

# MATH 529 - Lecture 8 (02/01/2024)

Today:

- \* orientation of a simplex
- \* orienting surfaces
- \* subdivision didn't get to it  $\textcircled{5}'$

Before we discuss orientation, we present one last illustration of how we use the Euler characteristic and its properties.

We saw that  $\chi(g\mathbb{R}P^2) = 2-g$  ( $g$ : # cross caps).

On the other hand, the classification theorem for compact connected 2-manifolds says that any non-orientable surface (2-manifold) is homeomorphic to the connected sum of copies of  $\mathbb{R}P^2$ , the projective plane. Recall that once we glue at least one cross cap, the surface becomes non-orientable. We could use the above result relating  $\chi$  and the # cross caps and the result on how  $\chi$  changes when we take the connected sum of two surfaces to identify the # copies of  $\mathbb{R}P^2$  whose connected sum is homeomorphic to a given nonorientable surface.

Consider  $\mathbb{II}^2 \# \mathbb{R}P^2$ . We get

$$\chi(\mathbb{II}^2 \# \mathbb{R}P^2) = \chi(\mathbb{II}^2) + \chi(\mathbb{R}P^2) - 2 = 0 + 1 - 2 = -1.$$

To get the # cross caps in the homeomorphic surface, we set  $\chi(g\mathbb{R}P^2) = 2-g = -1 \Rightarrow g=3$ . Hence we should have

$$\mathbb{II}^2 \# \mathbb{R}P^2 \approx \#(\mathbb{R}P^2)^3,$$

as we stated in Lecture 5!

Moving on, we now consider orientations of simplices, and how to extend them to possibly orient entire simplicial complexes.

**Def** Let  $\sigma$  be a simplex (geometric or abstract). Two orderings of its vertices are equivalent if they differ by an even permutation. If  $\dim(\sigma) > 0$ , the orderings fall into two equivalence classes. Each equivalence class is an **orientation** of  $\sigma$ .

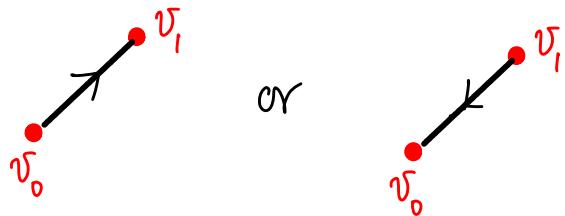
A 0-simplex has only one orientation.  
A simplex  $\sigma$  with an orientation of  $\sigma$ .

An even permutation is obtained by doing an even number of pairwise swaps.

An **oriented simplex** is a

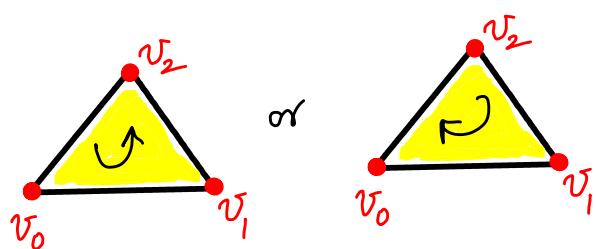
**Notation** For  $\sigma$  with vertices  $\{v_0, \dots, v_k\}$ ,  $\sigma = [v_0, \dots, v_k]$  denotes an oriented simplex. Note that we use  $\sigma$  to denote both the default and the oriented simplex.

### Examples



1-simplex

$[v_0 v_1]$  is opposite to  $[v_1 v_0]$   
(we can go forward or backward)

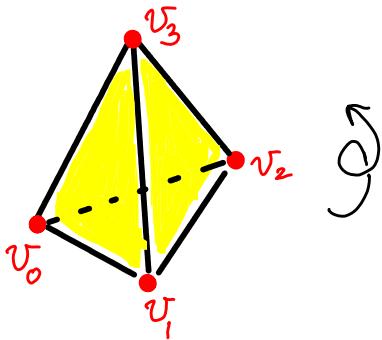


$[v_0 v_1 v_2]$  is the same orientation as  $[v_2 v_0 v_1]$   
— both go CCW (↑). But  $[v_1 v_0 v_2]$  goes clockwise.

$[v_0 v_1 v_2]$  and  $[v_2 v_0 v_1]$  are same orientations

$$[v_0 v_1 v_2] \xrightarrow{\text{swap}} [v_0 v_2 v_1] \xrightarrow{\text{swap}} [v_2 v_0 v_1]$$

two swaps, so they are even permutations of each other.



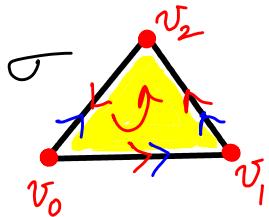
$[v_0 v_1 v_2 v_3]$  is the same orientation as  $[v_2 v_0 v_1 v_3]$  while  $[v_0 v_2 v_1 v_3]$  is the opposite orientation.

### Induced orientation

Let  $\sigma$  have vertices  $\{v_0, \dots, v_k\}$ . When  $\sigma$  is oriented, it induces an "induced orientation" on all its  $(k-1)$ -faces. Each  $(k-1)$ -face of  $\sigma$  can be denoted as  $\text{conv}\{v_0, \dots, \hat{v}_i, \dots, v_k\}$ , where the ' $\hat{\cdot}$ ' (hat) above a vertex indicates that it is excluded.

Let  $\sigma$  be oriented as  $[v_0, \dots, v_k]$ . Then the orientation induced by  $\sigma$  on  $\tau = \text{conv}\{v_0, \dots, \hat{v}_i, \dots, v_k\}$  is the same as that of  $[v_0, \dots, \hat{v}_i, \dots, v_k]$  if  $i$  is even. Else, it is the opposite orientation.

For illustration, consider the triangle oriented as  $\sigma = [v_0 v_1 v_2]$ .



$\sigma$  has three edges as faces, given by  $\{v_0, v_1\}$ ,  $\{v_0, v_2\}$ , and  $\{v_1, v_2\}$ .

The induced orientations on these edges are  $[v_0 v_1]$ ,  $[v_0 v_2]$ , and  $[v_1 v_2]$ , respectively.

We leave out  $v_i$  to get the edge  $\{v_0, v_2\}$ , and hence its induced orientation is the reverse of  $[v_0 v_2]$ , i.e., it is  $[v_2 v_0]$ .

">": induced orientations

">": edges' own orientations, which are independent of the induced orientations

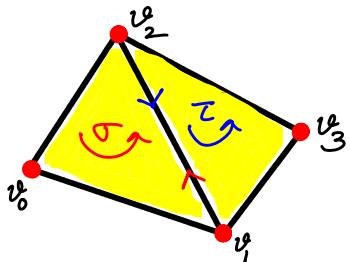
In the case of a triangle, the induced orientations indeed agree with the intuition of arrows "induced" on the edges by the  $\xrightarrow{\text{CCW}}$  (or CW) arrow of the triangle.  $\downarrow$  counterclockwise

"convex hull" here is used in general to cover both geometric and abstract cases.

## Comparing Orientations

Let  $\sigma, \tau$  be simplices. If  $\dim \sigma \neq \dim \tau$  we cannot compare their orientations. So let us consider the case when  $\dim \sigma = \dim \tau = k$ .

If  $\sigma$  and  $\tau$  share a common  $(k-1)$ -face, they are **consistently oriented**, or oriented the same way, if they induce **opposite** orientations on the common  $(k-1)$ -face.



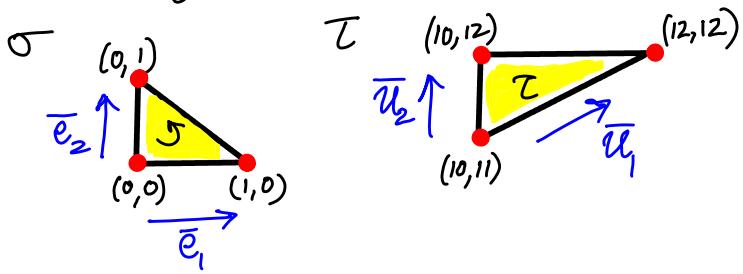
(induced orientations on  $\overrightarrow{v_1v_2}$  shown by  $\nwarrow$  and  $\nearrow$ )

orientations induced by  $\sigma = [v_0v_1v_2]$  and  $\tau = [v_1v_2v_3]$  on  $\overrightarrow{v_1v_2}$  are opposite.

Hence the two triangles are consistently oriented—both CCW here.

Note that the induced orientations are separate from the edge's own orientation. Here, we could have  $[v_1v_2]$  or  $[v_2v_1]$  as the inherent orientation of  $\overrightarrow{v_1v_2}$ . The induced orientations are still as shown in the figure.

Note: We could compare orientations of two  $k$ -simplices even if they do not share a common  $(k-1)$ -face, if they both are sitting in the same  $k$ -dimensional plane. For instance, consider two disjoint triangles in  $\mathbb{R}^2$ .



The 3 vertices of a triangle generate two vectors, whose cross-product can be used to calculate the **signed area** of the triangle.

For oriented triangle  $[v_0v_1v_2]$ , consider vectors  $\bar{e}_1 = \bar{v}_1 - \bar{v}_0$  and  $\bar{e}_2 = \bar{v}_2 - \bar{v}_0$ .

For  $\sigma$ , we can take  $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .  
The signed area of  $\sigma$  is given by  $\text{area}(\sigma) = \frac{1}{2} |\bar{e}_1 \times \bar{e}_2| = \frac{1}{2} \det([\bar{e}_1, \bar{e}_2])$ .

$$\text{Thus, } \text{area}(\sigma) = \frac{1}{2} \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \frac{1}{2}.$$

determinant

area of the parallelogram  
generated by  $\bar{e}_1, \bar{e}_2$

Similarly for  $\tau$ , we choose  $\bar{u}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\bar{u}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , giving  $\text{area}(\tau) = \frac{1}{2} |\bar{u}_1 \times \bar{u}_2| = \frac{1}{2} \left| \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \right| = 1$ .

Since  $\sigma$  and  $\tau$  have the same sign for their signed areas, they are consistently oriented. In this case, they are both oriented CCW. In fact, both sets of vectors  $\{\bar{e}_1, \bar{e}_2\}$  and  $\{\bar{u}_1, \bar{u}_2\}$  orient  $\mathbb{R}^2$  in the same way here.

These computations are naturally extended to  $d$ -dimensions for  $d \geq 3$ . We can compute the signed  $d$ -volume in the same fashion.

We could compare orientations in the abstract setting as well.

### Consistently oriented simplices

We consider an example in the abstract setting.

Let  $\sigma_1 = [2 \checkmark 5 \checkmark 12 \checkmark 19]$  and  $\sigma_2 = [\checkmark 12 \checkmark 19 \checkmark 7 \checkmark 2]$  be two oriented 3-simplices. Are they consistently oriented?

Notice that  $\tau = \{2, 12, 19\}$  is the common 2-face.

$$\sigma_1 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 5 & 12 & 19 \end{bmatrix} \quad \text{exclude to get } \tau \rightarrow \sigma_2 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 12 & 19 & 7 & 2 \end{bmatrix}$$

We get  $\tau$  by removing the first vertex from  $\sigma_1$ . Similarly, we get  $\tau$  by removing the second vertex from  $\sigma_2$ .

Hence, the orientation induced on  $\tau$  by  $\sigma_1$  is  $[12 \overset{(i=1)}{2} 19]$ , which is the opposite orientation to  $[2 12 19]$ .  $\rightarrow$  differ by 1 pairwise swap

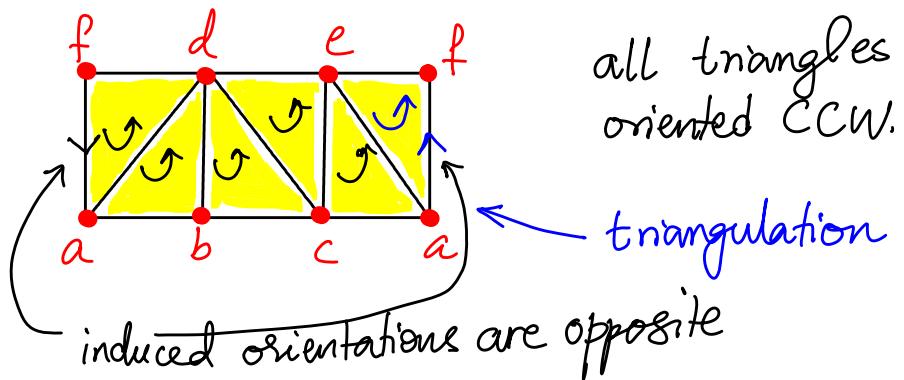
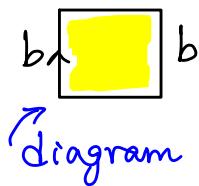
The orientation induced on  $\tau$  by  $\sigma_2$  is  $[12 19 \overset{(i=2)}{2}]$ .  
The two induced orientations are opposite here. Hence  $\sigma_1$  and  $\sigma_2$  are consistently oriented.

We extend the idea of when two d-simplices are consistently oriented to when the entire triangulation is consistently oriented.

**Def** A triangulable d-manifold (with or without boundary) is called **orientable** if all the d-simplices in any triangulation of the manifold can be consistently oriented. Else, it is a **nonorientable** manifold.

### Examples

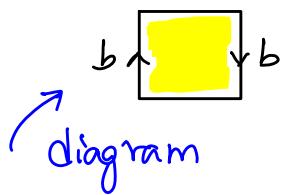
1. Cylinder



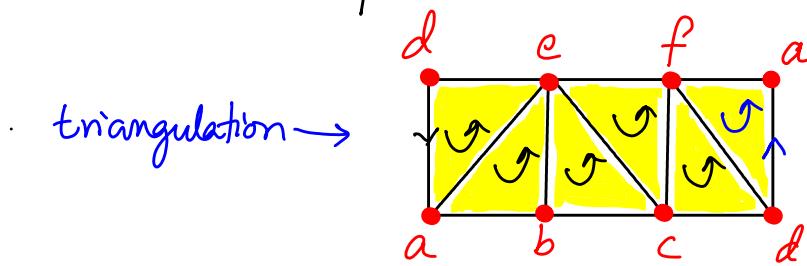
It can be checked that the orientations induced by each pair of triangles on their common (shared) edge are indeed opposite. In particular, notice that this is indeed the case for  $af$  — induced orientations from  $[adf]$  and  $[afe]$  are  $[fa]$  and  $[af]$ , respectively.

Thus, the cylinder is orientable.

## 2. Möbius strip



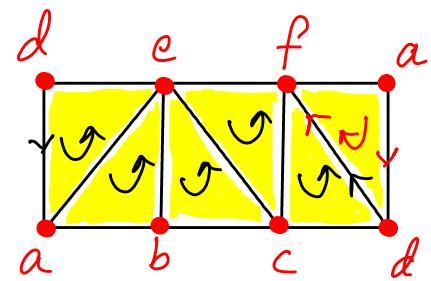
We had noted previously that the Möbius strip is non-orientable.



The orientations induced on edge  $\overline{ad}$  by  $[aed]$  and  $[afd]$  are the same here — both being  $[da]$ , or  $d \rightarrow a$ . Thus,  $\Delta_{aeb}$  and  $\Delta_{adb}$  are not consistently oriented.

Notice that the induced orientations on all remaining shared edges except  $\overline{ad}$  are indeed opposite — check the induced orientations on  $\overline{ae}$ ,  $\overline{be}$ ,  $\overline{ce}$ ,  $\overline{cf}$  and  $\overline{df}$ .

If we fix the orientations such that induced orientations on  $\overline{ad}$  are opposite, say, by orienting  $\Delta_{adf}$  clockwise, i.e.,  $[adf]$ , then the induced orientations on  $\overline{df}$  are now identical —  $[df]$ .

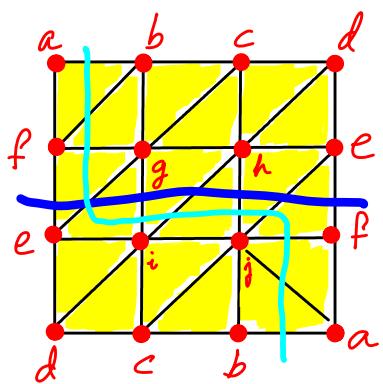


It turns out that however we orient the triangles, the induced orientations will be the same on one of the shared edges. As such, all triangles in the Möbius strip cannot be consistently oriented.

Hence the Möbius strip is non-orientable.

In fact, Möbius strips are the minimal non-orientable "objects" in 2D. For instance, you can identify Möbius strips in the triangulations of  $\mathbb{RP}^2$  and  $\mathbb{K}^2$  we introduced in Lecture 7!

Hence for surfaces, it is sufficient to identify a Möbius strip in the given triangulation to "certify" its nonorientability.

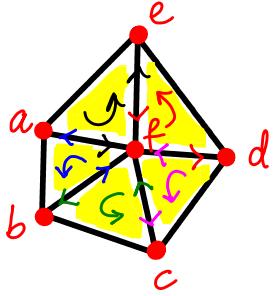
$\mathbb{RP}^2$ :

The middle strip of 6 triangles forms a Möbius strip! And there are several other instances of the Möbius strip here — another instance is identified by  $\sim$  curve.

In both the cylinder and the Möbius strip, the boundary edges can be oriented arbitrarily. These are the edges that are faces of only one triangle each. For example, in the Möbius strip above, edges  $\overline{ab}$ ,  $\overline{bc}$ ,  $\overline{cd}$ ,  $\overline{de}$ ,  $\overline{ef}$  and  $\overline{af}$  are boundary edges, and they can be assigned orientations arbitrarily without affecting the (non)-orientability of the manifold.

Checking orientability of a d-manifold Start by assigning an orientation to one d-simplex  $\sigma$  (pick one of the two possibilities). Then "propagate" this orientation to any other d-simplex  $\sigma'$  that shares a common  $(d-1)$ -simplex  $\tau$ , say, with  $\sigma$ . In other words, orient  $\sigma'$  such that the orientations induced on  $\tau$  by  $\sigma$  and  $\sigma'$  are opposite. Continue this process until all d-simplices are oriented. If we can consistently orient all d-simplices, the manifold is orientable. Else, it is non-orientable.

Example:



Start with  $[afe]$ , and propagate this orientation in the order  $[abf]$ ,  $[bef]$ ,  $[cdf]$ , and  $[def]$ . Notice that the shared edges  $\overline{af}$ ,  $\overline{bf}$ ,  $\overline{cf}$ ,  $\overline{df}$ , and  $\overline{ef}$  have opposite induced orientations.