.

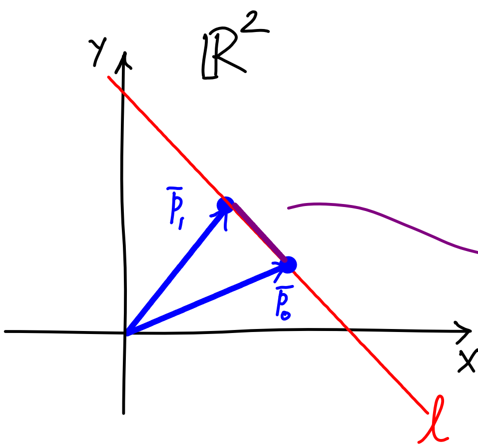
If $\sum_{i=0}^{k} \lambda_i = 1$, then $\bar{x}$ is an **affine combination** of $\bar{p}_i$'s.

In addition, if $\lambda_i \geqslant 0\ \forall i$, $\bar{x}$ is a **convex combination** of $\bar{p}_i$'s.

The set of all convex combinations of elements in S is the **convex hull** of S, denoted as

$$\operatorname{conv}(S) = \left\{ \sum_{i=0}^{k} \lambda_i \bar{p}_i \;\middle|\; \lambda_i \geqslant 0\ \forall i,\ \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

Illustration in 2D



set of all linear combinations $= \mathbb{R}^2$

set of all affine combinations $= \ell$, the line through $\bar{p}_0, \bar{p}_1$

$\text{conv}(\{\bar{p}_0, \bar{p}_1\})$ = line segment connecting $\bar{p}_0, \bar{p}_1$.

$\bar{p}_0, \bar{p}_1$ are not parallel here.

Def (Independence) S with $|S| \geq 2$ is **linearly** (**affinely**) **independent** if no point in S is a linear (affine) combination of other points in S.

We denote linearly independent in short as LI, and affinely independent as AI.

$|S|=1$ case: $\{\bar{p}_0\}$ is LI if $\bar{p}_0 \neq \bar{0}$ (zero vector), but $\{\bar{p}_0\}$ is AI for all $\bar{p}_0$ (even if $\bar{p}_0 = \bar{0}$).

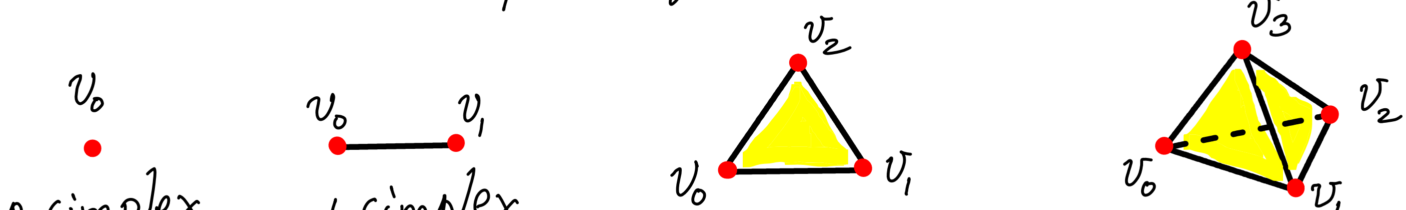
For example, 3 points in $\mathbb{R}^2$ are AI as long as they are not collinear.

We will use convex hulls of AI points as our building blocks — called simplices.

Def (simplex) The convex hull of $(k+1)$ affinely independent points $S=\{\bar{v}_0,\dots,\bar{v}_k\}$ is a **k-simplex**. The **dimension** of the simplex is k, and $\bar{v}_j$'s are the **vertices** of the k-simplex.

vertices, and hence $\bar{v}_i$, rather than $\bar{P}_i$.

Here are the k-simplices for small values of k: $0\le k\le 3$.



0-simplex (point) | 1-simplex (line segment) or edge | 2-simplex (triangle) | 3-simplex (tetrahedron)

Notice that the k-simplex is homeomorphic to the closed k-ball, i.e., $\bar{B}_k=\{\bar{x}\in\mathbb{R}^k \mid \|\bar{x}\|\le 1\}$. "=" gives S^{k-1}, the $(k-1)$-sphere.

Indeed, the boundary of the k-ball is the $(k-1)$-sphere, e.g., 2-ball (or 2-disc) has the circle (1-sphere) as the boundary.

Each p-simplex is made of lower dimensional simplices, i.e., k-simplices with $k\le p$. Thus, $\Delta v_0v_1v_2$ contains vertices v_0, v_1, v_2, edges $\overline{v_0v_1}, \overline{v_1v_2}, \overline{v_0v_2}$, and $\Delta v_0v_1v_2$ itself.

Def (face/coface). Let σ be the k-simplex defined on $S=\{\bar{v}_0,\bar{v}_1,\dots,\bar{v}_k\}$.
A simplex τ defined on a subset T of S, $|T|\neq\phi$, is a **face** of σ, and σ is a **coface** of τ. The notation is $\tau\leq\sigma$, $\sigma\geq\tau$
(some books use $\tau\preceq\sigma$, $\sigma\succeq\tau$ \succeq, \preceq in LaTeX.

Thus, $\overline{v_0v_1}$ is a face of $\Delta v_0v_1v_2$. So are v_0, v_1, v_2, $\overline{v_1v_2}$, and $\overline{v_0v_2}$.

A simplex is always a face of itself, i.e., $\sigma\leq\sigma$ and $\sigma\geq\sigma$.

We can attach simplices together "nicely" to form bigger objects called simplicial complexes.

Def A **simplicial complex** K is a set of simplices such that

1. $\sigma\in K$, $\tau\leq\sigma \Rightarrow \tau\in K$; and

   Every face of a simplex in K is also in K.

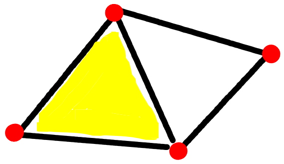
2. $\sigma,\sigma'\in K \Rightarrow \sigma\cap\sigma'\leq\sigma,\sigma'$ when $\sigma\cap\sigma'\neq\phi$.

   In particular, the non-empty intersection of two simplices in K is a face of both of them, and hence in K as well.

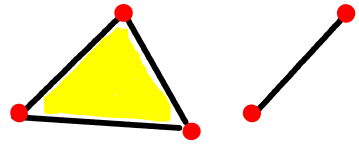
The above definition holds in the case of both finite and nonfinite K. In the latter case, K has infinitely many simplices satisfying the two conditions. But we will usually limit our attention in this course to finite simplicial complexes, unless mentioned otherwise.

Note that a simplex is the convex hull of a **finite** number of affinely independent points. So, we never talk about "infinite-dimensional" simplices. We could talk about each of these points sitting in infinite-dimensional space, but we restrict our attention to $\mathbb{R}^d$ for finite d.

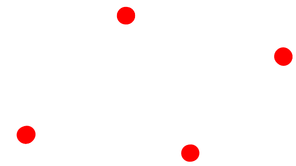
Examples Here are some simplicial complexes:



K_1



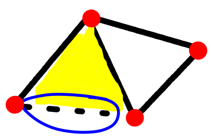
K_2



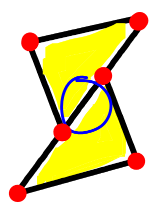
K_3

In particular, notice that a simplicial complex need not consist of just one connected component.

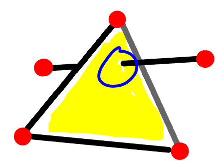
Here are some collections that are not simplicial complexes:



K_4



K_5



K_6

K_4 violates Condition 1, as one of the faces of the triangle in K_4 is not in the collection. K_5 and K_6 violate Condition 2, as the intersection of two simplices is not a face of either simplex in both cases.

Def The dimension of a simplicial complex is the same as that of the highest dimensional simplex in it, i.e.,

$$\dim K = \max\{\dim \sigma \mid \sigma \in K\}.$$

In the previous examples, $\dim(K_1) = 2$, $\dim(K_2) = 2$, and $\dim(K_3) = 0$.

Earlier we have been talking about continuous surfaces, e.g., the 2-sphere, torus, etc. And now we are talking about simplicial complexes as discrete objects. How do we "reconcile" the two notions? Indeed, we can formally define the "space" modeled by a simplicial complex. Later on, we will talk about a simplicial complex "triangulating", say, a torus, when this "space" is homeomorphic to $\mathbb{T}^2$.

Def The **underlying space** of a simplicial complex K the space made of all simplices in K together with the topology inherited from the ambient Euclidean space. We denote the underlying space of the simplicial complex K by $|K|$. Thus,

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

$|K|$ is also called the polyhedron (or polytope) of K.

$A \subseteq |K|$ is closed in $|K|$ iff $A \cap \sigma$ is closed $\forall \sigma \in K$.

MATH 529 : Lecture 6 (01/29/2026)

Today: * Abstract simplicial complexes (ASCs)
* geometric realization
* examples of ASCs

Def An **abstract simplicial complex** (ASC) is a collection $\mathcal{S}$ of finite non-empty sets such that if $A \in \mathcal{S}$, and $B \subseteq A$ with $B \neq \emptyset$, then $B \in \mathcal{S}$.

Note that the condition specified in the above definition of an abstract simplicial complex is equivalent to the first condition in the definition of a (regular) simplicial complex, which says that every face of a simplex in the complex is also in the complex.

The second intersection condition is trivially satisfied in the case of abstract simplicial complexes. The intersection of two sets is indeed a subset of both sets.

$\mathcal{S}$ itself can be finite or infinite, but each $A \subseteq \mathcal{S}$ is a finite set.

The sets in $\mathcal{S}$ are called the **simplices** of $\mathcal{S}$. The dimension of a simplex $A \in \mathcal{S}$ is $\dim(A) = |A| - 1$.

cardinality (# entries) of A

Note the correspondence of the above definition to the definition of simplices in the usual sense. Recall that a k-simplex is the convex hull of $(k+1)$ affinely independent points, which are its vertices. We maintain this correspondence by defining $\dim A = |A| - 1$ for any set $A \in \mathcal{S}$.

The **dimension** of $\mathcal{S}$ is $\dim(\mathcal{S}) = \max\{\dim(A) \mid A \in \mathcal{S}\}$.

In the definition of $\mathcal{S}$, we do assume that all $A \in \mathcal{S}$ are finite sets. And there exists a maximum dimensional simplex in $\mathcal{S}$.
which is finite

The singleton sets in $\mathcal{S}$ are called its **vertices**, and is denoted by $\text{Vert}(\mathcal{S})$.

Again note the correspondence of these singleton sets to the vertices (0-simplices) in (geometric) simplicial complexes.

Here is an example of an abstract simplicial complex.

$$\mathcal{S} = \{\underbrace{\{0\},\{1\},\{2\},\{3\}}_{\text{vertices}}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\}\}.$$

We can indeed check that the condition on inclusion of subsets is satisfied. For instance, consider the set $\{0,1,2\}$.

$$\{0,1,2\} \in \mathcal{S} \Rightarrow \text{need } \underset{\text{trivial}}{\{0,1,2\}}, \underbrace{\{0,1\}, \{0,2\}, \{1,2\}, \{0\}, \{1\}, \{2\}}_{\text{indeed!}} \in \mathcal{S}.$$

$\dim \mathcal{S} = 2$.

Given any (geometric) simplicial complex K, we can create an abstract simplicial complex $\mathcal{S}$ by taking just the sets of vertices in each simplex of K (and ignoring the geometry). $\mathcal{S}$ here is called the **vertex scheme** of K.

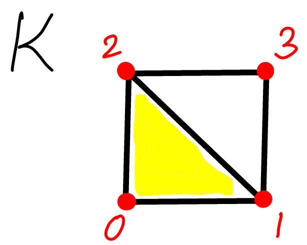
Symmetrically, K is a **geometric realization** of $\mathcal{S}$.

there could be other geometric realizations

Let's consider the ASC we saw previously:

$$\mathcal{S} = \{ \underbrace{\{0\}, \{1\}, \{2\}, \{3\}}_{\text{vertices}}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\} \}.$$

K shown here is a geometric realization of $\mathcal{S}$.



K can sit in $\mathbb{R}^2$

We have a vertex corresponding to each singleton set (i.e., vertex) in $\mathcal{S}$, an edge corresponding to each doublet, and a triangle for the triplet.

But this is just one geometric realization. In particular, if we were to specify the (x,y) coordinates of the vertices, we can imagine other realizations, e.g., by translating this one. We could also have a realization in $\mathbb{R}^3$, for instance.

It turns out that every ASC has a geometric realization.

Theorem (Geometric realization theorem) Every abstract simplicial complex $\mathcal{S}$ with $\dim \mathcal{S} = d$ has a geometric realization in $\mathbb{R}^{2d+1}$.

Idea of the Proof

set of vertices in $\mathcal{S}$

We map vertices of $\mathcal{S}$ injectively to points in $\mathbb{R}^{2d+1}$, say, $f: \mathrm{Vert}(\mathcal{S}) \to \mathbb{R}^{2d+1}$. Why $2d+1$?
We use the fact that $2d+2$ or fewer points in $\mathbb{R}^{2d+1}$ that are in **general position** are affinely independent (AI).

Def $(d+1)$ points in $\mathbb{R}^d$ are in **general position** if no hyperplane contains more than d of those points.

The idea is that the points do not satisfy any more linear relationships than they must.
For instance 3 points in $\mathbb{R}^2$ that are not collinear are in general position.

Recall that a d-simplex is the convex hull of $(d+1)$ AI points. We need to make sure that we will have "enough freedom", i.e., affine independence, among the mapped vertices so that we can map all the simplices in $\mathcal{S}$ to corresponding simplices in the geometric simplicial complex.

Consider $A, B \in \mathcal{S}$. Since $\dim(\mathcal{S}) = d$, $|A|, |B| \leq d+1$.

Hence, $|A \cup B| = |A| + |B| - |A \cap B| \leq d+1 + d+1 = 2d+2$.

Hence by going to $\mathbb{R}^{2d+1}$ and choosing points there in general position, we can ensure that (up to) $2d+2$ points are AI.

$\Rightarrow$ Any convex combination $\bar{x}$ in $A \cup B$ is unique.
$\Rightarrow$ $\bar{x} \in A$ and $\bar{x} \in B \iff \bar{x} \in A \cap B$.

$\hookrightarrow$ ensures the second requirement of nonempty intersections of two simplices being their faces.

We could often find geometric realizations in $\mathbb{R}^{d'}$ for d' smaller than $2d+1$.

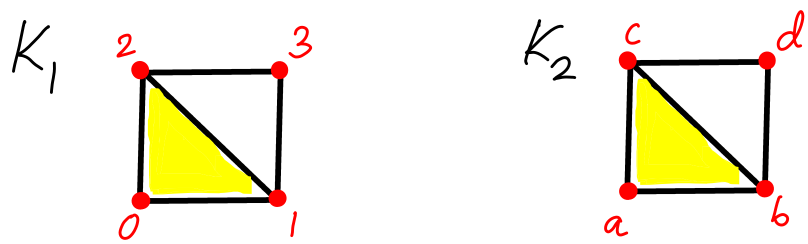
$\square$

How do we compare abstract simplicial complexes? We had previously defined the concept of homeomorphism to study when two topological spaces are "similar". We now define corresponding notions for simplicial complexes.
Recall that $\mathrm{Vert}(\mathcal{S})$ represents the vertex set of the ASC $\mathcal{S}$.

Def Two abstract simplicial complexes $\mathcal{S}_1$ and $\mathcal{S}_2$ are **isomorphic** if there exists a bijection $\varphi : \mathrm{Vert}(\mathcal{S}_1) \to \mathrm{Vert}(\mathcal{S}_2)$ such that $A \in \mathcal{S}_1$ iff $\varphi(A) \in \mathcal{S}_2$. φ is an isomorphism between $\mathcal{S}_1$ and $\mathcal{S}_2$. We write $\mathcal{S}_1 \approx \mathcal{S}_2$ here.

simplex

In this setting, every simplex in $\mathcal{S}_1$ has a unique corresponding simplex in $\mathcal{S}_2$.

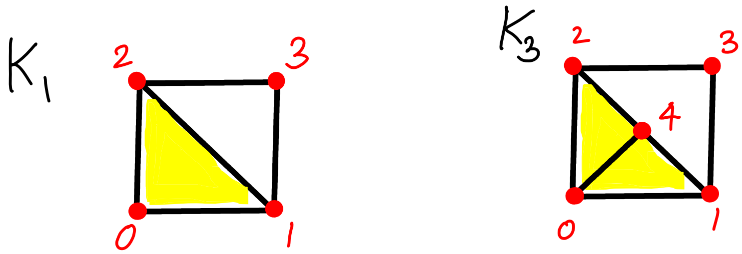


vertex schemes of K_1 and K_2 are isomorphic

Notice that the similarity here is defined for **abstract** simplicial complexes. Hence, K_1 and K_2 above need not be sitting in the same space. Still, they are isomorphic as ASCs. We make this notion precise in the following theorem.

Theorem Two (geometric) simplicial complexes K_1 and K_2 are isomorphic, or **simplicially homeomorphic**, iff their vertex schemes $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic as abstract simplicial complexes. We denote this fact by $K_1 \cong K_2$, which implies $|K_1| \approx |K_2|$, and $\mathcal{S}_1 \approx \mathcal{S}_2$.

The implication might not go the other way, though. For instance, $K_1 \cong K_2$ above. Now consider K_3 as shown below.



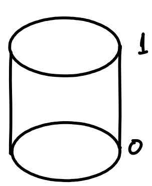
Notice that $K_1 \ncong K_3$, even though $|K_1| \approx |K_3|$. In fact, their underlying spaces could very well be identical!

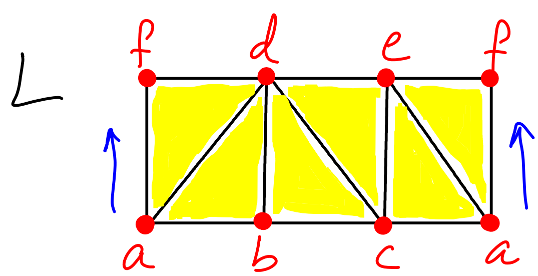
From the computational point of view, while simplicial complexes that are "smaller", i.e., have a smaller number of simplices while modeling the same topological space are usually preferred. At the same time, geometry might dictate that we need a large number of simplices to capture the complexity.

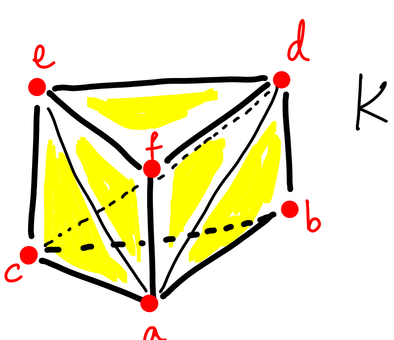
How do we use ASCs? We illustrate several examples

1. cylinder $\quad S^1 \times [0,1]$

geometric representation!







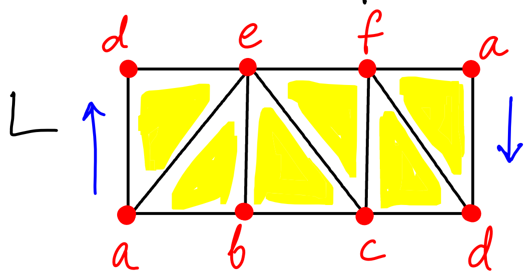
The underlying space of L appears to be rectangle, but notice how the vertex labels identify the left and right edges (both are af).

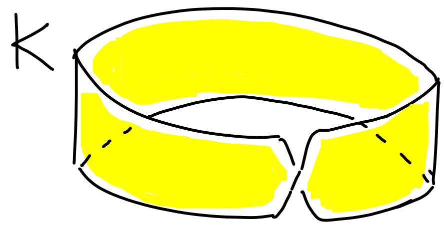
$$L = \{ \{a,b,d\}, \{a,d,f\}, \{b,c,d\}, \{c,d,e\}, \{a,c,e\}, \{a,e,f\}, \text{ and all nonempty subsets} \}$$

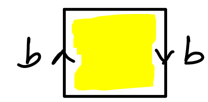
K is one geometric realization of L here.

2. Möbius strip

We start with the ASC L here







The left and right vertical edges are identified, after a "twist". Like in Example 1, the underlying space is a rectangle, but vertex labels are different.

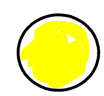
L represents the Möbius strip, i.e., K is a geometric realization of L.

We could also specify L abstractly:

$L = \{ \{a,d,e\}, \{a,b,e\}, \{b,c,e\}, \{c,e,f\}, \{c,d,f\}, \{a,d,f\}$, and all nonempty subsets of these triplets $\}$.

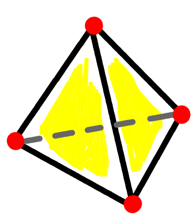
The abstract simplicial complexes shown above consist of triangles – indeed, they are triangulations. In general, triangulations consist of triangles (in 2D), and simplices in general as we formalize below.

Notice that each k-simplex $\approx$ k-ball (closed). 2-ball is the closed 2-disc.



Def (Triangulation). A **triangulation** of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.

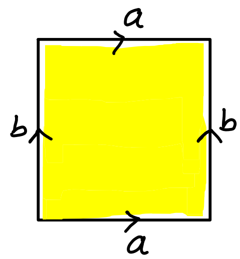
Example:



surface of a tetrahedron (triangles and their faces) is a triangulation of the 2-sphere S^2.

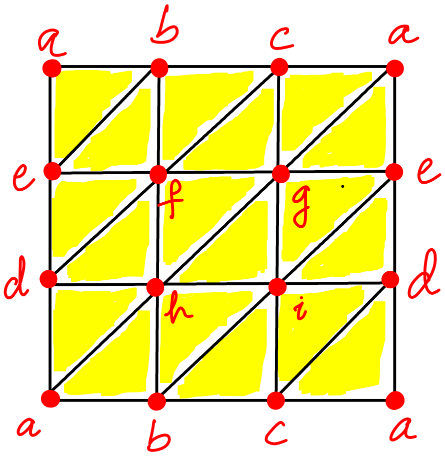
A triangulation is a piecewise linear representation of the topological space.

3. Torus ($\mathbb{T}^2$)



$\longrightarrow$ this diagram is not a triangulation!

K:

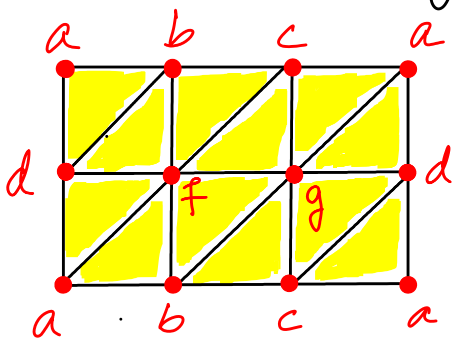


Why so many triangles?

This triangulation K of the torus has 18 (different) triangles. One might wonder if we could produce a triangulation using a much smaller number of triangles.

Consider the following candidate triangulation K':

K'



Is K' a triangulation of $\mathbb{T}^2$? No! For instance, consider edge $\overline{ad}$. It is a face of four triangles: adb, adf, adc, adg.

Hence, points on $\overline{ad}$ do not have neighborhoods homeomorphic to $\mathbb{R}^2$. We're doing "too much gluing" here.

Q. What is the minimum number of triangles needed to produce a triangulation of $\mathbb{T}^2$?

A. We need at least 14 triangles.

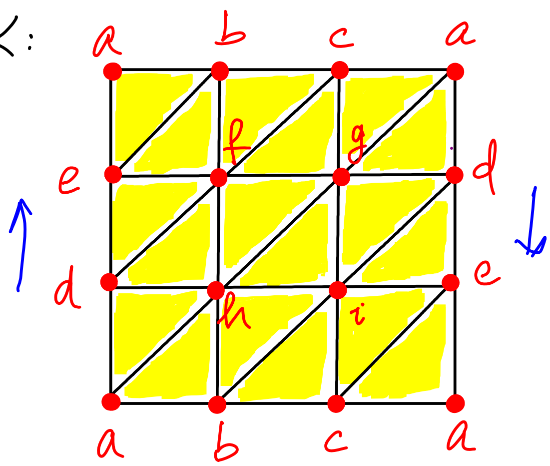
Rule: In a triangulation of a 2-manifold (with boundary), each edge must be part of (one or) two triangles.
Edges that are part of only one triangle each form the boundary of the 2-manifold.

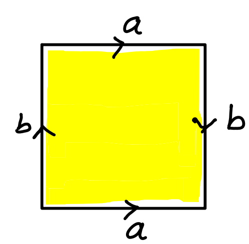
Recall that every point on a 2-manifold has a neighborhood homeomorphic to an open disc. The corresponding requirement for edges becomes that each edge has to be part of exactly two triangles. Similarly, points on the boundary of a 2-manifold have neighborhoods homeomorphic to half discs. Edges that are part of single triangles are indeed the boundary edges in the triangulation. The result extends to d-dimensions — every $(d-1)$-simplex is the face of one or two d-simplices.

The above rule could be used to check if a given simplicial complex is the triangulation of a manifold or not. But, be warned that satisfying this rule **alone** is not enough to identify the given simplicial complex as the triangulation of a specific manifold, e.g., the torus.

4. Klein bottle ($\mathbb{K}^2$)

K:





Same rule applies here — each edge is shared by two triangles exactly.

For example, edge $\overline{ae}$ is part of triangles Δabe and Δace.

We talked about distinguishing spaces that are not homeomorphic. In practice, spaces or objects are usually represented by triangulations. Checking for homeomorphisms between triangulations is not easy. How can we distinguish two topological spaces computationally? One option is to use an invariant.

MATH 529: Lecture 7 (02/03/2026)

Today:
* topological invariants
* Euler characteristic (χ)
* genus and ~~cross cap~~ → didn't get to it...

Topological Invariants

We want to define and use efficiently (and easily) computable functions that can help us to distinguish topological spaces of different topological types.

Def A **topological invariant** is a map that assigns the same object to spaces of the same topological type.

↳ usually, a number; but we could also have a "barcode", for instance

Let $f(\cdot)$ be an invariant.

$\mathbb{X} \approx \mathbb{Y} \Rightarrow f(\mathbb{X}) = f(\mathbb{Y})$.

So, $f(\mathbb{X}) \neq f(\mathbb{Y}) \Rightarrow \mathbb{X} \not\approx \mathbb{Y}$ → could be used for contrapositive arguments.

But $f(\mathbb{X}) = f(\mathbb{Y})$ does not necessarily mean $\mathbb{X} \approx \mathbb{Y}$.

If $f(\mathbb{X}) = f(\mathbb{Y}) \Rightarrow \mathbb{X} \approx \mathbb{Y}$, then $f(\cdot)$ is called a **complete invariant**.

Notice that an invariant could assign the same object to spaces of different topological types. The main way to use the invariant is in the contrapositive, i.e., if the invariant is different for a pair of spaces, then the two spaces have different topological types.

While the invariants are defined for topological spaces, we usually use triangulations to compute them. And since they're "invariants", the specific choice of the triangulation is not important.

We introduce a simple to evaluate/compute invariant — the Euler characteristic.

The Euler characteristic (χ) "chi" (originally defined for graphs)

Let K be a simplicial complex, and let s_i be the # i-simplices in K for $0 \leq i \leq \dim(K)$. Thus,

$$s_i = \left| \{ \sigma \in K \mid \dim \sigma = i \} \right|.$$

The **Euler characteristic** of K is defined as

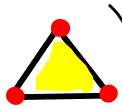
$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i .$$

Notice that $s_i = 0 \ \forall \ i > \dim K$.

Equivalently, $\chi(K) = v - e + f - t + \cdots$, where $v = \#$ vertices, $e = \#$ edges, $f = \#$ faces, $t = \#$ tetrahedra, ... and so on.

triangles

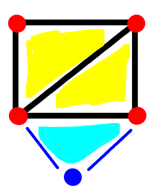
Let us find χ for a triangulation of the closed 2-disc.

χ () $= 3 - 3 + 1 = 1$

\# vertices \# edges \# faces (or \# triangles)

χ is an integer invariant, and it is an **invariant of the underlying space $|K|$**. So, χ is invariant over triangulations of a given space. Thus, any triangulation of a topological space $\mathbb{X}$ has the same $\chi(\mathbb{X})$ value.

Continuing with the example of the disc, we get the same χ using any other triangulation — see two examples below.

χ () $= 4 - 5 + 2 = 1.$

The triangulation is made of two triangles sharing an edge.

Now consider adding one more triangle to get another valid triangulation (as shown in blue).

$\Delta(V) = 1, \Delta(E) = 2, \Delta(F) = 1$, so $\Delta(\chi) = 1 - 2 + 1 = 0!$

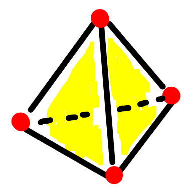
change in \# vertices, edges, triangles

change in χ

Q: Can we use χ to distinguish compact 2-manifolds?

Let us find χ for $\mathbb{S}^2, \mathbb{T}^2, \mathbb{RP}^2$ and $\mathbb{K}^2$, 2-sphere, torus, projective plane, and the Klein bottle.

1. $\mathbb{S}^2$ $\quad K$:

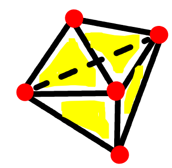


NOT the solid tetrahedron!

Surface of a tetrahedron is a triangulation of $\mathbb{S}^2$.

$\chi(K) = 4-6+4 = 2.$

K':

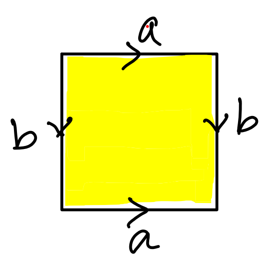


Let us consider another triangulation K' of $\mathbb{S}^2$, made of 3 triangles from top and 3 from bottom joined to form a "sphere".

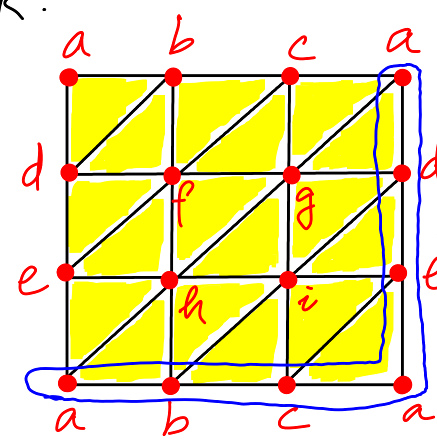
$$\chi(K') = 5-9+6 = 2.$$

Indeed, $\chi(\mathbb{S}^2) = 2$. One could take any triangulation of $\mathbb{S}^2$, χ will be the same.

2. $\mathbb{T}^2$ (torus)



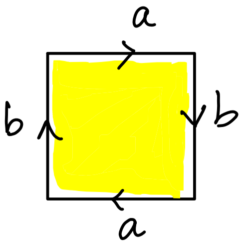
K:

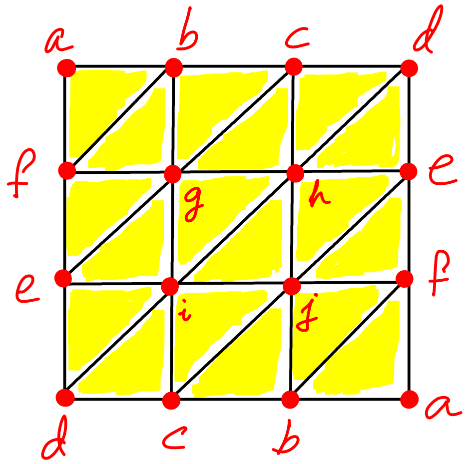


should not double count!

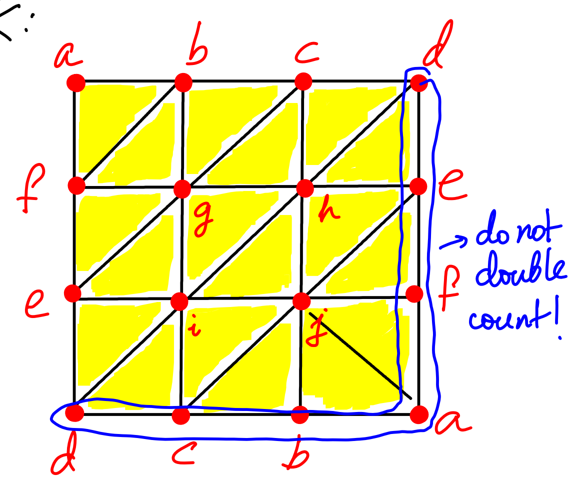
$$\chi(\mathbb{T}^2) = 9-27+18 = 0.$$

3. $\underline{\mathbb{RP}^2}$





K:



$\overline{bf} \in abf, jbf, gbf$! ×

$\overline{ab} \in abf$! ×

$\overline{ab} \in abj, abf$ ✓

$\overline{bf} \in bfa, bfg$ ✓

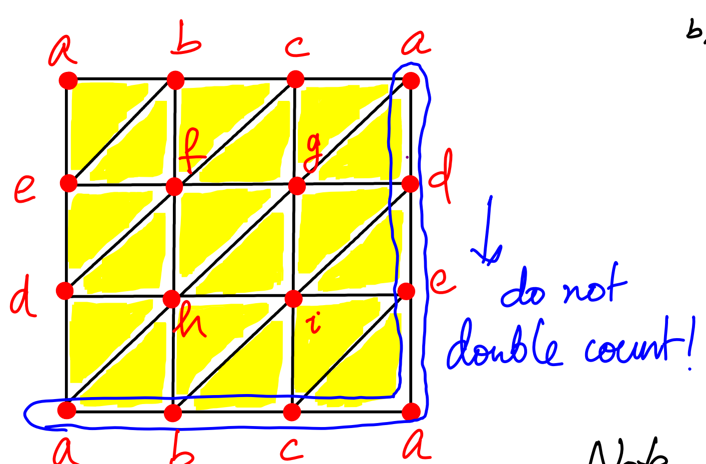
Our first attempt at finding a triangulation (left) of $\mathbb{RP}^2$ is not correct! In particular, edge $\overline{bf}$ (bottom right square) is part of 3 △les: abf, bfg, bfj! Also, $\overline{ab}$ and $\overline{af}$ are part of only one triangle each. But $\mathbb{RP}^2$ has no boundary! A correct triangulation is given on the right.

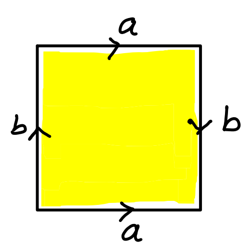
the left triangulation represents $\mathbb{RP}^2$ with a "flap" (△abf)

$$\chi(K) = \underbrace{10}_{\text{vertices } a\text{–}j} - 27 + 18 = 1.$$

4. $\mathbb{K}^2$ (Klein bottle)

K:





$$\chi(K) = 9 - 27 + 18 = 0.$$

Note that $\chi(\mathbb{K}^2) = \chi(\mathbb{T}^2)$!

Here is the summary of the χ values we have seen so far.

	2-manifold	χ
orientable	S^2	2
	$\mathbb{T}^2$	0
nonorientable	$\mathbb{RP}^2$	1
	$\mathbb{K}^2$	0

So, χ alone is not sufficient to distinguish between all these surfaces!

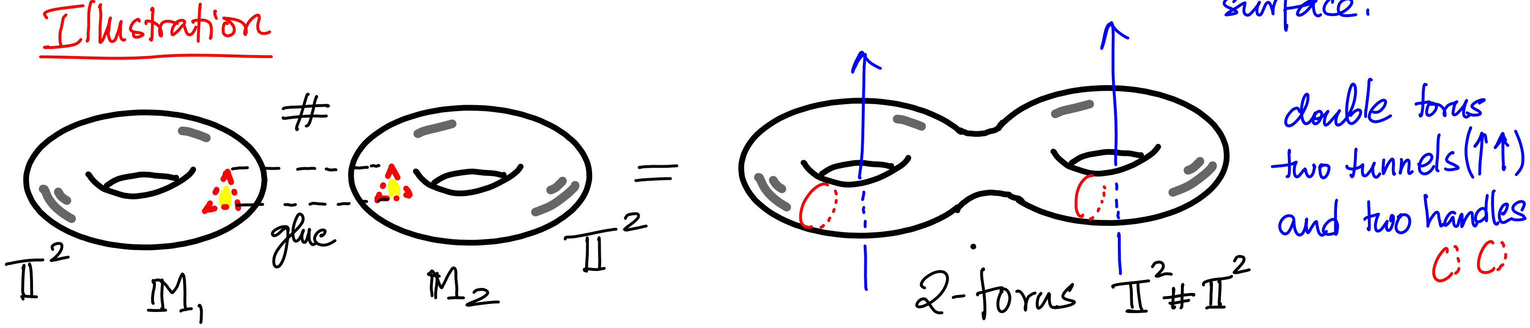
It turns out that if we add orientability to χ, we do get a complete invariant for all compact (connected) 2-manifolds (without boundary). Recall the original classification theorem, which states that every compact connected 2-manifold is homeomorphic to S^2, a connected sum of copies of $\mathbb{T}^2$, or a connected sum of copies of $\mathbb{RP}^2$. With this result in mind, let us first study how χ changes when we take the connected sum of two manifolds.

Theorem For compact, connected surfaces (2-manifolds) M_1 and M_2,

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2.$$

take one Δle out of each surface.

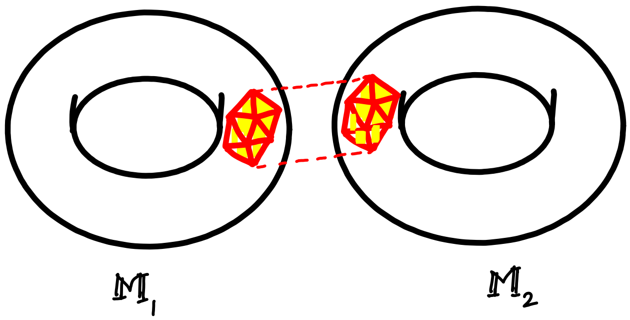
Illustration



We remove a triangle each from both M_1 and M_2, and glue along the boundaries of these triangles.

$\Delta(V) = -3$, $\Delta(E) = -3$, $\Delta(F) = -2$. So, $\Delta(\chi) = -3-(-3)+(-2) = -2$.

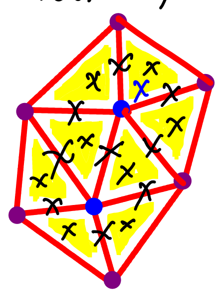
The result holds for the removal of a disc in general, and not just for the case of (the removal of) a triangle.



Here, we remove a patch (homeomorphic to the disc) from both $\mathbb{M}_1$ and $\mathbb{M}_2$, and identify the boundaries, which is composed of 6 edges and 6 vertices (to form a loop).

From the middle regions of each patch, we remove 2 vertices, 9 edges, and 8 triangles.

The change in $\chi(\mathbb{M}_1 \# \mathbb{M}_2)$ contributed by the simplices removed from $\mathbb{M}_1$ is $-(2-9+8) = -1$. A same change is contributed by the simplices removed from $\mathbb{M}_2$.



simplices marked with an 'x' are removed, and so are the two middle vertices.

There is no change in χ from identifying the boundaries — as they are both cycles (so have same # of vertices & edges).

As such, $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.

We get the same result even if we were to remove different "discs" from the two tori. Just that the homeomorphism defining the gluing would be more complicated there.

We could prove this result in general (for a removal of a general disk).

Theorem Two compact, closed, connected 2-manifolds $\mathbb{M}_1$ and $\mathbb{M}_2$ are homeomorphic **if and only if**

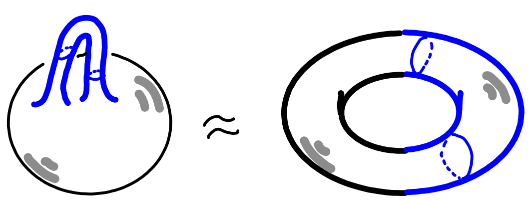
1. $\chi(\mathbb{M}_1) = \chi(\mathbb{M}_2)$ and
2. either $\mathbb{M}_1$ and $\mathbb{M}_2$ are both orientable, or are both nonorientable.

We could perform both checks 1 and 2 efficiently on a computer.

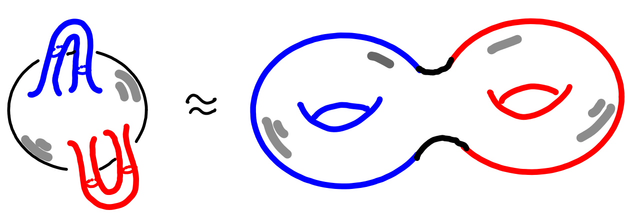
in polynomial time, to be precise

Genus and Cross-cap $\longrightarrow$ Two more terms used in the context of 2-manifolds.

Def The connected sum of g tori is called a surface with **genus** g. Equivalently, a 2-sphere with 1 tube is a surface with genus $g=1$.

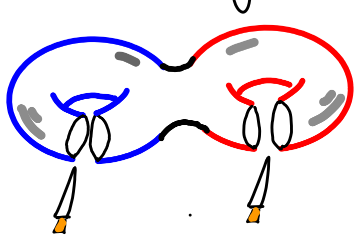


Sphere with one tube is homeomorphic to torus. (so, torus has genus = 1).



A sphere with two tubes is homeomorphic to the double torus.

$\mathbb{M}$ has genus $g \Rightarrow$ there are g disjoint closed curves on $\mathbb{M}$ along which you can cut without disconnecting $\mathbb{M}$.



$g=2$ here. If we cut along one more closed curve now, we get two pieces that are disconnected.

MATH 529: Lecture 8 (02/05/2026)

Today:
* cross caps
* orientation of a simplex
* orienting surfaces.

Recall The connected sum of g tori is an orientable surface with genus g.

How is the Euler characteristic connected to genus?

Euler characteristic and Genus

Recall $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.

$$\chi(g\mathbb{T}^2) = 2 - 2g.$$

$g\mathbb{T}^2$: connected sum of g tori

We could easily prove this result using induction, using the above fact about $\chi(\mathbb{M}_1 \# \mathbb{M}_2)$, and $\chi(\mathbb{T}^2) = 0$.

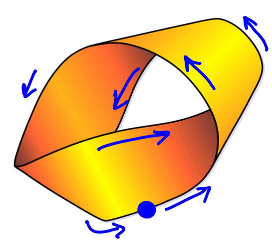
Recall, $\chi(\mathbb{S}^2) = 2$

Also, $\chi(\#(\mathbb{T}^2)^g)$

Cross cap

Recall that the Möbius strip has only one edge, i.e., its boundary is a single circle.

Starting from a point on the edge, we can traverse the entire boundary to come back to the same point (as shown by arrows).



(image: www)

If we remove an open disc from the 2-sphere, and glue a Möbius strip along its edge onto the boundary of this disc, we have added one **cross cap**.

A sphere with a single cross cap is homeomorphic to the real projective plane ($\mathbb{RP}^2$).
A sphere with two cross caps is homeomorphic to the Klein bottle ($\mathbb{K}^2$).

In general, a sphere with g cross caps is the connected sum of g projective planes, and we have

$$\chi(g\mathbb{RP}^2) = 2-g.$$

also, sphere with g cross caps

On the other hand, the classification theorem for compact connected 2-manifolds says that any non-orientable surface (2-manifold) is homeomorphic to the connected sum of copies of $\mathbb{RP}^2$, the projective plane. Recall that once we glue at least one cross cap, the surface becomes non-orientable. We could use the above result relating χ and the # cross caps and the result on how χ changes when we take the connected sum of two surfaces to identify the # copies of $\mathbb{RP}^2$ whose connected sum is homeomorphic to a given nonorientable surface.

Consider $\mathbb{T}^2 \# \mathbb{RP}^2$. We get

$$\chi(\mathbb{T}^2 \# \mathbb{RP}^2) = \chi(\mathbb{T}^2) + \chi(\mathbb{RP}^2) - 2 = 0+1-2 = -1.$$

To get the # cross caps in the homeomorphic surface, we set

$$\chi(g\mathbb{RP}^2) = 2-g = -1 \implies g = 3.$$

Hence we should have

$$\mathbb{T}^2 \# \mathbb{RP}^2 \approx \#(\mathbb{RP}^2)^3,$$

as we stated in Lecture 5!

Moving on, we now consider orientations of simplices, and how to extend them to possibly orient entire simplicial complexes.

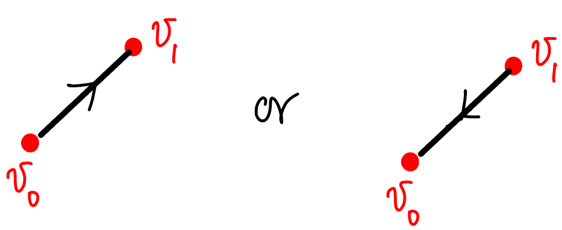
Def Let σ be a simplex (geometric or abstract). Two orderings of its vertices are equivalent if they differ by an even permutation. If $\dim(\sigma) > 0$, the orderings fall into **two** equivalence classes. Each equivalence class is an **orientation** of σ.
A 0-simplex has only one orientation.

An **oriented simplex** is a simplex σ with an orientation of σ.

An even permutation is obtained by doing an even number of pairwise swaps.

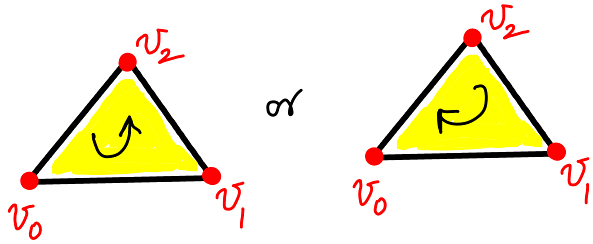
Notation For σ with vertices $\{v_0, \ldots, v_k\}$, $\sigma = [v_0, \ldots, v_k]$ denotes an oriented simplex. Note that we use σ to denote both the default and the oriented simplex.

Examples



1-simplex

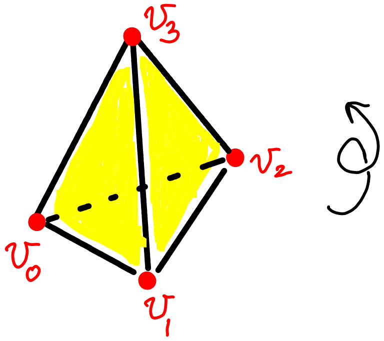
$[v_0 v_1]$ is opposite to $[v_1 v_0]$
(we can go forward or backward)



$[v_0 v_1 v_2]$ is the same orientation as $[v_2 v_0 v_1]$ — both go CCW. But $[v_1 v_0 v_2]$ goes clockwise.

$[v_0 v_1 v_2]$ and $[v_2 v_0 v_1]$ are same orientations

$[v_0 v_1 v_2] \longrightarrow [v_0 v_2 v_1] \longrightarrow [v_2 v_0 v_1]$ two swaps, so they are even permutations of each other.



$[v_0 v_1 v_2 v_3]$ is the same orientation as $[v_2 v_0 v_1 v_3]$ while $[v_0 v_2 v_1 v_3]$ is the opposite orientation.

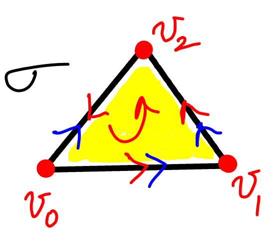
Induced orientation

"convex hull" here is used in general to cover both geometric and abstract cases.

Let σ have vertices $\{v_0, \ldots, v_k\}$. When σ is oriented, it induces an "induced orientation" on all its $(k-1)$-faces. Each $(k-1)$-face of σ can be denoted as conv$\{v_0, \ldots, \hat{v}_i, \ldots, v_k\}$, where the '^' (hat) above a vertex indicates that it is excluded.

Let σ be oriented as $[v_0, \ldots, v_k]$. Then the orientation induced by σ on $\tau = \text{conv}\{v_0, \ldots, \hat{v}_i, \ldots v_k\}$ is the same as that of $[v_0, \ldots, \hat{v}_i, \ldots, v_k]$ if i is even. Else, it is the opposite orientation.

For illustration, consider the triangle oriented as $\sigma = [v_0 v_1 v_2]$.



">": induced orientations

">": edges' own orientations, which are independent of the induced orientations

σ has three edges as faces, given by $\{v_0, v_1\}$, $\{v_0, v_2\}$, and $\{v_1, v_2\}$.

The induced orientations on these edges are $[v_0 v_1]$, $[v_2 v_0]$, and $[v_1 v_2]$, respectively.

We leave out v_1 to get the edge $\{v_0, v_2\}$, and hence its induced orientation is the reverse of $[v_0 v_2]$, i.e., it is $[v_2 v_0]$.

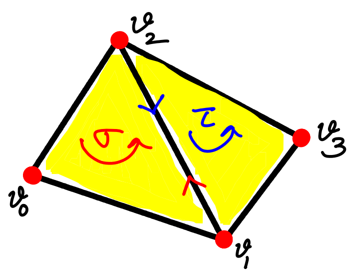
In the case of a triangle, the induced orientations indeed agree with the intuition of arrows "induced" on the edges by the CCW (or CW) arrow of the triangle.

CCW: counterclockwise

Comparing Orientations

Let σ, τ be simplices. If $\dim\sigma \neq \dim\tau$ we cannot compare their orientations. So let us consider the case when $\dim\sigma = \dim\tau = k$.

If σ and τ share a common $(k-1)$-face, they are **consistently oriented**, or oriented the same way, if they induce **opposite** orientations on the common $(k-1)$-face.



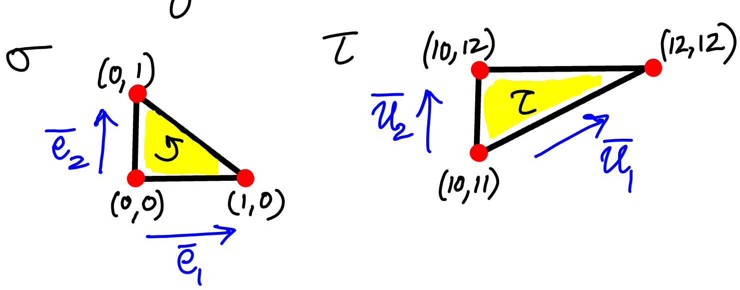
(induced orientations on $\overline{v_1v_2}$ shown by $\wedge$ and $\vee$)

orientations induced by $\sigma = [v_0v_1v_2]$ and $\tau = [v_1v_3v_2]$ on $\overline{v_1v_2}$ are opposite.

Hence the two triangles are consistently oriented — both CCW here.

Note that the induced orientations are separate from the edge's own orientation. Here, we could have $[v_1v_2]$ or $[v_2v_1]$ as the inherent orientation of $\overline{v_1v_2}$. The induced orientations are still as shown in the figure.

Note: We could compare orientations of two k-simplices even if they do not share a common $(k-1)$-face, if they both are sitting in the same k-dimensional plane. For instance, consider two disjoint triangles in $\mathbb{R}^2$.



The 3 vertices of a triangle generate two vectors, whose cross-product can be used to calculate the **signed area** of the triangle.

For oriented triangle $[v_0v_1v_2]$, consider vectors $\bar{e}_1 = \bar{v}_1 - \bar{v}_0$ and $\bar{e}_2 = \bar{v}_2 - \bar{v}_0$.

For σ, we can take $\bar{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\bar{e}_2 = \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

The signed area of σ is given by $\text{area}(\sigma) = \frac{1}{2}\underbrace{|\bar{e}_1 \times \bar{e}_2|}_{\text{area of the parallelogram generated by } \bar{e}_1 \,\&\, \bar{e}_2} = \frac{1}{2}\det([\bar{e}_1\, \bar{e}_2])$.

Thus, $\text{area}(\sigma) = \frac{1}{2}\left|\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right| = \frac{1}{2}$.

determinant

Similarly for τ, we choose $\bar{u}_1 = \begin{bmatrix}12\\12\end{bmatrix} - \begin{bmatrix}10\\11\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\bar{u}_2 = \begin{bmatrix}10\\12\end{bmatrix} - \begin{bmatrix}10\\11\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$,

giving $\text{area}(\tau) = \frac{1}{2}\left|[\bar{u}_1, \bar{u}_2]\right| = \frac{1}{2}\left|\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}\right| = 1$.

Since σ and τ have the same sign for their signed areas, they are consistently oriented. In this case, they are both oriented CCW. In fact, both sets of vectors $\{\bar{e}_1, \bar{e}_2\}$ and $\{\bar{u}_1, \bar{u}_2\}$ orient $\mathbb{R}^2$ in the same way here.

These computations are naturally extended to d-dimensions for $d \geq 3$. We can compute the signed d-volume in the same fashion. We could compare orientations in the abstract setting as well.

Consistently oriented simplices

We consider an example in the abstract setting.

Let $\sigma_1 = [2\ 5\ 12\ 19]$ and $\sigma_2 = [12\ 19\ 7\ 2]$ be two oriented 3-simplices. Are they consistently oriented?

Notice that $\tau = \{2, 12, 19\}$ is the common 2-face.

exclude to get τ

$$\sigma_1 = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 2 & 5 & 12 & 19 \end{bmatrix} \end{array} \qquad \sigma_2 = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 12 & 19 & 7 & 2 \end{bmatrix} \end{array}$$

We get τ by removing the first (i=1) vertex from σ_1. Similarly, we get τ by removing the second (i=2) vertex from σ_2.

Hence, the orientation induced on τ by σ_1 is $[12\ 2\ 19]$, which is the opposite orientation to $[2\ 12\ 19]$. differ by 1 pairwise swap

The orientation induced on τ by σ_2 is $[12\ 19\ 2]$.

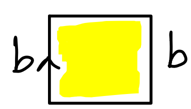
The two induced orientations are opposite here. Hence σ_1 and σ_2 are consistently oriented.

We extend the idea of when two d-simplices are consistently oriented to when the entire triangulation is consistently oriented.

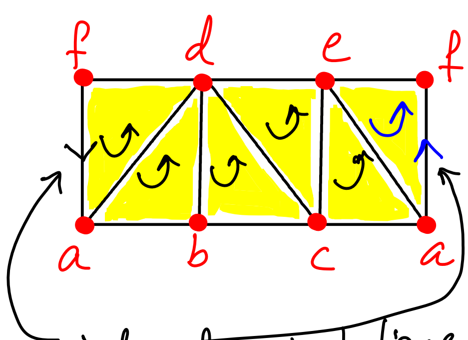
Def A triangulable d-manifold (with or without boundary) is called **orientable** if all the d-simplices in any triangulation of the manifold can be consistently oriented. Else, it is a **nonorientable** manifold.

<u>Examples</u>

1. Cylinder



diagram



all triangles oriented CCW.

triangulation

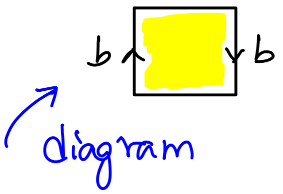
induced orientations are opposite

It can be checked that the orientations induced by each pair of triangles on their common (shared) edge are indeed opposite.

In particular, notice that this is indeed the case for $\overline{af}$ — induced orientations from $[adf]$ and $[afe]$ are $[fa]$ and $[af]$, respectively.

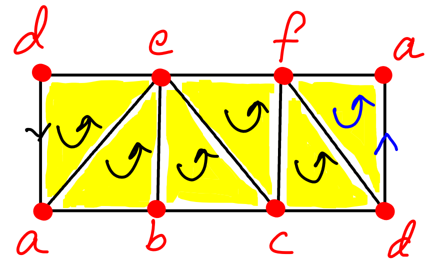
Thus, the cylinder is orientable.

2. Möbius strip



We had noted previously that the Möbius strip is non-orientable.

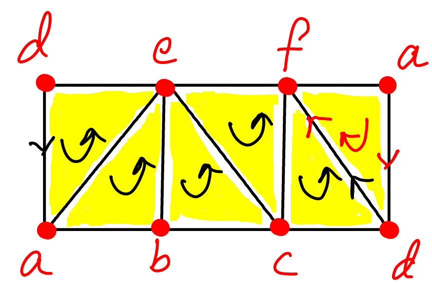
triangulation →



The orientations induced on edge $\overline{ad}$ by [aed] and [afd] are the same here — both being [da], or $d \to a$. Thus, Δaeb and Δadb are not consistently oriented.

Notice that the induced orientations on all remaining shared edges except $\overline{ad}$ are indeed opposite — check the induced orientations on $\overline{ae}$, $\overline{be}$, $\overline{ce}$, $\overline{cf}$ and $\overline{df}$.

If we fix the orientations such that induced orientations on $\overline{ad}$ are opposite, say, by orienting Δadf clockwise, i.e., [adf], then the induced orientations on $\overline{df}$ are now identical — [df].



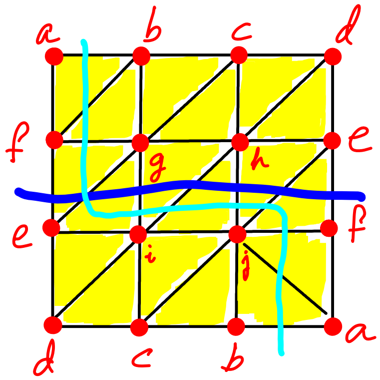
It turns out that however we orient the triangles, the induced orientations will be the same on one of the shared edges. As such, all triangles in the Möbius strip cannot be consistently oriented.

Hence the Möbius strip is non-orientable.

In fact, Möbius strips are the minimal non-orientable "objects" in 2D. For instance, you can identify Möbius strips in the triangulations of $\mathbb{RP}^2$ and $\mathbb{K}^2$ we introduced in Lecture 7!

Hence for surfaces, it is sufficient to identify a Möbius strip in the given triangulation to "certify" its nonorientability.

$\mathbb{RP}^2$:



The middle strip of 6 triangles forms a Möbius strip! And there are several other instances of the Möbius strip here — another instance is identified by ~ curve.

In both the cylinder and the Möbius strip, the boundary edges can be oriented arbitrarily. These are the edges that are faces of only one triangle each. For example, in the Möbius strip above, edges $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, $\overline{de}$, $\overline{ef}$ and $\overline{af}$ are boundary edges, and they can be assigned orientations arbitrarily without affecting the (non-)orientability of the manifold.

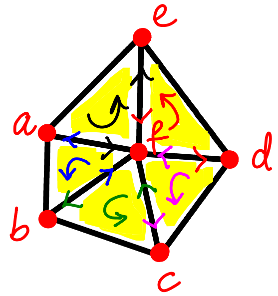
MATH 529: Lecture 9 (02/10/2026)

Today:
* propagating orientation
* subdivision
* star, closed star, link

To algorithmically check the orientability of a manifold, we try to propagate an orientation.

Checking orientability of a d-manifold Start by assigning an orientation to one d-simplex σ (pick one of the two possibilities). Then "propagate" this orientation to any other d-simplex σ' that shares a common (d-1)-simplex τ, say, with σ. In other words, orient σ' such that the orientations induced on τ by σ and σ' are opposite. Continue this process until all d-simplices are oriented. If we can consistently orient all d-simplices, the manifold is orientable. Else, it is non-orientable.

Example:



Start with [afe], and propagate this orientation in the order [abf], [bcf], [cdf], and [def]. Notice that the shared edges $\overline{af}$, $\overline{bf}$, $\overline{cf}$, $\overline{df}$, and $\overline{ef}$ have opposite induced orientations.

To certify non-orientability of a 2-manifold, it's best to identify a Möbius strip in it (as a subcomplex).

We now define several constructs on simplicial complexes that are simplicial analogues of standard constructs on (continuous) spaces, such as subspaces, open sets, neighborhoods of points, boundary of a neighborhood, etc.

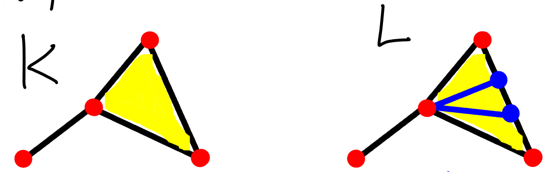
Subdivision

Def A simplicial complex L is a **subdivision** of another complex K if $|K| = |L|$, and $\forall \sigma \in L$, $\sigma \subseteq \tau \in K$.

not $\approx$!

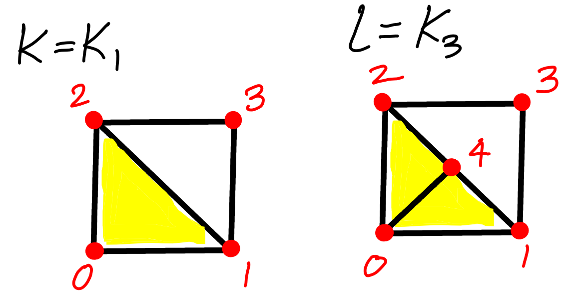
In words, every simplex in L is contained in a simplex in K.

Example:



the 3 triangles are contained in the one triangle in K, for instance.

Here is another example we saw in a different context (in Lecture 6):



Barycentric Subdivision

One way to create a subdivision of K is by forming the **barycentric subdivision**, denoted $\operatorname{Sd} K$ (this is a "standard" way to subdivide a complex).

The **barycenter** of a simplex is the centroid of its vertices.

We define (or construct) the barycentric subdivision inductively on the dimension of the simplices.

Inductive construction of Sd K

all simplices in K with dimension $\leq j$

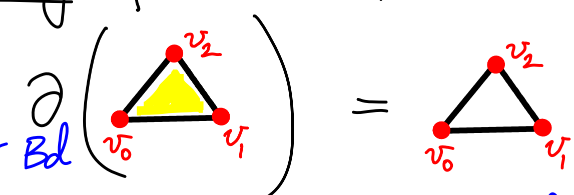
Notation $K^{(j)} = \{\sigma \in K \mid \dim \leq j\}$ is the j-skeleton of K.

Thus, $K^{(0)}$ is the set of vertices of K.

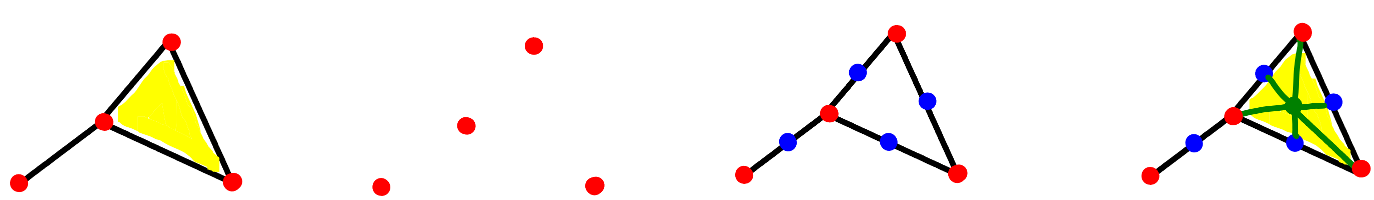
We start by defining $\operatorname{Sd} K^{(0)} = K^{(0)}$. (barycenter of a vertex is the vertex itself).

If we have $\operatorname{Sd} K^{(j-1)}$, we construct $\operatorname{Sd} K^{(j)}$ by adding the barycenter of every j-simplex as a new vertex, and connecting that vertex to each simplex that subdivides the boundary of that simplex.

$$\operatorname{Bd}\sigma = \bigcup_{\tau < \sigma} \tau$$

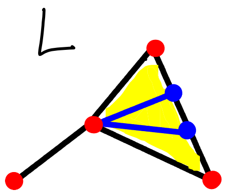


We illustrate this construction on the same simplicial complex we used to illustrate subdivisions.



K $\qquad$ $\operatorname{Sd} K^{(0)} = K^{(0)}$ $\qquad$ $\operatorname{Sd} K^{(1)}$ $\qquad$ $\operatorname{Sd} K^{(2)} = \operatorname{Sd} K$

Note that $\mathrm{Sd}\,K^{(0)} = K^{(0)}$ as the barycenter of a vertex is the vertex itself. Each edge in K is replaced by 2 edges, while each triangle is replaced by 6 triangles in $\mathrm{Sd}\,K$.



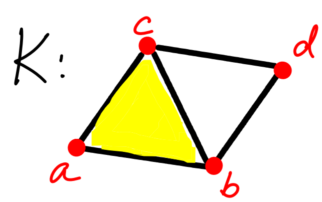
In the previous example, L is a subdivision of of K, but is not a barycentric subdivision.

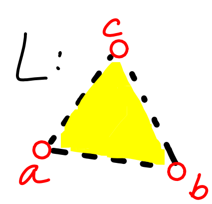
In certain applications, we may want not to subdivide certain simplices, e.g., keep a subset of original edges in tact.

We now present definitions on simplicial complexes corresponding to open neighborhoods around points in $\mathbb{R}^d$. We start with some preliminary definitions.

Def A **subcomplex** of K is a simplicial complex L, such that $L \subseteq K$.
A subset that is a simplicial complex by itself

Def The smallest subcomplex containing a subset $L \subseteq K$ is its **closure**, $\mathrm{Cl}\,L = \{\tau \in K \mid \tau \leq \sigma \in L\}$. $\to$ could also use $\bar{L}$





L is not a subcomplex of K here!

$\mathrm{Cl}\,\{\overline{ac}, d\} = \{\overline{ac}, d, a, c\}$. $\quad \mathrm{Cl}\,\{\Delta abc\} = \{\Delta abc, \overline{ab}, \overline{ac}, \overline{bc}, a, b, c\}$.

Notice that $\mathrm{Cl}\,L$ is a simplicial complex by itself.

Def For a simplex $\sigma \in K$, we define its **boundary** and **interior** as follows.

$$\mathrm{Bd}\,\sigma = \bigcup_{\tau < \sigma} \tau \quad \text{and} \quad \mathrm{Int}\,\sigma = \sigma \setminus \mathrm{Bd}\,\sigma.$$

or $\partial\sigma$ $\qquad$ or $\mathring{\sigma}$

Def For a vertex $\bar{v}$ of K, the **star** of $\bar{v}$, denoted $St\,\bar{v}$, is

$$St\,\bar{v} = \bigcup_{\sigma \geq \bar{v}} \operatorname{Int} \sigma.$$

coface

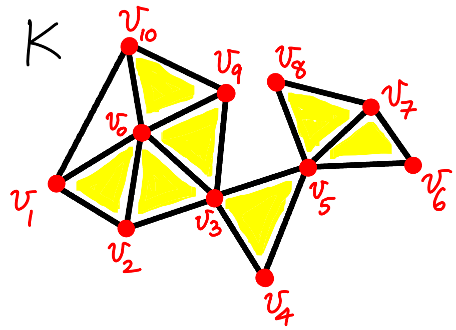
Union of interiors of all simplices that have $\bar{v}$ as a vertex

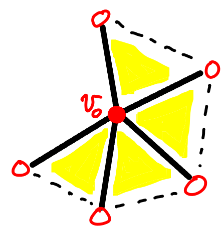
Also, $Cl\,St\,\bar{v}$ (or $\overline{St\,\bar{v}}$) is the **closed star** of $\bar{v}$ → take the closure of $St\,\bar{v}$, i.e., throw in all faces.

The set $Cl\,St\,\bar{v} \setminus St\,\bar{v}$ is the **link** of $\bar{v}$, denoted $Lk\,\bar{v}$.

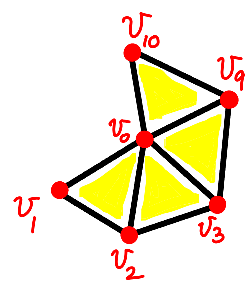
Here is an example — consider the complex K shown.

K

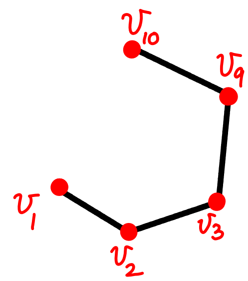




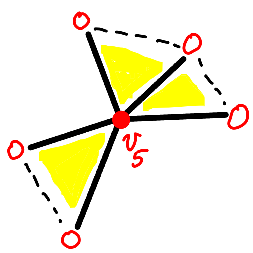
$St\,v_0$



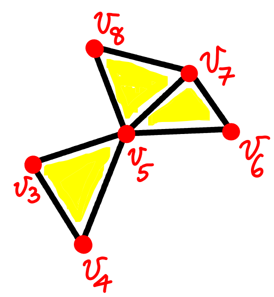
$Cl\,St\,v_0$



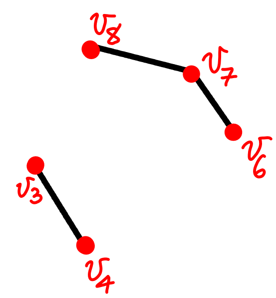
$Lk\,v_0$



$St\,v_5$



$Cl\,St\,v_5$



$Lk\,v_5$

Following these examples, we could make the following observations.

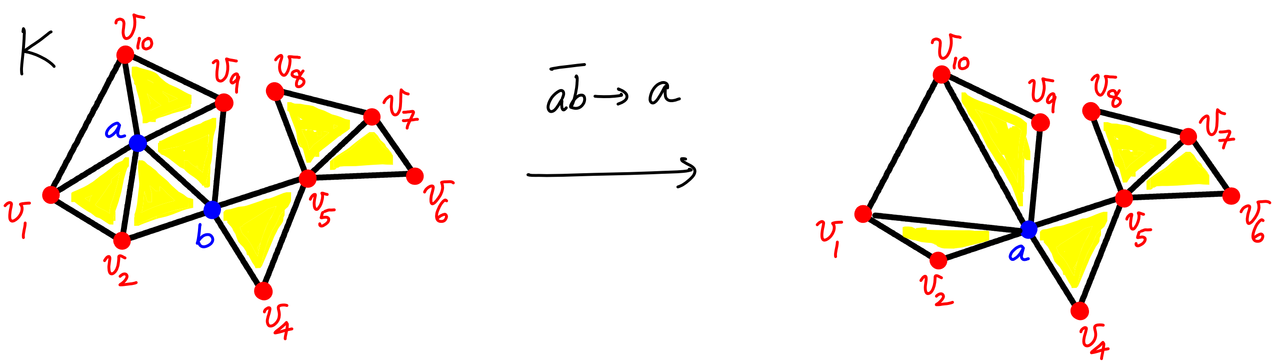
$St\,\bar{v}$ and $Cl\,St\,\bar{v}$ are both path connected, but $Lk\,\bar{v}$ need not be be connected.

there is a path between any two points in the set

Also, $Cl\,St\,\bar{v}$ and $Lk\,\bar{v}$ are subcomplexes of K, but $St\,\bar{v}$ is typically not a subcomplex.

An aside on mesh simplification

Links of vertices are employed in mesh simplification — you want a simplicial complex with a smaller number of simplices while preserving topology. A standard operation used in this context is edge contraction, where we replace edge $\overline{ab}$, say, with vertex a. We also make associated changes to other simplices connected to edge $\overline{ab}$. Here is an illustration.



Here, contracting $\overline{ab}$ to a preserves the topology — at least, we do not close any holes in K. But if we were to contract edge $\overline{v_1 v_{10}}$, we will close a hole!

We could define a condition on how the links of a, b, and of $\overline{ab}$ (to be defined next) are related. This is a local condition, and can be checked quickly. If it is satisfied, we can contract $\overline{ab}$ without the fear of closing any holes.

Such operations are critical to efficient computations on simplicial complexes. We will talk about them later in the semester.

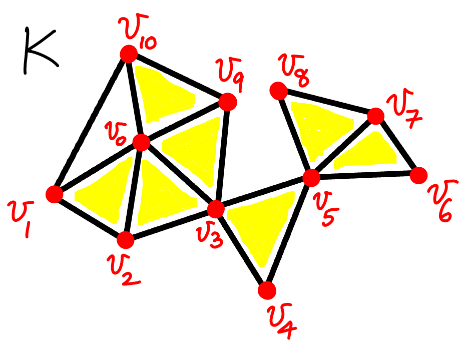
We now extend the definitions of star and link to simplices (of any dimension) and then to collections of simplices.

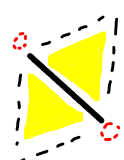
Def For a simplex $\sigma \in K$, we let

$$\mathrm{St}\,\sigma = \bigcup_{\tau \geq \sigma} \mathrm{Int}\,\tau \,, \qquad \mathrm{Cl}\,\mathrm{St}\,\sigma \text{ is its closed star,}$$

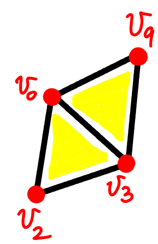
and $$\mathrm{Lk}\,\sigma = \{ \cup \tau \mid \tau \in \mathrm{Cl}\,\mathrm{St}\,\sigma,\ \tau \cap \sigma = \phi \}.$$

In words, the link of σ is the set of simplices in its closed star which are disjoint from σ.

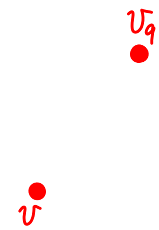




$\mathrm{St}\,\overline{v_0 v_3}$



$\mathrm{Cl}\,\mathrm{St}\,\overline{v_0 v_3}$



$\mathrm{Lk}\,\overline{v_0 v_3}$

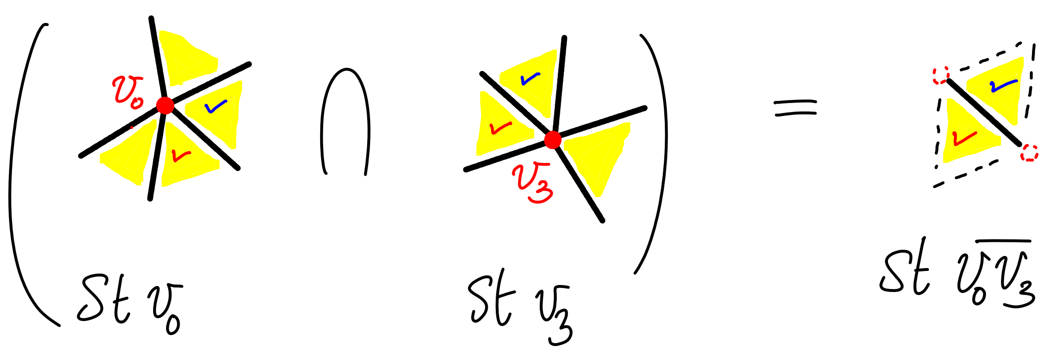
MATH 529 : Lecture 10 (02/12/2026)

Today :
* star of $X \subset K$
* poset representation
* retraction, homotopy equivalence

Recall : $\operatorname{St} \bar{v} = \bigcup_{\sigma \geqslant v} \operatorname{Int} \sigma$, $\quad \operatorname{St} \sigma = \bigcup_{\tau \geqslant \sigma} \operatorname{Int} \tau$. How are these two concepts related?

For $\sigma = [v_0, \dots, v_k]$, $\quad \operatorname{St} \sigma = \left(\bigcap_{i=0}^{k} \operatorname{St} v_i \right)$

For instance, with $\sigma = v_0 v_3$, we get



Def For $X \subseteq K$, we define

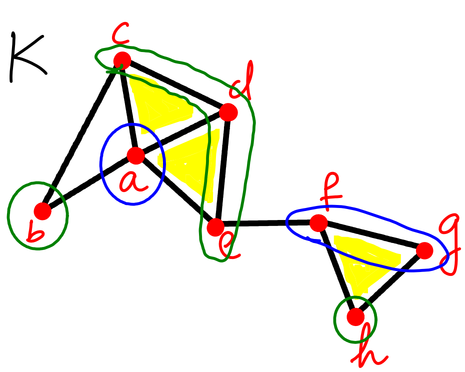
$$\operatorname{St} X = \left\{ \bigcup \tau \,\middle|\, \tau \geq \sigma, \sigma \in X \right\}.$$

Set of cofaces of each simplex in the set X.

$\operatorname{Cl} \operatorname{St} X$ is the closure of $\operatorname{St} X$. We define

$\operatorname{Lk} X$ = Set of simplices in $\operatorname{Cl} \operatorname{St} X$ that do not belong to $\operatorname{St}(\bar{X})$.

The intuition of the star being the open neighborhood of X and link being the boundary of the closed neighborhood still holds.



Let $X = \{a, \overline{fg}\}$. Then we intuitively want

$$\operatorname{Lk} X = \{b, c, d, e, \overline{cd}, \overline{de}, h\}.$$

We follow the definitions to get the following sets.

$$St\, X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \Delta aed, \Delta ade, \overline{fg}, \Delta fgh\}.$$

$$Cl\,St\, X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \Delta aed, \Delta ade, \overline{fg}, \Delta fgh, \ldots$$
$$b, c, d, e, \overline{cd}, \overline{de}, f, g, h, \overline{fh}, \overline{gh}, \}$$

the simplices added as proper faces

We also get

$$Cl\, X = \{a, \overline{fg}, f, g\}\text{, and}$$

faces added to close X

$$St\, Cl\, X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \Delta aed, \Delta ade, \overline{fg}, \Delta fgh, \ldots$$
$$\underline{f, g, \overline{ef}, \overline{fh}, \overline{gh}}\ \}$$

cofaces of the elements added to close X

$$\Rightarrow Lk\, X = Cl\,St\, X - St\,Cl\, X = \{b, c, d, e, \overline{cd}, \overline{de}, h\},$$

as expected!

Note that $\overline{ef} \in St\, Cl\, X$, but is not in $Lk\, X$ (as per definition).

For a small example, we can easily eye-ball these sets. But how do you handle large simplicial complexes with, say, 10^4 simplices?

We describe a way to efficiently store simplicial complexes and to read off $St\,X$, $Lk\,X$, $Cl\,X$, etc. from that representation.

Def (poset) Given a finite set S, a **partial order** is a binary relation $\leq$ on S that is reflexive, antisymmetric, and transitive, i.e., $\forall\, x,y,z \in S$,

partial, as not every $x,y \in S$ are related by $\leq$.

(a) $x \leq x$;

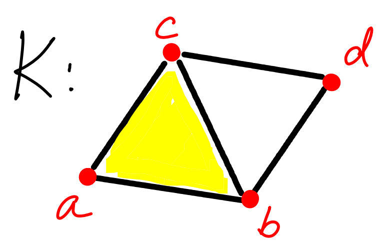
(b) $x \leq y$ and $y \leq x \Rightarrow x = y$; and

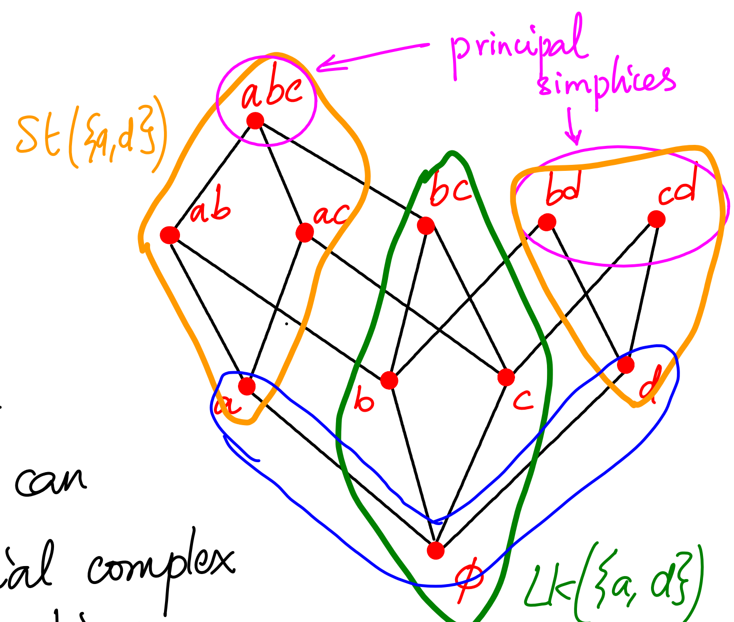
(c) $x \leq y,\ y \leq z \Rightarrow x \leq z$.

A set S with a partial order is called a partially ordered set, or a poset.

The face relationships of a simplicial complex is a partial order. So the vertex scheme of a simplicial complex with face relationships is a poset.

Illustration





The simplices "at the top" are called principal simplices. We can determine the entire simplicial complex if we know the principal simplices.

To find Star ($St\,X$), take X and everything above. For instance,

$St(\{a,d\}) = \{a, ab, ac, abc, d, bd, cd\}$.

To find $Cl\,X$, take X and everything below; e.g., $Cl(\{a,d\}) = \{a, d, \emptyset\}$.

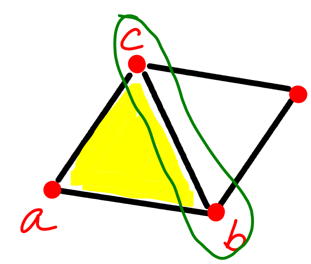
Notice that $Cl\,St(\{a,d\}) = K \cup \{\emptyset\}$ here.

As a convention, the empty simplex (or null set) is added at the bottom of this poset representation. It plays the role of the "root node" from which the poset representation "grows up".

Hence, we include the empty set $\emptyset$ in our definitions and discussions of closure, star, and link. In particular, we modify the definition of link slightly as follows:

$$Lk\,X = Cl\,St\,X - St(Cl\,X - \{\emptyset\}).$$

With $X = \{a,d\}$, we expect $Lk\,X$ to be $\{bc, b, c\}$.



Recall, $Cl\,X = \{a, d, \emptyset\}$. So,

$St(Cl\,X - \{\emptyset\}) = St(\{a,d\})$ here. Hence we indeed get

$$Lk\,X = \{bc, b, c, \emptyset\}.$$

We now define a notion of topological similarity that is weaker than homeomorphism. We then use this notion to define how to build simplicial complexes on data sets of points in $\mathbb{R}^d$.

Homotopy

Def Let $f, g : \mathbb{X} \to \mathbb{Y}$ be continuous maps from topological space $\mathbb{X}$ to space $\mathbb{Y}$. A **homotopy** between f and g is another continuous map

$H : \mathbb{X} \times \underbrace{[0,1]}_{t} \longrightarrow \mathbb{Y}$ such that H agrees with f at $t=0$, and with g at $t=1$. In other words,

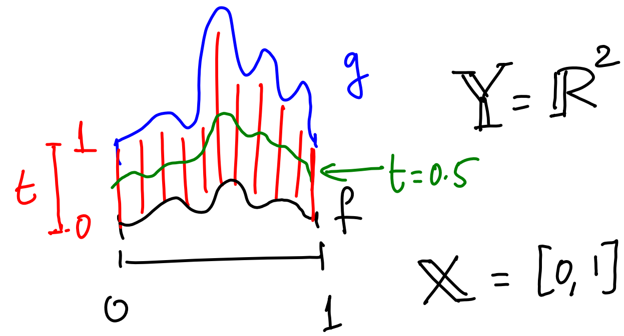
$$H(x,0) = f(x) \ \forall x \in \mathbb{X}, \text{ and}$$
$$H(x,1) = g(x) \ \forall x \in \mathbb{X}.$$

The index t can be thought of as time, varying from 0 to 1.

H could be thought of a time-series of functions $f_t : \mathbb{X} \to \mathbb{Y}$ for $t \in [0,1]$, with $f_t(x) = H(x,t)$, where $f_0 = f$ and $f_1 = g$.

We say that f is **homotopy equivalent** to g, or that f is homotopic to g. We denote this equivalence relation by $f \simeq g$. reflexive, symmetric, and transitive.

Here is an illustration, with $\mathbb{X} = [0,1]$ and $\mathbb{Y} = \mathbb{R}^2$. The homotopy H is a 2D strip of functions going from f to g. All of f, g, and H are continuous.



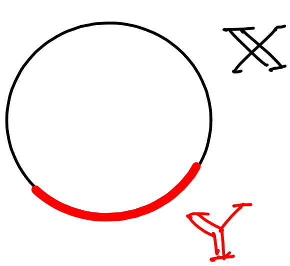
We extend the definition of *homotopy* to topological spaces.
First we consider a special case.

Def $\mathbb{Y} \subseteq \mathbb{X}$ is a **retract** of $\mathbb{X}$ if there is a continuous map $r : \mathbb{X} \to \mathbb{Y}$ with $r(y) = y \;\forall y \in \mathbb{Y}$.
r is called a **retraction**.

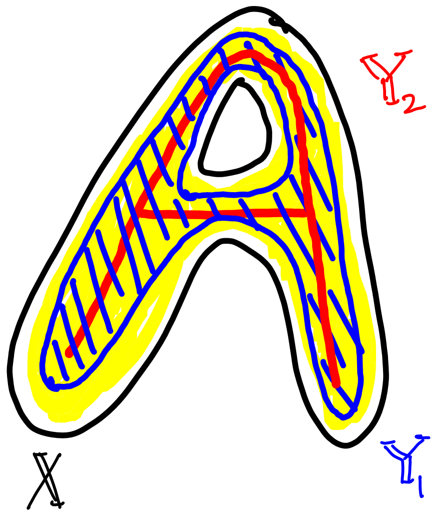
Def $\mathbb{Y}$ is a **deformation retract** of $\mathbb{X}$, and r is a **deformation retraction**, if there is a homotopy between the retract r and the identity map $id_{\mathbb{X}}$ on $\mathbb{X}$, i.e., $r \simeq id_{\mathbb{X}}$.

$id_{\mathbb{X}}(x) = x \;\forall x \in \mathbb{X}$.

We also say that $\mathbb{X}$ deformation retracts to $\mathbb{Y}$.

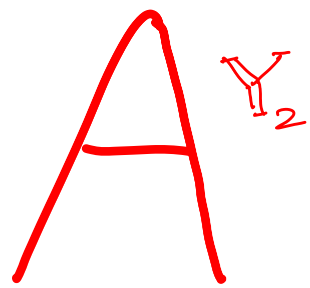


Here is an example of a retract that is not a deformation retract.
Notice that $\mathbb{X}$ is S^1 (circle), while $\mathbb{Y}$ is just an open arc.



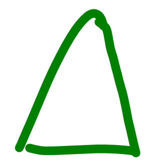
$\mathbb{Y}_1$ is a deformation retract of $\mathbb{X}$.

Continue to deform to obtain



$(\mathbb{Y}_2 \subset \mathbb{X})$.
"skeleton" sitting inside the "fat A".

$\mathbb{Y}_2 \not\approx \mathbb{X}$, but $\mathbb{Y}_2$ and $\mathbb{X}$ have the same homotopy type.
we'll define it formally soon!

Deforming even further, we can get $\mathbb{Y}_3$  $(\mathbb{Y}_3 \subset \mathbb{Y}_2)$.

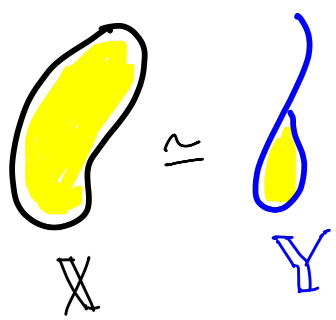
$\mathbb{X}$, and $\mathbb{Y}_j$, $j = 1, 2, 3$ are all homotopy equivalent. Also, each $\mathbb{Y}_j$ is a deformation retract of $\mathbb{Y}_k$ for $k < j$, and also of $\mathbb{X}$.

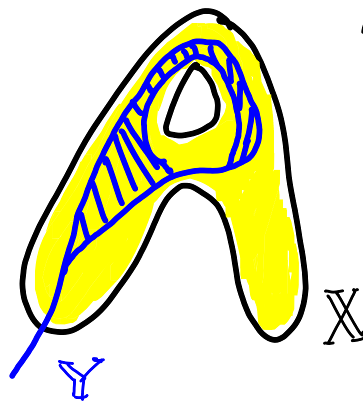
Notice that while $\mathbb{X}$ and $\mathbb{Y}_2$, for instance, are not homeomorphic, they both are forms of the letter 'A'. $\mathbb{Y}_2$ is, in some sense, the "skeleton" of $\mathbb{X}$. These types of transformations are allowed in the less tight notion of topological similarity termed homotopy equivalence, which is not as strict as homeomorphism.

Def $\mathbb{X}$ and $\mathbb{Y}$ are homotopy equivalent, or have the **same homotopy type**, if there exists continuous maps $f: \mathbb{X} \to \mathbb{Y}$ and $g: \mathbb{Y} \to \mathbb{X}$ such that $f \circ g \simeq id_{\mathbb{Y}}$ and $g \circ f \simeq id_{\mathbb{X}}$.

We denote $\mathbb{X} \simeq \mathbb{Y}$.

Note we have $f \circ g \simeq id_{\mathbb{Y}}$ and $g \circ f \simeq id_{\mathbb{X}}$, and not **equal to** in each case.





notice that $\mathbb{X} \simeq \mathbb{Y}$ here, but $\mathbb{Y}$ is not a retract of $\mathbb{X}$.

If two spaces are homeomorphic, they have the same homotopy type.

So, $\mathbb{X} \approx \mathbb{Y} \implies \mathbb{X} \simeq \mathbb{Y}$.

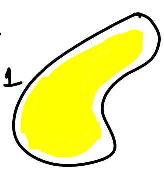
The implication does not go the other way, as many of the above examples show. For instance, $\mathbb{X}$ (fat 'A') is a 2-manifold with boundary, and $\mathbb{Y}_2$ (1-D 'A'), its 'skeleton', is a 1-manifold with boundary.

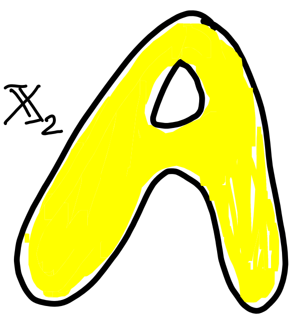
MATH 529: Lecture 11 (02/17/2026)

Today:
- * Nerve, Nerve theorem
- * Čech complex
- * Vietoris-Rips complex

Recall $\mathbb{X} \simeq \mathbb{Y}$ (homotopic) ...

Def If $\mathbb{Y}$ is a single point, and $\mathbb{X} \simeq \mathbb{Y}$, then we say that $\mathbb{X}$ has the homotopy type of a point, and we say that $\mathbb{X}$ is **contractible**.

$\mathbb{X}_1$  is contractible

$\mathbb{X}_2$  $\mathbb{X}_2$ is not contractible.

Our next goal is to study how to construct simplicial complexes from sets of points (in some space $\mathbb{R}^d$). Most applications analyze data in this format. We would like to construct the simplicial complex such that it captures the topology of the point set — if not up to homeomorphism, up to homotopy, or even up to a weaker level (to be defined later). We need one more concept to introduce such constructions.

Def (Nerves) Let F be a finite collection of sets in $\mathbb{R}^d$. The **nerve** of F consists of all subcollections of F with nonempty intersections.

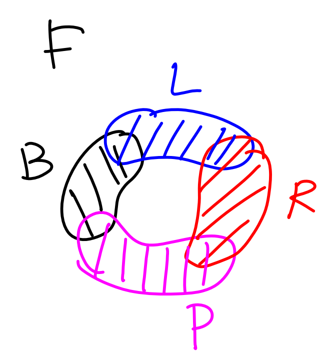
$$\operatorname{Nrv} F = \{ X \subseteq F \mid \bigcap X \neq \emptyset \}.$$

abstract simplicial complex

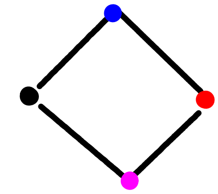
$\operatorname{Nrv} F$ is always an ASC, as $\bigcap X \neq \emptyset$ and $Y \subseteq X \Rightarrow \bigcap Y \neq \emptyset$.

Example

Consider an instance of F consisting of four sets, shaded Black, BLue, Red, and Pink. The four sets intersect in four pairs, as shown. Then $\operatorname{Nrv} F$ consists of the following intersecting subsets of $\{B, L, R, P\}$.

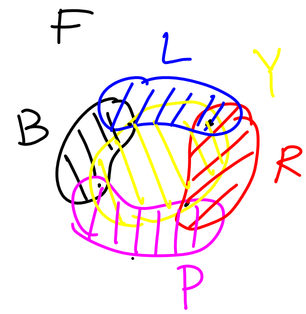


$$\text{Nrv}\, F = \{ \{B\}, \{L\}, \{R\}, \{P\},$$
$$\{B, L\}, \{L, R\}, \{R, P\}, \{B, P\} \}$$



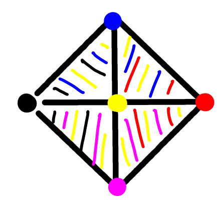
One geometric realization of Nrv F

Nrv F has a geometric realization in the same space ($\mathbb{R}^2$) as F here.



Now consider adding another set to F, shaded Yellow, such that Y intersects each pair of intersections already present, as shown.

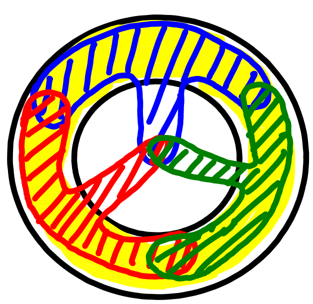
Now, Nrv F has a geometric realization as a disc made of four triangles, as shown here.



One geometric realization of Nrv F

In this example, F and Nrv F are homotopy equivalent. But does this result hold in general? Let's consider another example...

$|F|$ is a closed disc with three holes



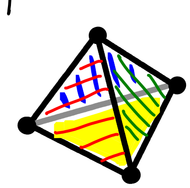
F is a collection of four regions, such that every subset of three regions has a common intersection.

With $F = \{R, B, G, Y\}$ for Red, Blue, Green, Yellow, we can write

$$\text{Nrv}\, F = \{R, B, G, Y, \{R,B\}, \{R,G\}, \{R,Y\}, \{B,G\}, \{B,Y\}, \{G,Y\},$$
$$\{R,B,G\}, \{R,B,Y\}, \{R,G,Y\}, \{B,G,Y\}\}.$$

Indeed, Nrv F has a geometric realization as the surface of a tetrahedron as shown.

Nrv F



$\approx S^2$

So, $\mathrm{Nrv}\, F \not\simeq |F|$ here!

underlying space, disk with 3 holes.

But if the sets in F are "nice", we do get homotopy equivalence with $\mathrm{Nrv}\, F$, as specified by the following theorem.

Nerve theorem Let F be a finite collection of closed **convex** sets in $\mathbb{R}^d$. Then $\mathrm{Nrv}\, F$ has the same homotopy type as the collection of sets in F.

Our goal is to build simplicial complexes out of collections of points. We could consider a collection of convex sets, each containing one point from the set, and then form its nerve. A default convex set containing a point is a closed ball centered at that point. We will consider a few different ways of forming simplicial complexes out of points using balls centered on them.

Čech Complex Let S be a finite set of points in $\mathbb{R}^d$.

pronounced as "Check"

We write $B_{\bar{x}}(r) = \bar{x} + r\mathbb{B}^d = \{\bar{y} \in \mathbb{R}^d \,/\, \|\bar{y}-\bar{x}\| \leq r\}$, for the closed ball of radius r and center $\bar{x}$.

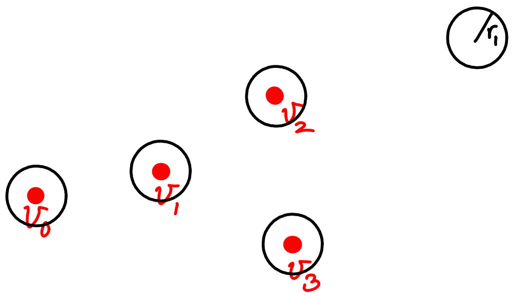
The **Čech complex** at radius r of the points in set S is the nerve of the collection of closed r-balls centered at the points.

$$\check{C}ech(r) = \{\sigma \subseteq S \mid \bigcap_{\bar{x}\in\sigma} B_{\bar{x}}(r) \neq \emptyset\}.$$

one could write $\check{C}ech_S(r)$ to be complete, but S is understood, and hence omitted, typically.

to be exact, one should say $\mathrm{conv}(\sigma)$ here. We do mean the simplex spanned by vertices in σ; $\sigma = \mathrm{conv}\{\bar{v}_0, \ldots, \bar{v}_k\}$, $\bar{v}_i \in S$.

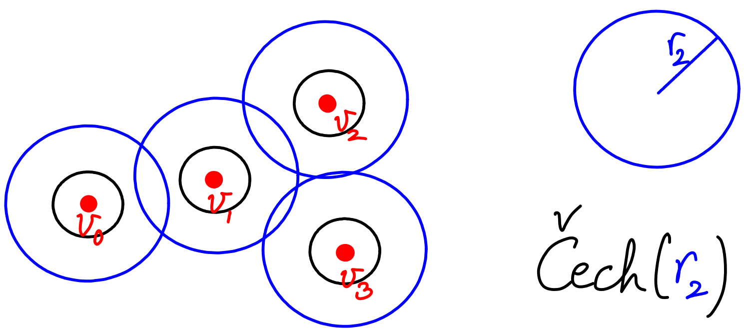
Consider an example with four points in $\mathbb{R}^2$ as shown.



$$\check{C}ech_S(r_1) \simeq \{v_0, v_1, v_2, v_3\}$$

Čech complex is homotopic to the union of balls centered at $\bar{v}_i$ — at all radii (and not just for small values such as r_1 shown here)

r_1 is small enough that no two of the balls centered at v_i intersect. Hence, $\check{C}ech(r_1)$ has just the four points. Let's consider a bigger radius now.

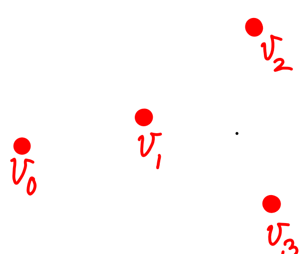


(the balls at v_1 & v_2 intersect

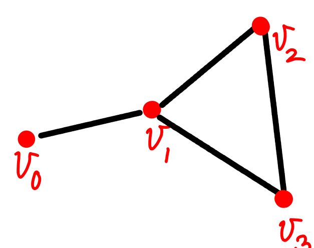
$$\check{C}ech(r_2) = \{v_0, v_1, v_2, v_3, \overline{v_0 v_1}, \overline{v_1 v_2}, \overline{v_2 v_3}, \overline{v_1 v_3}\}.$$

Geometric realizations of $\check{C}ech(r_1)$ and $\check{C}ech(r_2)$:

$\check{C}ech(r_1)$:

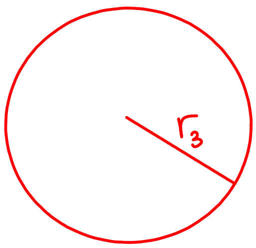


$\check{C}ech(r_2)$:

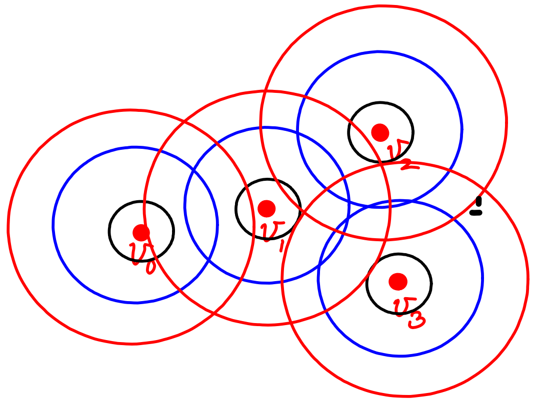


Notice that the balls centered at v_1, v_2, v_3 do not all intersect. Thus, there is a "hole" in between these three balls, which is represented by the empty triangle $v_1v_2v_3$ in $\check{C}ech(r_2)$.

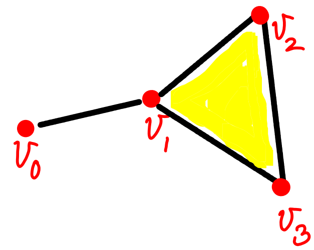
Increasing the radius a bit more brings in $\Delta v_1 v_2 v_3$:



$r_3 > r_2 > r_1$



Čech(r_3):

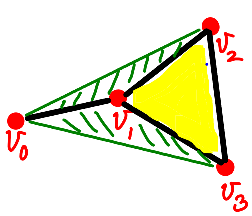


Note: The circles shown here are solid discs — shading is avoided for clarity.

Notice that Čech(r_1) is a subcomplex of Čech(r_2), which in turn is a subcomplex of Čech(r_3).

In general, **Čech$(r_i) \subseteq$ Čech(r_j)** when $r_i \le r_j$.

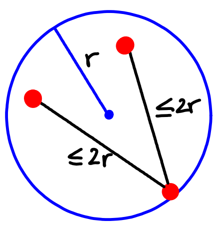
Also, Čech(r) of a set S of points in $\mathbb{R}^d$ may not have a geometric realization in $\mathbb{R}^d$ itself. But you can always treat it as an abstract simplicial complex.



At larger radii (r_4), triangles $\Delta v_0 v_1 v_2$ and $\Delta v_0 v_1 v_3$ are included in Čech(r_4), and at a still higher radius, tetrahedron $v_0 v_1 v_2 v_3$ is included. But, of course, $v_0 v_1 v_2 v_3$ cannot be embedded in $\mathbb{R}^2$.

We will consider this aspect — the complex having a geometric realization in the input space itself — later on. First, we look at more properties of the Čech complex.

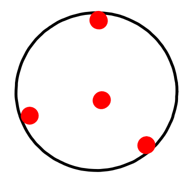
Another property central to the Čech complex is that balls of radius r have a common intersection iff their centers lie inside a ball of radius r.



So, $\sigma \subseteq S \in \check{C}ech(r) \iff$ smallest ball enclosing σ has radius $\leq r$.

Def The **miniball** of a set $\sigma \subseteq S$ is the smallest closed ball containing σ.

similar to circumsphere/circumcircle



Hence, radius of miniball of $\sigma \leq r \iff \sigma \in \check{C}ech(r)$.

To build (or define) $\check{C}ech(r)$, we need to check intersections of multiple (≥ 3) balls. This step could be computationally expensive, especially in large sets of points, since we have to go upto checking all points together in the data set! But, here is a better option.

Vietoris-Rips Complexes Instead of checking the intersection of all balls, if we check just pairwise intersections, and add 2- or higher dimensional simplices whenever all edges are in, we get the **Vietoris-Rips** or VR complex.

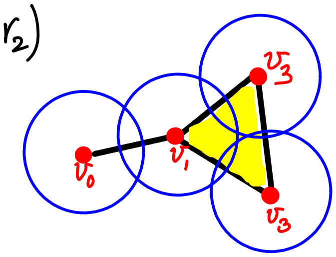
We write $VR_S(r) = \{\sigma \subseteq S \mid \text{diam}\, \sigma \leq 2r\}$.

or $\text{Vietoris-Rips}_S(r)$

diameter of σ

Def The **diameter** of σ is the supremum of all pairwise distances between points in σ.

$VR(r_2)$



Compared to $\check{C}ech(r_2)$, we add $\Delta v_1 v_2 v_3$ to the Vietoris-Rips complex at $r = r_2$.

How do $\check{C}ech(r)$ and $VR(r)$ compare?

Naturally, $\check{C}ech_S(r) \subseteq VR_S(r)$. But notice that $VR_S(r_2)$ does not have a hole, as $\Delta v_1 v_2 v_3$ is included. At the same time, $\left|\bigcup_{i=0}^{4} B_{v_i}(r_2)\right|$ does have a hole, and so does $\check{C}ech(r_2)$.

So, homotopy is not preserved in $VR_S(r_2)$. Nonetheless, we get an inclusion going the other way, i.e., $VR(r) \subseteq \check{C}ech(r')$, at a larger radius r'.

<u>Vietoris-Rips Lemma</u> Let S be a finite set of points in $\mathbb{R}^d$, and let $r \geq 0$. Then

$$VR_S(r) \subseteq \check{C}ech(\sqrt{2}r).$$

The inclusions going both ways mean that VR and Čech complexes are "quite comparable" when we consider all possible radii $(-\infty < r < \infty)$. While we may not get the same series of complexes, either family would be sufficient for most topological computations of interest. Hence, VR complexes are almost always preferred for computations, while Čech complexes are sometimes preferred when used in proofs.

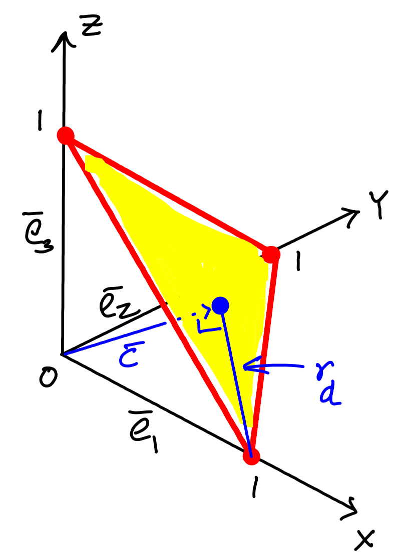
Proof (IDEA)

Consider Δ^d, the regular d-simplex in $\mathbb{R}^{d+1}$. Each vertex is a unit vector in this space. Thus,

$$\Delta^d = \text{conv}(\bar{e}_1, \ldots, \bar{e}_{d+1}), \quad \text{where } \bar{e}_j \text{ is the } j^{th} \text{ unit vector in } \mathbb{R}^{d+1}.$$

Regular simplices are the "limiting" cases to consider here, due to their symmetry.

Let $\bar{c}$ be the barycenter of Δ^d.



$$\bar{c} = \begin{bmatrix} \frac{1}{d+1} \\ \vdots \\ \frac{1}{d+1} \end{bmatrix} \qquad \|\bar{c}\| = \frac{1}{\sqrt{d+1}}$$ is the length from origin of Δ^d.

→ perpendicular distance

We compute $r_d = \sqrt{\frac{d}{d+1}} \left(= \sqrt{1 - \|\bar{c}\|^2} \right)$.

Note: $r_d \longrightarrow 1$ as $d \to \infty$.

The pairwise distance between $\bar{e}_i$ and $\bar{e}_j$ in σ is $\sqrt{2}$.

We'll finish the argument in the next lecture...

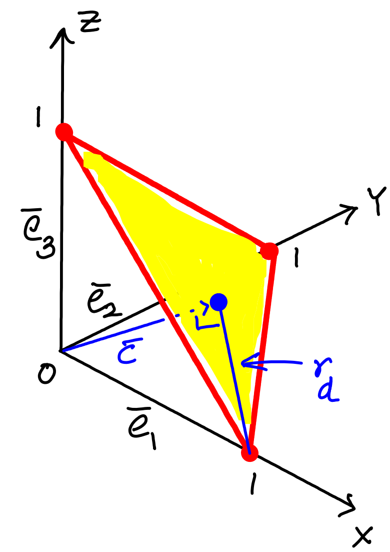
MATH 529: Lecture 12 (02/19/2026)

Today:
* Voronoi diagram
* Delaunay triangulation
* filtration

Recall VR Lemma: $VR_S(r) \subseteq \check{C}ech(\sqrt{2}r)$

Proof idea continued...

Let $\bar{c}$ be the barycenter of Δ^d. → regular d-simplex



$$\bar{c} = \begin{bmatrix} \frac{1}{d+1} \\ \vdots \\ \frac{1}{d+1} \end{bmatrix} \qquad \|\bar{c}\| = \frac{1}{\sqrt{d+1}}$$ is the length from origin of Δ^d.

↳ perpendicular distance

We compute $r_d = \sqrt{\frac{d}{d+1}} \left(= \sqrt{1 - \|\bar{c}\|^2} \right)$.

Note: $r_d \longrightarrow 1$ as $d \to \infty$.

The pairwise distance between $\bar{e}_i$ and $\bar{e}_j$ in σ is $\sqrt{2}$.
Also, the miniball of Δ^d has radius r_d.
Hence, simplex Δ^d of diameter $\sqrt{2}$ also belongs to $\check{C}ech(r_d)$. Multiplying by $\sqrt{2}r$, we get that,

$$VR(r) \subseteq \check{C}ech(\sqrt{2}\, r\, r_d).$$ But $r_d \leq 1$,

and hence, $VR(r) \subseteq \check{C}ech(\sqrt{2}r)$. $\square$

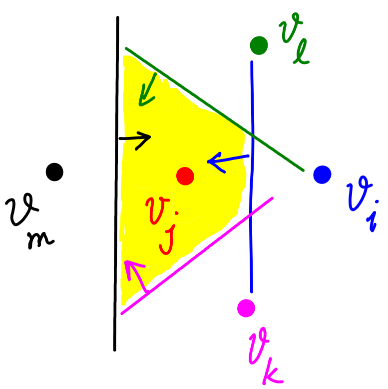
We saw $\check{C}ech_S(r)$ and $VR_S(r)$ of S (points) in $\mathbb{R}^d$.
Can we limit the dimension of the simplices we get from Now S? Yes!
We can build the Delaunay complex. We first describe its dual construction.

Voronoi Diagram

Recall: $S = \{\bar{v}_1, \ldots, \bar{v}_n\}$ is a finite set of points in $\mathbb{R}^d$.

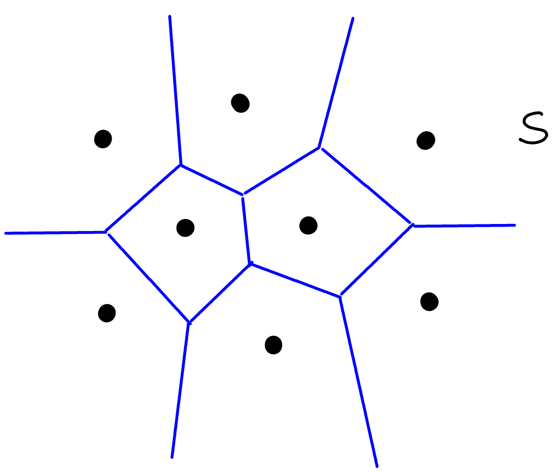
The Voronoi cell of $\bar{v}_j \in S$ is the set of points in $\mathbb{R}^d$ closest to $\bar{v}_j$:

$$V_{\bar{v}_j} = \{\bar{x} \in \mathbb{R}^d \mid \|\bar{x} - \bar{v}_j\| \leq \|\bar{x} - \bar{v}_i\| \ \forall\ \bar{v}_i \in S\}.$$



When we have just two points, say, v_i and v_j, the perpendicular bisector between them is the set of points equidistant from both of them. The half plane on the side of v_j then is V_{v_j}, its Voronoi cell.

$V_{\bar{v}_j}$ is a convex polyhedron, as it is the intersection of a set of half spaces, each being convex. $V_{\bar{v}_j}$ for all $\bar{v}_j \in S$ together tile or cover all of $\mathbb{R}^d$.



The collection of $V_{\bar{v}_j}$ for all $\bar{v}_j \in S$ is called the **Voronoi diagram** of S.

$V_{\bar{v}_i}$ and $V_{\bar{v}_j}$ meet at most in a common boundary. In $\mathbb{R}^2$, Voronoi cells meet at points or edges.

Notice that $V_{\bar{v}_j}$ can be open or closed. Intuitively, the boundary of $V_{\bar{v}_j}$ can be thought of as the "fence" around $\bar{v}_j$'s "house" — everything within the fence "belongs" to $\bar{v}_j$.

Delaunay Triangulation

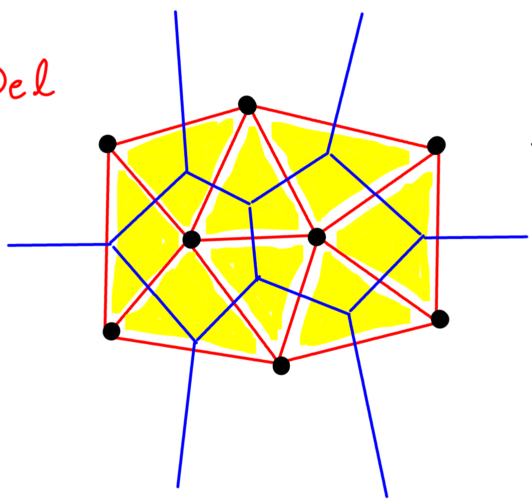
The **Delaunay complex** of S is (isomorphic to) the nerve of its Voronoi diagram.

$$\mathrm{Del}_S = \{\sigma \subseteq S \mid \bigcap_{\bar{v}_j \in \sigma} V_{\bar{v}_j} \neq \phi\}.$$

or $\mathrm{Delaunay}_S$

Similar to the Čech complex, we start with a convex set or cell associated with each point in S, and then take the nerve. But instead of balls, we use the Voronoi cells for each vertex.

Del

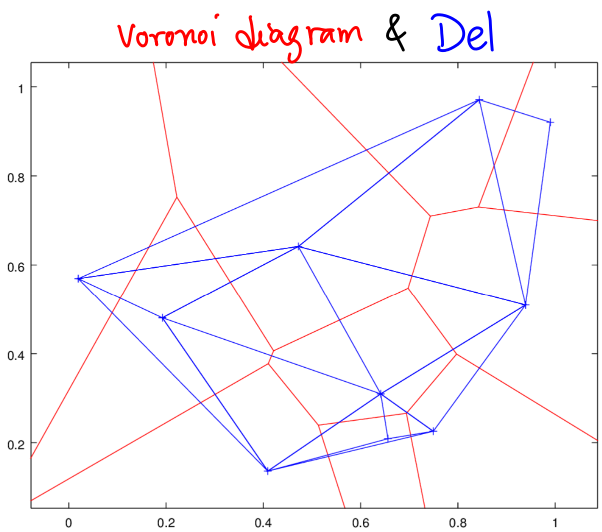


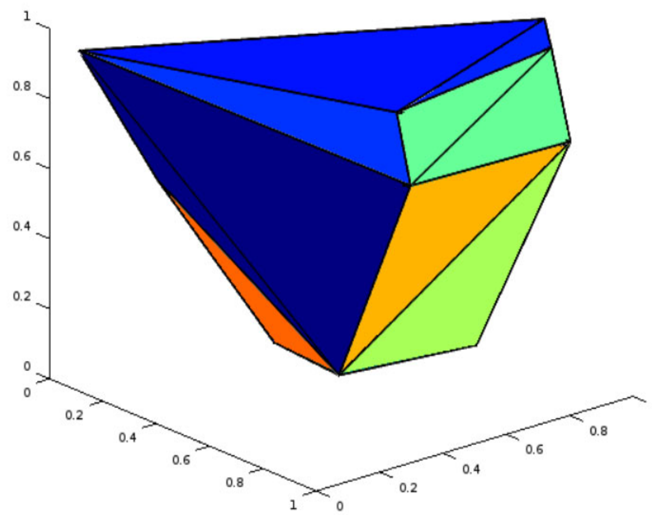
S

Shown here is one geometric realization of $\mathrm{Delaunay}_S$. This is the "natural" realization, as well.

Check out the commands voronoi, delaunay, delaunayn, and DelaunayTriangulation, as well as related commands in Matlab. Similar commands are available in Python as well.

Here are a couple sample pictures of 2D and 3D Delaunay complexes produced in Matlab. The 2D picture shows the Voronoi diagram as well.

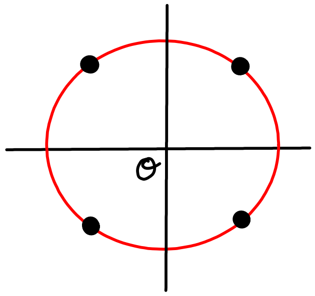




Looks like we only get (upto) triangles here. Recall one of the main motivations we stated for introducing Delaunay complexes — that we wanted to get only upto d-simplices for point sets S in $\mathbb{R}^d$.

Q. Do we always get only triangles in Del_S is 2D?

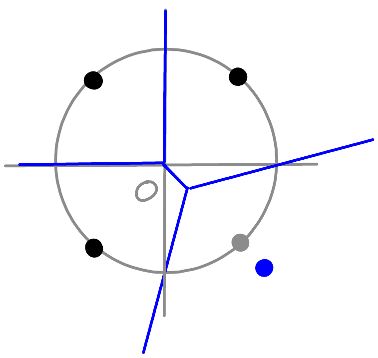
No!



The Voronoi cells V_{v_j} of the four points meet at the central point o (origin) here.

Hence, Del_S contains the tetrahedron!

But, if we move even one of the four points just ever so slightly away from the circle, we can avoid the 4-way intersection of their Voronoi cells.



Mathematically, we need to move only one (out of the four) points by $\epsilon > 0$ in one of the coordinate directions; ϵ could be really small, as long as it is > 0.

Def The set of points S in $\mathbb{R}^d$ is in **general position** if no $(d+2)$ points in S lie on a common $(d-1)$-sphere.

e.g., $d=2 \Rightarrow$ no 4 points lie on a circle.

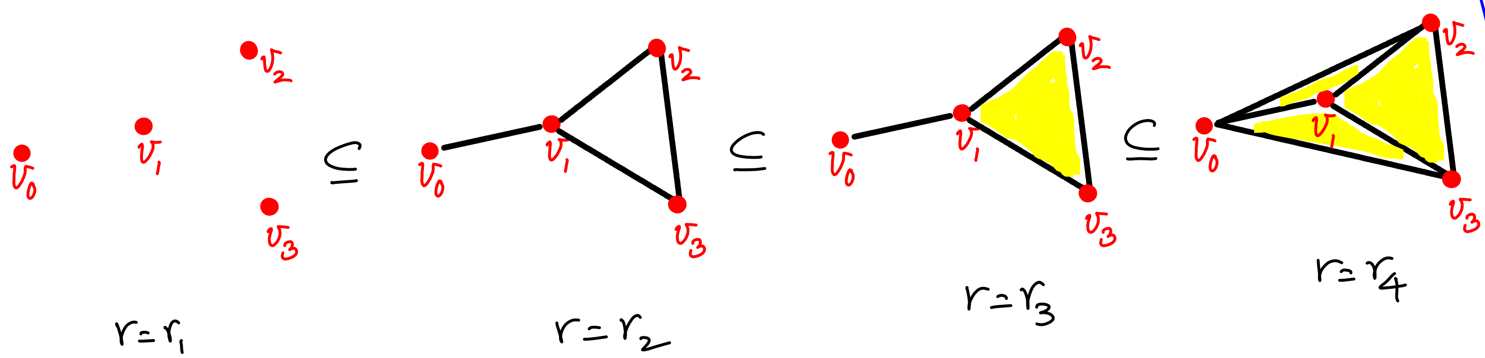
No $(d+2)$ V_{v_j}'s have a common intersection $\Rightarrow$ ($\sigma \in Del_S$ means $\dim \sigma \leq d$)

We assume general position usually. If this condition is not satisfied, we could perturb a single coordinate of a single point from any $(d+2)$ such points by a small $\epsilon > 0$.

There are very efficient (polynomial time) algorithms for constructing Delaunay tessellations, at least in 2D and 3D.

We have seen 3 families of simplicial complexes: $\check{C}ech_S(r)$, $VR_S(r)$, and Del_S. Note that the Delaunay complex is independent of any distance cutoffs (or radii of balls). But, what do we gain by varying r?

Given S, we could study the family of $\check{C}ech_S(r)$ or $VR_S(r)$ as r varies from 0 to ∞ (or, even from $-\infty$ to $+\infty$). Since S is finite, we will only a finite number of such complexes.



$\subseteq$ tetrahedron $v_0v_1v_2v_3$ (at $r = r_5$)

→ Since there are only four points here, their tetrahedron is the largest dimensional simplex we get in $\check{C}ech(r)$, even if we keep increasing r beyond r_5.

We capture the fact that v_1, v_2, v_3 are closer to each other than, say, $\{v_2, v_3\}$ are to v_0, since $\Delta v_1v_2v_3$ comes in to $\check{C}ech(r)$ before the other triangles.

We will study such families of complexes in detail.

Def A **filtration** of a simplicial complex K is a nested sequence of subcomplexes of K

$$\phi = K^0 \subseteq K^1 \subseteq \cdots \subseteq K^m = K.$$

The simplicial complex K along with a filtration is a **filtered simplicial complex**.

A **filtration ordering** is a full ordering of all simplices in K such that each prefix of the ordering is a subcomplex.

Filtration ordering: Example

$K =$  $v_0 v_1 v_2 v_3$

a subcomplex

$$v_0 < v_1 < v_2 < v_3 < v_0v_1 < v_0v_2 < v_0v_3 < v_1v_2 \,|\, < v_1v_3 < v_2v_3$$
$$< v_0v_1v_2 < \cdots < v_0v_1v_2v_3.$$

More generally, '$<$' could assign the same rank for several simplices, e.g., all vertices $<$ all edges $<$ all triangles $< \cdots$. In this case, $<$ is not a full ordering but we can convert it to one by breaking ties arbitrarily.

Q. What do we use filtrations for?

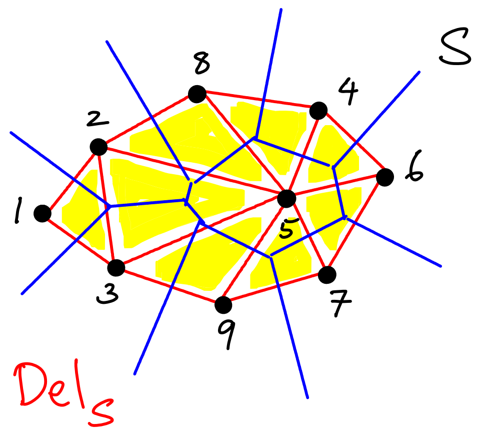
We could study signature functions: $\lambda : \{0, 1, \ldots, m\} \longrightarrow \mathbb{R}^d, d \geq 1$. λ assigns a value in $\mathbb{R}^d$ for each $k \in \{0, \ldots, m\}$. We could compare the signatures for two point sets S_1 and S_2 to distinguish them — by comparing $\lambda(S_1)$ and $\lambda(S_2)$.

For instance, χ (Euler characteristic). We could compute $\chi(K^i)$ for each $i \in \{0, \ldots, m\}$, and study the Euler characteristic curve (or vector).

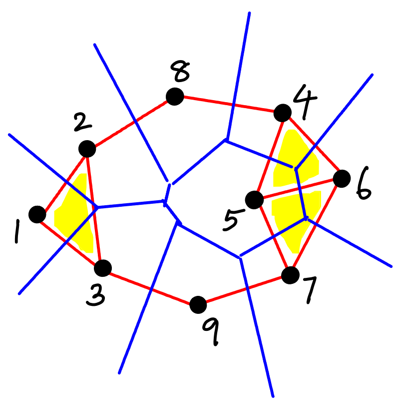
But we will study other more involved signatures soon.

Could we vary r to create a family of nested Delaunay complexes? Equivalently, could we create a filtration for Del_S?

Consider S with 9 points shown here:



Observe: $\{1,2,3\}$ and $\{4,5,6,7\}$ form two clusters of nearby points, which are further away from each other.



A subcomplex of Del_S as shown here would capture the topology of S "better".

For the given set of 9 points, how do we define the complex shown here?

We'll introduce the alpha complex in the next lecture...