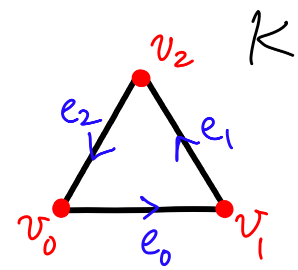
MATH 524 – Lecture 28 (11/30/2023)

Today: * Cohomology of $\mathbb{K}^2$

Example 3 (continued...)

Consider the 1-cochain $\psi^1 = \sum_{i=0}^{2} m_i e_i^*$. It is a cocycle (trivially), as there are no 2-cochains. We show that $\psi^1 \underset{c}{\sim}$ some multiple of e_0^*.



We show $e_1^* \underset{c}{\sim} e_0^*$ and $e_2^* \underset{c}{\sim} e_0^*$.

But we get these results from $\delta v_0^* = e_2^* - e_0^*$ and $\delta v_1^* = e_0^* - e_1^*$.

$\Rightarrow \psi^1 \underset{c}{\sim} m e_0^*$ for some $m \in \mathbb{Z}$, $m \neq 0$.

$m e_0^*$ is not a coboundary unless $m=0$.

Suppose $m e_0^* = \delta\left(\sum_{i=0}^{2} n_i' v_i^*\right) = \sum_{i=0}^{2} n_i' (\delta v_i^*)$

$$= \underbrace{(n_1' - n_0')}_{=0} e_0^* + \underbrace{(n_2' - n_1')}_{=0} e_1^* + \underbrace{(n_0' - n_2')}_{=0} e_2^* \longrightarrow \text{needed.}$$

$\Rightarrow n_0' = n_1' = n_2'$. $\Rightarrow m = 0$ if $m e_i^*$ is a coboundary.

Hence we conclude that $H^1(K) \simeq \mathbb{Z}$, and is generated by $\{e_0^*\}$, or by $\{e_1^*\}$ or $\{e_2^*\}$.

Here $H^i(K) \simeq H_i(K)\ \forall i$ (they are both trivial for $i \geq 2$).

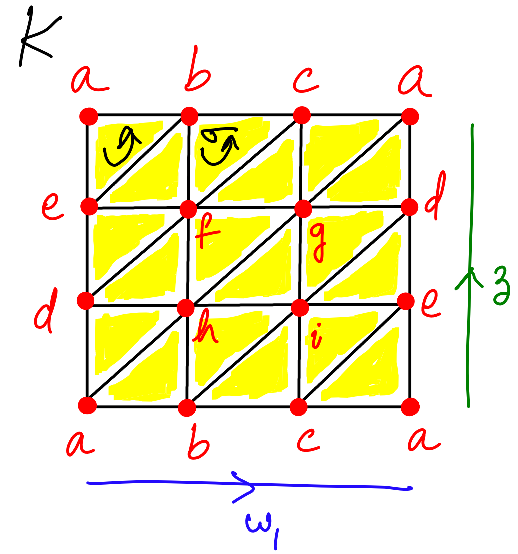
But in general, $H^i(K) \not\simeq H_i(K)$.

Here, $H^1(K) \simeq H_0(K)$ and $H^0(K) \simeq H_1(K)$, actually.

Example 4 (Klein bottle)

We show that $H^2(K)$ is nontrivial.

Recall, $H_2(K)=0$.



Orient all triangles CCW. Let $\bar{\gamma} = \sum f_i$ (all elementary 2-chains).

Then, $\bar{\gamma}$ is not a 2-cycle.

$$\partial \bar{\gamma} = 2\bar{z}_1, \quad \text{where} \quad \bar{z}_1 = [a,e] + [e,d] + [d,a].$$

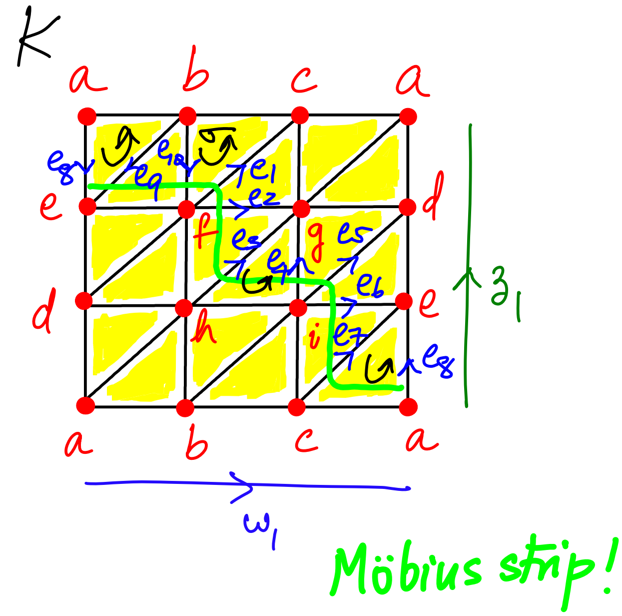
Let σ be a 2-simplex, $[bfc]$ here. Then σ^* is a 2-cocycle (as there are no 3-simplices). Also, σ^* is not a 2-coboundary.

For, if ϕ^1 is an arbitrary 1-cochain, then

$$\langle \delta\phi^1, \bar{\gamma} \rangle = \langle \phi^1, \partial\bar{\gamma} \rangle = \langle \phi^1, 2\bar{z}_1 \rangle = 2\underbrace{\langle \phi^1, \bar{z}_1 \rangle}_{\text{even integer}}.$$

But $\langle \sigma^*, \bar{\gamma} \rangle = 1$, which is odd.

$\Rightarrow$ σ^* represents a nontrivial member of $H^2(K)$.



In fact, σ^* represents an element of order 2 in $H^2(K)$.

Indeed, for $\psi^1 = \sum_{i=1}^{10} e_i^*$ as shown in the figure,

$$\delta\psi^1 = 2\sigma^*.$$

The CCW orientation of σ agrees with that of both $\bar{e}_1$ and e_{10}. But all other triangles in the "band" appear twice in the expression for $\delta\psi^1$, once with +1 and once with −1 (as part of δe_i^* and δe_{i+1}^* ...).

Thus, for the Klein bottle, $H^2(K;\mathbb{Z}) \not\cong H_2(K)$, which is yet another example where homology and cohomology groups differ in their structure.

In fact, we can show that $H^2(\mathbb{K}^2;\mathbb{Z}) \simeq \mathbb{Z}/2$.