MATH 524 : Lecture 1 (08/19/2025)

This is Algebraic Topology.
I'm Bala Krishnamoorthy (Call me Bala).

Today:
* syllabus, logistics
* neighborhoods, continuous functions
* topology using neighborhoods
* homeomorphism

I will be teaching computational topology (Math 529) next semester. The two classes – Math 524 and Math 529 will be kept independent. In particular, we will spend nearly no focus on computational aspects in Math 524.

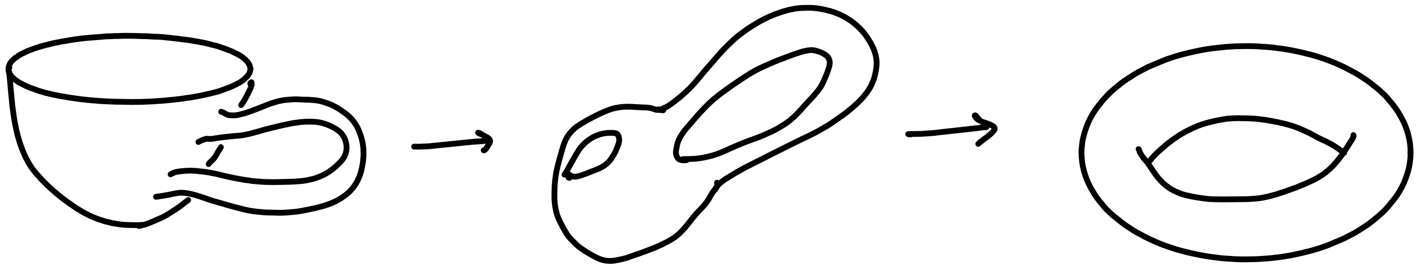
Check the course web page at

https://bala-krishnamoorthy.github.io/Math524.html

All documents, important updates, homework assignments etc. will be posted there. Check the class page frequently.

More about the video assignment to come soon. But you're encouraged to start looking for topics that you might want to make the video on as we proceed in the course.

Topology studies how "space is connected". You might have heard the (true!) joke that the topologist cannot distinguish between a coffee cup and a donut! Indeed, they both are connected the same way.



In algebraic topology, we cast problems on how space is connected as equivalent problems on algebraic objects — groups, rings, etc., and maps between them (homomorphisms).

As a subfield of mathematics, algebraic topology started in late 19^{th} and early 20^{th} century. Poincaré introduced the fundamental group first. Later Betti introduced homology groups, which are much easier to compute (both by hand as well as algorithmically) than the former.

We will spend a lot of time talking about homology groups, and the dual concept of cohomology. We will not be spending much attention on the fundamental group.

There are several (equivalent) ways to define homology groups. Perhaps the "nicest" way to do so is using simplicial complexes. We will spend a fair bit on time studying simplicial homology.

We will introduce/refresh background concepts as needed. First, we will talk about continuous functions and topological spaces, defined in terms of neighborhoods.

Continuous functions

We first give the classical ε-δ definition in Euclean spaces.

Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$. f is continuous at $\bar{x} \in \mathbb{R}^n$ if there exists $\delta > 0$ for every $\varepsilon > 0$ such that $\|f(\bar{y}) - f(\bar{x})\| < \varepsilon$ whenever $\|\bar{y} - \bar{x}\| < \delta$ for $\bar{y} \in \mathbb{R}^n$. f is continuous (in all of $\mathbb{R}^n$) if it is so at every $\bar{x} \in \mathbb{R}^n$.

my notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\mu}$, etc., are all vectors — lower case letters with a bar.

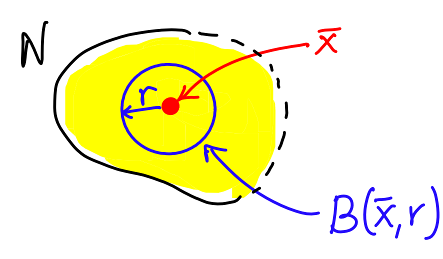
We give an equivalent definition based on neighborhoods.

Def A subset N of $\mathbb{R}^n$ is a **neighborhood** of $\bar{x} \in \mathbb{R}^n$ if for some $r > 0$, the closed ball $B(\bar{x}, r)$ centered at $\bar{x}$ is contained entirely within N.

Notice that neighborhood N can be open or closed.

$$B(\bar{x}, r) = \{\bar{y} \in \mathbb{R}^n \mid \|\bar{x} - \bar{y}\| \leq r\}$$

closed Ball of radius r centered at $\bar{x}$



Def $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous if given any $\bar{x} \in \mathbb{R}^n$ and a neighborhood N of $f(\bar{x})$ in $\mathbb{R}^m$, $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in $\mathbb{R}^n$.

Now we define what a topological space (or topology) is. We give the definition in terms of neighborhoods first. In most textbooks, you will see the definition given in terms of open sets. Later, we will see that both definitions are equivalent.

Topological space (or topology)

more notation: Upper case letters, e.g., A, B, X, Y, etc, denote sets or matrices.

Def I We are given a set X and a nonempty collection of subsets of X for each $\bar{x} \in X$ called the neighborhoods of $\bar{x}$. This is a topological space if it satisfies the following axioms.

(a) $\bar{x}$ lies in each of its neighborhood.

(b) Intersection of two neighborhoods of $\bar{x}$ is itself a neighborhood of $\bar{x}$.

(c) If N is a neighborhood of $\bar{x}$, and $U \subseteq X$ contains N, then U is a neighborhood of $\bar{x}$.

(d) If N is a neighborhood of $\bar{x}$, $\mathring{N}$, the **interior** of N is also a neighborhood of $\bar{x}$.

The interior of N is $\mathring{N} = \{\bar{y} \in N \mid N \text{ is a neighborhood of } \bar{y}\}$.

Intuitively, every point of N not on its boundary is in its interior.

We can extend the definition of continuous functions to functions defined between topological spaces.

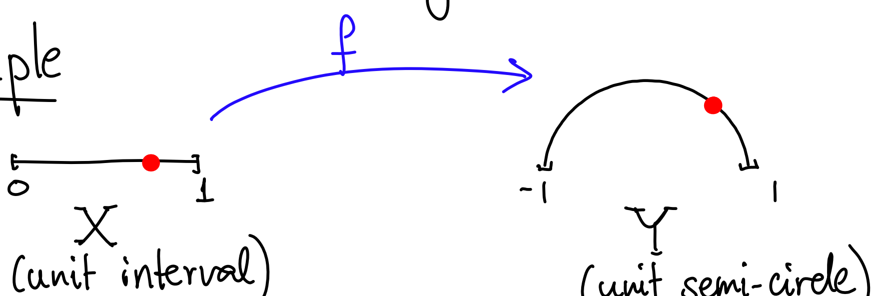
Def Let X, Y be topological spaces. $f: X \to Y$ is continuous if $\forall\, \bar{x} \in X$ and for every neighborhood N of $f(\bar{x})$ in Y, the set $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in X.

We are interested in studying when two topological spaces are similar. There are a few different notions of topological similarity, and the strongest notion is that of homeomorphism. For two spaces to be homeomorphic, we need a function between them that is "nicer" than just a continuous function.

Def A function $f: X \to Y$ is a **homeomorphism** if it is one-to-one, onto, continuous, and has a continuous inverse.

When such a function exists between two spaces X and Y, we say they are **homeomorphic**, or are topologically equivalent. We denote this fact by $X \approx Y$.

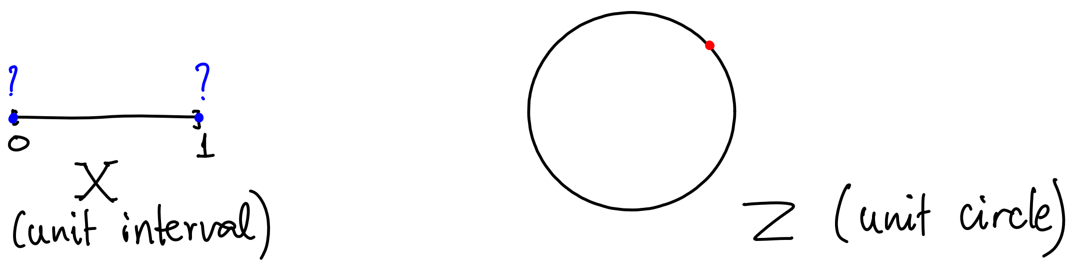
Example



$X \approx Y$. Can you define the function f?

Think of X & Y as subsets of $\mathbb{R}^2$, and write down the form of f, as well as f^{-1}. You can show f satisfies all requirements for being a homeomorphism.

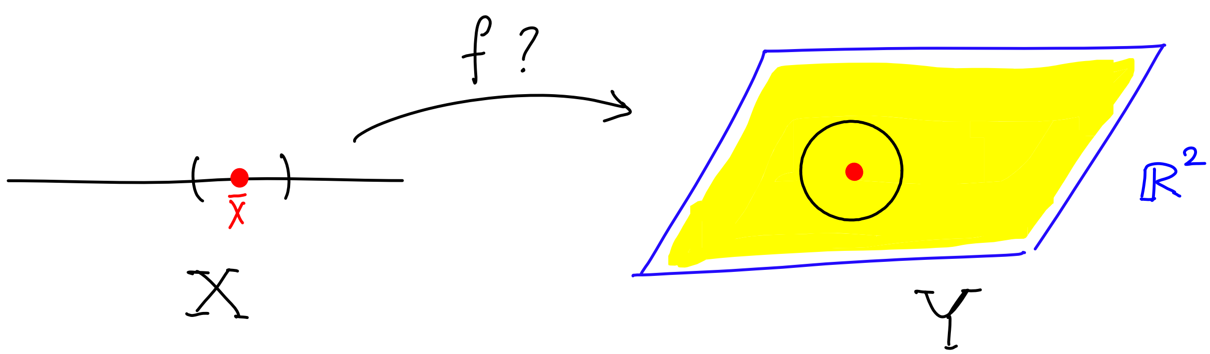
Showing two spaces are **not** homeomorphic could be harder — we need to show that no such function exists between X and Y.



Here, $X \not\cong Z$. Where do things breakdown?

Intuitively, one can notice the two end points of X behave distinctly from any point in Y.

Here is another example. Perhaps the simplest example of a topological space is $\mathbb{R}^d$ under the usual definition of neighborhoods, which specifies that any set $N \subseteq \mathbb{R}^d$ big enough to contain $B(\bar{x}, r)$ for some $r > 0$ is a neighborhood of $\bar{x} \in \mathbb{R}^d$. But notice that $\mathbb{R}^1 \not\cong \mathbb{R}^2$, for instance. It is not straightforward to prove this fact rigorously. But, how would one "argue" for it?



One method is to appeal to how the two spaces are connected. Recall that topologically similar spaces are "connected" the same way. Here, if we remove one point from both $X = \mathbb{R}^1$ and $Y = \mathbb{R}^2$, we can see that it affects the connectivity differently. Removing one point leaves X disconnected (into two pieces). But removing a point from Y still leaves it connected – its just like poking a hole in the "sheet" that is $\mathbb{R}^2$, which remains connected.

More formally, we could try to define a homeomorphism from X to Y. But we can observe that neighborhoods in X are 1-dimensional, while those in Y are 2D. Hence we cannot define a bijection between them.

We will talk about open sets in the next lecture, and define a topology using open sets. That definition is equivalent to the one introduced earlier today, i.e., Def I.

MATH 524 : Lecture 2 (08/21/2025)

Today: * open sets, topology using open sets
* simplices, properties of simplices

We now consider topology defined in terms of open sets. This is the default approach taken in most textbooks. We first define open sets using the concept of neighborhoods.

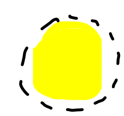
Def $O \subseteq X$ is **open** if it is a neighborhood of each of its points. By (c) of **Def I**, union of any collection of open sets is also open. Also, by (b) of **Def I**, the intersection of any finite number of open sets is open.

We mention unions and finite intersections of open sets as they are both required to be open in a topology. See below.

Notice, $\mathring{N}$ (interior of neighborhood N) is always open.

Alternatively, we can start by defining open sets directly.

Def A set $A \subseteq \mathbb{R}^n$ is **open** if each $\bar{x} \in A$ can be surrounded by a ball of positive radius that lies entirely inside the set.



We can also define open sets more generally, starting with collections of subsets of some set X.

We could define neighborhoods in terms of open sets.

Def A subset $N \subseteq X$ is a neighborhood of $\bar{x}$ if there exists an open set O s.t. $\bar{x} \in O \subseteq N$.

We now formally state the definition of topology in terms of open sets. This definition sees more use than the one using neighborhoods.

Def II A **topology** on a set X is a collection of open sets of X such that any union and finite intersection of open sets is open, and ϕ (empty set) and X are open.
The set X along with the topology is called a **topological space**.

We can define continuous functions also in terms of open sets.

Def $f: X \to Y$ is **continuous** iff (if and only if) the inverse image of each open set of Y is open in X.

We now start the discussion of homology, which is a less strict version of topological similarity than homeomorphism. We study in detail simplicial homology, where the spaces are made of "gluing" "nice" objects called simplices together, and are hence are very "regular".

As we will see, it is also much easier to algebraize questions about homology (than those about homeomorphism).

There is a "continuous" version of homology defined on spaces not composed to regular pieces (simplices), termed singular homology. It turns out singular homology is equivalent to simplicial homology.

We start by defining simplices, which are the building blocks.

Simplices

We define simplices in the usual geometric setting first, and then define them abstractly. We need some concepts from geometry first.

Def The set $\{\bar{a}_0, \dots, \bar{a}_n\}$ of points in $\mathbb{R}^d$ is **geometrically independent** (**GI**) if for any scalars $t_i \in \mathbb{R}$, the equations
$\sum_{i=0}^{n} t_i = 0$, $\sum_{i=0}^{n} t_i \bar{a}_i = \bar{0}$ imply that $t_0 = t_1 = \cdots = t_n = 0$.

Here are some observations about GI sets.

* $\{\bar{a}_i\}$ is GI $\forall i$. (singleton sets)

* $\{\bar{a}_0, \dots, \bar{a}_n\}$ is GI $\Longleftrightarrow$ ← if and only if
$\{\bar{a}_1 - \bar{a}_0, \bar{a}_2 - \bar{a}_0, \dots, \bar{a}_n - \bar{a}_0\}$ is linearly independent (LI).

$\bar{a}_0$ is chosen as the "origin", so to speak. But any $\bar{a}_i$ could play the role of $\bar{a}_0$ here.

IDEA: $\sum_{i=1}^{n} t_i(\bar{a}_i - \bar{a}_0) = \bar{0} \implies t_i = 0 \; \forall i$ (LI)

$$\sum_{i=1}^{n} t_i \bar{a}_i + \underbrace{\left(-\sum_{i=1}^{n} t_i\right)}_{t_0} \bar{a}_0 = \bar{0} \quad \Bigg\} \quad \sum_{i=0}^{n} t_i \bar{a}_i = 0 \;\&\; \sum_{i=0}^{n} t_i = 0 \implies t_i = 0 \; \forall i$$

* 2 distinct points in $\mathbb{R}^d$ are GI,
3 non-collinear points are GI,
4 non-coplanar points are GI, and so on.

Notice the relationship/correspondence to LI vectors. For instance, $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ is GI, but of course the set is not LI.

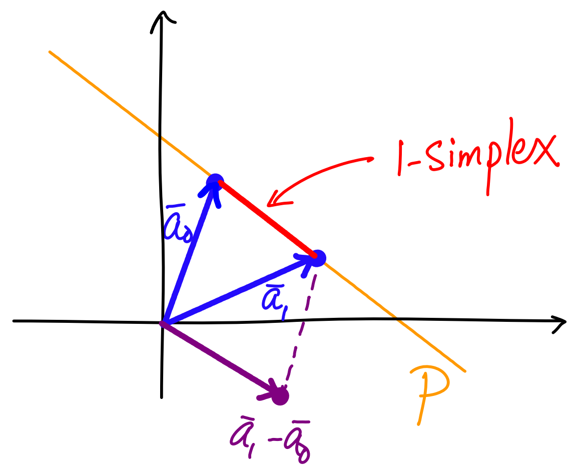
Def Given GI set $\{\bar{a}_0, \dots, \bar{a}_n\}$, the **n-plane** P spanned by these points consists of all $\bar{x}$ such that $\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i$ for scalars t_i with $\sum_{i=0}^{n} t_i = 1$.

The scalars t_i are uniquely determined by $\bar{x}$.

Notice that t_i could be ≥ 0 or ≤ 0 here.

P can also be described as the set of $\bar{x}$ such that

$$\bar{x} = \bar{a}_0 + \sum_{i=1}^{n} t_i(\bar{a}_i - \bar{a}_0).$$



Hence P is the plane through $\bar{a}_0$ parallel to the vectors $\bar{a}_i - \bar{a}_0$.

Going back to the previous example with $\{[\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}2\\2\end{smallmatrix}]\}$, the plane P is the line generated by one of the two vectors.

Q. What is the set described by $\bar{x} = \sum_{i=0}^{n} t_i a_i$, $\sum t_i = 0$?

e.g., consider $n=1$: $\bar{x} = t_0\bar{a}_0 + t_1\bar{a}_1$ with $t_0 + t_1 = 0 \Rightarrow t_0 = -t_1$.

$\Rightarrow \bar{x} = t_0(\bar{a}_0 - \bar{a}_1)$, i.e., it's the line generated by $\bar{a}_0 - \bar{a}_1$.

We now define a simplex as the set "spanned" by a set of GI points.

Def Let $\{\bar{a}_0, \cdots, \bar{a}_n\}$ be a GI set in $\mathbb{R}^d$. The **n-simplex** σ spanned by $\bar{a}_0, \cdots, \bar{a}_n$ is the set of points $\bar{x} \in \mathbb{R}^d$ s.t. $\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i$ with $\sum_{i=0}^{n} t_i = 1$, $t_i \geq 0 \; \forall i$.

The t_i are uniquely determined by $\bar{x}$, and are called the **barycentric coordinates** of $\bar{x}$ (in σ) w.r.t. $\bar{a}_0, \cdots, \bar{a}_n$.

we will later extend definition of t_i to $\bar{x} \notin \sigma$. the

0-simplex : a point

1-simplex : line segment

2-simplex

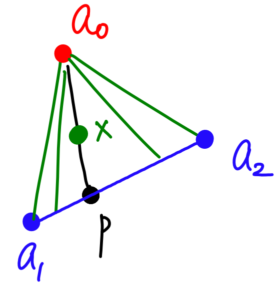
$\bar{x} = \bar{a}_0$ is trivial to consider.

Assume $\bar{x} \neq \bar{a}_0$, i.e., $t_0 \neq 1$. Now consider

$$\bar{x} = \sum_{i=0}^{2} t_i a_i = t_0 \bar{a}_0 + (1-t_0) \underbrace{\left[\frac{t_1}{1-t_0} \bar{a}_1 + \frac{t_2}{1-t_0} \bar{a}_2 \right]}_{\bar{p}}$$

Since $\sum_{i=0}^{2} t_i = 1$, $1 - t_0 = t_1 + t_2$. Hence $\left(\frac{t_1}{1-t_0}\right) \bar{a}_1 + \left(\frac{t_2}{1-t_0}\right) \bar{a}_2$ is a point $\bar{p}$ on the line segment $\overleftrightarrow{\bar{a}_1\bar{a}_2}$, and $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{p}$ is a point on the line segment $\overleftrightarrow{\bar{a}_0 \bar{p}}$.

Hence the 2-simplex is the union of such line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\overleftrightarrow{a_1 a_2}$, i.e., the triangle $a_0 a_1 a_2$ $(\triangle a_0 a_1 a_2)$.



This result extends to higher order simplices. For instance, a tetrahedron is the union of all line segments $\overleftrightarrow{a_0 p}$ for all $\bar{p}$ in $\triangle a_1 a_2 a_3$.

Properties of Simplices

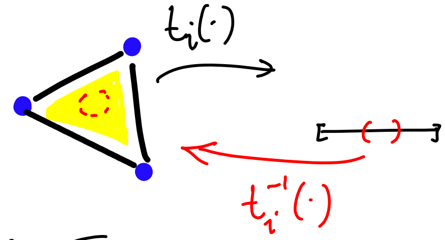
(1) $t_i(\bar{x})$ are continuous functions of $\bar{x}$.

IDEA: $t_i : \mathbb{R}^d \to \mathbb{R}$

convex hull $\{\bar{x} \mid \bar{x} = \sum_{i=0}^{n} t_i a_i,\ t_i \geq 0,\ \sum_i t_i = 1\}$

domain $\to$ $\mathrm{Dom}(t_i) = \mathrm{conv}(\{\bar{a}_0, \dots, \bar{a}_n\})$

$\mathrm{Range}(t_i) = [0, 1]$



Prove that $t_i^{-1}(\text{open set in } [0,1])$ is open in σ.

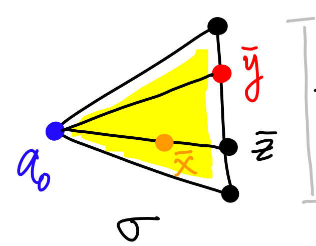
(2) σ is the union of all line segments joining $\bar{a}_0$ to points of the simplex spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$.

Two such line segments intersect only at $\bar{a}_0$.

proof ?

Assume two such line segments from $\bar{a}_0$ to $\bar{y}, \bar{z} \in \tau$, the simplex spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$, meet at $\bar{x} \neq \bar{a}_0$.

$\bar{y} \neq \bar{z}$



Then $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{y} = s_0 \bar{a}_0 + (1-s_0)\bar{z}$, for $t_0, s_0 \in [0,1]$,

where $t_0 \neq s_0$ by assumption (else $\bar{y} = \bar{z}$!).

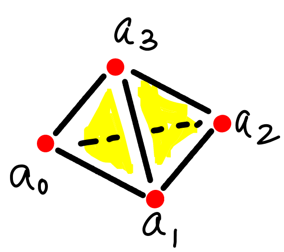
$\Rightarrow \bar{a}_0 = u\bar{y} + v\bar{z}$, where $u, v \in \mathbb{R}$ with $u+v=1$.

$\Rightarrow \bar{a}_0 \in P(\{\bar{y}, \bar{z}\}) \in P(\tau)$ $\to$ $(n-1)$-plane spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$.

which contradicts the GI of $\{\bar{a}_0, \dots, \bar{a}_n\}$.

Def The points $\bar{a}_0, \dots, \bar{a}_n$ which span σ are called its **vertices**. The **dimension** of σ is n ($\dim(\sigma) = n$).
A simplex spanned by a non empty subset of $\{\bar{a}_0, \dots, \bar{a}_n\}$ is a **face** of σ. The face spanned by $\{\bar{a}_0, \dots, \hat{\bar{a}}_i, \dots \bar{a}_n\}$ where $\hat{\bar{a}}_i$ means $\bar{a}_i$ is not included, is the **face opposite $\bar{a}_i$**. Faces of σ distinct from σ itself are its **proper faces**, their union is its **boundary**, $\mathrm{Bd}\,\sigma$ or $\partial\sigma$.

$\partial(\bar{a}_0) = \phi \longrightarrow$ there are no proper faces of a vertex.



tetrahedron $a_0 a_1 a_2 a_3 = \sigma$ → a 3-simplex

proper faces : $\Delta a_0 a_1 a_2, \Delta a_0 a_2 a_3, \dots$ (4)

edges → $\overline{a_0 a_1}, \overline{a_0 a_2}, \dots$ (6)

vertices → $\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3$ (4)

$\partial\sigma = \bigcup(\text{proper faces})$ (triangles, edges, vertices)

→ the "hollow" tetrahedron

Def The **interior** of σ, $\mathrm{Int}(\sigma)$ or $\mathring{\sigma}$, is $\mathrm{Int}(\sigma) = \sigma - \mathrm{Bd}\,\sigma$.

$\mathrm{Int}(\sigma)$ is called an open simplex.

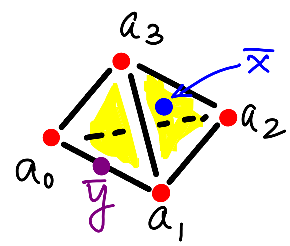
$\mathrm{Int}(\bar{a}_0) = \bar{a}_0 \longrightarrow$ as $\partial a_0 = \phi$.

$\mathrm{Bd}\,\sigma$ consists of all $\bar{x} \in \sigma$ with at least one $t_i(\bar{x}) = 0$.
$\mathrm{Int}\,\sigma$ consists of all $\bar{x} \in \sigma$ with $t_i(\bar{x}) > 0$ $\forall i$.

could be σ itself!

Given $\bar{x} \in \sigma$, there is exactly one face τ s.t. $\bar{x} \in \operatorname{Int} \tau$.
τ is that face of σ spanned by those $\bar{a}_i$ for which $t_i(\bar{x}) > 0$.

$\bar{x}$ is interior to $\Delta a_1 a_2 a_3$
$\bar{y}$ is interior to $\overleftrightarrow{a_0 a_1}$



(3) σ is a compact, convex set in $\mathbb{R}^d$, and is the intersection of all convex sets in $\mathbb{R}^d$ containing $\bar{a}_0, \dots, \bar{a}_n$.

(4) There exists one and only one GI set of points $\{\bar{a}_0, \dots, \bar{a}_n\}$ spanning σ.

(5) $\operatorname{Int} \sigma$ is convex, and is open in P, and $Cl(\operatorname{Int} \sigma) = \sigma$. $\operatorname{Int} \sigma$ is the union of all "open line segments" joining $\bar{a}_0$ with points in $\operatorname{Int} \tau$, where τ is the face opposite $\bar{a}_0$.

closure

MATH 524: Lecture 3 (08/26/2025)

Today: * simplicial complexes
* underlying space

One more property of simplices first...

Def Unit ball: $\mathbb{B}^n = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| \leq 1\}$

Unit sphere: $\mathbb{S}^{n-1} = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| = 1\}$

upper/lower hemisphere: $\mathbb{E}_+^{n-1} / \mathbb{E}_-^{n-1} =$

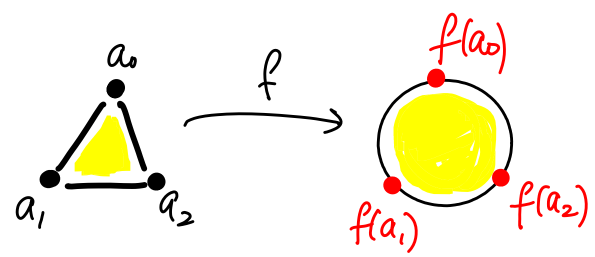
$\{\bar{x} \in \mathbb{S}^{n-1} \mid x_n \geq 0 / x_n \leq 0\}$.

We will use these definitions later on

e.g., $\mathbb{B}^0 = \{\bar{0}\}$, $\mathbb{B}^1 = [-1, 1]$, $\mathbb{S}^0 = \{-1, 1\}$. two points

(6) There is a homeomorphism of σ (n-simplex) with $\mathbb{B}^n$ that carries $\partial\sigma$ to $\mathbb{S}^{n-1}$.

(proof in Munkres [M] EAT)



See [M] (Munkres - Elements of Algebraic Topology) for the proof.

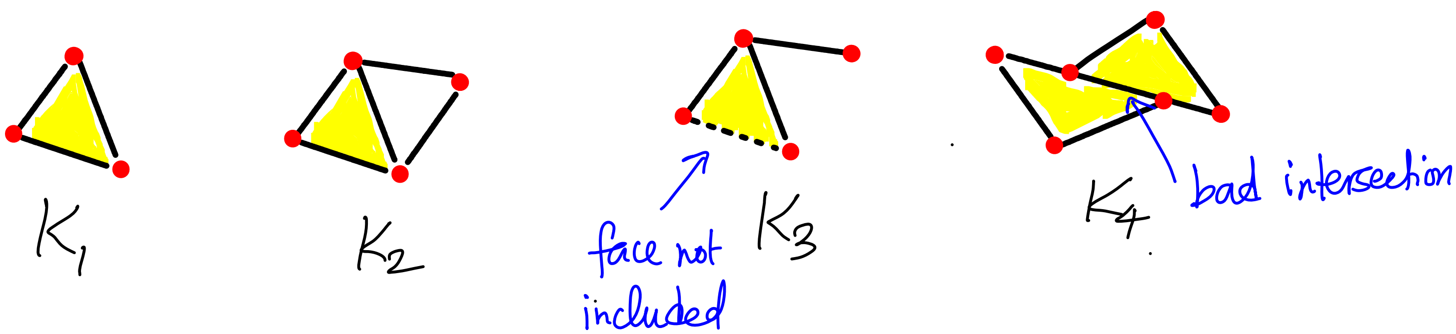
In summary, simplices are "nice" elementary objects that can be used as building blocks to build larger spaces or objects. We will now introduce these larger objects, which are quite general, but are still "nice" since we "glue" simplices together nicely to build them.

Simplicial Complexes

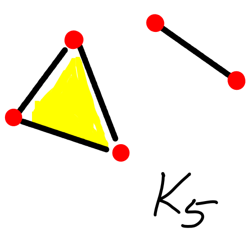
Def A simplicial complex K in $\mathbb{R}^d$ is a collection of simplices in $\mathbb{R}^d$ such that

(1) every face of a simplex in K is in K, and

(2) the intersection of any two simplices of K, when non-empty is a face of each of them.



K_1, K_2 are simplicial complexes, while K_3, K_4 are not.



K_5 is a simplicial complex – in particular, a simplicial complex need not be a single connected component.

Here is another equivalent definition:

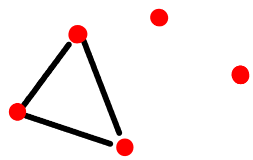
Lemma 2.1 [M] A collection of simplices K is a simplicial complex iff

(1) every face of a simplex in K is in K; and

(2) every pair of distinct simplices in K have disjoint interiors.

A simplex σ and all its proper faces together is a simplicial complex.

Def If L is a subcollection of K that contains all faces of its elements, then it is a simplicial complex on its own, called a **subcomplex** of K.

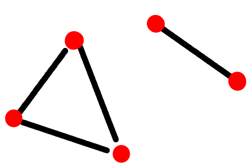


L_5.

A subcomplex of K_5

Def The subcomplex of K that is the collection of all simplices in K of dimension at most p is the **p-skeleton** of K, denoted $K^{(p)}$.

$K^{(0)}$ are the vertices of K.

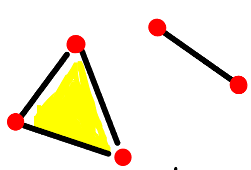


$K_5^{(1)}$ (the 1-skeleton of K_5).

Def The **dimension** of a simplicial complex K is the largest dimension of any simplex in K.

$$\dim(K) = \max_{\sigma \in K} \{\dim(\sigma)\}.$$

e.g.,



K_5

$\dim(K_5) = 2$, also referred to as a 2-complex.

A p-dimensional simplicial complex is referred to, in short, as a p-complex.

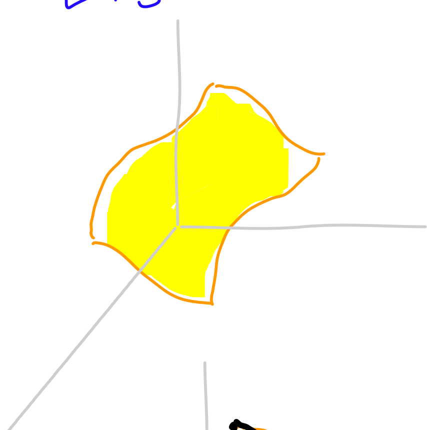
Q. What is $\dim\left(K^{(p)}\right)$? $\longrightarrow$ p-skeleton of K

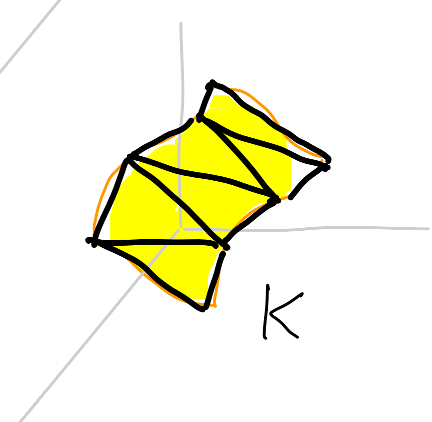
One can immediately conclude $\dim\left(K^{(p)}\right) \leq p$. But notice that $\dim\left(K^{(p)}\right)$ need not always be $= p$. For instance, $\dim\left(K_5^{(3)}\right) = 2$, since $K_5^{(3)} = K_5$ itself. But if we avoid this somewhat trivial case, $\dim\left(K^{(p)}\right) = p$, typically.

Or, more generally, $\dim\left(K^{(p)}\right) = \min\left(p, \dim(K)\right)$.

Recall that we want to use simplicial complexes as a "nice" structured way to model spaces. We now outline the somewhat subtle distinction between the simplicial complex and the (sub)space that it models.

Let's start with an illustration.





Consider a subspace of, say, $\mathbb{R}^3$ modeled by a sheet of paper. We could capture this space by a simplicial complex K consisting of six triangles.

Complementarily, if we start with K, we could talk about the subspace of $\mathbb{R}^3$ that it captures. We can specify the usual topology on this subspace (as inherited from $\mathbb{R}^3$).

Def Let $|K|$ be the subset of $\mathbb{R}^d$ which is the union of all simplices in K. Give each simplex its natural topology as a subspace of $\mathbb{R}^d$. Then we can topologize $|K|$ by declaring a subset A of $|K|$ is closed in $|K|$ if $A \cap \sigma$ is closed in K $\forall \sigma \in K$. $|K|$ is called the **underlying space** of K, or the **polytope** of K.

also referred to as "polyhedron"

Some people use the word polytope only when K is finite, i.e., it has a finite number of simplices, while using the word polyhedron more generally, i.e., even for the case where K is not finite.

In convex geometry, $P = \{\bar{x} \in \mathbb{R}^d \mid A\bar{x} \leq \bar{b}\}$ is a polyhedron, and a closed polyhedron is referred to as a polytope.

The two topologies — one as a subspace of $\mathbb{R}^d$, and the other defined using the simplices as above — need not be identical in all cases. But if K is finite, they usually concide. In fact, typical examples where they differ come from infinite simplicial complexes K.

$|K|$ topologized in two different ways: here is an example where the two topologies are different.

<u>Example</u> $K = \{ \bigcup [m, m+1] \ \forall m \in \mathbb{Z} \setminus \{0\} \} \cup$ notice that $[-1,0]$ is included

$\{ [\frac{1}{n+1}, \frac{1}{n}] \ \forall n \in \mathbb{Z}_{>0} \}$ and all faces.

K is an infinite 1-complex. infinitely many simplices

$|K| = \mathbb{R}$ as a set, but not as a topological space. Indeed,
$A = \{ \frac{1}{n}, n \in \mathbb{Z}_{>0} \}$ is closed in $|K|$, but not in $\mathbb{R}$.
A does not include 0.

But if K is finite, the topologies are the same.

Properties of $|K|$

Munkres – Elements of Algebraic Topology

<u>Lemma 2.2 [M]</u> If $L \subseteq K$ is a subcomplex, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.
$|\sigma|$ to be precise, but notice σ and $|\sigma|$ are identical!

<u>Lemma 2.3 [M]</u> A map $f : |K| \to X$ is continuous iff $f|_\sigma$ is continuous for each $\sigma \in K$.
f restricted to σ

Recall the barycentric coordinates of $\bar{x} \in \sigma$ ($t_{\bar{a}_i}(\bar{x})$ for vertices $\bar{a}_i$).
We can naturally extend the barycentric coordinates to $\bar{x} \notin \sigma$.

Def If $\bar{x} \in |K|$, then $\bar{x}$ is interior to precisely one simplex in K, whose vertices are, say, $\bar{a}_0, \dots, \bar{a}_n$. Then

$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i, \quad \text{where} \quad t_i > 0 \ \forall i, \ \sum_{i=0}^{n} t_i = 1.$$

If $\bar{v}$ is an arbitrary vertex of K, then the barycentric coordinate of $\bar{x}$ w.r.t $\bar{v}$, $t_{\bar{v}}(\bar{x})$, is defined as $t_{\bar{v}}(\bar{x}) = 0$ if $\bar{v} \notin \{\bar{a}_0, \dots, \bar{a}_n\}$, and $t_{\bar{v}}(\bar{x}) = t_i$ if $\bar{v} = \bar{a}_i$.

Notice that $t_{\bar{v}}(\bar{x})$ is continuous on $|K|$, as $t_{\bar{a}_i}(\bar{x})$ are continuous, as we noted in the last lecture, and then by Lemma 2.3.

Lemma 2.4[M] $|K|$ is Hausdorff.

A space X is Hausdorff if every pair of distinct points $\bar{x}, \bar{y} \in X$ can be surrounded by open sets $\mathcal{U}, \mathcal{V} \subseteq X$ s.t. $\bar{x} \in \mathcal{U}$, $\bar{y} \in \mathcal{V}$, $\mathcal{U} \cap \mathcal{V} = \phi$.

Proof For $\bar{x}_i \neq \bar{x}_j$ in $|K|$, by definition, there exists at least one $\bar{v}$ (vertex) s.t. $t_{\bar{v}}(\bar{x}_i) \neq t_{\bar{v}}(\bar{x}_j)$. Choose r in between $t_{\bar{v}}(\bar{x}_i)$ and $t_{\bar{v}}(\bar{x}_j)$ and define $\mathcal{U} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) < r\}$ and $\mathcal{V} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) > r\}$ as the required open sets.

We now study some important subspaces of $|K|$.

MATH 524: Lecture 4 (08/28/2025)

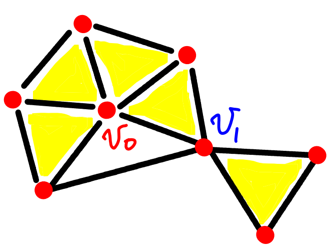
Today:
* star, closed star, link
* simplicial maps
* abstract simplicial complexes

We now study some important subspaces of $|K|$.

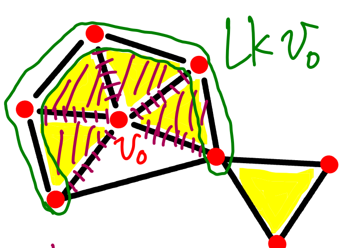
Three Subspaces of $|K|$

Def If $\bar{v}$ is a vertex of K, then the **star** of $\bar{v}$ in K, denoted $\mathrm{St}\,\bar{v}$ (or $\mathrm{St}(\bar{v}, K)$) is the union of the interiors of all simplices in K that contain $\bar{v}$ as a vertex. The closure of $\mathrm{St}\,\bar{v}$, denoted $\overline{\mathrm{St}\,\bar{v}}$ or $\mathrm{Cl}\,\mathrm{St}\,\bar{v}$, is the **closed star** of $\bar{v}$. It is the union of all simplices of K which have $\bar{v}$ as a vertex. $\mathrm{Cl}\,\mathrm{St}\,\bar{v}$ is a polytope of a subcomplex of K. $\mathrm{Cl}\,\mathrm{St}\,\bar{v} - \mathrm{St}\,\bar{v}$ is called the **link** of $\bar{v}$, denoted $\mathrm{Lk}\,\bar{v}$.

K_6

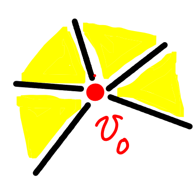


We illustrate these subcomplexes on K_6 for vertices v_0 and v_1. Note that the unshaded triangle below v_0 is not part of K_6.

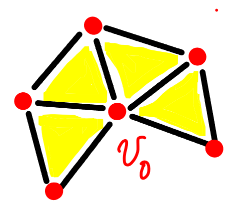


$\mathrm{St}\,v_0$

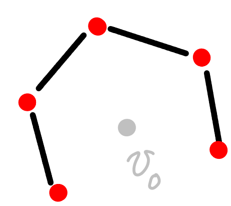
add to get $\mathrm{Cl}\,\mathrm{St}\,v_0$



$\mathrm{St}\,v_0$



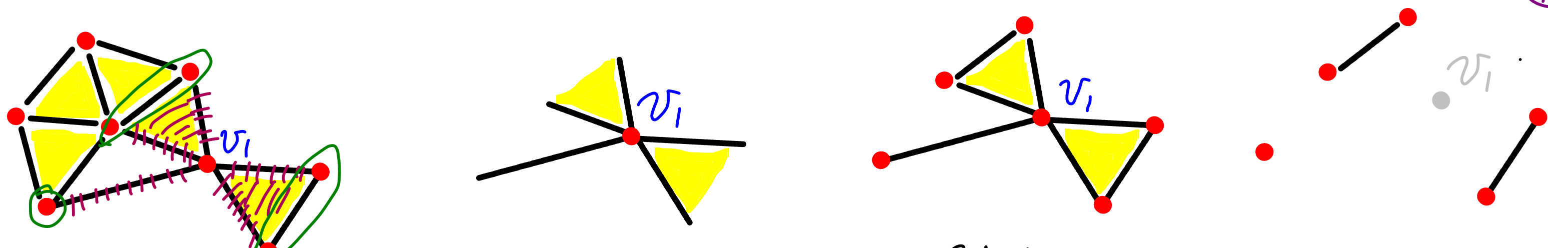
$\mathrm{Cl}\,\mathrm{St}\,v_0$



$\mathrm{Lk}\,v_0$

Note that $\mathrm{Lk}\,v_0 = \mathrm{Cl}\,\mathrm{St}\,v_0 - \mathrm{St}\,v_0$.

Also note that $v_0 \in \mathrm{St}\,v_0$ (indeed, $\mathrm{Int}\,v_0 = v_0$, and v_0 is a simplex that contains v_0 as a vertex, trivially).



$St\, v_1$ $\quad$ $st\, v_1$ $\quad$ $ClSt\, v_1$ $\quad$ $Lk\, v_1$

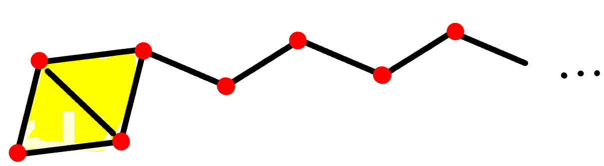
Properties of star, closed star, link

* $St\, \bar{v}$ is open in $|K|$. → We could use $t_{\bar{v}}(\cdot)$ to prove.
* The complement of $St\, \bar{v}$ is the union of all simplices that do not contain $\bar{v}$ as a vertex, and hence it is the polytope of a subcomplex of K.
* $Lk\, \bar{v}$ is the polytope of a subcomplex of K.
* $Lk\, \bar{v} = Cl\, St\, \bar{v} \cap (\text{complement of } St\, \bar{v})$.
* $St\, \bar{v}$ and $Cl\, St\, \bar{v}$ are both path-connected.
  X is path-connected if $\forall \bar{u}, \bar{v} \in X, \bar{u} \neq \bar{v}$, $\exists$ a path connecting $\bar{u}$ and $\bar{v}$ in X.
* $Lk\, \bar{v}$ need not be connected.

Def A simplicial complex K is **locally finite** if each vertex of K belongs to only finitely many simplices of K.
Equivalently, K is locally finite iff each closed star is the polytope of a finite subcomplex of K.

Note: A locally finite simplicial complex could be infinite, e.g., K_7.

K_7



(the edges continue forever)

Simplicial Maps

We study maps between simplicial complexes as a first step toward developing the tools to compare spaces modeled by the simplicial complexes.

Def Let K, L be simplicial complexes. A function $f: |K| \to |L|$ is a (linear) **simplicial map** if it takes simplices of K linearly onto onto simplices of L. In other words, if $\sigma \in K$, then $f(\sigma) \in L$.

linearly: If $\sigma = \text{conv}\{\bar{v}_0, \ldots, \bar{v}_n\}$ and $\bar{x} = \sum_{i=0}^{n} t_i \bar{v}_i$, $t_i \geq 0$, $\sum_{i=0}^{n} t_i = 1$, then $f(\bar{x}) = \sum_{i=0}^{n} t_i f(\bar{v}_i)$.

Note that $\{f(\bar{v}_0), \ldots, f(\bar{v}_n)\}$ span a simplex τ of L, which could be of a lower dimension than σ.

Munkres takes a slightly different approach in defining simplicial maps.

[M]: Starts with $f: K^{(0)} \to L^{(0)}$, then insist that when $\{\bar{v}_0, \ldots, \bar{v}_n\}$ span $\sigma \in K$, $\{f(\bar{v}_0), \ldots f(\bar{v}_n)\}$ span $\tau \in L$.

f is a continuous map of σ onto τ, and hence as a map of σ onto $|L|$. Then by Lemma 2.3, it is a continuous map from $|K|$ to $|L|$.

If $g: |K| \to |L|$ and $h: |L| \to |M|$ are simplicial maps, then $f = h \circ g$ is a simplicial map from $|K|$ to $|M|$.

If we further insist that $f: K^{(0)} \to L^{(0)}$ is a **bijective** correspondence such that vertices $\bar{v}_0, \ldots, \bar{v}_n$ of K span a simplex of K iff $f(\bar{v}_0), \ldots, f(\bar{v}_n)$ span a simplex of L, then the induced simplicial map $g: |K| \to |L|$ is a homeomorphism. We call this map an **isomorphism** of K with L (or a simplicial homeomorphism).

Abstract Simplicial Complexes (ASCs)

Def An abstract simplicial complex (ASC) is a collection $\mathcal{S}$ of finite nonempty sets such that if $A \in \mathcal{S}$, then so is every nonempty subset of A.

Note: $\mathcal{S}$ itself could be infinite, but each $A \in \mathcal{S}$ is finite.

Example: $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ is an ASC.

We specify several more definitions related to ASCs.

Def A (any element of $\mathcal{S}$) is a **simplex** of $\mathcal{S}$. Its **dimension** is given as $\dim(A) = |A| - 1$.

↳ # elements in A, or size of A

The **dimension of the ASC** is defined as follows.

$\dim(\mathcal{S})$ = largest dimension of any simplex in $\mathcal{S}$, or ∞ if no such largest dimension exists.

The **vertex set** V of $\mathcal{S}$ (or $V(\mathcal{S})$) is the union of all singleton elements (simplices) of $\mathcal{S}$. We do not distinguish between the individual vertices and the singleton sets they represent.

$$v_0 \text{ (vertex)} \equiv \{v_0\} \quad \text{0-simplex of } \mathcal{S}.$$

A subcollection of $\mathcal{S}$ that is a simplicial complex by itself is a **subcomplex** of $\mathcal{S}$.

We can now talk about when two ASCs are "similar".

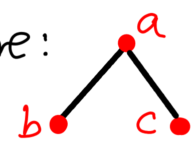
Def Two ASCs $\mathcal{S}$ and $\mathcal{T}$ are **isomorphic** if there exists a bijective correspondence f mapping $V(\mathcal{S})$ to $V(\mathcal{T})$ such that $\{a_0, \dots, a_n\} \in \mathcal{S}$ iff $\{f(a_0), \dots, f(a_n)\} \in \mathcal{T}$.

e.g., With $\mathcal{T} = \{\{d\}, \{e\}, \{f\}, \{d,e\}, \{d,f\}\}$, $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.

It turns out the previous notion of simplicial complexes (in $\mathbb{R}^d$) and ASC are directly related.

Def Let K be a (geometric) simplicial complex. Let V be its vertex set. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, \dots, a_n\}$ of V such that $\bar{a}_0, \dots, \bar{a}_n$ span a simplex of K. Then $\mathcal{K}$ is an ASC called the **vertex scheme** of K. Symmetrically, we call K a **geometric realization** of $\mathcal{K}$.

e.g., (continued) $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ has a geometric realization S as shown here:



This complex could be sitting in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Theorem 3.1 [M] (a) Every ASC $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K.

A version of this result is given as the **geometric realization theorem** which states that every abstract d-complex has a geometric realization in $\mathbb{R}^{2d+1}$.

IDEA: If $\dim(\mathcal{S}) = d$ then let $f: V(\mathcal{S}) \to \mathbb{R}^{2d+1}$ be an injective function whose image is a set of G.I. points in $\mathbb{R}^{2d+1}$. Specify that for each abstract simplex $\{a_0, \dots, a_n\} \in \mathcal{S}$, $\{f(a_0), \dots, f(a_n)\} \in K$. Then $\mathcal{S}$ is isomorphic to the vertex scheme of K.