MATH 401: Lecture 22 (10/30/2025)

Today: * pointwise and uniform convergence

Recall: uniform continuity uses same δ for continuity at all $x \in X$.

We now generalize the definition of continuity to use the same δ for a collection of functions (and hence for all points in the domain for each function in the collection).

Def 4.1.3 Let (X, d_X), (Y, d_Y) be metric spaces. $\mathcal{F}$, a collection of functions $f: X \to Y$, is **equicontinuous** if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall f \in \mathcal{F}$, $\forall x, y \in X$ with $d_X(x,y) < \delta$ we have $d_Y(f(x), f(y)) < \epsilon$.

same δ for all $f \in \mathcal{F}$

Problem 3, LSIRA pg 80 $f: X \to Y$ is Lipschitz continuous with Lipschitz constant K if $d_Y(f(x), f(y)) \le K\, d_X(x,y)$ $\forall x, y \in X$. Assume $\mathcal{F}$ is a collection of functions $f: X \to Y$ that are Lipschitz continuous with the same Lipshitz constant K. Show that $\mathcal{F}$ is equicontinuous.

Want to show: $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x,y) < \delta$ $\forall f \in \mathcal{F}$ $\forall x, y \in X$.

Given: $\forall f \in \mathcal{F}$, f is Lipschitz continuous with Lipschitz constant K

$$\Rightarrow d_Y(f(x), f(y)) \le K\, d_X(x,y).$$

Only $K > 0$ makes sense here ($K = 0$ gives result trivially).

Choose $\delta = \dfrac{\epsilon}{K}$, and we get $d_Y(f(x), f(y)) < \epsilon$. $\square$

LSIRA 4.2 Modes of Convergence for Functions

Similar to how we generalized modes of continuity, we now generalize notions of convergence.

Def 4.2.1 Let $(X, d_X), (Y, d_Y)$ be metric spaces, and $\{f_n\}$ be a sequence of functions $f_n : X \to Y$. We say that $\{f_n\}$ **converges pointwise** to $f: X \to Y$ if $f_n(x)$ converges to $f(x)$ $\forall x \in X$.
This means $\forall \epsilon > 0, x \in X, \exists N_x \in \mathbb{N}$ s.t. $d_Y(f_n(x), f(x)) < \epsilon$ whenever $n \geq N_x$.

N_x depends on x

We get uniform convergence if the same N can be used $\forall x \in X$.

x

Def 4.2.2 Let $(X, d_X), (Y, d_Y)$ be metric spaces, and $\{f_n\}$ be a sequence of functions $f_n : X \to Y$. We say that $\{f_n\}$ **converges uniformly** to $f: X \to Y$ if $\forall \epsilon > 0, x \in X, \exists N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon$ whenever $n \geq N$.

$\hookrightarrow$ N is independent of $x \in X$.

Let's do a problem on pointwise v/s uniform convergence.

Problem 1, LSIRA pg 85 Let $f_n(x) = \frac{x}{n}$. Show $\{f_n\}$ converges pointwise, but not uniformly to 0.

Pointwise Convergence

Need to show: $\forall x, \epsilon > 0 \quad \exists N_x \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \epsilon$ when $n \geq N_x$.

But what is $f(x)$?

For a given x, $\lim_{n \to \infty} \frac{x}{n} = 0 \Rightarrow f(x) = 0$.

$\Rightarrow$ We want $|f_n(x)| = \left|\frac{x}{n}\right| = \frac{|x|}{n} < \epsilon. \Rightarrow N_x > \frac{|x|}{\epsilon}$ works.

Choose $N_x = \left\lceil \frac{|x|}{\epsilon} \right\rceil + 1$, for instance.

Not converging uniformly: Show that $\forall N \in \mathbb{N}$, and given $\epsilon > 0$, $\exists x$ s.t. $|f_N(x)| > \epsilon$. → thus violating the definition of uniform convergence

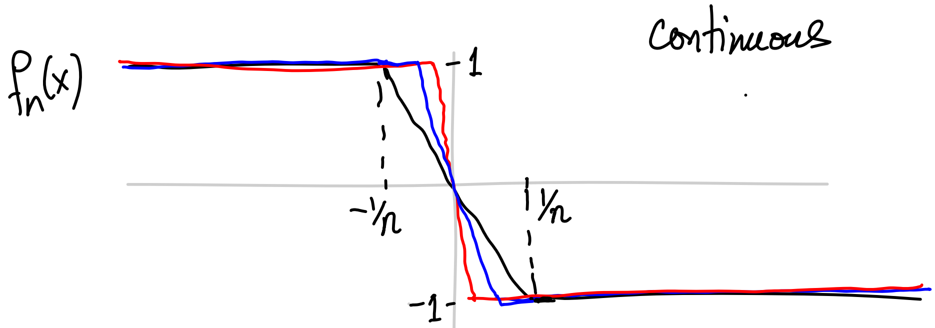
Pick $\epsilon = 1$. For any $N \in \mathbb{N}$, pick $x > N$. Then

$$f_N(x) = \frac{x}{N} > 1 = \epsilon.$$

$\square$

We can have a sequence $\{f_n\}$ of continuous functions converging pointwise to function f that is not continuous!

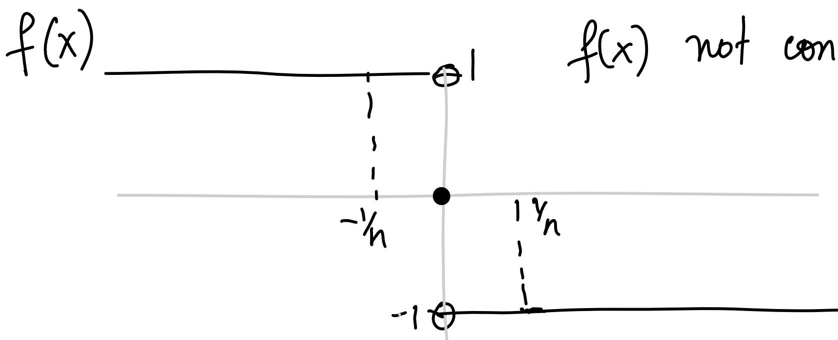
Consider the sequence of functions shown below for $n \in \mathbb{N}$. Note that each $f_n(x)$ is continuous.



Continuous

$$f_n(x) = \begin{cases} 1, & x \leq -1/n \\ -nx, & -1/n < x < \frac{1}{n} \\ -1, & x \geq \frac{1}{n} \end{cases}$$

$\{f_n\}$ converges pointwise to the following function $f(x)$, which has a discontinuity at $x=0$!



$f(x)$ not continuous at $x=0$!

$$f(x) = \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ -1, & x > 0 \end{cases}$$

But if we insist on uniform convergence, then continuity is preserved.

Proposition 4.2.4 Let (X, d_X), (Y, d_Y) be metric spaces, and $\{f_n\}$ be a sequence of continuous functions $f_n: X \to Y$ converges uniformly to $f: X \to Y$. Then f is continuous.

Proof: See LSIRA. Uses triangle inequality & $\frac{\epsilon}{3}$-technique.

Problem 5, LSIRA pg 85 Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and the sequence $\{f_n\}$ converges uniformly to $f : \mathbb{R} \to \mathbb{R}$ on all intervals $[-k, k]$, $k \in \mathbb{N}$. Show that $f(x)$ is continuous.

Need to show: f is continuous at all $x \in \mathbb{R}$.

Note that $\forall x \in \mathbb{R}$, $\exists\, k \in \mathbb{N}$ s.t. $x \in [-k, k]$.

By Proposition 4.2.4, $f(x)$ is continuous on $[-k, k]$, and hence at x.
But x is arbitrary, and hence $f(x)$ is continuous. $\square$

We now present a way to show uniform convergence by presenting an iff characterization of the same

Proposition 4.2.3 Let (X, d_X), (Y, d_Y) be metric spaces, and $\{f_n\}$ be a sequence of functions $f_n : X \to Y$, with $f : X \to Y$ being another function. The following statements are equivalent.

(i) $\{f_n\}$ converges uniformly to f.

(ii) $\sup \{ d_Y(f_n(x), f(x)) \mid x \in X \} \longrightarrow 0$ as $n \to \infty$.

In words, unif. convergence $\iff$ "max distance" between f_n and $f \to 0$.
This result gives us a way to show uniform convergence.

Problem 3, LSIRA pg 85 Let $f_n : [0,\infty) \to \mathbb{R}$ be defined as $f_n(x) = e^{-x}\left(\frac{x}{n}\right)^{ne}$.

a) Show $\{f_n\}$ converges pointwise.

b) Find maximum of $f_n(x)$. Does $\{f_n\}$ converge uniformly?

a) Pointwise convergence to $f(x)$. what is $f(x)$?

$x \in [0,\infty)$, $\epsilon > 0$ are given.

As $n \to \infty$, $\frac{x}{n} \to 0$ and $\left(\frac{x}{n}\right)^n \to 0$ if $\frac{x}{n} < 1$.

$\Rightarrow f(x) = 0$.

Need to show $\forall x \geq 0, \epsilon > 0$, $\exists N_x \in \mathbb{N}$ s.t. $|f_n(x)| < \epsilon$ $\forall n \geq N_x$.

Note: when $n > x$, $\frac{x}{n} < 1 \Rightarrow \left(\frac{x}{n}\right)^{ne} < \left(\frac{x}{n}\right)$.

$\Rightarrow$ Choose n s.t. $\frac{x}{n} < \min(1, e^x\epsilon)$ to get

$$e^{-x}\left(\frac{x}{n}\right)^{ne} < e^{-x}\left(\frac{x}{n}\right) < e^{-x}e^x\epsilon = \epsilon.$$

$$\Rightarrow N_x = \left\lceil \frac{x}{\min(1, e^x\epsilon)} \right\rceil + 1 \text{ will do.}$$

(b) To find $\max\{f_n(x)\}$, solve $f_n'(x) = 0$.

$f_n(x) = e^{-x}\left(\frac{x}{n}\right)^{ne}$ product rule!

$$\Rightarrow f_n'(x) = e^{-x}\cancel{n}e\left(\frac{x}{n}\right)^{(ne-1)}\left(\frac{1}{\cancel{n}}\right) - e^{-x}\left(\frac{x}{n}\right)^{ne}$$

$$= e^{-x}\left(\frac{x}{n}\right)^{ne-1}\left(e - \frac{x}{n}\right) = 0 \Rightarrow x = 0,\ x = ne.$$

$$f_n(x)\big|_{x=0}=0 \quad , \quad f_n(x)\big|_{x=ne} = e^{-ne}\left(\frac{\cancel{n}e}{\cancel{n}}\right)^{ne} = 1.$$

Check $f''(x)\big|_{x=ne}<0$ to ensure it is a local maximum.

$$f''(x) = \underbrace{e^{-x}\left(\frac{x}{n}\right)^{ne-1}\left(-\frac{1}{n}\right)}_{<0 \text{ at } x=ne} + \overbrace{\cancel{\left(e-\frac{x}{n}\right)}}^{=0 \text{ at } x=ne}\left(e^{-x}(ne-1)\left(\frac{x}{n}\right)^{ne-2}\cdot\frac{1}{n} - e^{-x}\left(\frac{x}{n}\right)^{ne-1}\right)$$

$\Rightarrow \quad \max f_n(x) = 1.$

$\Rightarrow \quad \sup\{|f_n(x)-\overset{=0}{f(x)}|\} = 1 \not\to 0 \quad$ as $n\to\infty$

Hence by Proposition 4.2.3, $\{f_n\}$ does not converge uniformly.