

# MATH 529: Lecture 1 (01/13/2026)

Today:  $\begin{cases} \text{* syllabus, logistics} \\ \text{* two motivating applications} \end{cases}$

Call me Bala!

Introduction to Computational topology  $\xrightarrow{\text{focus for this course}}$

This course will be offered completely electronically:

- scribes will be posted as "course notes"; videos will also be posted.
- assignments to be turned in electronically  
(you could submit scanned versions of handwritten assignments).
- web page has all the docs/info.

## Topology

"Topo"  $\rightarrow$  place or space  
"logos"  $\rightarrow$  study or talk } in Greek

Topology talks about how space is connected.

topology

```

graph LR
    A[Topology] --> B[point set topology]
    A --> C[algebraic topology]
    
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point set topology (open/closed, connected, ...)

algebraic topology (groups, addition, basis, ...)

We will concentrate on algebraic topology.

## Computational topology

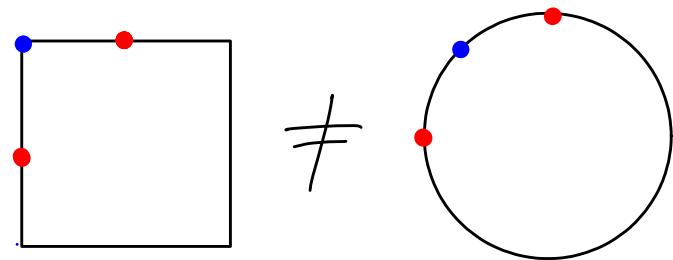
combine efficient algorithms and data structures with results from topology to analyze real-life data.

let's start with an intuition for what we mean by connectivity of spaces.

### An Example

According to geometry, the square and circle are not equal.

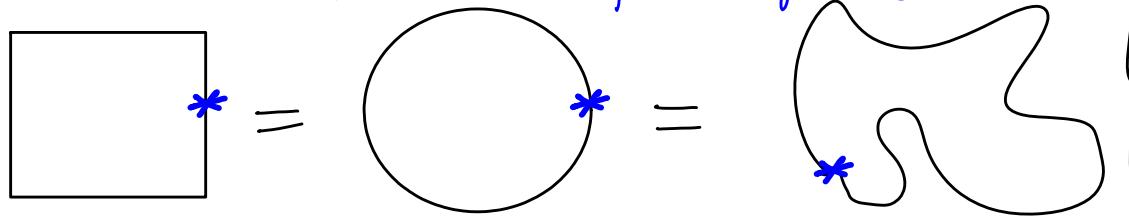
Size (length, here) is critical in geometry, but not so much in topology.



But topology says they are same as far as how they are connected!

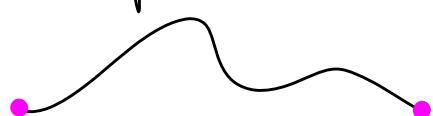
For instance, take a string, and tie a knot to make a loop.

We want to study connectivity irrespective of size here!



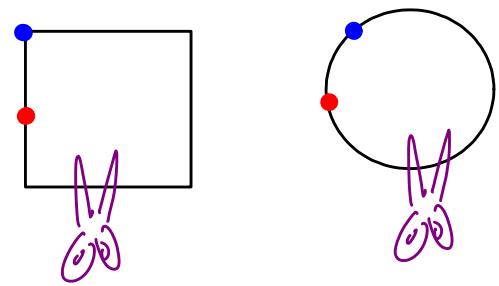
All these objects are same, i.e., they are connected the same way.

But, if you did not tie the knot, the loose string (open thread) differs from any of the above tied loops in connectivity.

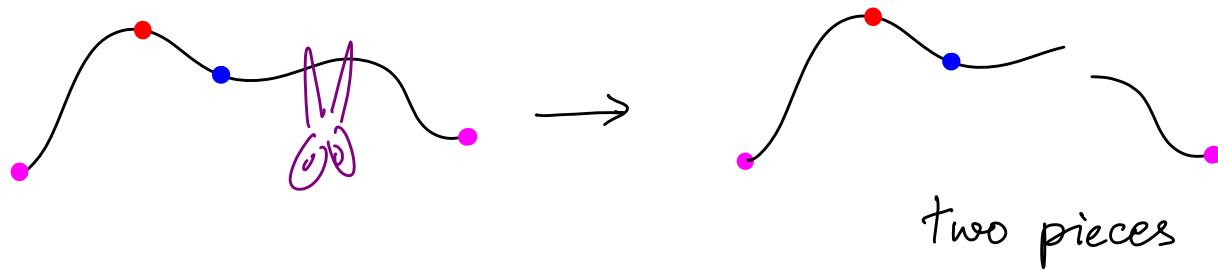


Note that the two end points have different "neighborhoods."

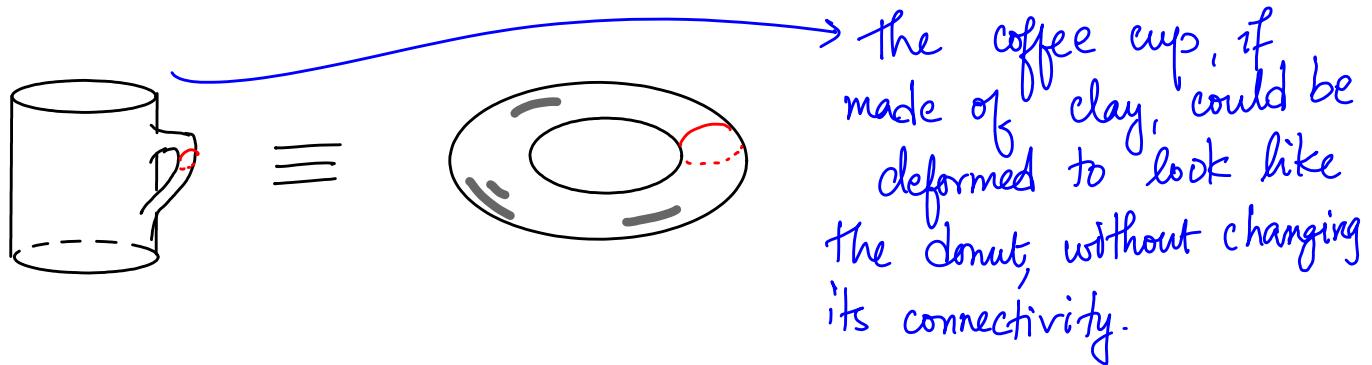
Here is another way to understand connectivity. Consider cutting the string (tied into a loop) once. Such a cut leaves the string in one piece, i.e., connected.



But cutting the open thread once leaves two pieces, i.e., it is disconnected.



A popular quote : "A topologist cannot distinguish the coffee cup from a donut!"



A more practical example:  
how we are able to read (recognize) letters of  
the alphabet in different fonts.

A **A**  $\neq$  B **B**  $\neq$  C **C**

# Two Illustrations of Computational Topology

## 1. Patient antibiotic trajectories

<https://doi.org/10.1145/3307339.3342143>

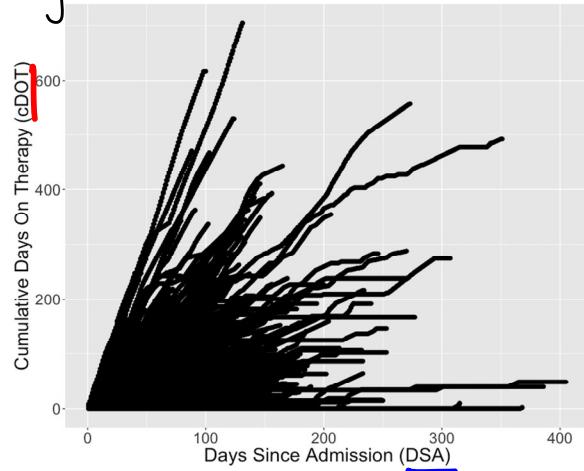
? How do agents and doses affect length of stay?

Number of hospitals	25
Number of hospital unit-categories	9
Number of distinct patient-admission records	349,610
Number of adult patients	334,207
Number of male patients	148,540
Number of female patients	201,052
Average LOS per admission	7 days
Longest LOS → length of stay	405 days
Number of antibacterials used	66
Most used antibacterial	Vancomycin
Average DOT per admission	6
Number of agent ranks	4
Most used agent rank	rank 3

Days On Therapy

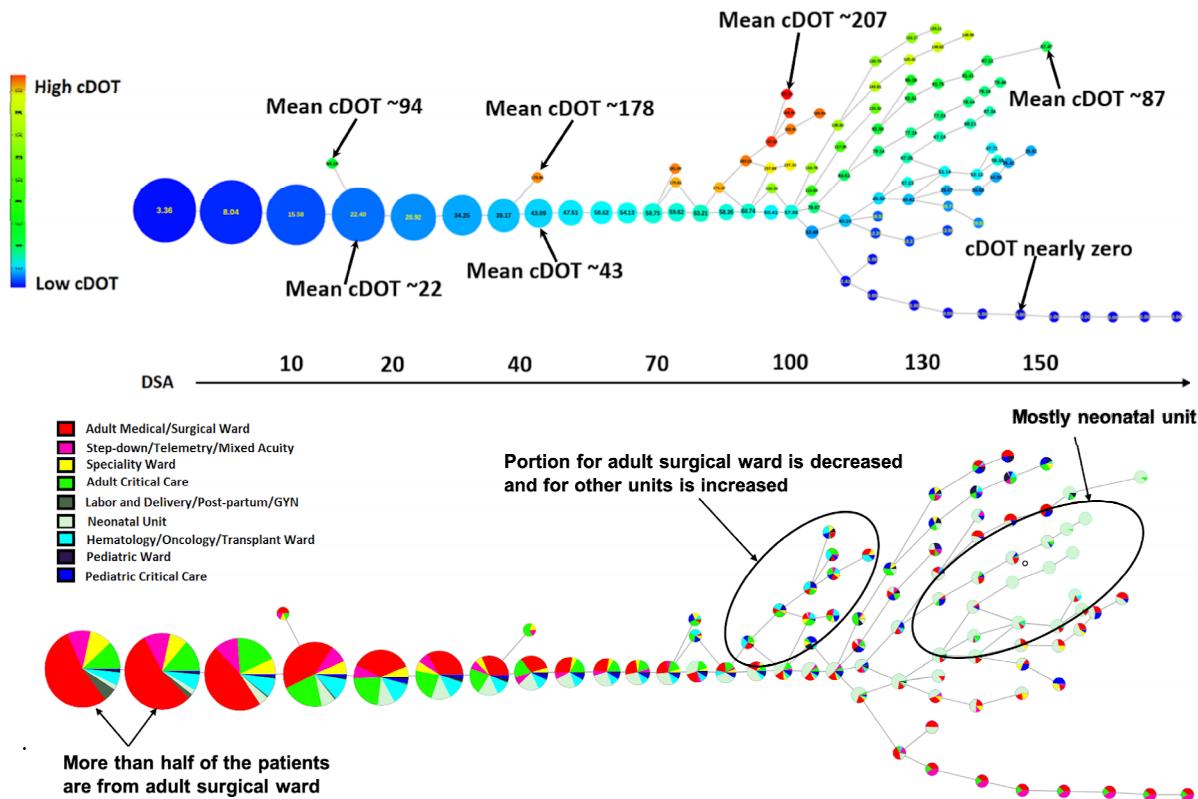
If a patient gets one dose of an agent (antibiotic) that is counted as 1 Day On therapy (DOT).

cDOT : cumulative Days On Therapy  
 DSA : Days Since Admission.



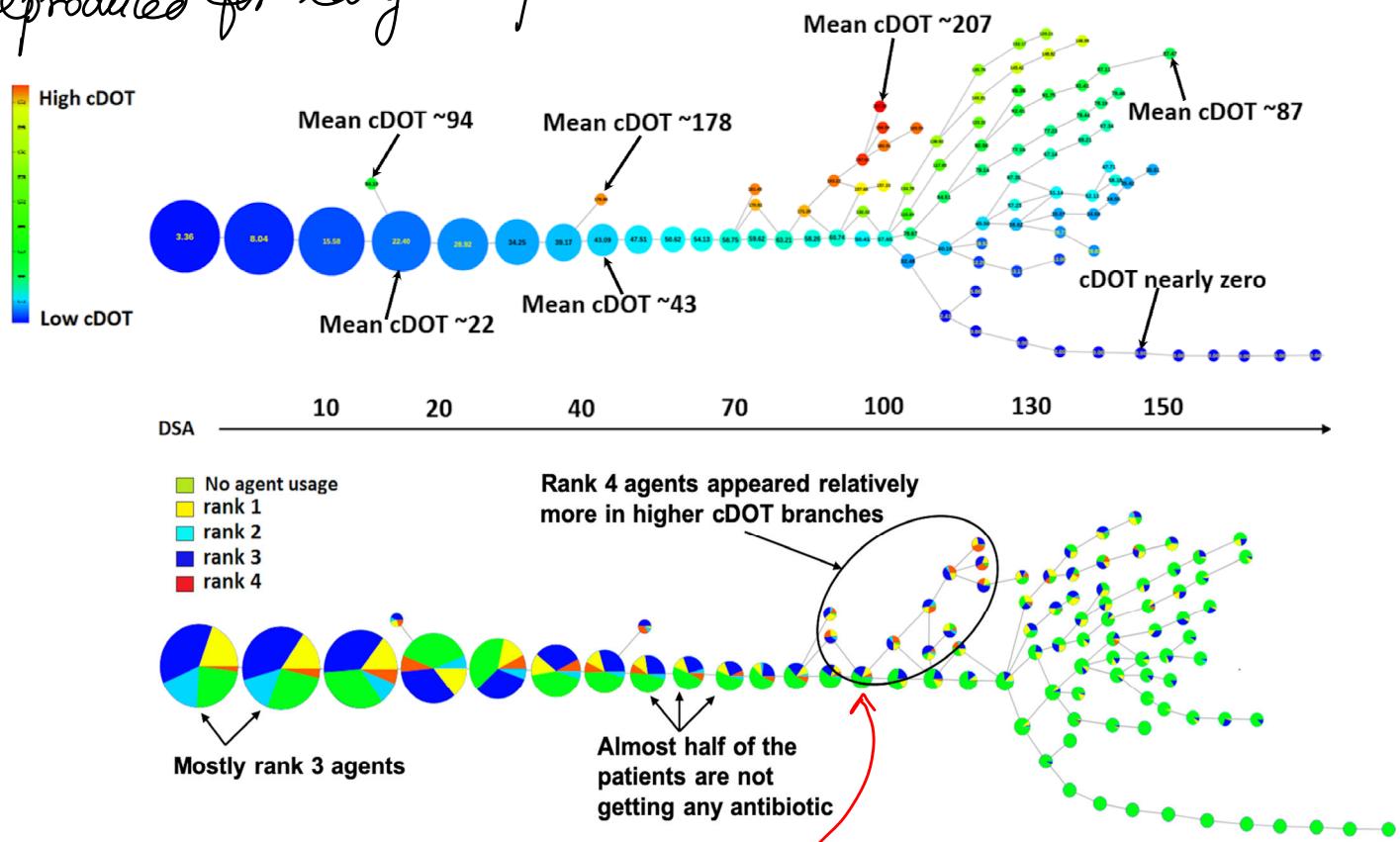
This plot by itself is not very informative or insightful (even if we were to use color...)

Here are two versions of a Mapper representation of the same data:



Each node represents a cluster of patient trajectories. A lot of the patients got low cDOTs and they had small(er) DSAs — as captured by the big clusters on the left. As the second Mapper shows, many of these patients were treated in the adult surgical ward — one of the most common types of admissions to hospitals.

Here is another version of The Mapper showing ranks (1-4, 4 is strongest) of The antibiotics! The first mapper (using cDOT) is reproduced for easy comparison.



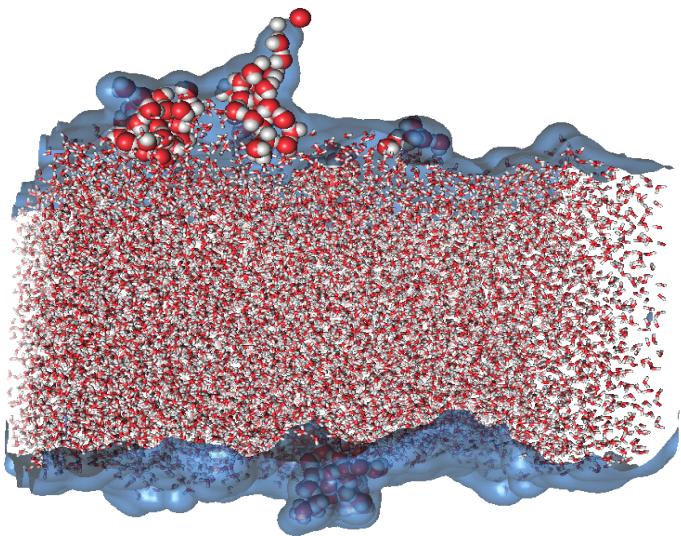
The high cDOT + high rank sub-branches had more patients in the other (higher risk) wards. Similarly, the much higher DSA group (120+) on the right end with relatively smaller cDOT values turned out to be patients in neonatal ICUs.

Note that these nontrivial subgroups are identified in an unsupervised manner — no learning is involved!

## 2. Interface features in Chemistry

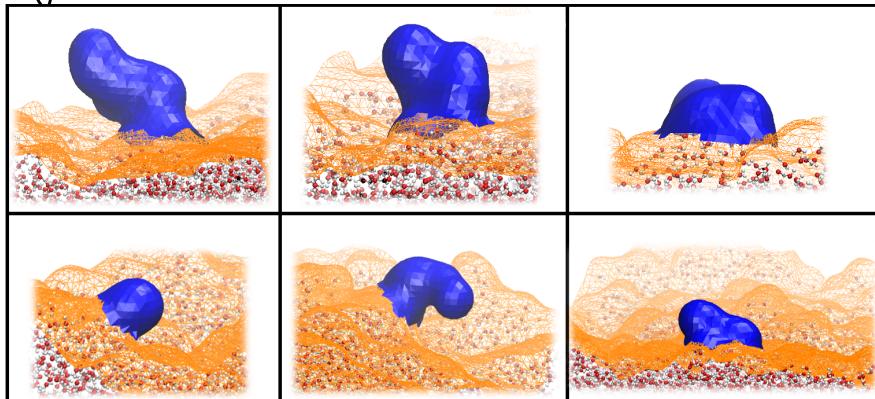
<https://doi.org/10.1021/acs.jctc.0c00260> ( <https://doi.org/10.26434/chemrxiv.11988048.v1>)

→ preprint



An interface surface separates a water layer from a hexane (organic) layer. When a reagent is added, the reaction is initiated and the water molecules escape to the hexane layer through finger-like features in the interface called "protrusions". These features were identified manually (by observation!).

The goal was to identify and characterize protrusions using geometric measure theory and computational topology.



Which of these six features do you think are protrusions?  
It is not easy to guess!

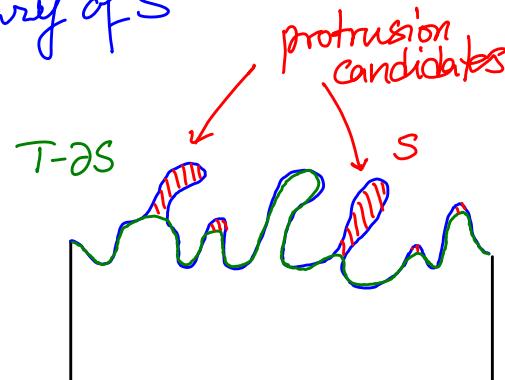
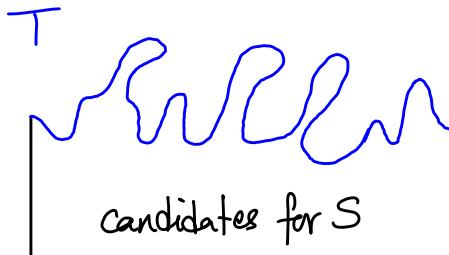
We use the notion of multiscale flat norm of surface  $T$ :

$$F_\lambda(T) = \min_S \{ \text{Area}(T - \lambda S) + \lambda \text{Volume}(S) \}, \quad \lambda \geq 0$$

scale parameter

$\xrightarrow{\text{3D volume}}$   $\xrightarrow{\text{boundary of } S}$

Illustration in 2D:



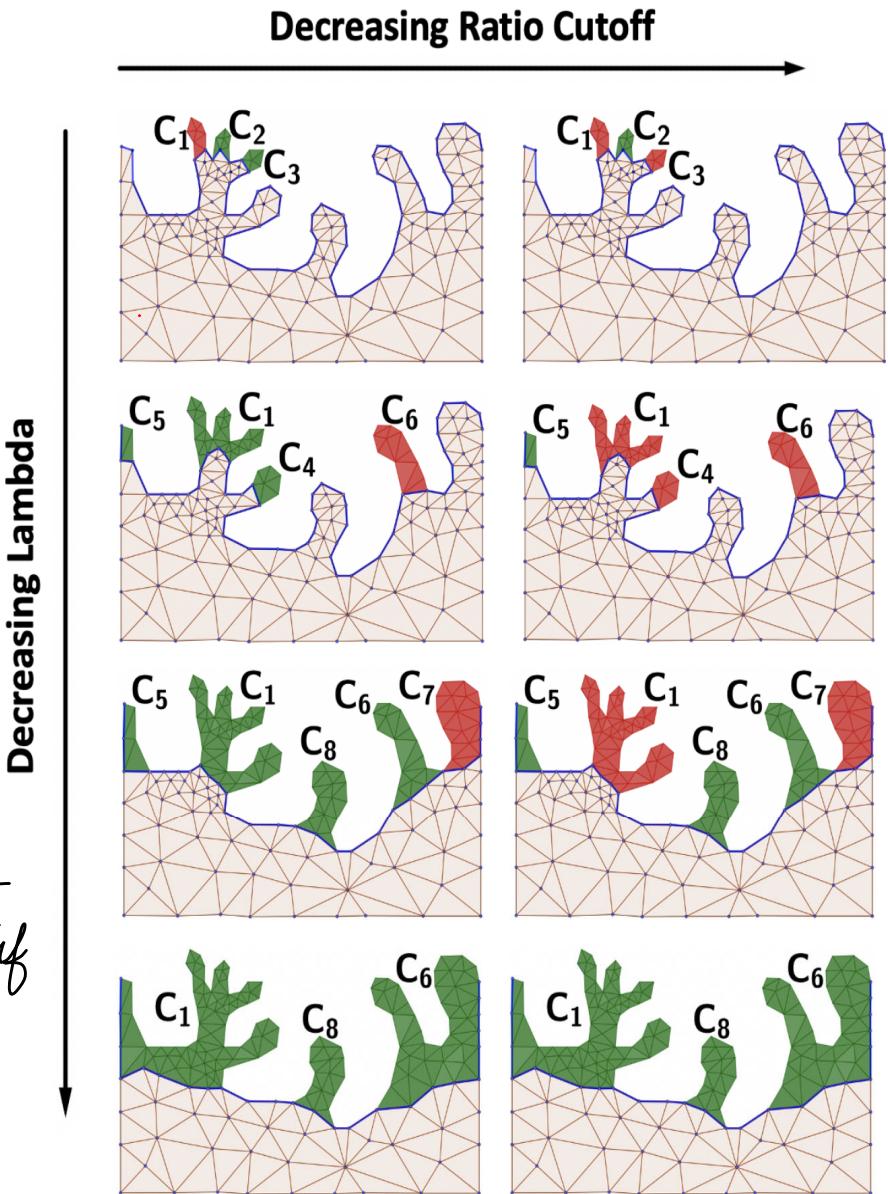
We keep track of connected components in  $S$  as  $\lambda \downarrow$ . We relabel them and also keep track of merging behavior.

We also track the ratio  $\frac{\text{vol}(C)}{\text{vol}(B(\lambda))}$  for each

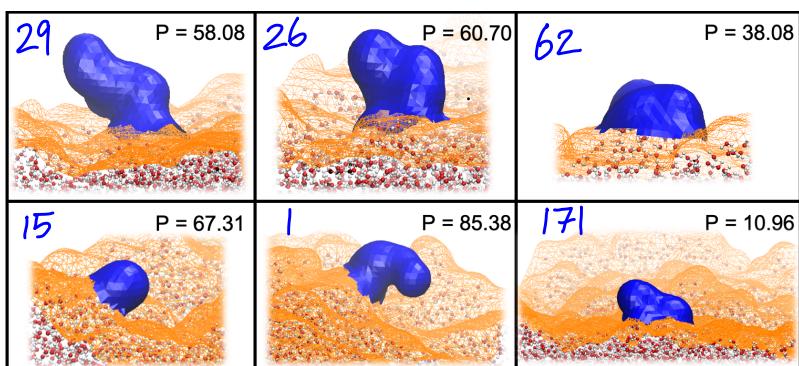
component  $C$ , where  $B(\lambda)$  is the ball with radius  $\lambda$  and  $\text{Vol}(C)$  is the volume of component  $C$ .

A component  $C$  is "alive" at ratio cutoff  $r$  and scale  $\lambda$  if

$$\frac{\text{Vol}(C)}{\text{Vol}(B(\lambda))} > r.$$



The longer a component is alive, the more likely it is to be a protrusion.



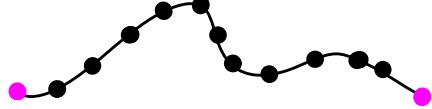
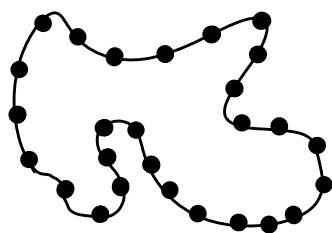
It turned out that all six of these features were protrusions! The probabilities (as %'s) along with their ranks among 195 candidate features (lower rank  $\Rightarrow$  more likely to be a protrusion) are shown here.

While this example is described in 3D, the underlying concepts are more general, and in fact generate certain key fundamental questions in geometric measure theory (GMT).

In fact, when we talk about applied algebraic topology the application could be to pure mathematics! We will talk about this aspect toward the end of the semester.

Note that we are showing a discrete version of the surface - in the form of a triangular mesh. Indeed, we need to discretize continuous spaces to perform computations!

Here is a notion of Connectivity in the "discrete setting":



The neighborhood, i.e., the set of nearby points, of the two • points are different - they each have only one neighbor, while the • points all have two neighbors each.

# MATH 529 : Lecture 2 (01/15/2026)

Today: \* topology, open/closed sets  
 \* homeomorphism, examples

We define topology as a mathematical method to define and study how a space is connected.

**Notation** For a set  $X$ , we denote by  $2^X$  the power set of  $X$ , which is the set of all subsets of  $X$ .

**Def** A **topology** on a set  $X$  is a subset  $T$  of  $2^X$  such that the following conditions hold.

1.  $A_1, A_2 \in T \Rightarrow A_1 \cap A_2 \in T$  (finite intersections)
2.  $\{A_j \mid j \in J\} \in T \Rightarrow \bigcup_{j \in J} A_j \in T$  (infinite unions)  
↑ index set  
infinite or finite
3.  $\emptyset, X \in T$  empty set

$(X, T)$  is a topological space, denoted  $\mathbb{X}$  ( $T$  is understood from context).

$A \in T$  is an **open set** of  $\mathbb{X}$ .

The complement of  $A$ , i.e.,  $X - A$  (or  $X \setminus A$ ) is a **closed set**.

Some sets can be both open and closed at the same time, e.g.,  $\emptyset, X$  are both open and closed in any topology.

We typically specify a topology by specifying its open sets.

interior  $\text{int } A$  of  $A \subseteq X$ :  $\text{int } A = \bigcup^{\text{union}} (\text{open sets contained in } A)$

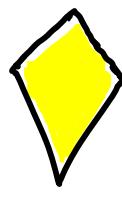
closure  $\bar{A}$  of  $A \subseteq X$ :  $\bar{A} = \bigcap^{\text{intersection}} (\text{closed sets containing } A)$ .  
 minimal closed set that contains  $A$ .

boundary  $\partial A$  of  $A \subseteq X$ :  $\partial A = \bar{A} - \text{int } A$ .

$\partial A = \{ \text{points in } A \text{ that intersect both } \bar{A} \text{ and } \overline{(X-A)} \}$ .

### Examples

1.

 $A \subseteq X$  $\text{int } A$  $\bar{A}$  $\partial A$ 

2. A discrete example. Let  $X = \{a, b, c\}$ .

We can define different topologies on  $X$ .

Let  $T_1 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$  and

$T_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ .

Under  $T_1$ ,  $\{a, b\}$  is open, its complement  $\{c\}$  is closed. With

$A = \{a, b\}$ ,  $\text{int } A = \bigcup \{\emptyset, \{b\}, \{a, b\}\} = \{a, b\} = A$ .

We can specify other topologies on  $X$ , e.g.,  $T_3 = 2^X$ , where each set in  $T$  is both open and closed. But  $T_4 = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}\}$  is not a topology, as, e.g.,  $\{a\} \cup \{b\} = \{a, b\} \notin T_4$ .

Neighborhood Let  $\mathbb{X} = (X, T)$ . A neighborhood of  $x \in X$  is any  $A \in T$  such that  $x \in \overset{\circ}{A}$ .

More generally, some books define a neighborhood as any set that includes, i.e., contains as a subset, an open set which contains  $X$ . Under this definition, the neighborhood could be a closed set (or neither open nor closed).

Now that we have defined topology, we consider the natural next question of comparing two spaces — how do we say two given spaces have the "same topology"? We introduce the notion of homeomorphism as a (strong) notion of topological similarity.

## Homeomorphism

In geometry, we can study transformations that preserve "shape" of a rigid body, e.g., rotation and translation. These transformations "do not change the geometry of the body".

In topology, we permit more types of transformations — e.g., stretch, shrink, expand, twist, etc., as long as you do not cut one piece into two or more, or join two pieces into one, or poke a hole in your object. All such permitted transformations "preserve topology".

A series of such permitted transformations that preserve topology constitute a homeomorphism. And two spaces are topologically "similar" if such "nice" functions exist from one space to the other and also back. We define what we mean by "nice" here.

We start with some background and definitions on functions.

**Def** let  $A, B$  be sets. A function  $f: A \rightarrow B$  is a rule that assigns exactly one  $b \in B$  for every  $a \in A$ .

$\text{dom } f$ : domain of  $f = A$ ,  $\text{cod } f$ : codomain of  $f = B$

$\text{im } f$ : image of  $f = \{b \in B \mid f(a) = b \text{ for some } a \in A\} = \{f(a) \mid a \in A\}$ .  
 $\text{im } f$  is also called the range of  $f$ . Note that  $\text{im } f \subseteq \text{cod } f$ .

$f: A \rightarrow B$  is 1-to-1 or injective if  $\forall b \in B$ , there exists at most one  $a \in A$  with  $f(a) = b$ :  
can be none

$f: A \rightarrow B$  is onto or surjective if  $\forall b \in B$ , there exists at least one  $a \in A$  with  $f(a) = b$ .  
can be more

If  $f$  is both injective and surjective, we say that  $f$  is bijective, or that  $f$  is a bijection.

**Def** A function  $f: \mathbb{X} \rightarrow \mathbb{Y}$  is continuous if for every open set  $B \subseteq \mathbb{Y}$ ,  $f^{-1}(B)$  is open in  $\mathbb{X}$ . "takes" open sets to open sets.

A continuous function is also called a map.

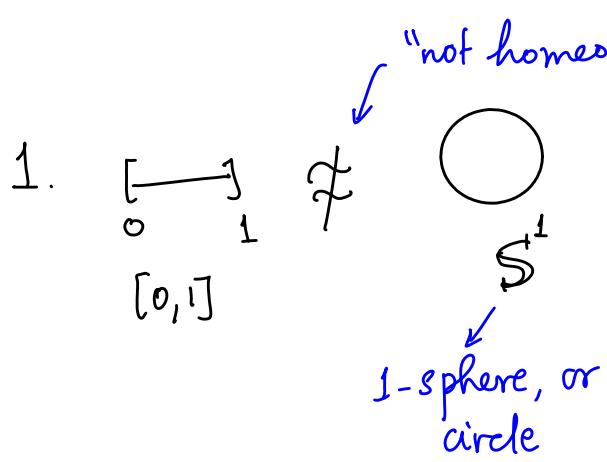
**Def** A homeomorphism  $f: \mathbb{X} \rightarrow \mathbb{Y}$  is a bijective function such that both  $f$  and  $f^{-1}$  are continuous.

We say  $\mathbb{X}$  is homeomorphic to  $\mathbb{Y}$ , or  $\mathbb{X} \approx \mathbb{Y}$ .

We also say that  $\mathbb{X}$  and  $\mathbb{Y}$  have the same topological type.

Examples

It's often easier to argue why two spaces are not homeomorphic — we just identify one (or more) place(s) where things don't work.



We would need a map that assigns both end points of  $[0, 1]$  to a single point in  $S^1$ .

But the inverse of any map that takes both end points of  $[0, 1]$  to one point in  $S^1$  is not bijective.

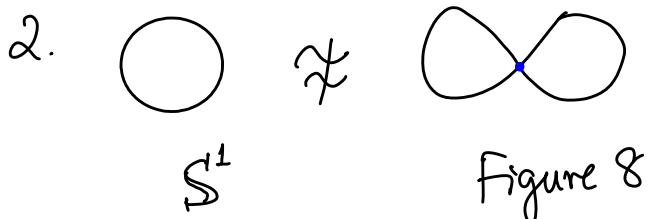
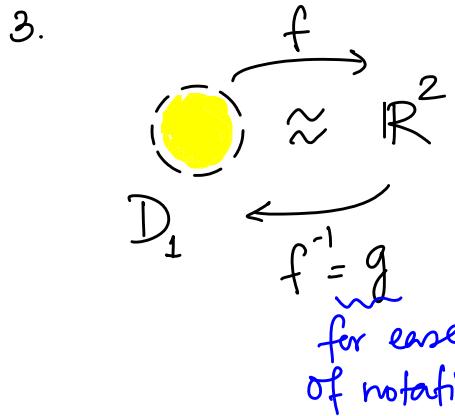


Figure 8

The crossing point in  $\infty$  ( $x$ ) cannot be mapped to a corresponding point in  $S^1$ .

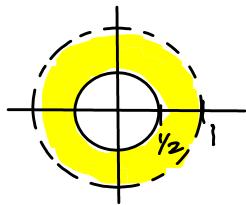
Also, we could map  $S^1$  to one of the two circles in figure-8, but not both.

On the other hand, to show that two spaces are homeomorphic, we need to specify the maps  $f$  and  $f^{-1}$ :



$$D_1 = \{ \bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1 \} \rightarrow \text{open unit disc}$$

Intuitively, we can shrink all of  $\mathbb{R}^2$  into  $D_1$ . Similarly, we can stretch  $D_1$  to fill all of  $\mathbb{R}^2$ .



$$g(\bar{x}) = \frac{\bar{x}}{1 + \|\bar{x}\|_2} \quad g: \mathbb{R}^2 \rightarrow D$$

Euclidean norm

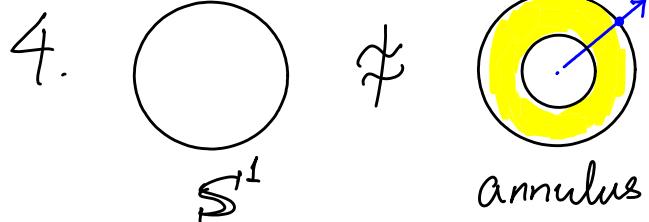
$g$  maps all of  $D_1$  (in  $\mathbb{R}^2$ ) to fit within  $D_{\frac{r_2}{2}} = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < \frac{1}{2}\}$ , and then fits all of  $\mathbb{R}^2$  outside  $D_1$  within the half open annulus with radii  $\frac{1}{2}$  and 1.

The continuous function going from  $D_1$  to  $\mathbb{R}^2$  can be similarly defined:

$$f: D \rightarrow \mathbb{R}^2 \text{ where } f(\bar{x}) = \frac{\bar{x}}{1 - \|\bar{x}\|}. \quad f \text{ is an "infinite stretch".}$$

Note that points  $\bar{x}$  in  $D$  that are close to the edge, i.e., have  $\|\bar{x}\|$  close to 1, are mapped so as to fill up the entire  $\mathbb{R}^2$  outside  $D$ . We stretch the open disc so as to fill the entire plane, and hence it is called an infinite stretch.

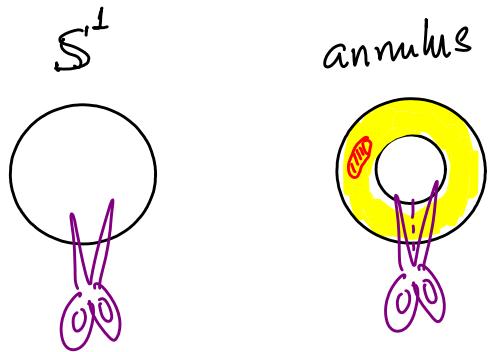
Usually, we try to define the continuous maps  $f$  and  $f^{-1}$  to show that two spaces  $X$  and  $Y$  are homeomorphic. At the same time, the intuition (geometric when possible) is also important to grasp. On the other hand, to show that  $X$   $\not\cong$   $Y$ , it is often sufficient to identify subset(s) that create the obstructions, e.g., the X in figure-8 v/s  $S^1$ .



Both these spaces have the shape of a "hole".

We could shrink the annulus so that it reduces to the circle. The corresponding function maps every point on the annulus radially onto the outer circle, for instance. But we cannot uniquely map the circle back to the annulus - would need to "map" each point on the circle to (infinitely) many points on the thick strip of the annulus.

Another observation highlights the neighborhoods of points in the circle and the annulus. Every point on the circle has open neighborhoods that look like the number line ( $\mathbb{R}'$ ). On the other hand, points in the annulus have neighborhoods that look like the open disc ( $\mathbb{R}^2$ ) or open half disc (the points on the boundary). Intuitively, the annulus is 2-dimensional, while  $S^1$  is one-dimensional.



Notice that the two spaces behave the same way under a "cut" as we had been talking about earlier with the string.

In particular, a straight cut along one "edge" of either space would leave them both connected. At the same time, one could "carve out" a 2D disc (red) from the annulus, but not from the circle.

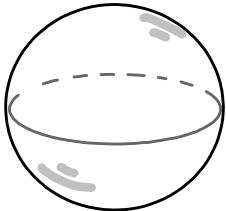
If we "relax" our definition of topological similarity, the two spaces would be considered the same - they both look like a hole, after all. Indeed we will see that checking for homeomorphism is difficult (both theoretically and computationally). We'll work with looser concepts of topological similarity later on - homology!

# MATH 529 : Lecture 3 (01/20/2026)

Today: \* 1 more example of homeomorphism  
\* manifolds

## Examples of homeomorphism (continued...)

5. sphere  $\not\cong \mathbb{R}^2$   
 $S^2$  (2-sphere)



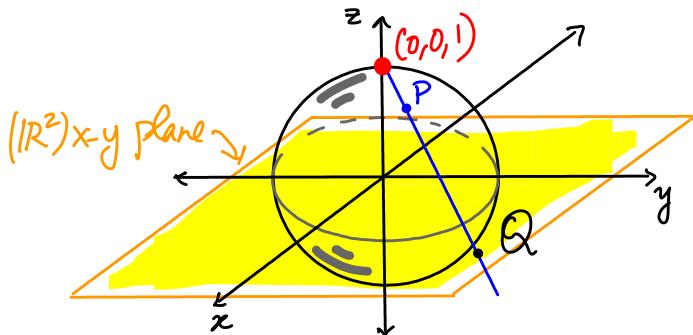
Here is an observation: the sphere encloses a 3D pocket (or void), while  $\mathbb{R}^2$  does not do so.

Such enclosed voids are the 3D analogues of holes (which are 2D).

But  $\mathbb{R}^2 \setminus \{\infty\} \approx S^2$   
 "point at infinity"

By stereographic projection, which is used to map out the surface of the earth onto a (planar) map, for instance.

Recall the equation for  $S^2$ :  $x^2 + y^2 + z^2 = 1$ .



If you poke a hole in the sphere, you can spread it out on the 2D plane, like a pierced balloon.

The equation of the line connecting  $(0,0,1)$  and  $P(x,y,z)$  is given by  
 $\bar{x} = (0,0,1) + t(x-0, y-0, z-1)$ ,  $t \in \mathbb{R}$ .

This line intersects the  $x-y$  plane at  $Q$ , which has  $z=0$ . Hence we get  $t(z-1)+1=0 \Rightarrow t = \frac{1}{1-z}$ .

Thus,  $Q$  is  $(\frac{x}{1-z}, \frac{y}{1-z}, 0)$ .

$P(x,y,z)$  on  $S^2$  projected from  $(0,0,1)$  to  $\mathbb{R}^2$  is

$$Q \left( \frac{x}{1-z}, \frac{y}{1-z}, 0 \right).$$

This formula is valid for all points on  $S^2$ , except the north pole  $(0,0,1)$ .

→ North pole

The lower hemisphere of  $S^2$  gets mapped to  $D$  (unit disc), and the upper hemisphere gets mapped to the rest of  $\mathbb{R}^2$ .

According to a topologist, "a sphere is nothing but the plane with a point added at infinity"!

Note that every point on  $S^2$  has a neighborhood that looks like  $\mathbb{R}^2$ , i.e., "it feels locally Euclidean". Such objects are called **manifolds** and are among the most commonly studied spaces in computational topology.

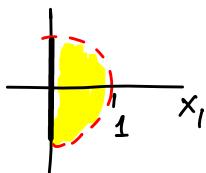
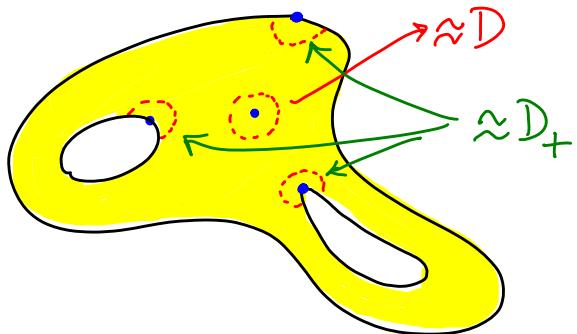
**Def** A topological space  $M$  is a **2-manifold** if all points in  $M$  lie in open discs, i.e., every point has a neighborhood

$$\approx D = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1\}. \quad \text{these are 2-manifolds without boundary.}$$

e.g.,  $S^2$ ,  $\mathbb{R}^2$ .

A **2-manifold with boundary** is a topological space  $M$  whose every point has a neighborhood homeomorphic to  $D$  or to  $D_+ = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1, x_1 \geq 0\}$  (but not both), and there exist some points of the latter type.

1st entry

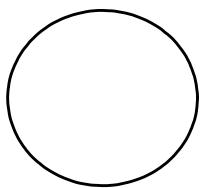


points on the boundary have neighborhoods that are half discs.

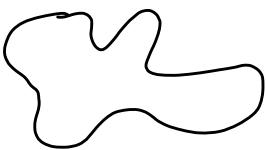
**Def** The **boundary** of a 2-manifold with boundary  $M$  is the set of points in  $M$  that have neighborhoods homeomorphic to the half disc.

(The definition of  $\partial A$  used for sets  $A$  is equivalent to this definition).

Notice that the 1-manifold is just the circle ( $S^1$ ),

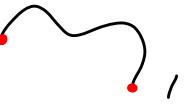


or

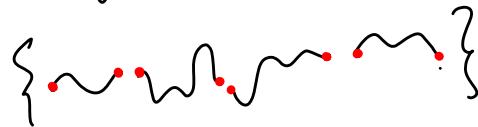


or  $\{ \textcircled{1} \textcircled{2} \dots \textcircled{n} \}$ ,

a collection of disjoint circles.

A 1-manifold with boundary:  or

 only one end point is included here!



boundary is indeed the set of end points.

0-manifold: Any collection of distinct points (discrete set of points).

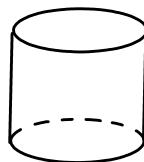
2-manifolds, also called surfaces, are a well studied class of spaces (objects), both from the theoretical as well as applied points of view. We will present several details of the properties of 2-manifolds first. To define and study  $d$ -manifolds for  $d \geq 2$ , we will need a few more definitions and concepts from analysis/point set topology.

By default, we assume a manifold (w/ or w/o boundary) is connected.

A 2-manifold (for that matter,  $d$ -manifold for  $d \geq 2$ ) is orientable or non-orientable.



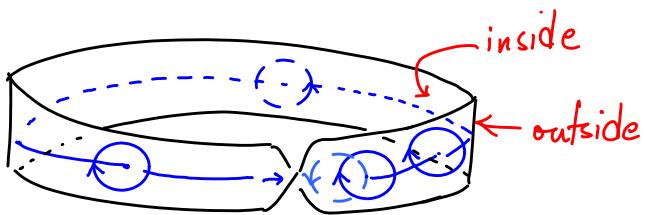
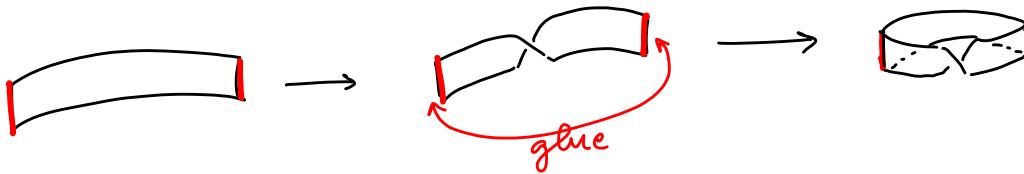
Möbius strip is non-orientable



cylinder is orientable

The Möbius strip has only one "side", while the cylinder has two "sides"—inside and outside.

We obtain the Möbius strip by taking a rectangular strip of paper, and gluing its short edges together after twisting the strip once.



there are two "sides" at each point (locally) on the Möbius strip—front/back or inside/outside.

Consider sliding an oriented loop, or a clock  $\odot$  along the surface of the Möbius strip. Look at the path followed by the center of the clock. Once the center comes back to where it started, its orientation is reversed (as it goes over the "twist").

The path traced by the center of the clock here is hence an **orientation reversing** closed curve. If the orientation is not reversed this way, the curve is said to be **orientation preserving**.

Def If all closed curves in a 2-manifold (with or without boundary) are orientation preserving, then the 2-manifold is **orientable**, else it is **nonorientable**.  
 $\mathbb{R}^2 \downarrow, S^1, \text{torus, etc.}$

$\hookrightarrow$  Möbius strip,  
Klein Bottle, etc.

# Classification of Manifolds

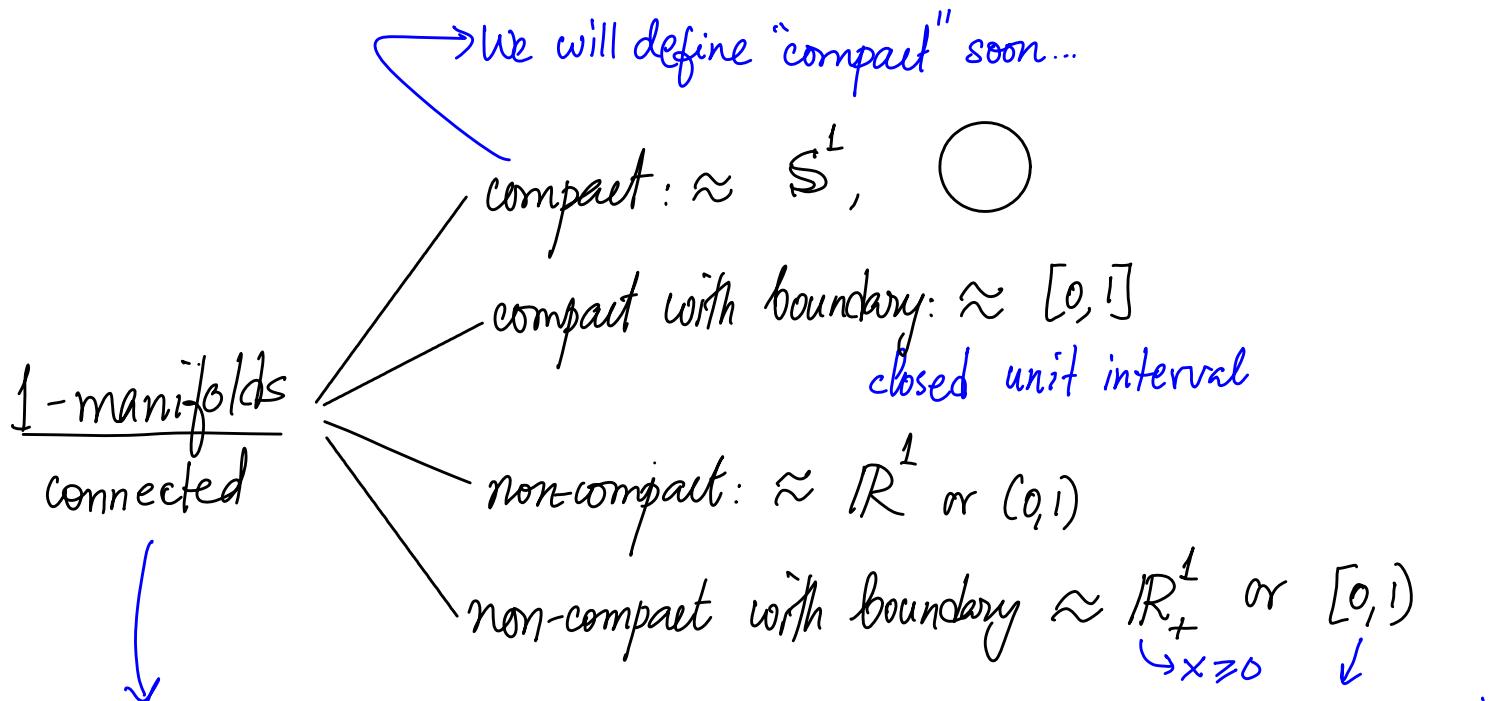
3.5

Enumerate all possible manifolds up to homeomorphisms for each dimension.

We mention the classification for 0- and 1-dimensional manifolds, but will come back to give some more definitions to finish the discussion for 2-manifolds.

0-manifolds: a discrete space, e.g.,  $\mathbb{Z}^2$  - all points with integer coordinates in  $\mathbb{R}^2$ .  
each point has to have a neighborhood  $\approx \mathbb{R}^0$ , i.e., a point.

Notice that  $\mathbb{Z}^2$ , all points with even integer coordinates in  $\mathbb{R}^2$  is homeomorphic to  $\mathbb{Z}^2$ .



If a 1-manifold is not a single connected space, each connected subspace has one of these structures.

# MATH 529: Lecture 4 (01/22/2026)

Today: \*  $d$ -manifolds in general  
\* Classification of 2-manifolds

We will now introduce some concepts which we will use to define  $d$ -manifolds in general (in particular, for  $d \geq 2$ ).

**Def** A **cover** of  $A \subseteq X$  is a family  $\{C_j | j \in J\}$  in  $2^X$  such that  $A \subseteq \bigcup_{j \in J} C_j$ .  
↳ index set

An **open cover** is a cover made of open sets.  
↳ index set is a subset

A **subcover** of  $A$  is a cover  $\{C_k | k \in K\}$  such that  $K \subseteq J$ .

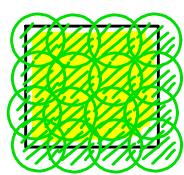
**Def** A set  $A$  is **compact** if every open cover of  $A$  has a finite subcover. Correspondingly, a topological space  $A \subseteq X$  is compact if every open cover of  $A$  has a finite subcover.

Note: In  $\mathbb{R}^d$ , closed + bounded  $\Leftrightarrow$  compact.

e.g.,  $S^2$  is compact, but  $\mathbb{R}^2$  is not.

○  $D_1$  is not compact.

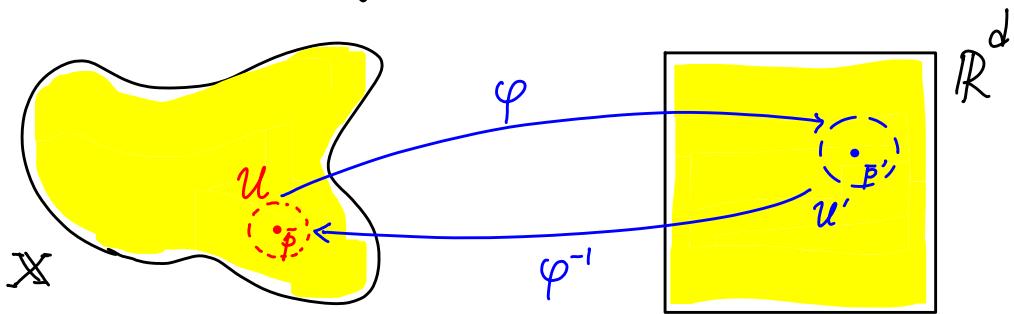
○  $\bar{D}_1$  (closed disc) is a compact 2-manifold with boundary



Example of finite subcover: consider a unit square, which is a subset of  $\mathbb{R}^2$ . Consider open discs of radius  $\frac{1}{4}$  centered at each rational point within the square (denoted here as ).

There are infinitely many such discs, which together cover the square. But a finite subset of those discs also covers the square.

Def A chart at  $\bar{P} \in \mathbb{X}$  is a homeomorphism  $\varphi: U \rightarrow \mathbb{R}^d$  for  $U \in \mathbb{X}$  an open set containing  $\bar{P}$ . The dimension of the chart is  $d$ .



Def (Hausdorff) A topological space  $\mathbb{X}$  is Hausdorff if  $\forall x, y \in X, x \neq y$ , there exist neighborhoods  $U, V$  of  $x, y$ , respectively, such that  $U \cap V = \emptyset$ .  
e.g.,  $\mathbb{R}^2$ .

Example of a non-Hausdorff space:  $X = L \cup \{a, b\}$  where  $a$  and  $b$  are both used in place of the origin.  
Open sets are the usual open intervals in  $\mathbb{R}$ .

$$\xleftarrow{\hspace{1cm}} \xrightarrow{\hspace{1cm}} \quad a \cdot b \quad \downarrow \quad L = \mathbb{R} \setminus \{0\}$$

and such that

$$L \cup \{a\} \approx \mathbb{R} \text{ and } L \cup \{b\} \approx \mathbb{R}.$$

But every pair of open sets  $U$  and  $V$  containing  $a$  and  $b$ , respectively, intersect!

**Def** A topological space is completely separable if it has a countable basis, i.e., it has a countable collection of open sets such that every open set can be written as a union of open sets from this collection (basis). think  $\mathbb{Z}$ , integers, as opposed to  $\mathbb{R}$ , which is uncountable

e.g.,  $\mathbb{R}$  is completely separable - it can be shown that open intervals with rational lengths centered at only rational points works as a countable basis.

A space that is not completely separable: take uncountably many copies of  $[0,1]$ , e.g., with the 0 of  $[0,1]$  anchored at all irrational points - called the long line or Alexandroff line.

**Def** (Manifold) → d-dimensional manifold

A completely separable, Hausdorff space  $\mathbb{X}$  is a d-manifold if there exists a d-dimensional chart at every  $\bar{x} \in \mathbb{X}$ , i.e.,  $\bar{x}$  has a neighborhood homeomorphic to  $\mathbb{R}^d$ .

$\mathbb{X}$  is a d-manifold with boundary if every  $\bar{x} \in \mathbb{X}$  has a neighborhood homeomorphic to  $\mathbb{R}^d$  or  $H^d = \{\bar{x} \in \mathbb{R}^d \mid x_1 \geq 0\}$  (d-dimensional half space).

The boundary of  $\mathbb{X}$ , denoted by  $\partial \mathbb{X}$ , is the set of  $\bar{x} \in \mathbb{X}$  with a neighborhood homeomorphic to  $H^d$ .

The dimension of the manifold is  $d$  here.

Notice the correspondence between the definition of  $d$ -manifolds introduced previously, and the general definition here. The main condition is the existence of neighborhoods  $\approx \mathbb{R}^d$  around each point.

Def (Embedding) An **embedding** of  $\mathbb{X}$  in  $\mathbb{Y}$  is a map  $g: \mathbb{X} \rightarrow \mathbb{Y}$  whose restriction to  $g(\mathbb{X})$  is a homeomorphism.

Manifolds are manifolds irrespective of their embedding!  
 $S^2$  is a  $2$ -manifold even if it is not sitting in  $\mathbb{R}^3$ .

We will introduce alternative representations of manifolds to highlight this point. In fact, in many cases, we can study the manifold easily using such representations.

## Classification of Manifolds (continued)

Enumerate all possible manifolds of a given dimension up to homeomorphism. We already listed the classifications for 0- and 1-dimensional manifolds.

We now consider the case of compact, connected, closed  $d$ -manifolds. We first list the "basic building blocks", so to speak, which include the 2-sphere, torus, Möbius strip, and the real projective plane. We can build larger  $d$ -manifolds by gluing these building blocks together.

## 2-Manifolds (we consider compact 2-manifolds)

First, let us study several typical 2-manifolds, some of which we already saw previously.

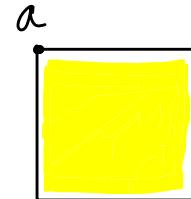
1.  $S^2$

2-sphere



$$\{\bar{x} \in \mathbb{R}^3 \mid \|\bar{x}\|_2 = 1\}$$

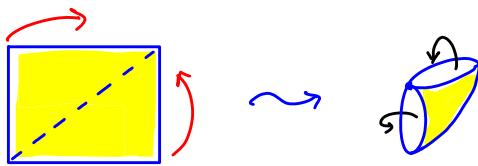
both are 2-spheres!



"identify" all points on boundary with the point a.

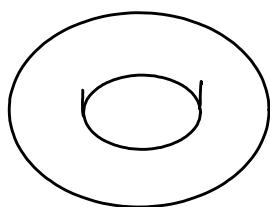
This is a "diagram" of  $S^2$ .

Start with a square sheet of paper and glue its all its edges together to make a sphere.



Arrows capture how edges are glued - with or without twist.

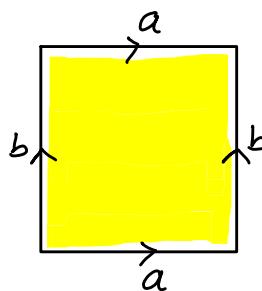
2.



$\mathbb{T}^2$

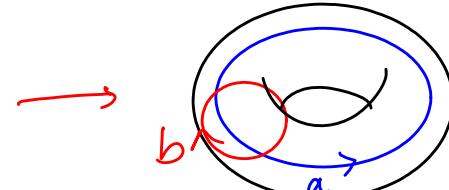
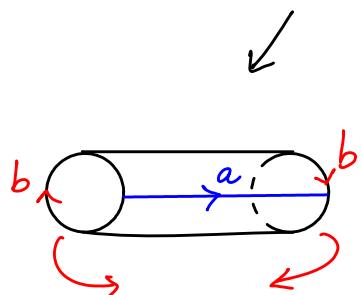
torus

$\approx$

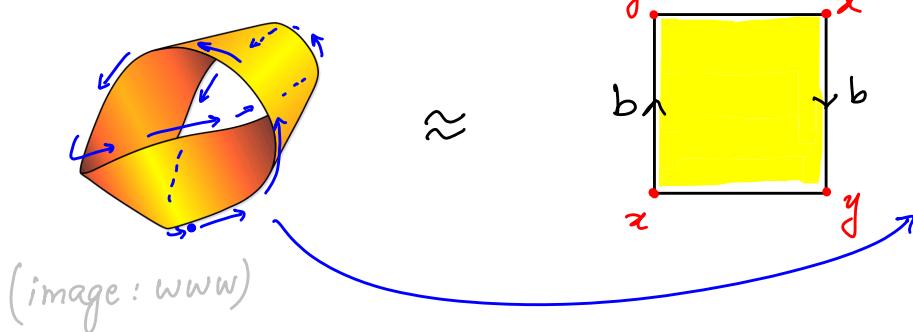


→ we could mathematically study this representation

Imagine folding a rectangular sheet of paper first into an open cylinder, and then gluing its end circles to form a torus.



### 3. Möbius strip $\rightarrow$ 2-manifold with boundary

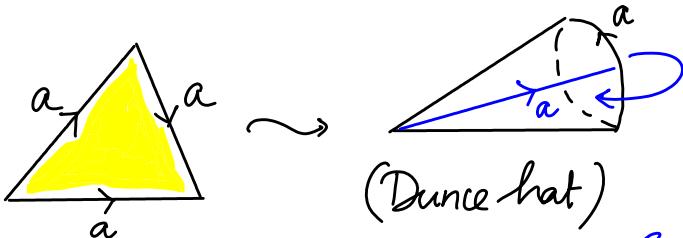


Notice that we can traverse the "edge" of the Möbius Strip in one go — it's one big circle!

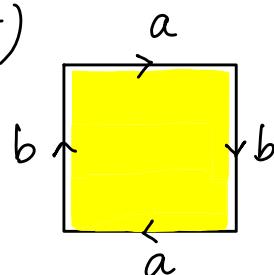
Notice that we are not identifying the horizontal edges. So they remain as boundaries. At the same time all four edges are identified pairwise in the case of the torus. Indeed, the Möbius strip is a manifold with boundary, while the torus is a manifold (without boundary).

$\rightarrow$  cannot be embedded in  $\mathbb{R}^3$ !

### 4. (Real) Projective plane ( $\mathbb{RP}^2$ ) (also, Dunce hat)



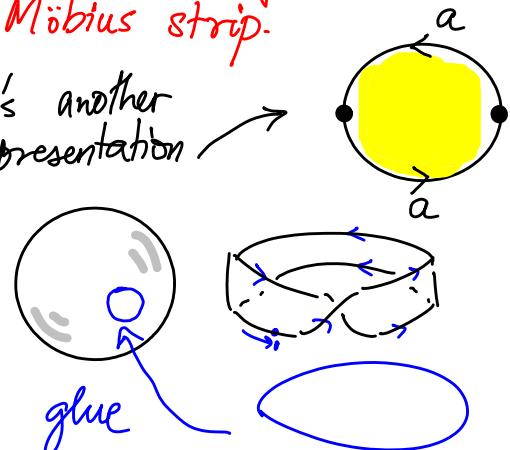
Yet another way to make  $\mathbb{RP}^2$ : Cut an open disc out of  $S^2$  and glue a Möbius strip along the edge left by the cut.



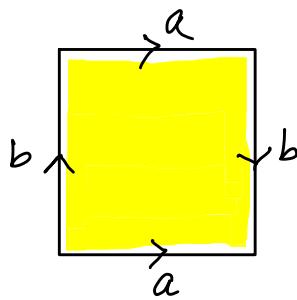
identify the free edges of Möbius strip in an opposing sense.

Same as gluing the boundary of a disc to the boundary of a Möbius strip.

Here's another representation



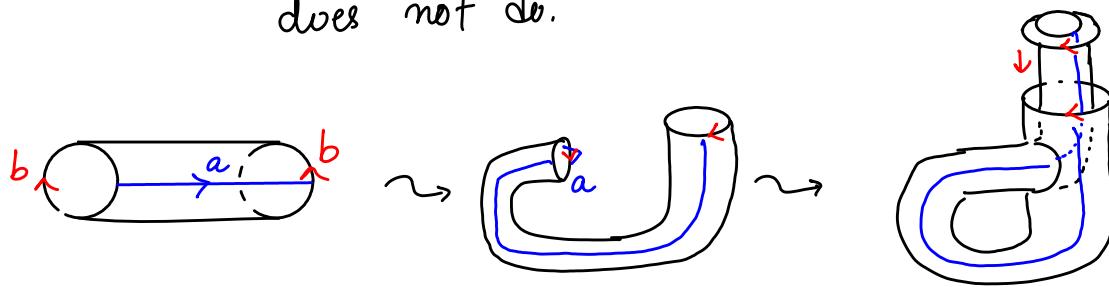
## 5. Klein bottle ( $\mathbb{K}^2$ )



Identify free edges of the Möbius strip in the same direction.

An "immersion" in 3D:

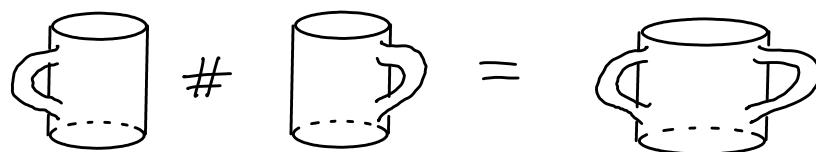
→ allows self intersection, which an embedding does not do.



We get  $\mathbb{K}^2$  also by gluing together two Möbius strips along their boundary circles. Or, cut two discs out of  $S^2$  and glue a Möbius strip each along the edges of both cuts.

Note that  $S^2$  and  $T^2$  are orientable manifolds, while the Möbius strip,  $\mathbb{RP}^2$ , and  $\mathbb{K}^2$  are non-orientable manifolds (with or without boundary).

We can obtain more general 2-manifolds by "gluing" these basic shapes together. For example, we can connect two coffee cups to get one coffee cup with two handles!



We can do this kind of "gluing" to join manifolds in any dimension (as long as the manifolds being joined have the same dimension). This is formally termed connected sum.

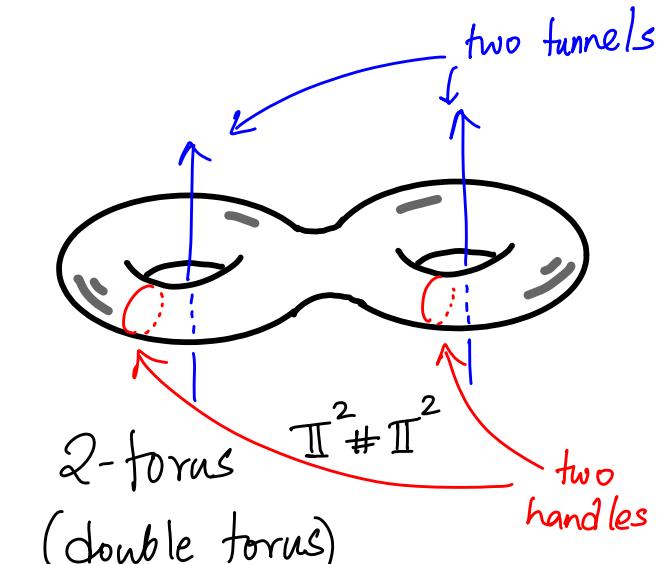
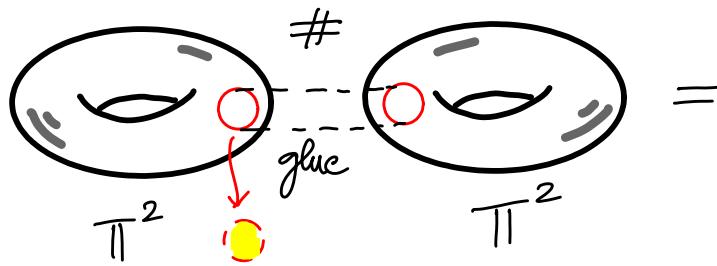
Def (Connected sum). Let  $M_1, M_2$  be  $d$ -manifolds. The connected sum of these  $d$ -manifolds is another  $d$ -manifold defined as follows.

$$M_1 \# M_2 = (M_1 - D_1^d) \cup_{\partial D_1^d \cong \partial D_2^d} (M_2 - D_2^d)$$

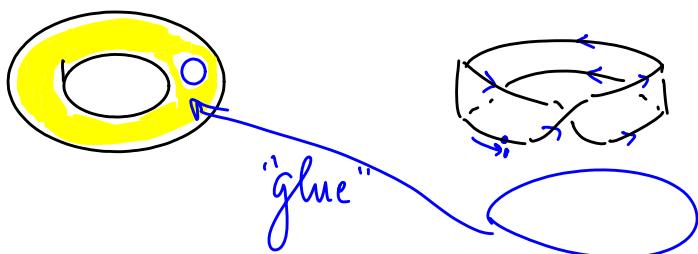
↳ identified by homeomorphism

$D_1^d, D_2^d$  are  $d$ -dimensional open discs in  $M_1, M_2$ , respectively.

Here is an illustration:



Remove open discs from both tori, and "glue" along the boundaries of these circular holes.



Illustrating how to glue a Möbius strip to a hole in a torus.

body of Möbius strip

# MATH 529 : Lecture 5 (01/27/2026)

\* classification of 2-manifolds  
 Today: \* simplices and simplicial complexes  
 \* abstract simplicial complexes ~~didn't get to it...~~

## Classification of compact, connected 2-manifolds

**Result** Every compact, connected 2-manifold is homeomorphic to  $S^2$ , or a connected sum of copies of  $\mathbb{H}^2$ , or a connected sum of copies of  $\mathbb{RP}^2$ . If a 2-manifold is not connected, each component has this structure.

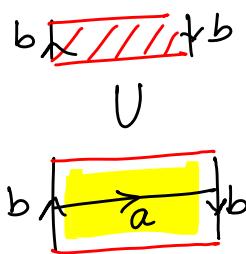
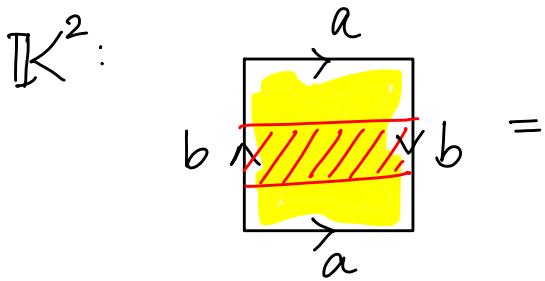
### Examples

$$1. S^2 \# \mathbb{RP}^2 \approx \mathbb{RP}^2$$

→ you just close back the open disc cut out from  $\mathbb{M}$ !

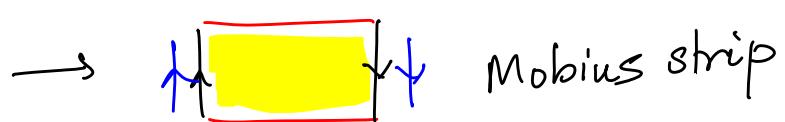
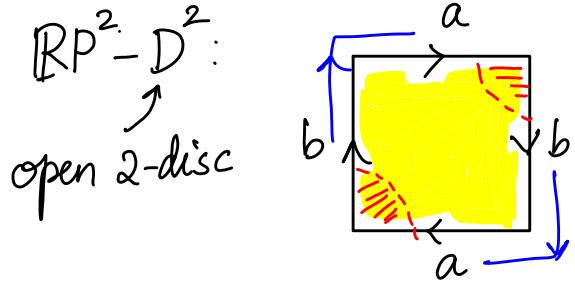
In fact,  $S^2 \# \mathbb{M} \approx \mathbb{M}$  for any 2-manifold  $\mathbb{M}$ .

$$2. \mathbb{RP}^2 \# \mathbb{RP}^2 \approx \mathbb{K}^2 \quad \text{Here's a "proof" by pictures:}$$



Connected sum  
of 2 Möbius  
strips?

And  $\mathbb{RP}^2 - D^2$ :



So,  $\mathbb{K}^2$  is  $\approx$  connected sum of 2  $\mathbb{RP}^2$ 's.

Also, we get  $\mathbb{RP}^2$  as  $S^2 \# \mathbb{M}$ , for a Möbius strip  $\mathbb{M}$ .

$$3. \mathbb{H}^2 \# \mathbb{RP}^2 \approx \#(\mathbb{RP}^2)^3$$

orientable      nonorientable

Once you join a non-orientable 2-manifold, the result stays non-orientable.

As one would expect, the corresponding result for 3-manifolds is much more complicated. Thurston's geometrization conjecture states that all compact 3-manifolds can be canonically decomposed into submanifolds that have geometric structure. This conjecture implies the famous Poincaré conjecture, which states that every compact, simply connected 3-manifold is homeomorphic to  $S^3$ , the 3-sphere.

(Informally, an object is simply connected if there are no "holes" passing through the object).

Perelman presented a proof of the Poincaré conjecture, almost 100 years after it was originally proposed (in 1904; Perelman's proof appeared in 2003). The corresponding result for  $n$ -manifolds with  $n \geq 4$  turns out to be easier to prove, informally because of the "increased geometric freedom" one can afford in higher dimensions.

The main concepts used in Perelman's proof can be used to provide a proof for Thurston's geometrization conjecture (Perelman presented such a proof in 2003, along with his proof of the Poincaré conjecture).

## Simplices

While we can study simple 2-manifolds as is, we cannot do computations on them. For this purpose, we need a discretized version of the spaces in question, which could be stored and handled naturally by a computer. Can we, for instance, use some sort of "counting arguments" to distinguish  $S^2$  from  $T^2$ ?

We introduce the concept of simplicial complexes in this context, and use concepts from combinatorial algebraic topology. The idea is that we can handle the combinatorics using efficient algorithms. Similarly, there are efficient data structures that can be used to work with simplicial complexes modeling the spaces. We also want to separate the topology from the geometry of the object. → we use the geometry in many applications.

We first introduce some definitions.

**Def** (Combinations) Let  $S = \{\bar{p}_0, \dots, \bar{p}_k\} \subseteq \mathbb{R}^d$ . A

linear combination of  $\bar{p}_i$  is  $\bar{x} = \sum_{i=0}^k \lambda_i \bar{p}_i$ ,  $\lambda_i \in \mathbb{R}$   $\forall i$ .

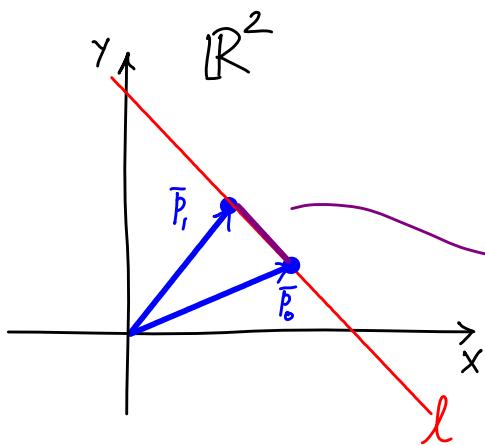
If  $\sum_{i=0}^k \lambda_i = 1$ , then  $\bar{x}$  is an affine combination of  $\bar{p}_i$ 's.

In addition, if  $\lambda_i \geq 0 \forall i$ ,  $\bar{x}$  is a convex combination of  $\bar{p}_i$ 's.

The set of all convex combinations of elements in  $S$  is the convex hull of  $S$ , denoted as

$$\text{conv}(S) = \left\{ \sum_{i=0}^k \lambda_i \bar{p}_i \mid \lambda_i \geq 0 \forall i, \sum_{i=0}^k \lambda_i = 1 \right\}.$$

## Illustration in 2D



set of all linear combinations =  $\mathbb{R}^2$   
 set of all affine combinations =  $l$ ,  
 the line through  $\bar{P}_0, \bar{P}_1$   
 $\text{conv}(\{\bar{P}_0, \bar{P}_1\})$  = line segment connecting  $\bar{P}_0, \bar{P}_1$ .  
 $\bar{P}_0, \bar{P}_1$  are not parallel here.

Def (Independence)  $S$  with  $|S| \geq 2$  is **linearly (affinely) independent** if no point in  $S$  is a linear (affine) combination of other points in  $S$ .

We denote linearly independent in short as LI, and affinely independent as AI.

$|S|=1$  case:  $\{\bar{P}_0\}$  is LI if  $\bar{P}_0 \neq \bar{0}$  (zero vector) but  $\{\bar{P}_0\}$  is AI for all  $\bar{P}_0$  (even if  $\bar{P}_0 = \bar{0}$ ).

For example, 3 points in  $\mathbb{R}^2$  are AI as long as they are not collinear.

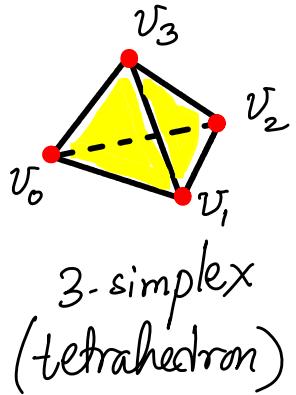
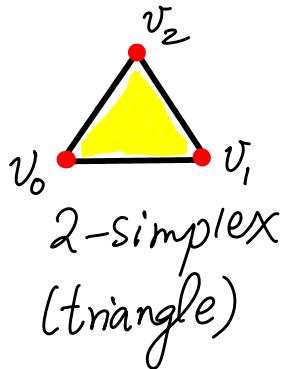
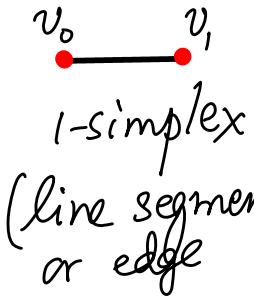
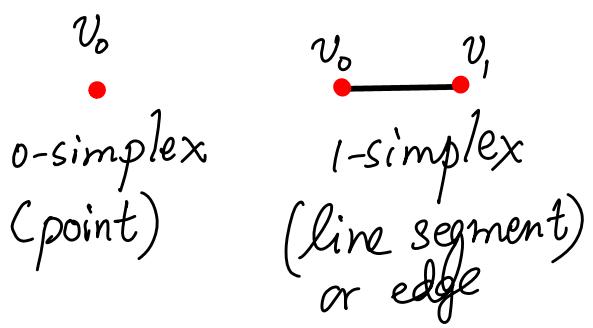
We will use convex hulls of AI points as our building blocks – called simplices.

**Def**

(simplex) The convex hull of  $(k+1)$  independent points  $S = \{\bar{v}_0, \dots, \bar{v}_k\}$  is a  $k$ -simplex. The dimension of the simplex is  $k$ , and  $\bar{v}_j$ 's are the vertices of the  $k$ -simplex.

vertices, and  
hence  $\bar{v}_i$ ,  
rather than  $P_i$ .  
(5-5)

Here are the  $k$ -simplices for small values of  $k$ :  $0 \leq k \leq 3$ .



Notice that the  $k$ -simplex is homeomorphic to the closed  $k$ -ball, i.e.,  $\bar{B}_k = \{\bar{x} \in \mathbb{R}^k \mid \|\bar{x}\| \leq 1\}$ . "gives  $S^{k-1}$  the  $(k-1)$ -sphere.

Indeed, the boundary of the  $k$ -ball is the  $(k-1)$ -sphere, e.g., 2-ball (or 2-disc) has the circle (1-sphere) as the boundary.

Each  $p$ -simplex is made of lower dimensional simplices, i.e.,  $k$ -simplices with  $k \leq p$ . Thus,  $\Delta v_0 v_1 v_2$  contains vertices  $v_0, v_1, v_2$ , edges  $\bar{v}_0 \bar{v}_1, \bar{v}_1 \bar{v}_2, \bar{v}_0 \bar{v}_2$ , and  $\Delta v_0 v_1 v_2$  itself.

**Def** (face/coface). Let  $\sigma$  be the  $k$ -simplex defined on  $S = \{\bar{v}_0, \bar{v}_1, \dots, \bar{v}_k\}$ . A simplex  $\tau$  defined on a subset  $T$  of  $S$ ,  $|T| \neq \emptyset$ , is a **face** of  $\sigma$ , and  $\sigma$  is a **coface** of  $\tau$ . The notation is  $\tau \leq \sigma$ ,  $\sigma \geq \tau$  (some books use  $\tau \preceq \sigma$ ,  $\sigma \succeq \tau$ ) \succcurlyeq, \preccurlyeq in LaTeX.

Thus,  $\bar{v}_0\bar{v}_1$  is a face of  $\Delta v_0v_1v_2$ . So are  $v_0, v_1, v_2, \bar{v}_0\bar{v}_2$ , and  $\bar{v}_0\bar{v}_1$ .

A simplex is always a face of itself, i.e.,  $\sigma \leq \sigma$  and  $\sigma \geq \sigma$ .

We can attach simplices together "nicely" to form bigger objects called simplicial complexes.

**Def** A **simplicial complex**  $K$  is a set of simplices such that

1.  $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$ ; and

Every face of a simplex in  $K$  is also in  $K$ .

2.  $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$

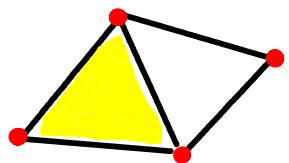
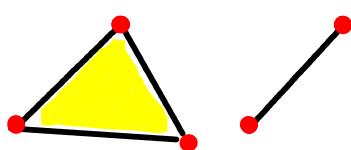
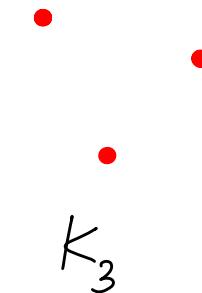
when  $\sigma \cap \sigma' \neq \emptyset$ .

In particular, the non-empty intersection of two simplices in  $K$  is a face of both of them, and hence in  $K$  as well.

The above definition holds in the case of both finite and nonfinite  $K$ . In the latter case,  $K$  has infinitely many simplices satisfying the two conditions. But we will usually limit our attention in this course to finite simplicial complexes, unless mentioned otherwise.

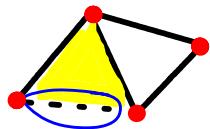
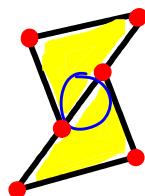
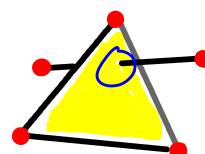
Note that a simplex is the convex hull of a **finite** number of affinely independent points. So, we never talk about "infinite-dimensional" simplices. We could talk about each of these points sitting in infinite-dimensional space, but we restrict our attention to  $\mathbb{R}^d$  for finite  $d$ .

Examples Here are some simplicial complexes:

 $K_1$  $K_2$  $K_3$ 

In particular, notice that a simplicial complex need not consist of just one connected component.

Here are some collections that are *not* simplicial complexes:

 $K_4$  $K_5$  $K_6$ 

$K_4$  violates Condition 1, as one of the faces of the triangle in  $K_4$  is not in the collection.  $K_5$  and  $K_6$  violate Condition 2, as the intersection of two simplices is not a face of either simplex in both cases.

**Def** The dimension of a simplicial complex is the same as that of the highest dimensional simplex in it, i.e.,

$$\dim K = \max \{ \dim \sigma \mid \sigma \in K \}.$$

In the previous examples,  $\dim(K_1) = 2$ ,  $\dim(K_2) = 2$ , and  $\dim(K_3) = 0$ .

Earlier we have been talking about continuous surfaces, e.g., the 2-sphere, torus, etc. And now we are talking about simplicial complexes as discrete objects. How do we "reconcile" the two notions? Indeed, we can formally define the "space" modeled by a simplicial complex. Later on, we will talk about a simplicial complex "triangulating", say, a torus, when this "space" is homeomorphic to  $\mathbb{T}^2$ .

**Def** The underlying space of a simplicial complex  $K$  is the space made of all simplices in  $K$  together with the topology inherited from the ambient Euclidean space. We denote the underlying space of the simplicial complex  $K$  by  $|K|$ . Thus,

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

$|K|$  is also called the polyhedron (or polytope) of  $K$ .

$A \subseteq |K|$  is closed in  $|K|$  iff  $A \cap \sigma$  is closed  $\forall \sigma \in K$ .

# MATH 529 : Lecture 6 (01/29/2026)

Today:

- \* Abstract simplicial complexes (ASCs)
- \* geometric realization
- \* Examples of ASCs

**Def** An **abstract simplicial complex** (ASC) is a collection  $\mathcal{S}$  of finite non-empty sets such that if  $A \in \mathcal{S}$ , and  $B \subseteq A$  with  $B \neq \emptyset$ , then  $B \in \mathcal{S}$ .

Note that the condition specified in the above definition of an abstract simplicial complex is equivalent to the first condition in the definition of a (regular) simplicial complex, which says that every face of a simplex in the complex is also in the complex.

The second intersection condition is trivially satisfied in the case of abstract simplicial complexes. The intersection of two sets is indeed a subset of both sets.  $\mathcal{S}$  itself can be finite or infinite, but each  $A \subseteq \mathcal{S}$  is a finite set.

The sets in  $\mathcal{S}$  are called the **simplices** of  $\mathcal{S}$ . The dimension of a simplex  $A \in \mathcal{S}$  is  $\dim(A) = |A| - 1$ .

→ cardinality (# entries) of  $A$

Note the correspondence of the above definition to the definition of simplices in the usual sense. Recall that a  $k$ -simplex is the convex hull of  $(k+1)$  affinely independent points, which are its vertices. We maintain this correspondence by defining  $\dim A = |A| - 1$  for any set  $A \in \mathcal{S}$ .

The dimension of  $\mathcal{S}$  is  $\dim(\mathcal{S}) = \max \{\dim(A) \mid A \in \mathcal{S}\}$ .

In the definition of  $\mathcal{S}$ , we do assume that all  $A \in \mathcal{S}$  are finite sets.  
And there exists a maximum dimensional simplex in  $\mathcal{S}$ .  
→ which is finite

The singleton sets in  $\mathcal{S}$  are called its *vertices*, and is denoted by  $\text{Vert}(\mathcal{S})$ .

Again note the correspondence of these singleton sets to the vertices (0-simplices) in (geometric) simplicial complexes.

Here is an example of an abstract simplicial complex.

$$\mathcal{S} = \underbrace{\{\{0\}, \{1\}, \{2\}, \{3\}\}}_{\text{vertices}}, \{\{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\}\}.$$

We can indeed check that the condition on inclusion of subsets is satisfied. For instance, consider the set  $\{0,1,2\}$ .

$$\{0,1,2\} \in \mathcal{S} \Rightarrow \text{need } \underbrace{\{0,1,2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0\}, \{1\}, \{2\}}_{\substack{\text{trivial} \\ \text{indeed!}}} \in \mathcal{S}.$$

$$\dim \mathcal{S} = 2.$$

Given any (geometric) simplicial complex  $K$ , we can create an abstract simplicial complex  $S$  by taking just the sets of vertices in each simplex of  $K$  (and ignoring the geometry).  $S$  here is called the **vertex scheme** of  $K$ .

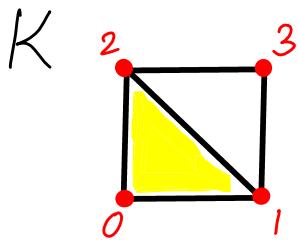
Symmetrically,  $K$  is a **geometric realization** of  $S$ .

$\hookrightarrow$  there could be other geometric realizations

Let's consider the ASC we saw previously:

$$S = \left\{ \underbrace{\{0\}, \{1\}, \{2\}, \{3\}}_{\text{vertices}}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\} \right\}.$$

$K$  shown here is a geometric realization of  $S$ .



$K$  can sit in  $\mathbb{R}^2$

We have a vertex corresponding to each singleton set (i.e., vertex) in  $S$ , an edge corresponding to each doublet, and a triangle for the triplet.

But this is just one geometric realization. In particular, if we were to specify the  $(x,y)$  coordinates of the vertices, we can imagine other realizations, e.g., by translating this one. We could also have a realization in  $\mathbb{R}^3$ , for instance.

If turns out that every ASC has a geometric realization.

Theorem (Geometric realization theorem) Every abstract simplicial complex  $S$  with  $\dim S = d$  has a geometric realization in  $\mathbb{R}^{2d+1}$ .

### Idea of the Proof

We map vertices of  $S$  injectively to points in  $\mathbb{R}^{2d+1}$ , say,  $f: \text{Vert}(S) \rightarrow \mathbb{R}^{2d+1}$ . Why  $2d+1$ ? We use the fact that  $2d+2$  or fewer points in  $\mathbb{R}^{2d+1}$  that are in general position are affinely independent (AI).

Def  $(d+1)$  points in  $\mathbb{R}^d$  are in **general position** if no hyperplane contains more than  $d$  of those points.

The idea is that the points do not satisfy any more linear relationships than they must. For instance 3 points in  $\mathbb{R}^2$  that are not collinear are in general position.

Recall that a  $d$ -simplex is the convex hull of  $(d+1)$  AI points. We need to make sure that we will have "enough freedom", i.e., affine independence, among the mapped vertices so that we can map all the simplices in  $S$  to corresponding simplices in the geometric simplicial complex.

Consider  $A, B \in S$ . Since  $\dim(S) = d$ ,  $|A|, |B| \leq d+1$ .

$$\text{Hence, } |A \cup B| = |A| + |B| - |A \cap B| \leq d+1 + d+1 = 2d+2.$$

Hence by going to  $\mathbb{R}^{2d+1}$  and choosing points there in general position, we can ensure that (up to)  $2d+2$  points are AI.

$\Rightarrow$  Any convex combination  $\bar{x}$  in  $A \cup B$  is unique.

$$\Rightarrow \bar{x} \in A \text{ and } \bar{x} \in B \iff \bar{x} \in A \cap B.$$

$\hookrightarrow$  ensures the second requirement of nonempty intersections of two simplices being their faces.

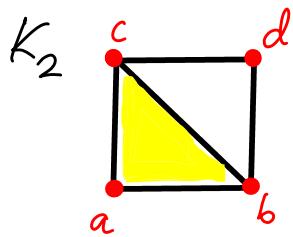
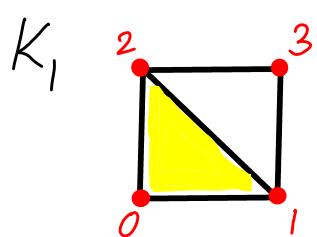
We could often find geometric realizations in  $\mathbb{R}^d$  for  $d'$  smaller than  $2d+1$ . □

How do we compare abstract simplicial complexes? We had previously defined the concept of homeomorphism to study when two topological spaces are "similar". We now define corresponding notions for simplicial complexes.

Recall that  $\text{Vert}(\mathcal{S})$  represents the vertex set of the ASC  $\mathcal{S}$ .

**Def** Two abstract simplicial complexes  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are **isomorphic** if there exists a bijection  $\varphi: \text{Vert}(\mathcal{S}_1) \rightarrow \text{Vert}(\mathcal{S}_2)$  such that  $A \in \mathcal{S}_1$  iff  $\varphi(A) \in \mathcal{S}_2$ .  $\varphi$  is an isomorphism between  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . We write  $\mathcal{S}_1 \approx \mathcal{S}_2$  here.  $\xrightarrow{\text{simplex}}$

In this setting, every simplex in  $\mathcal{S}_1$  has a unique corresponding simplex in  $\mathcal{S}_2$ .

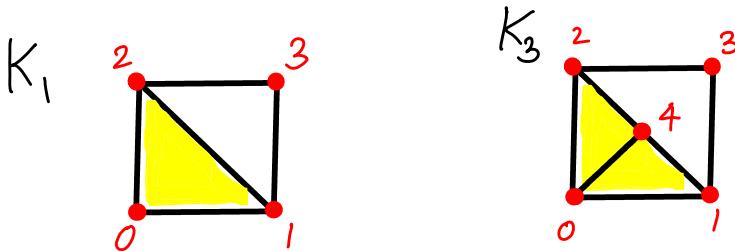


vertex schemes of  $K_1$  and  $K_2$  are isomorphic

Notice that the similarity here is defined for **abstract** simplicial complexes. Hence,  $K_1$  and  $K_2$  above need not be sitting in the same space. Still, they are isomorphic as ASCs. We make this notion precise in the following theorem.

**Theorem** Two <sup>(geometric)</sup> simplicial complexes  $K_1$  and  $K_2$  are isomorphic, or **simplicially homeomorphic**, iff their vertex schemes  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are isomorphic as abstract simplicial complexes. We denote this fact by  $K_1 \cong K_2$ , which implies  $|K_1| \approx |K_2|$ , and  $\mathcal{S}_1 \approx \mathcal{S}_2$ .

The implication might not go the other way, though. For instance,  $K_1 \cong K_2$  above. Now consider  $K_3$  as shown below.



Notice that  $K_1 \not\cong K_3$ , even though  $|K_1| \approx |K_3|$ . In fact, their underlying spaces could very well be identical!

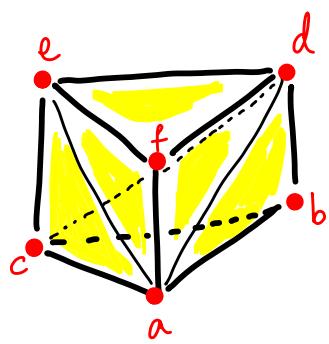
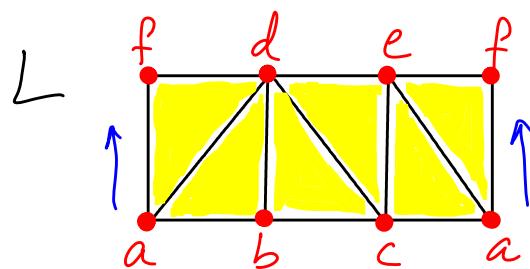
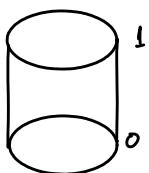
From the computational point of view, while simplicial complexes that are "smaller", i.e., have a smaller number of simplices while modeling the same topological space are usually preferred. At the same time, geometry might dictate that we need a large number of simplices to capture the complexity.

How do we use ASCs? We illustrate several examples

1. cylinder

geometric representation:

$$\mathbb{S}^1 \times [0, 1]$$



K

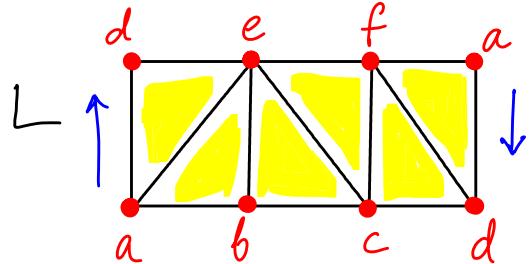
The underlying space of L appears to be rectangle, but notice how the vertex labels identify the left and right edges (both are  $\overline{af}$ ).

$$L = \left\{ \{a, b, d\}, \{a, d, f\}, \{b, c, d\}, \{c, d, e\}, \{a, c, e\}, \{a, c, f\}, \text{ and all nonempty subsets} \right\}$$

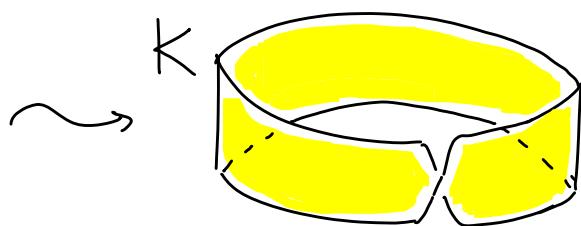
K is one geometric realization of L here.

## 2. Möbius strip

We start with the ASC  $L$  here



The left and right vertical edges are identified, after a "twist". Like in Example 1, the underlying space is a rectangle, but vertex labels are different.



$L$  represents the Möbius strip, i.e.,  $K$  is a geometric realization of  $L$ .

We could also specify  $L$  abstractly:

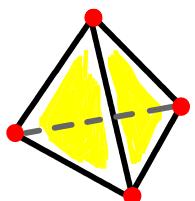
$$L = \left\{ \{a, d, e\}, \{a, b, e\}, \{b, c, e\}, \{c, e, f\}, \{c, d, f\}, \{a, d, f\} \right\} \text{ and} \\ \text{all nonempty subsets of these triplets} \right\}.$$

The abstract simplicial complexes shown above consist of triangles – indeed, they are triangulations. In general, triangulations consist of triangles (in 2D), and simplices in general as we formalize below.

Notice that each  $k$ -simplex  $\approx k$ -ball (closed). 2-ball is the closed 2-disc.

**Def** (Triangulation). A **triangulation** of a topological space  $X$  is a simplicial complex  $K$  such that  $|K| \approx X$ .

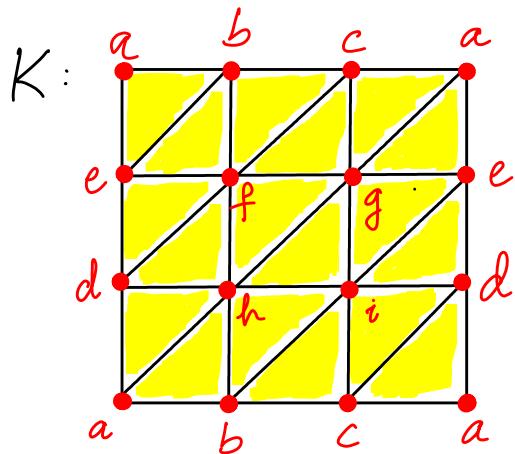
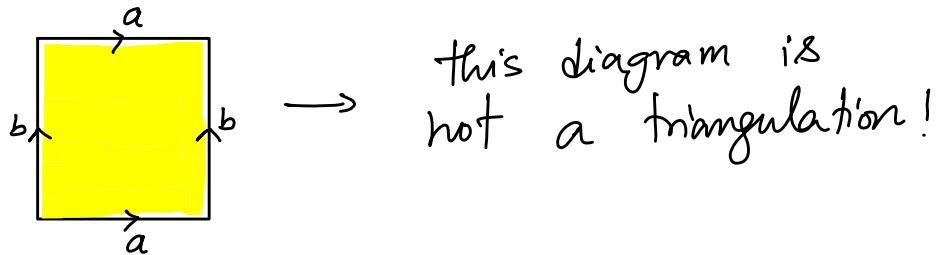
Example:



surface of a tetrahedron (triangles and their faces) is a triangulation of the 2-sphere  $S^2$ .

A triangulation is a piecewise linear representation of the topological space.

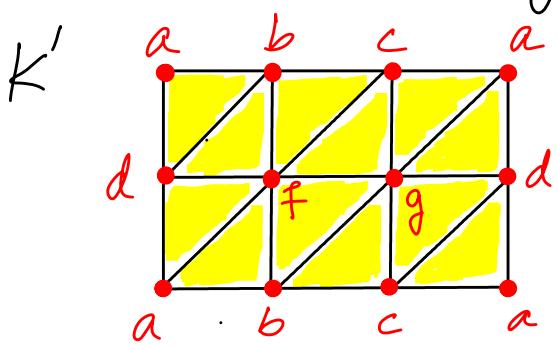
### 3. Torus ( $\mathbb{T}^2$ )



Why so many triangles?

This triangulation  $K$  of the torus has 18 (different) triangles. One might wonder if we could produce a triangulation using a much smaller number of triangles.

Consider the following candidate triangulation  $K'$ :



Is  $K'$  a triangulation of  $\mathbb{T}^2$ ? No! For instance, consider edge  $ad$ . It is a face of four triangles:  $adb, adf, adc, adg$ .

Hence, points on  $ad$  do not have neighborhoods homeomorphic to  $\mathbb{R}^2$ . We're doing "too much gluing" here.

Q. What is the minimum number of triangles needed to produce a triangulation of  $\mathbb{T}^2$ ?

A. We need at least 14 triangles.

Rule: In a triangulation of a 2-manifold (with boundary),

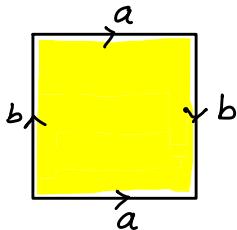
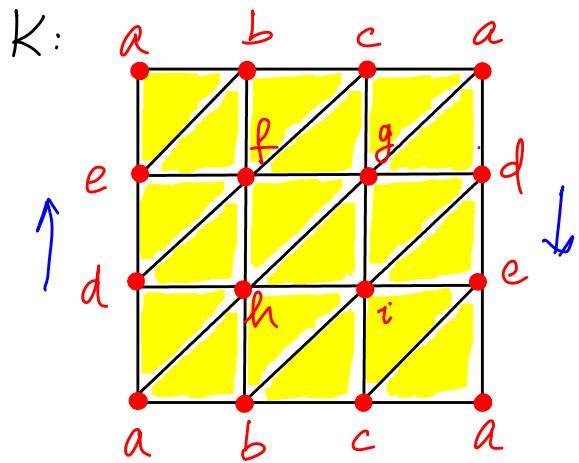
each edge must be part of (one or) two triangles.

Edges that are part of only one triangle each form the boundary of the 2-manifold.

Recall that every point on a 2-manifold has a neighborhood homeomorphic to an open disc. The corresponding requirement for edges becomes that each edge has to be part of exactly two triangles. Similarly, points on the boundary of a 2-manifold have neighborhoods homeomorphic to half discs. Edges that are part of single triangles are indeed the boundary edges in the triangulation. The result extends to d-dimensions – every  $(d-1)$ -simplex is the face of one or two  $d$ -simplices.

The above rule could be used to check if a given simplicial complex is the triangulation of a manifold or not. But, be warned that satisfying this rule **alone** is not enough to identify the given simplicial complex as the triangulation of a specific manifold, e.g., the torus.

#### 4. Klein bottle ( $K^2$ )



Same rule applies here – each edge is shared by two triangles exactly.

For example, edge  $\overline{ae}$  is part of triangles  $\triangle abe$  and  $\triangle ace$ .

We talked about distinguishing spaces that are not homeomorphic. In practice, spaces or objects are usually represented by triangulations. Checking for homeomorphisms between triangulations is not easy. How can we distinguish two topological spaces computationally? One option is to use an invariant.

# MATH 529 : Lecture 7 (02/03/2026)

Today: \* topological invariants  
 \* Euler characteristic ( $\chi$ )  
 \* genus and ~~cross cap~~ → didn't get to it...

## Topological Invariants

We want to define and use efficiently (and easily) computable functions that can help us to distinguish topological spaces of different topological types.

**Def** A **topological invariant** is a map that assigns the same object to spaces of the same topological type.

(usually, a number; but we could also have a "barcode", for instance)

Let  $f(\cdot)$  be an invariant.

$$\mathbb{X} \approx \mathbb{Y} \Rightarrow f(\mathbb{X}) = f(\mathbb{Y}).$$

$$\text{So, } f(\mathbb{X}) \neq f(\mathbb{Y}) \Rightarrow \mathbb{X} \not\approx \mathbb{Y}$$

could be used for contrapositive arguments

But  $f(\mathbb{X}) = f(\mathbb{Y})$  does not necessarily mean  $\mathbb{X} \approx \mathbb{Y}$ .

If  $f(\mathbb{X}) = f(\mathbb{Y}) \Rightarrow \mathbb{X} \approx \mathbb{Y}$ , then  $f(\cdot)$  is called a **complete invariant**.

Notice that an invariant could assign the same object to spaces of different topological types. The main way to use the invariant is in the contrapositive, i.e., if the invariant is different for a pair of spaces, then the two spaces have different topological types.

While the invariants are defined for topological spaces, we usually use triangulations to compute them. And since they're "invariants", the specific choice of the triangulation is not important.

We introduce a simple to evaluate/compute invariant - the Euler characteristic.

The Euler characteristic ( $\chi$ ) (originally defined for graphs)

Let  $K$  be a simplicial complex, and let  $s_i$  be the # i-simplices in  $K$  for  $0 \leq i \leq \dim(K)$ . Then,

$$s_i = |\{\sigma \in K \mid \dim \sigma = i\}|.$$

The Euler characteristic of  $K$  is defined as

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i.$$

Notice that  $s_i = 0 \nabla i > \dim K$ .

Equivalently,  $\chi(K) = v - e + f - t + \dots$ , where  $v = \# \text{vertices}$ ,  $e = \# \text{edges}$ ,  $f = \# \text{faces}$ ,  $t = \# \text{tetrahedra}$ , ... and so on.

$\xrightarrow{\text{triangles}}$

Let us find  $\chi$  for a triangulation of the closed 2-disc.

$$\chi \left( \begin{array}{c} \text{Yellow triangle} \\ \text{with 3 vertices, 3 edges, 1 face} \end{array} \right) = 3 - 3 + 1 = 1$$

↓      ↓      ↓  
 # vertices    # edges    # faces (or # triangles)

$\chi$  is an integer invariant, and it is an invariant of the underlying space  $|K|$ . So,  $\chi$  is invariant over triangulations of a given space. Thus, any triangulation of a topological space  $X$  has the same  $\chi(X)$  value.

Continuing with the example of the disc, we get the same  $\chi$  using any other triangulation — see two examples below.

$$\chi \left( \begin{array}{c} \text{Yellow rectangle} \\ \text{with 4 vertices, 5 edges, 2 faces} \end{array} \right) = 4 - 5 + 2 = 1.$$

The triangulation is made of two triangles sharing an edge.

Now consider adding one more triangle to get another valid triangulation (as shown in blue).

$$\Delta(v)=1, \Delta(E)=2, \Delta(F)=1, \text{ so } \Delta(X)=1-2+1=0!$$

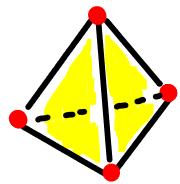
↓ change in # vertices, edges, triangles → change in  $\chi$

Q: Can we use  $\chi$  to distinguish compact 2-manifolds?

Let us find  $\chi$  for  $S^2$ ,  $T^2$ ,  $RP^2$  and  $K^2$ , 2-sphere, torus, projective plane, and the Klein bottle.

$$1. S^2$$

$K:$



Surface of a tetrahedron  
is a triangulation of  $S^2$ .

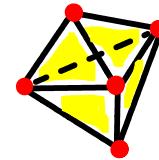
NOT the solid tetrahedron!

$$\chi(K) = 4 - 6 + 4 = 2.$$

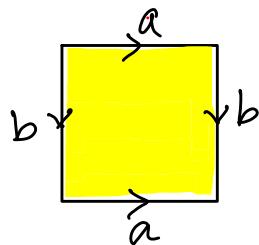
let us consider another triangulation  $K'$  of  $S^2$ , made of 3 triangles from top and 3 from bottom joined to form a "sphere".

$$\chi(K') = 5 - 9 + 6 = 2.$$

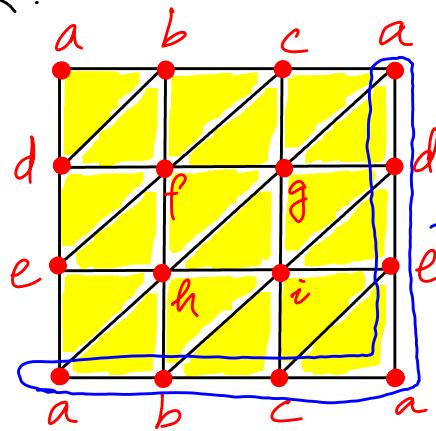
$K':$



$$2. T^2 \text{ (torus)}$$

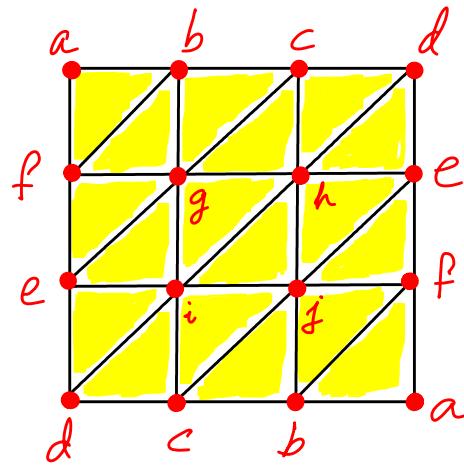
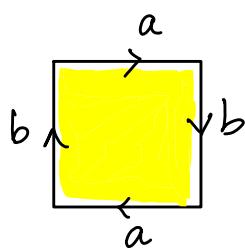


$K:$

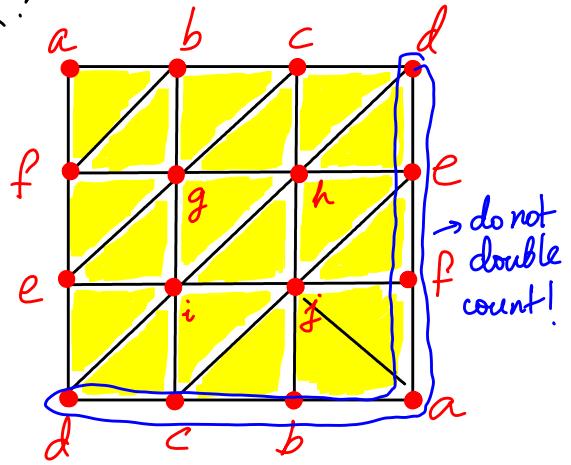


should not  
double count!

$$\chi(T^2) = 9 - 27 + 18 = 0.$$

3.  $\mathbb{RP}^2$ 

K:



$$\overline{bf} \in abf, jbf, gbf ! \times$$

$$\overline{ab} \in abf ! \times$$

$$\overline{ab} \in abj, abf \checkmark$$

$$\overline{bf} \in bfa, bfg \checkmark$$

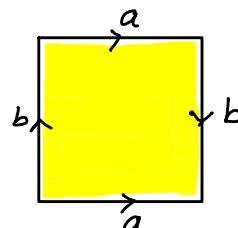
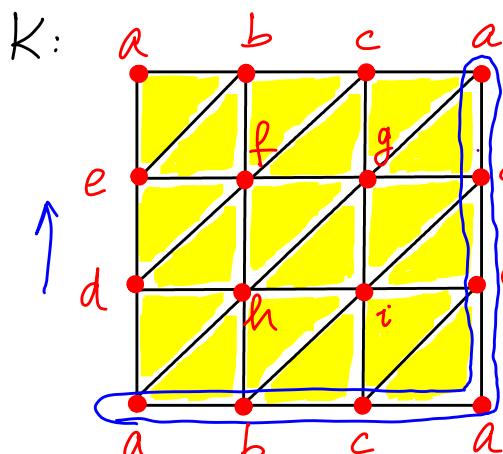
Our first attempt at finding a triangulation (left) of  $\mathbb{RP}^2$  is not correct! In particular, edge  $\overline{bf}$  (bottom right square) is part of 3 triangles:  $abf, bfg, bfi$ ! Also,  $\overline{ab}$  and  $\overline{af}$  are part of only one triangle each. But  $\mathbb{RP}^2$  has no boundary!

A correct triangulation is given on the right.

the left triangulation represents  $\mathbb{RP}^2$  with a "flap" ( $\triangle abf$ )

$$\chi(K) = 10 - 27 + 18 = 1.$$

vertices a-j

4.  $\mathbb{K}^2$  (Klein bottle)

$$\chi(K) = 9 - 27 + 18 = 0.$$

Note that  $\chi(\mathbb{K}^2) = \chi(\mathbb{P}^2) /$

Here is the summary of the  $\chi$  values we have seen so far.

2-manifold	$\chi$
orientable	$S^2$ 2
	$T^2$ 0
nonorientable	$RP^2$ 1
	$TK^2$ 0

So,  $\chi$  alone is not sufficient to distinguish between all these surfaces!

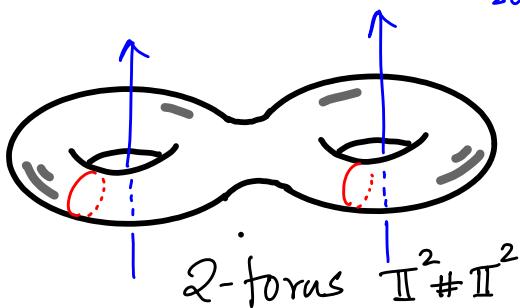
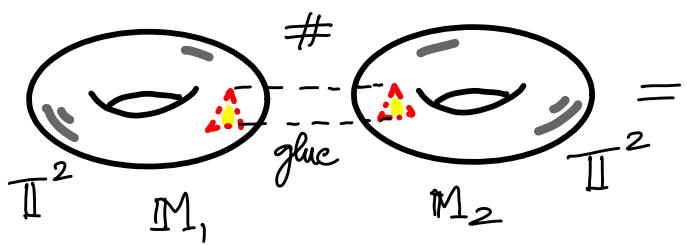
It turns out that if we add orientability to  $\chi$ , we do get a complete invariant for all compact (connected) 2-manifolds (without boundary). Recall the original classification theorem, which states that every compact connected 2-manifold is homeomorphic to  $S^2$ , a connected sum of copies of  $T^2$ , or a connected sum of copies of  $RP^2$ . With this result in mind, let us first study how  $\chi$  changes when we take the connected sum of two manifolds.

**Theorem** For compact, connected surfaces  $M_1$  and  $M_2$ ,

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2.$$

take one triangle out of each surface.

### Illustration

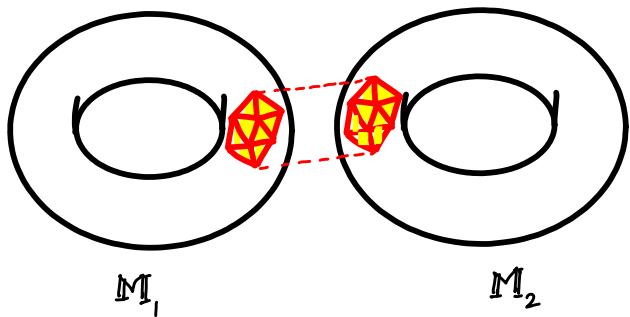


double torus  
two tunnels ( $\uparrow\uparrow$ )  
and two handles  
 $C; C$

We remove a triangle each from both  $M_1$  and  $M_2$ , and glue along the boundaries of these triangles.

$$\Delta(V) = -3, \Delta(E) = -3, \Delta(F) = -2. \text{ So, } \chi(X) = -3 - (-3) + (-2) = -2.$$

The result holds for the removal of a disc in general, and not just for the case of (the removal of) a triangle.



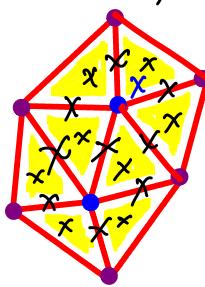
Here, we remove a patch (homeomorphic to the disc) from both  $\underline{M}_1$  and  $\underline{M}_2$ , and identify the boundaries, which is composed of 6 edges and 6 vertices (to form a loop).

from the middle regions of each patch, we remove 2 vertices, 9 edges, and 8 triangles.

The change in  $\chi(\underline{M}_1 \# \underline{M}_2)$  contributed by the simplices removed from  $\underline{M}_1$  is

$-(2-9+8) = -1$ . A same change is contributed by the simplices removed from  $\underline{M}_2$ .

$$\text{As such, } \chi(\underline{M}_1 \# \underline{M}_2) = \chi(\underline{M}_1) + \chi(\underline{M}_2) - 2.$$



simplices marked with an 'x' are removed, and so are the two middle vertices.

There is no change in  $\chi$  from identifying the boundaries — as they are both cycles (so have same # of vertices & edges).

We get the same result even if we were to remove different "discs" from the two tori. Just that the homeomorphism defining the gluing would be more complicated there.

We could prove this result in general (for a removal of a general disk).

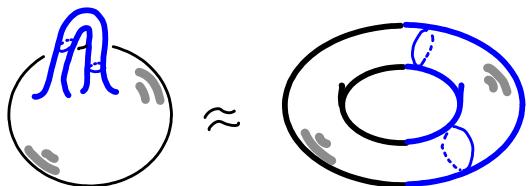
Theorem Two compact, closed, connected 2-manifolds  $M_1$  and  $M_2$  are homeomorphic if and only if

1.  $\chi(M_1) = \chi(M_2)$  and
  2. either  $M_1$  and  $M_2$  are both orientable,  
or are both nonorientable.
- in polynomial time,  
to be precise

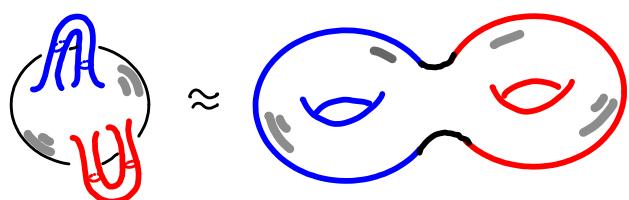
We could perform both checks 1 and 2 efficiently on a computer.

Genus and Cross-cap → Two more terms used in the context of 2-manifolds.

Def The connected sum of  $g$  tori is called a surface with **genus**  $g$ . Equivalently, a 2-sphere with 1 tube is a surface with genus  $g=1$ .

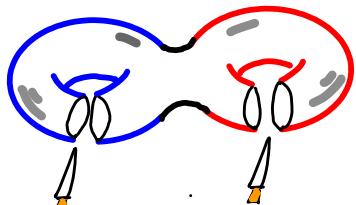


Sphere with one tube is homeomorphic to torus.  
(so, torus has genus = 1).



A sphere with two tubes is homeomorphic to the double torus.

$M$  has genus  $g \Rightarrow$  there are  $g$  disjoint closed curves on  $M$  along which you can cut without disconnecting  $M$ .



$g=2$  here. If we cut along one more closed curve now, we get two pieces that are disconnected.