MATH 529: Lecture 1 (01/13/2026)

Today: * syllabus, logistics
* two motivating applications

Call me Bala!

Introduction to **Computational** topology → focus for this course

This course will be offered completely electronically:
- scribes will be posted as "course notes"; videos will also be posted.
- assignments to be turned in electronically (you could submit scanned versions of handwritten assignments).
- web page has all the docs/info.

Topology

"Topo" → place or space
"logos" → study or talk
} in Greek

Topology talks about how space is connected.

topology
- point set topology (open/closed, connected, ...)
- algebraic topology (groups, addition, basis, ...)

We will concentrate on algebraic topology.

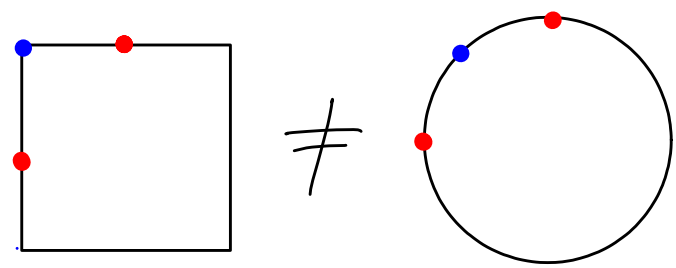
Computational topology

combine efficient algorithms and data structures with results from topology to analyze real-life data.

Let's start with an intuition for what we mean by connectivity of spaces.

An Example

According to geometry, the square and circle are not equal.

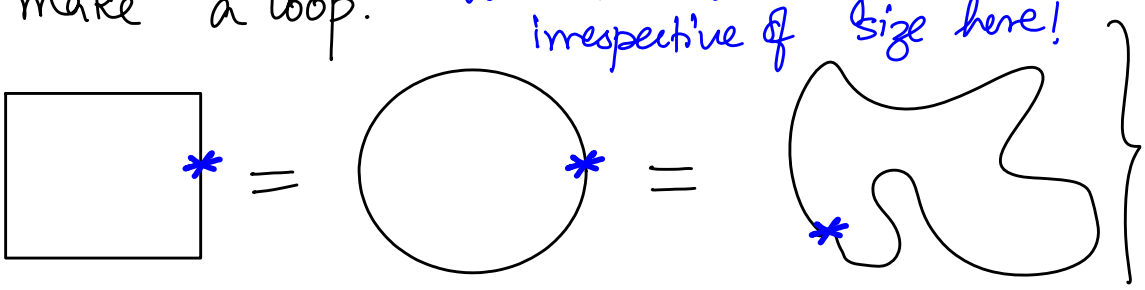


Size (length, here) is critical in geometry, but not so much in topology.

But topology says they are same as far as how they are connected!

For instance, take a string and tie a knot to make a loop.

We want to study connectivity irrespective of size here!

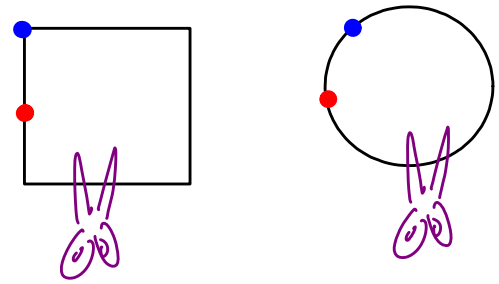


All these objects are same, i.e., they are connected the same way.

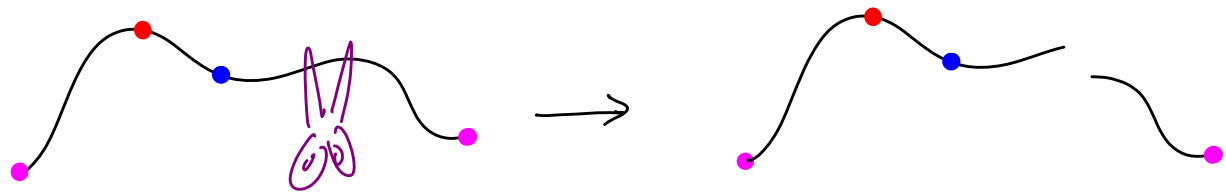
But, if you did not tie the knot, the loose string (open thread) differs from any of the above tied loops in connectivity.

Note that the two end points have different "neighborhoods."

Here is another way to understand connectivity. Consider cutting the string (tied into a loop) once. Such a cut leaves the string in one piece, i.e., connected.

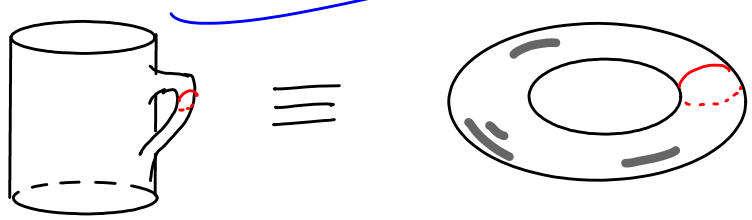


But cutting the open thread once leaves two pieces, i.e., it is disconnected.



two pieces

A popular quote: "A topologist cannot distinguish the coffee cup from a donut!"



the coffee cup, if made of clay, could be deformed to look like the donut, without changing its connectivity.

A more practical example:
how we are able to read (recognize) letters of the alphabet in different fonts.

A **A** $\mathcal{A}$ $\neq$ B **B** $\mathcal{B}$ $\neq$ C $\mathcal{C}$ **C**

Two Illustrations of Computational Topology

1. Patient antibiotic trajectories

https://doi.org/10.1145/3307339.3342143

? How do agents and doses affect length of stay?

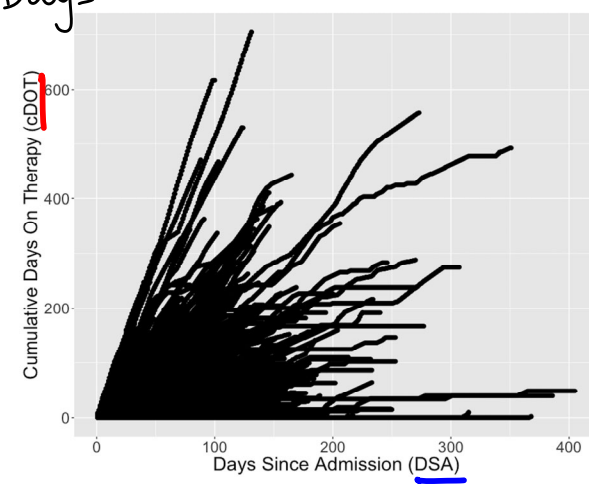
Number of hospitals	25
Number of hospital unit-categories	9
Number of distinct patient-admission records	349, 610
Number of adult patients	334, 207
Number of male patients	148, 540
Number of female patients	201, 052
Average LOS per admission	7 days
Longest LOS → length of stay	405 days
Number of antibacterials used	66
Most used antibacterial	Vancomycin
Average DOT per admission	6
Number of agent ranks	4
Most used agent rank	rank 3

→ Days On Therapy

If a patient gets one dose of an agent (antibiotic) that is counted as 1 Day On Therapy (DOT).

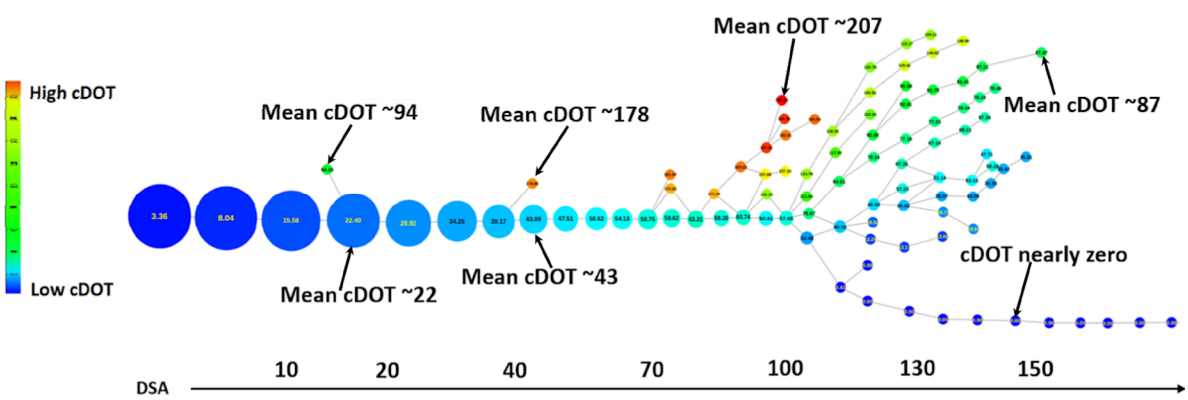
cDOT : cumulative Days On Therapy

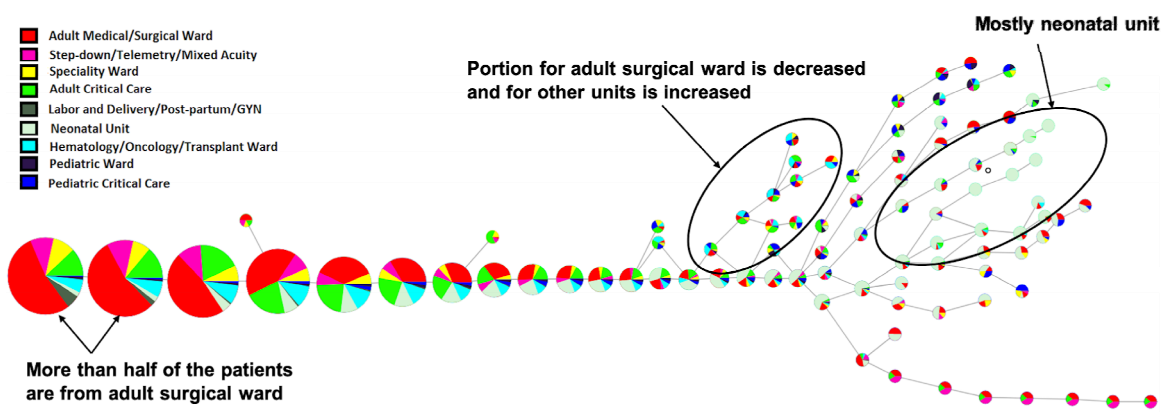
DSA : Days Since Admission.



This plot by itself is not very informative or insightful (even if we were to use color...)

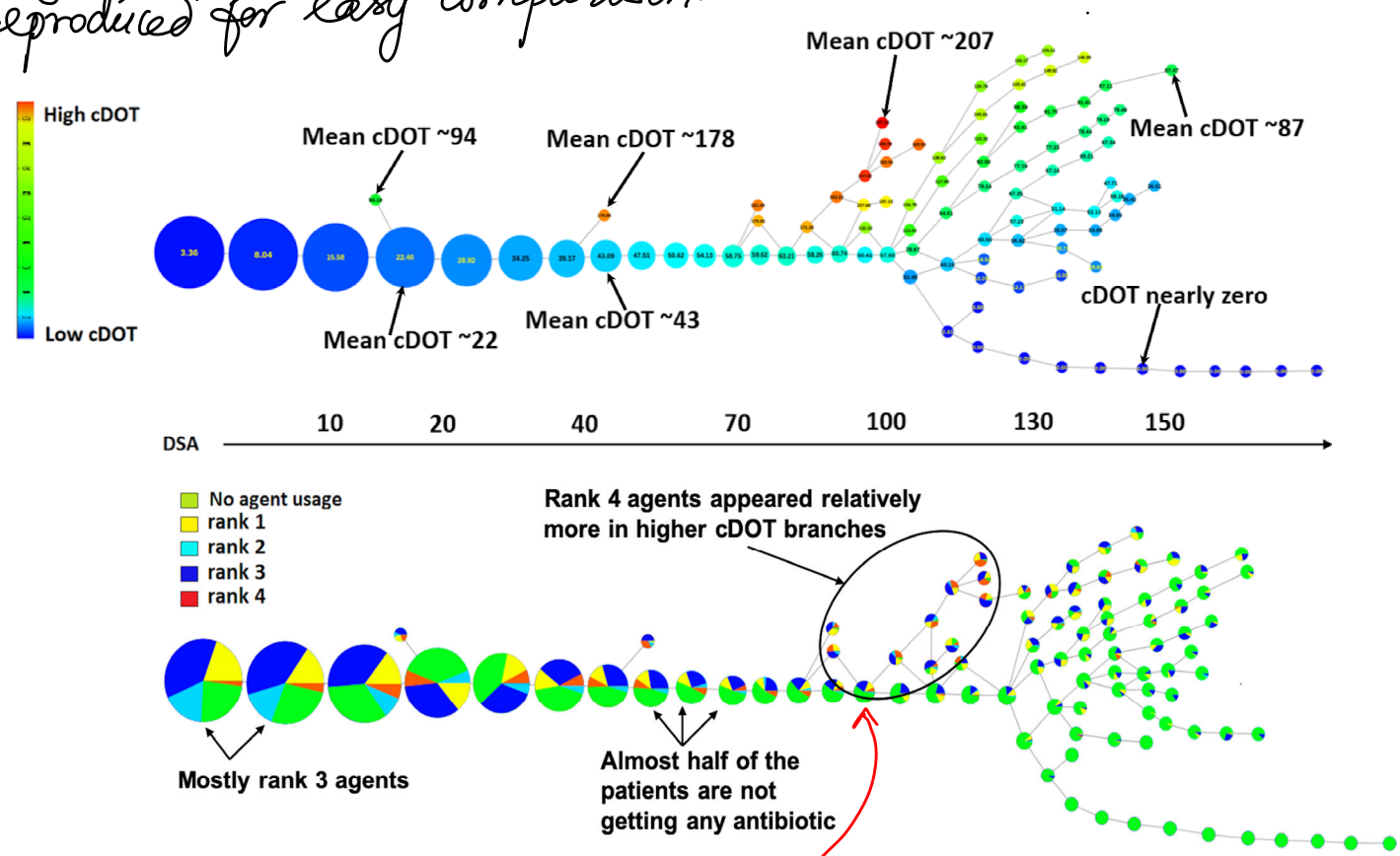
Here are two versions of a Mapper representation of the same data:





Each node represents a cluster of patient trajectories. A lot of the patients got low cDOTs and they had small(er) DSAs – as captured by the big clusters on the left. As the second Mapper shows, many of these patients were treated in the adult surgical ward — one of the most common types of admissions to hospitals.

Here is another version of the Mapper showing ranks (1–4, 4 is strongest) of the antibiotics. The first mapper (using cDOT) is reproduced for easy comparison.

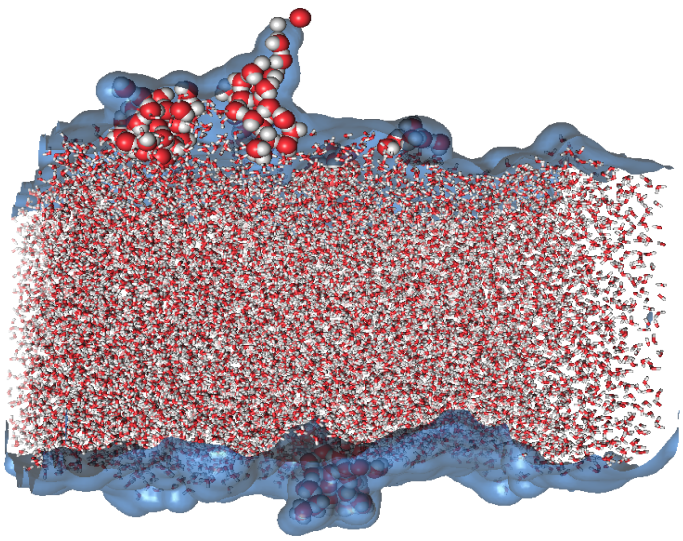


The high cDOT + high rank sub-branches had more patients in the other (higher risk) wards. Similarly, the much higher DSA group (120+) on the right end with relatively smaller cDOT values turned out to be patients in neonatal ICUs.

Note that these nontrivial subgroups are identified in an unsupervised manner — no learning is involved!

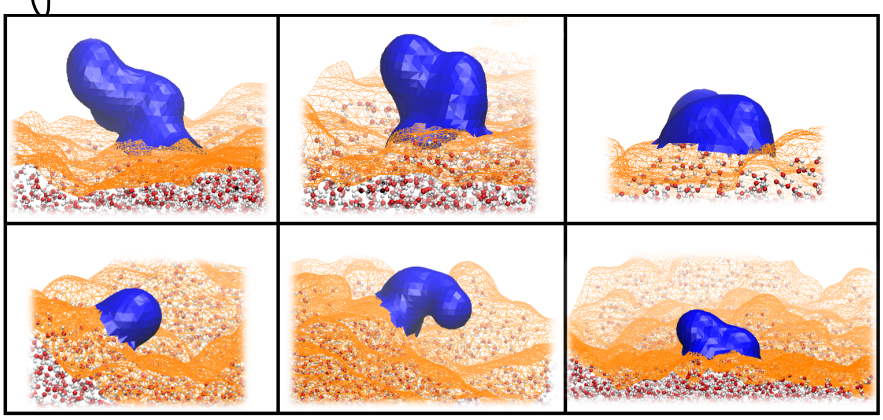
2. Interface features in Chemistry

https://doi.org/10.1021/acs.jctc.0c00260 (https://doi.org/10.26434/chemrxiv.11988048.v1) → preprint



An interface surface separates a water layer from a hexane (organic) layer. When a reagent is added, the reaction is initiated and the water molecules escape to the hexane layer through finger-like features in the interface called "protrusions". These features were identified manually (by observation!)

The goal was to identify and characterize protrusions using geometric measure theory and computational topology.



Which of these six features do you think are protrusions?

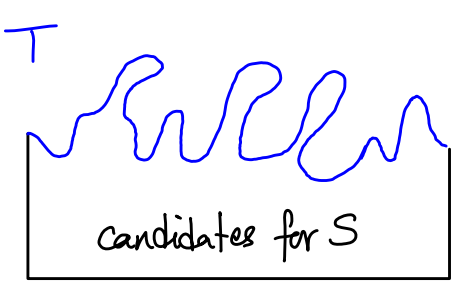
It is not easy to guess!

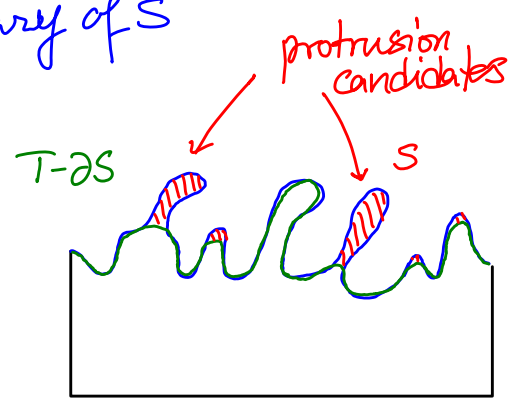
We use the notion of multiscale flat norm of surface T:

$$F_\lambda(T) = \min_S \{\text{Area}(T - \partial S) + \lambda \, \text{Volume}(S)\}, \quad \lambda \geq 0$$

S → 3D volume; ∂S → boundary of S; λ → scale parameter

Illustration in 2D:



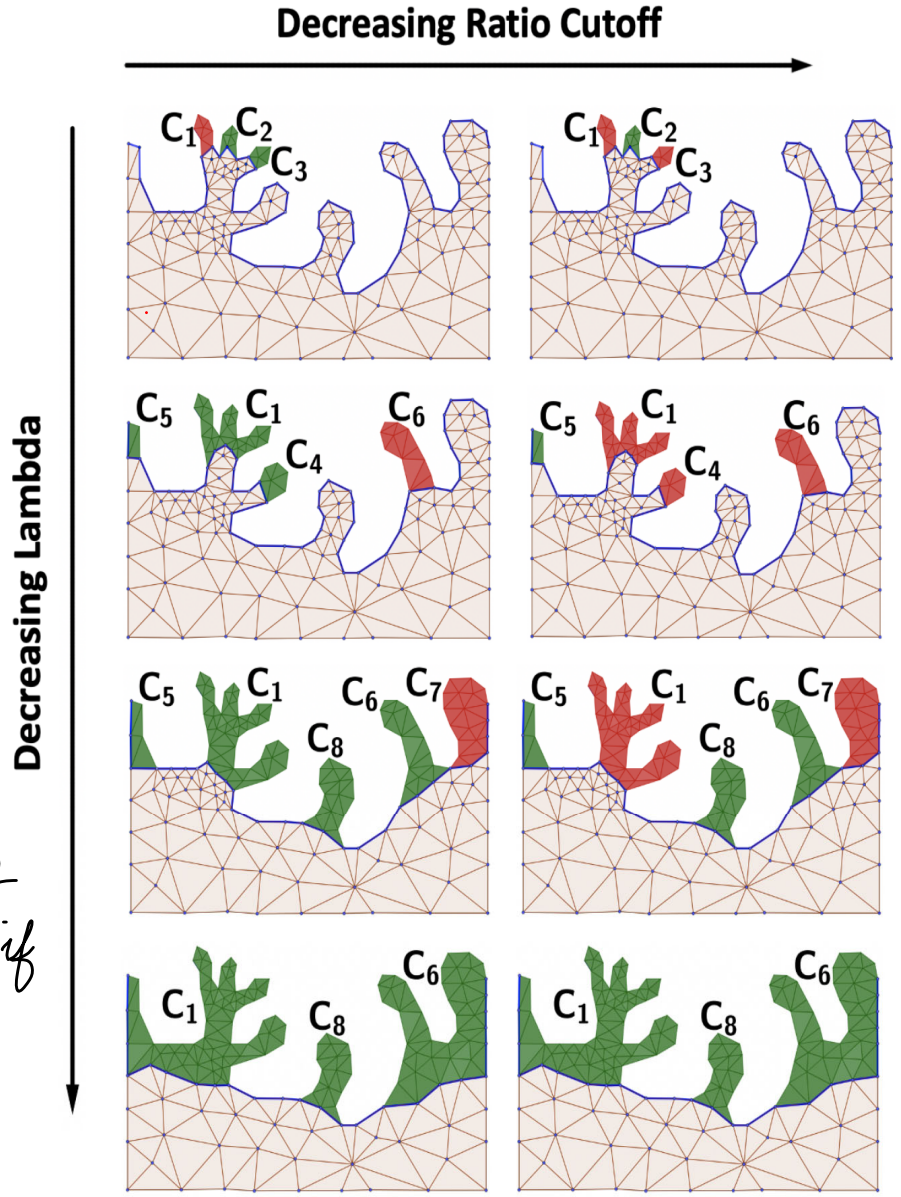


We keep track of connected components in S as $\lambda \downarrow$. We relabel them and also keep track of merging behavior.

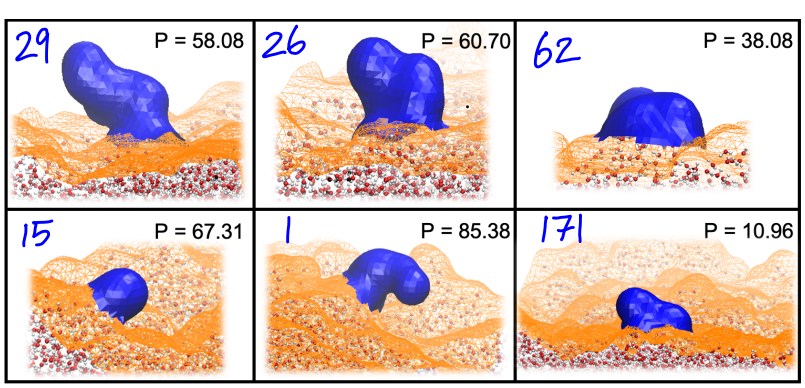
We also track the ratio $\frac{Vol(C)}{Vol(B(1/\lambda))}$ for each component C, where $B(1/\lambda)$ is the ball with radius $1/\lambda$ and $Vol(C)$ is the volume of component C.

A component C is "alive" at ratio cutoff r and scale λ if

$$\frac{Vol(C)}{Vol(B(1/\lambda))} > r.$$



The longer a component is alive, the more likely it is to be a protrusion.



It turned out that all six of these features were protrusions! The probabilities (as %'s) along with their ranks among 195 candidate features (lower rank $\Rightarrow$ more likely to be a protrusion) are shown here.

While this example is described in 3D, the underlying concepts are more general, and in fact generate certain key fundamental questions in geometric measure theory (GMT).

In fact, when we talk about applied algebraic topology, the application could be to pure mathematics! We will talk about this aspect toward the end of the semester.

Note that we are showing a discrete version of the surface – in the form of a triangular mesh. Indeed, we need to discretize continuous spaces to perform computations!

Here is a notion of Connectivity in the "discrete setting":

The neighborhood, i.e., the set of nearby points, of the two • points are different – they each have only one neighbor, while the • points all have two neighbors each.

MATH 529: Lecture 2 (01/15/2026)

Today: * topology, open/closed sets
* homeomorphism, examples

We define topology as a mathematical method to define and study how a space is connected.

Notation For a set X, we denote by 2^X the power set of X, which is the set of all subsets of X.

Def A **topology** on a set X is a subset T of 2^X such that the following conditions hold.

1. $A_1, A_2 \in T \Rightarrow A_1 \cap A_2 \in T$ (finite intersections)
2. $\{A_j \mid j \in J\} \in T \Rightarrow \bigcup_{j \in J} A_j \in T$ (infinite unions) or finite

   index set
   infinite or finite
3. $\emptyset, X \in T$ empty set

(X, T) is a topological space, denoted $\mathbb{X}$ (T is understood from context).

$A \in T$ is an **open set** of $\mathbb{X}$.

The complement of A, i.e., $X - A$ (or $X \setminus A$) is a **closed set**. "set minus"

Some sets can be both open and closed at the same time, e.g., $\emptyset, X$ are both open and closed in any topology.

We typically specify a topology by specifying its open sets.

interior $\mathring{A}$ of $A \subseteq \mathbb{X}$: $\mathring{A} = \bigcup$ (open sets contained in A)

union

closure $\bar{A}$ of $A \subseteq \mathbb{X}$: $\bar{A} = \bigcap$ (closed sets containing A).

intersection

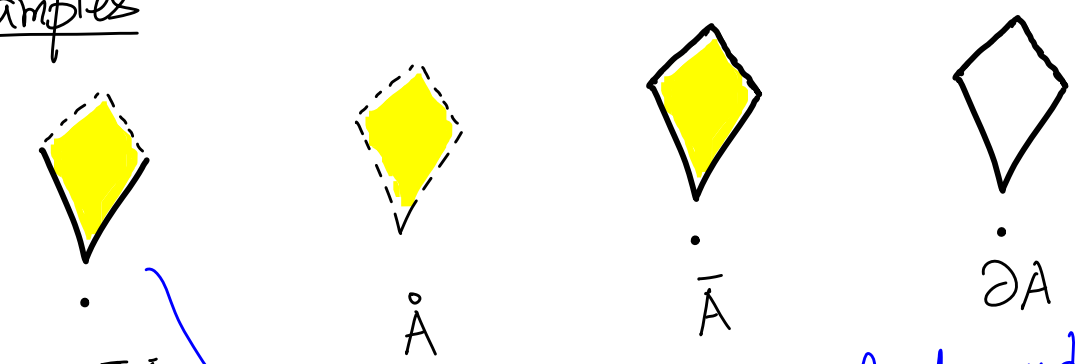
minimal closed set that contains A.

boundary ∂A of $A \subseteq \mathbb{X}$: $\partial A = \bar{A} - \mathring{A}$.

$\partial A = \{$points in A that intersect both $\bar{A}$ and $\overline{(X-A)}\}$.

<u>Examples</u>

1.



A consists of the "half-open" rhombus and a separate point.

2. A discrete example. Let $X = \{a, b, c\}$.

We can define different topologies on X.

Let $T_1 = \{\phi, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ and

$T_2 = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$.

Under T_1, $\{a,b\}$ is open, its complement $\{c\}$ is closed. With $A = \{a,b\}$, $\mathring{A} = \bigcup\{\phi, \{b\}, \{a,b\}\} = \{a,b\} = A$.

We can specify other topologies on X, e.g., $T_3 = 2^X$, where each set in T is both open and closed. But $T_4 = \{\phi, \{a\}, \{b\}, \{a,b,c\}\}$ is not a topology, as, e.g., $\{a\} \cup \{b\} = \{a,b\} \notin T_4$.

Neighborhood Let $\mathbb{X} = (X, T)$. A neighborhood of $x \in X$ is any $A \in T$ such that $x \in \mathring{A}$.

More generally, some books define a neighborhood as any set that includes, i.e., contains as a subset, an open set which contains X. Under this definition, the neighborhood could be a closed set (or neither open nor closed).

Now that we have defined topology, we consider the natural next question of comparing two spaces — how do we say two given spaces have the "same topology"? We introduce the notion of homeomorphism as a (strong) notion of topological similarity.

Homeomorphism

In geometry, we can study transformations that preserve "shape" of a rigid body, e.g., rotation and translation. These transformations "do not change" the geometry of the body.

In topology, we permit more types of transformations — e.g., stretch, shrink, expand, twist, etc., as long as you do not cut one piece into two or more, or join two pieces into one, or poke a hole in your object. All such permitted transformations "preserve topology".

A series of such permitted transformations that preserve topology constitute a homeomorphism. And two spaces are topologically "similar" if such "nice" functions exist from one space to the other and also back. We define what we mean by "nice" here.

We start with some background and definitions on functions.

Def Let A, B be sets. A function $f: A \to B$ is a rule that assigns exactly one $b \in B$ for every $a \in A$.

dom f: domain of $f = A$, cod f: codomain of $f = B$

im f: image of $f = \{b \in B \mid f(a) = b \text{ for some } a \in A\} = \{f(a) \mid a \in A\}$.

im f is also called the range of f. Note that im $f \subseteq$ cod f.

"for every"

$f: A \to B$ is 1-to-1 or **injective** if $\forall b \in B$, there exists **at most** one $a \in A$ with $f(a) = b$.

can be none

$f: A \to B$ is onto or **surjective** if $\forall b \in B$, there exists **at least** one $a \in A$ with $f(a) = b$.

can be more

If f is both injective and surjective, we say that f is **bijective**, or that f is a bijection.

Def A function $f: \mathbb{X} \to \mathbb{Y}$ is **continuous** if for every open set $B \subseteq \mathbb{Y}$, $f^{-1}(B)$ is open in $\mathbb{X}$. "takes" open sets to open sets.

A continuous function is also called a **map**.

Def A **homeomorphism** $f: \mathbb{X} \to \mathbb{Y}$ is a bijective function such that both f and f^{-1} are continuous.

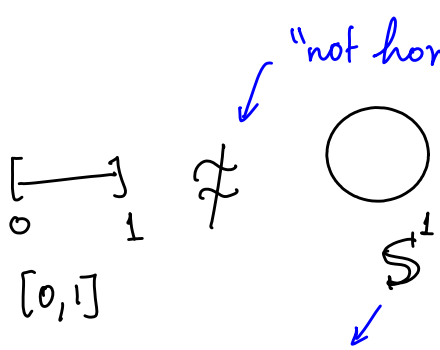
We say $\mathbb{X}$ is homeomorphic to $\mathbb{Y}$, or $\mathbb{X} \approx \mathbb{Y}$.

We also say that $\mathbb{X}$ and $\mathbb{Y}$ have the same topological type.

Examples

It's often easier to argue why two spaces are not homeomorphic — we just identify one (or more) place(s) where things don't work.

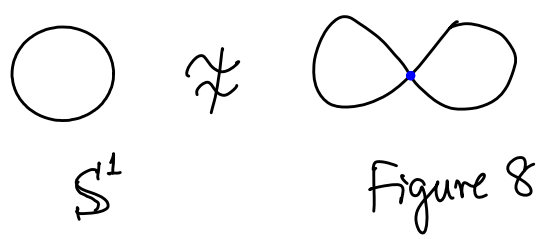
1.



We would need a map that assigns both end points of $[0,1]$ to a single point in S^1.

But the inverse of any map that takes both end points of $[0,1]$ to one point in S^1 is not bijective.

2.

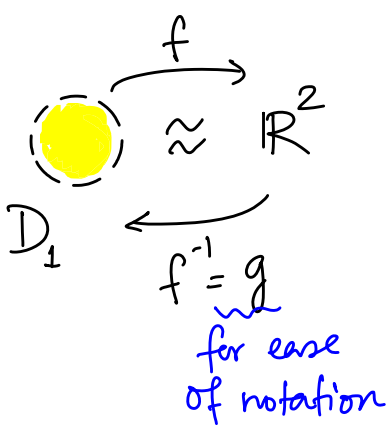


The crossing point in ∞ (x) cannot be mapped to a corresponding point in S^1.

Also, we could map S^1 to one of the two circles in figure-8, but not both.

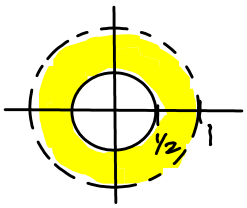
On the other hand, two show that two spaces are homeomorphic, we need to specify the maps f and f^{-1}.

3.



$D_1 = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1\} \rightarrow$ open unit disc

Intuitively, we can shrink all of $\mathbb{R}^2$ into D_1. Similarly, we can stretch D_1 to fill all of $\mathbb{R}^2$.



$$g(\bar{x}) = \frac{\bar{x}}{1+\|\bar{x}\|_2} \qquad g: \mathbb{R}^2 \to D$$

Euclidean norm

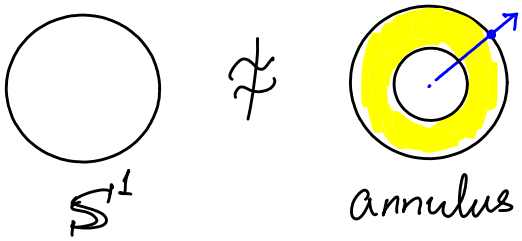
g maps all of D_1 (in $\mathbb{R}^2$) to fit within $D_{1/2} = \{\bar{x} \in \mathbb{R} \mid \|\bar{x}\| < \frac{1}{2}\}$, and then fits all of $\mathbb{R}^2$ outside D_1 within the half open annulus with radii $\frac{1}{2}$ and 1.

The continuous function going from D_1 to $\mathbb{R}^2$ can be similarly defined:

$f: D \to \mathbb{R}^2$ where $f(\bar{x}) = \dfrac{\bar{x}}{1-\|\bar{x}\|}$. f is an "infinite stretch".

Note that points $\bar{x}$ in D that are close to the edge, i.e., have $\|\bar{x}\|$ close to 1, are mapped so as to fill up the entire $\mathbb{R}^2$ outside $\bar{D}$. We stretch the open disc so as to fill the entire plane, and hence it is called an infinite stretch.

Usually, we try to define the continuous maps f and f^{-1} to show that two spaces $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic. At the same time, the intuition (geometric when possible) is also important to grasp. On the other hand, to show that $\mathbb{X} \not\approx \mathbb{Y}$, it is often sufficient to identify subset(s) that create the obstructions, e.g., the $\times$ in figure-8 v/s S^1.

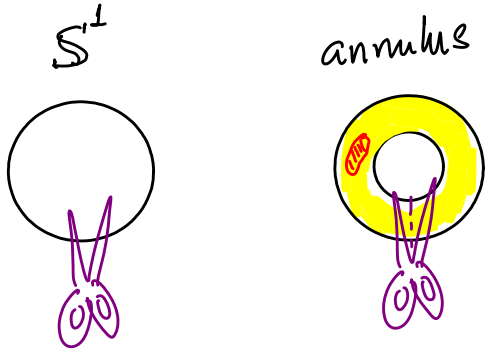
4. 

Both these spaces have the shape of a "hole".

We could shrink the annulus so that it reduces to the circle. The corresponding function maps every point on the annulus radially onto the outer circle, for instance. But we cannot uniquely map the circle back to the annulus – would need to "map" each point on the circle to (infinitely) many points on the thick strip of the annulus.

Another observation highlights the neighborhoods of points in the circle and the annulus. Every point on the circle has open neighborhoods that look like the number line ($\mathbb{R}^1$). On the other hand, points in the annulus have neighborhoods that look like the open disc ($\mathbb{R}^2$) or open half disc (the points on the boundary).

Intuitively, the annulus is 2-dimensional, while S^1 is one-dimensional.



Notice that the two spaces behave the same way under a "cut" as we had been talking about earlier with the string.

In particular, a straight cut along one "edge" of either space would leave them both connected. At the same time, one could "carve out" a 2D disc ( ) from the annulus, but not from the circle.

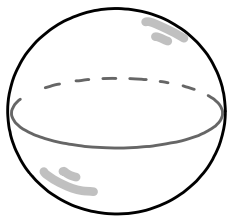
If we "relax" our definition of topological similarity, the two spaces would be considered the same – they both look like a hole, after all. Indeed we will see that checking for homeomorphism is difficult (both theoretically and computationally). We'll work with looser concepts of topological similarity later on – homology!

MATH 529: Lecture 3 (01/20/2026)

Today: * 1 more example of homeomorphism
* manifolds

Examples of homeomorphism (continued...)

5. sphere $\not\approx \mathbb{R}^2$
$\mathbb{S}^2$ (2-sphere)



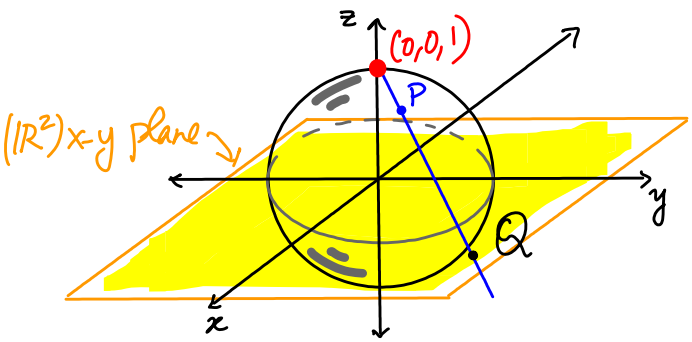
Here is an observation: the sphere encloses a 3D pocket (or void), while $\mathbb{R}^2$ does not do so.

Such enclosed voids are the 3D analogues of holes (which are 2D).

But $\mathbb{R}^2 \cup \{\infty\} \approx \mathbb{S}^2$ → By stereographic projection, which is used to map out the surface of the earth on to a (planar) map, for instance.

"point at infinity"

Recall the equation for $\mathbb{S}^2$: $x^2+y^2+z^2=1$.



If you poke a hole in the sphere, you can spread it out on the 2D plane, like a pierced balloon.

The equation of the line connecting $(0,0,1)$ and $P(x,y,z)$ is given by
$\bar{x} = (0,0,1) + t\,(x-0, y-0, z-1)$, $t \in \mathbb{R}$.

This line intersects the x-y plane at Q, which has $z=0$. Hence we get $t(z-1)+1=0 \Rightarrow t = \frac{1}{1-z}$.

Thus, Q is $\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$.

$P(x,y,z)$ on $\mathbb{S}^2$ projected from $(0,0,1)$ → North pole to $\mathbb{R}^2$ is

$$Q\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

This formula is valid for all points on $\mathbb{S}^2$, except the north pole $(0,0,1)$.

The lower hemisphere of $\mathbb{S}^2$ gets mapped to D (unit disc), and the upper hemisphere gets mapped to the rest of $\mathbb{R}^2$.

According to a topologist, "a sphere is nothing but the plane with a point added at infinity"!

Note that every point on $\mathbb{S}^2$ has a neighborhood that looks like $\mathbb{R}^2$ i.e., "it feels locally Euclidean". Such objects are called **manifolds** and are among the most commonly studied spaces in computational topology.

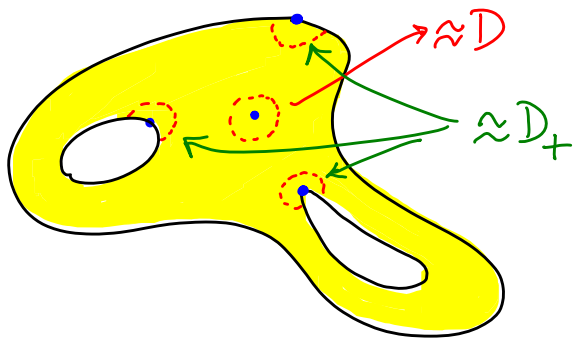
Def A topological space $\mathbb{M}$ is a **2-manifold** if all points in $\mathbb{M}$ lie in open discs, i.e., every point has a neighborhood $\approx D = \{\bar{x} \in \mathbb{R}^2 \mid \|x\| < 1\}$.

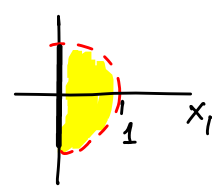
e.g., $\mathbb{S}^2$, $\mathbb{R}^2$.

these are 2-manifolds without boundary.

A **2-manifold with boundary** is a topological space $\mathbb{M}$ whose every point has a neighborhood homeomorphic to D or to $D_+ = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1, x_1 \geq 0\}$ (but not both), and there exist some points of the latter type.

1st entry



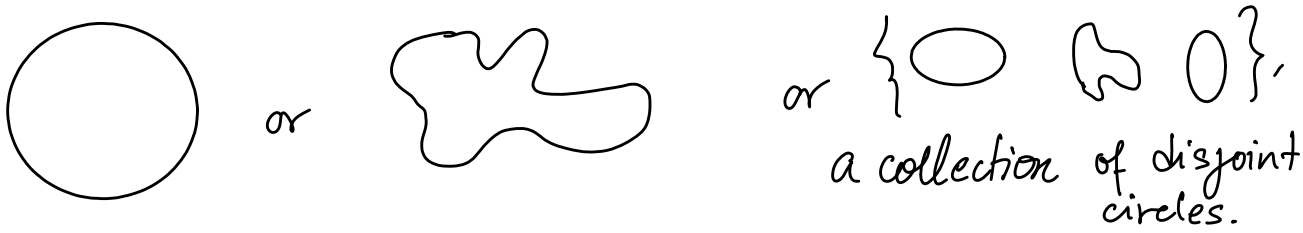


points on the boundary have neighborhoods that are half discs.

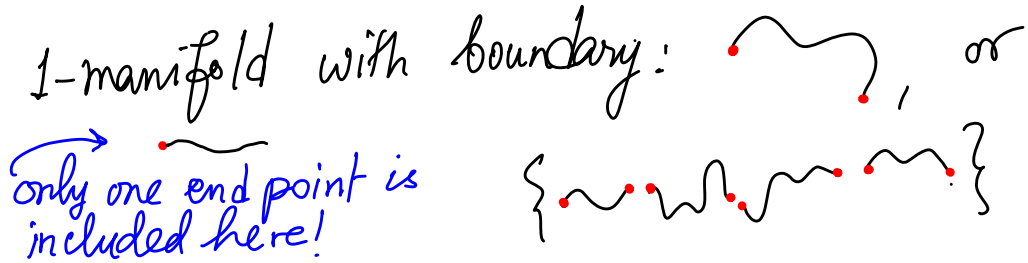
Def The **boundary** of a 2-manifold with boundary $\mathbb{M}$ is the set of points in $\mathbb{M}$ that have neighborhoods homeomorphic to the half disc.

(The definition of ∂A used for sets A is equivalent to this definition).

Notice that the 1-manifold is just the circle (S^1),



A 1-manifold with boundary:



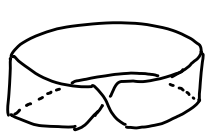
boundary is indeed the set of end points.

0-manifold: Any collection of distinct points (discrete set of points).

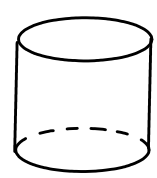
2-manifolds, also called surfaces, are a well studied class of spaces (objects), both from the theoretical as well as applied points of view. We will present several details of the properties of 2-manifolds first. To define and study d-manifolds for $d \geq 2$, we will need a few more definitions and concepts from analysis/point set topology.

By default, we assume a manifold (w/ or w/o boundary) is connected.

A 2-manifold (for that matter, d-manifold for $d \geq 2$) is orientable or non-orientable.



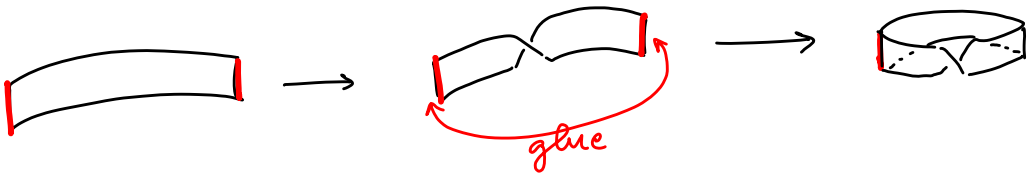
Möbius strip is non-orientable

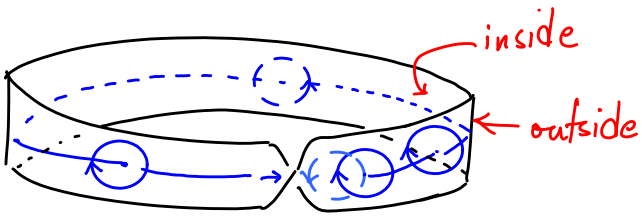


cylinder is orientable

The Möbius strip has only one "side", while the cylinder has two "sides" — inside and outside.

We obtain the Möbius strip by taking a rectangular strip of paper, and gluing its short edges together after twisting the strip once.





there are two "sides" at each point (locally) on the Möbius strip — front/back or inside/outside.

Consider sliding an oriented loop, or a clock along the surface of the Möbius strip. Look at the path followed by the center of the clock. Once the center comes back to where it started, its orientation is reversed (as it goes over the "twist").

The path traced by the center of the clock here is hence an **orientation reversing** closed curve. If the orientation is not reversed this way, the curve is said to be **orientation preserving**.

Def If all closed curves in a 2-manifold (with or without boundary) are orientation preserving, then the 2-manifold is **orientable**, else it is **nonorientable**.

$\mathbb{R}^2$, $\mathbb{S}^2$, torus, etc.

Möbius strip, Klein bottle, etc.

Classification of Manifolds

Enumerate all possible manifolds upto homeomorphisms for each dimension.

We mention the classification for 0- and 1-dimensional manifolds, but will come back to give some more definitions to finish the discussion for 2-manifolds.

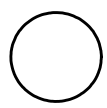
each point has to have a neighborhood $\approx \mathbb{R}^0$, i.e., a point.

0-manifolds: a discrete space, e.g., $\mathbb{Z}^2$ – all points with integer coordinates in $\mathbb{R}^2$.

Notice that $2\mathbb{Z}^2$, all points with even Integer coordinates in $\mathbb{R}^2$ is homeomorphic to $\mathbb{Z}^2$.

We will define "compact" soon...

1-manifolds connected

- compact: $\approx S^1$, 
- compact with boundary: $\approx [0,1]$ closed unit interval
- non-compact: $\approx \mathbb{R}^1$ or $(0,1)$
- non-compact with boundary $\approx \mathbb{R}^1_+$ or $[0,1)$

$\mathbb{R}^1_+$: $x \geq 0$

$[0,1)$: half-open unit interval

if a 1-manifold is not a single connected space, each connected subspace has one of these structures.

MATH 529: Lecture 4 (01/22/2026)

Today: * d-manifolds in general
* Classification of 2-manifolds

We will now introduce some concepts which we will use to define d-manifolds in general (in particular, for $d \geq 2$).

Def A **cover** of $A \subseteq X$ is a family $\{C_j \mid j \in J\}$ in 2^X such that $A \subseteq \bigcup_{j \in J} C_j$.

↳ index set

An **open cover** is a cover made of open sets.

index set is a subset

A **subcover** of A is a cover $\{C_k \mid k \in K\}$ such that $K \subseteq J$.

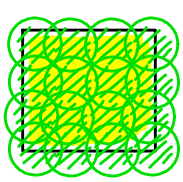
Def A set A is **compact** if every open cover of A has a finite subcover. Correspondingly, a topological space $\mathbb{A} \subseteq \mathbb{X}$ is compact if every open cover of $\mathbb{A}$ has a finite subcover.

Note: In $\mathbb{R}^d$, closed + bounded $\Longleftrightarrow$ compact.

e.g., S^2 is compact, but $\mathbb{R}^2$ is not.

D_1 is not compact.

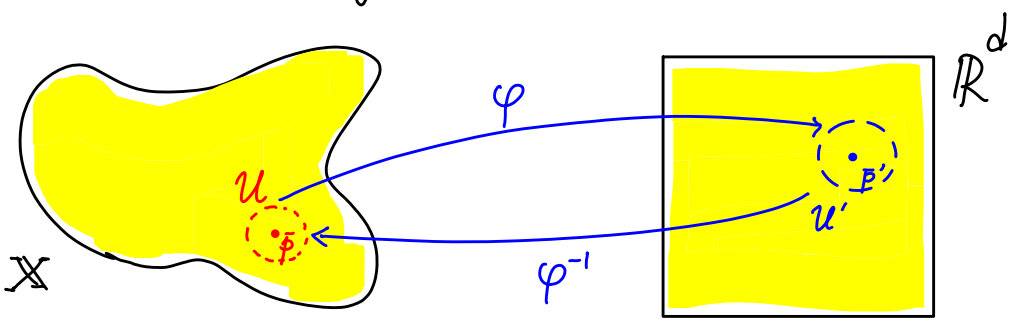
$\bar{D}_1$ (closed disc) is a compact 2-manifold with boundary



Example of finite subcover: consider a unit square, which is a subset of $\mathbb{R}^2$. Consider open discs of radius $1/4$ centered at each rational point within the square (denoted here as).

There are infinitely many such discs, which together cover the square. But a finite subset of those discs also covers the square.

Def A **chart** at $p \in \mathbb{X}$ is a homeomorphism $\varphi : \mathcal{U} \to \mathbb{R}^d$ for $\mathcal{U} \in \mathbb{X}$ an open set containing p. The **dimension** of the chart is d.



Def (Hausdorff) A topological space $\mathbb{X}$ is **Hausdorff** if $\forall x, y \in X$, $x \neq y$, there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of x, y, respectively, such that $\mathcal{U} \cap \mathcal{V} = \phi$.

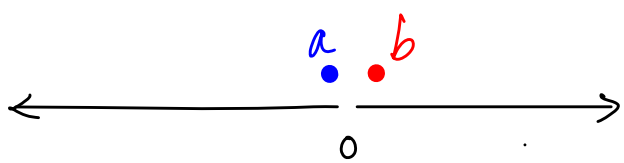
an open set containing the point in question

e.g., $\mathbb{R}^2$.

Example of a non-Hausdorff space: $X = L \cup \{a, b\}$ where

$L = \mathbb{R} \setminus \{0\}$

a and b are both used in place of the origin. Open sets are the usual open intervals in $\mathbb{R}$.



and such that

$$L \cup \{a\} \approx \mathbb{R} \text{ and } L \cup \{b\} \approx \mathbb{R}.$$

But every pair of open sets $\mathcal{U}$ and $\mathcal{V}$ containing a and b, respectively, intersect!

Def A topological space is **completely separable** (or second countable) if it has a countable basis, i.e., it has a countable collection of open sets such that every open set can be written as a union of open sets from this collection (basis).

(think $\mathbb{Z}$, integers, as opposed to $\mathbb{R}$, which is uncountable)

e.g., $\mathbb{R}$ is completely separable – it can be shown that open intervals with rational lengths centered at only rational points works as a countable basis.

A space that is not completely separable: take uncountably many copies of $[0,1)$, e.g., with the 0 of $[0,1)$ anchored at all irrational points – called the long line or Alexandroff line.

Def (Manifold) A completely separable, Hausdorff space $\mathbb{X}$ is a **d-manifold** (d-dimensional manifold) if there exists a d-dimensional chart at every $\bar{x} \in \mathbb{X}$, i.e., $\bar{x}$ has a neighborhood homeomorphic to $\mathbb{R}^d$.

$\mathbb{X}$ is a d-manifold with boundary if every $\bar{x} \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$ or $\mathbb{H}^d = \{\bar{x} \in \mathbb{R}^d \mid x_1 \geq 0\}$ (d-dimensional half space).

The boundary of $\mathbb{X}$, denoted by $\partial\mathbb{X}$, is the set of $\bar{x} \in \mathbb{X}$ with a neighborhood homeomorphic to $\mathbb{H}^d$.

The **dimension** of the manifold is d here.

Notice the correspondence between the definition of 2-manifolds introduced previously, and the general definition here. The main condition is the existence of neighborhoods $\approx \mathbb{R}^d$ around each point.

Def (Embedding) An **embedding** of $\mathbb{X}$ in $\mathbb{Y}$ is a map $g: \mathbb{X} \to \mathbb{Y}$ whose restriction to $g(\mathbb{X})$ is a homeomorphism.

Manifolds are manifolds irrespective of their embedding! $\mathbb{S}^2$ is a 2-manifold even if it is not sitting in $\mathbb{R}^3$.

We will introduce alternative representations of manifolds to highlight this point. In fact, in many cases, we can study the manifold easily using such representations.

Classification of Manifolds (continued)

Enumerate all possible manifolds of a given dimension upto homeomorphism. We already listed the classifications for 0- and 1-dimensional manifolds.

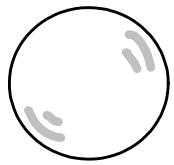
We now consider the case of compact, connected, closed 2-manifolds. We first list the "basic building blocks", so to speak, which include the 2-sphere, torus, Möbius strip, and the real projective plane. We can build larger 2-manifolds by gluing these building blocks together.

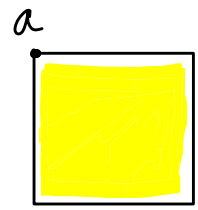
2-Manifolds (We consider compact 2-manifolds)

First, let us study several typical 2-manifolds, some of which we already saw previously.

1. S^2

2-sphere





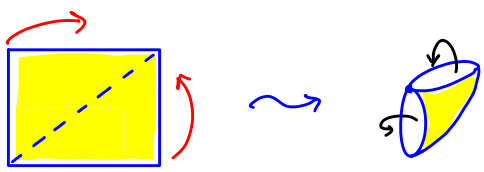
$\{\bar{x} \in \mathbb{R}^3 \mid \|\bar{x}\|_2 = 1\}$

both are 2-spheres!

"identify" all points on boundary with the point a.

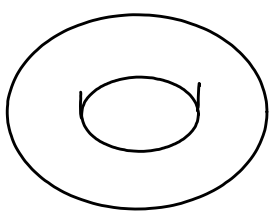
This is a "diagram" of S^2.

Start with a square sheet of paper and glue its all its edges together to make a sphere.



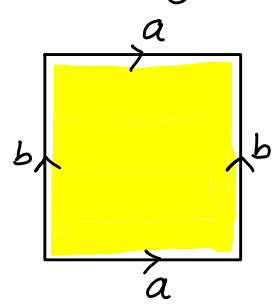
Arrows capture how edges are glued – with or without twist.

2.



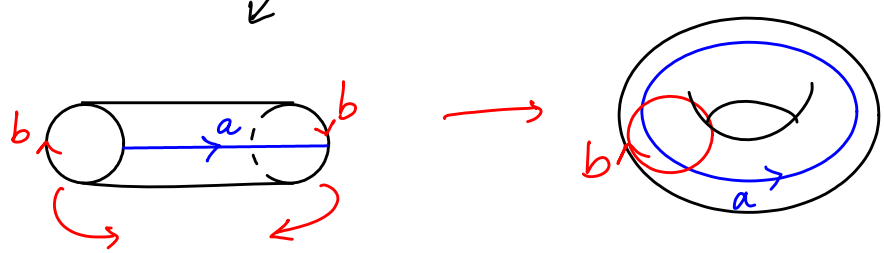
$\mathbb{T}^2$ torus

$\approx$

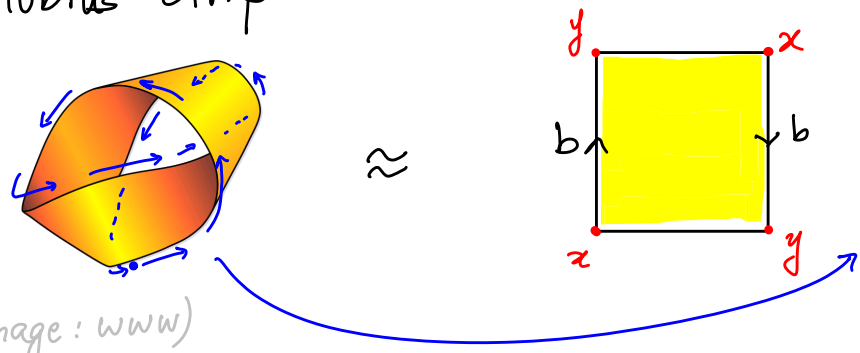


we could mathematically study this representation

Imagine folding a rectangular sheet of paper first into an open cylinder, and then gluing its end circles to form a torus.



3. Möbius strip → 2-manifold with boundary



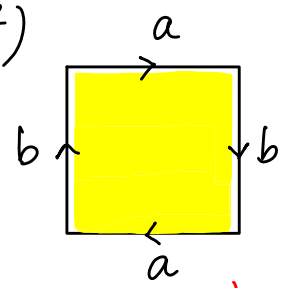
(image: www)

Notice that we can traverse the "edge" of the Möbius strip in one go — it's one big circle!

Notice that we are not identifying the horizontal edges. So they remain as boundaries. At the same time all four edges are identified pairwise in the case of the torus. Indeed, the Möbius strip is a manifold with boundary, while the torus is a manifold (without boundary).

cannot be embedded in $\mathbb{R}^3$!

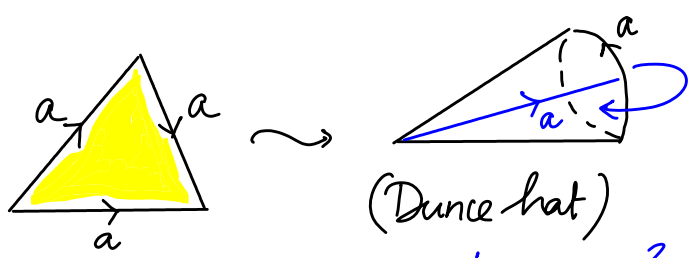
4. (Real) Projective plane ($\mathbb{RP}^2$)
(also, Dunce hat)



identify the free edges of Möbius strip in an opposing sense.

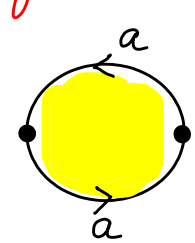
hat (or cone)

Same as gluing the boundary of a disc to the boundary of a Möbius strip.

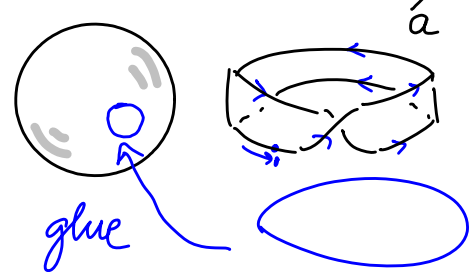


(Dunce hat)

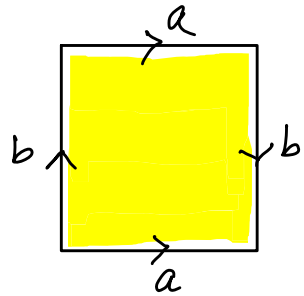
Here's another representation →



Yet another way to make $\mathbb{RP}^2$: Cut an open disc out of S^2 and glue a Möbius strip along the edge left by the cut.

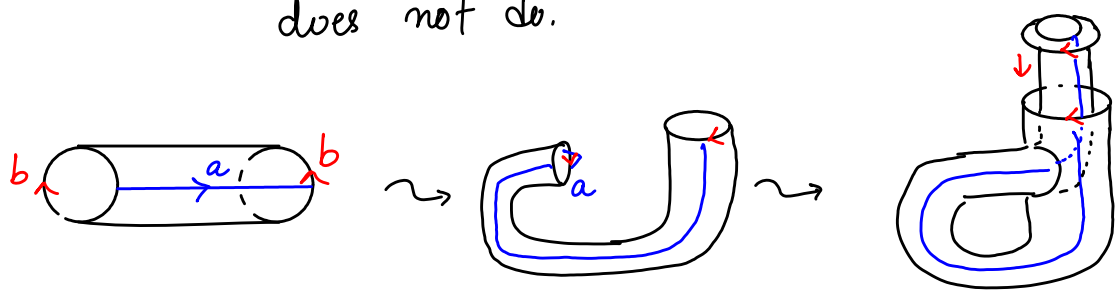


5. Klein bottle ($\mathbb{K}^2$)



Identify free edges of the Möbius strip in the same direction.

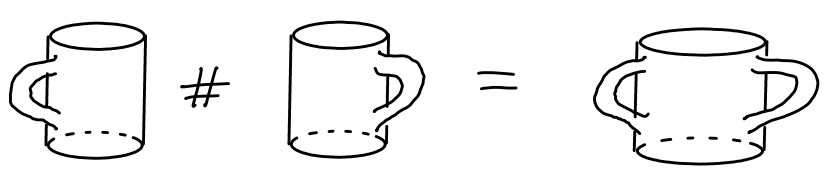
An "immersion" in 3D:
→ allows self intersection, which an embedding does not do.



We get $\mathbb{K}^2$ also by gluing together two Möbius strips along their boundary circles. Or, cut two discs out of $\mathbb{S}^2$ and glue a Möbius strip each along the edges of both cuts.

Note that $\mathbb{S}^2$ and $\mathbb{T}^2$ are orientable manifolds, while the Möbius strip, $\mathbb{RP}^2$, and $\mathbb{K}^2$ are non-orientable manifolds (with or without boundary).

We can obtain more general 2-manifolds by "gluing" these basic shapes together. For example, we can connect two coffee cups to get one coffee cup with two handles!



We can do this kind of "gluing" to join manifolds in any dimension (as long as the manifolds being joined have the same dimension). This ing is formally termed connected sum.

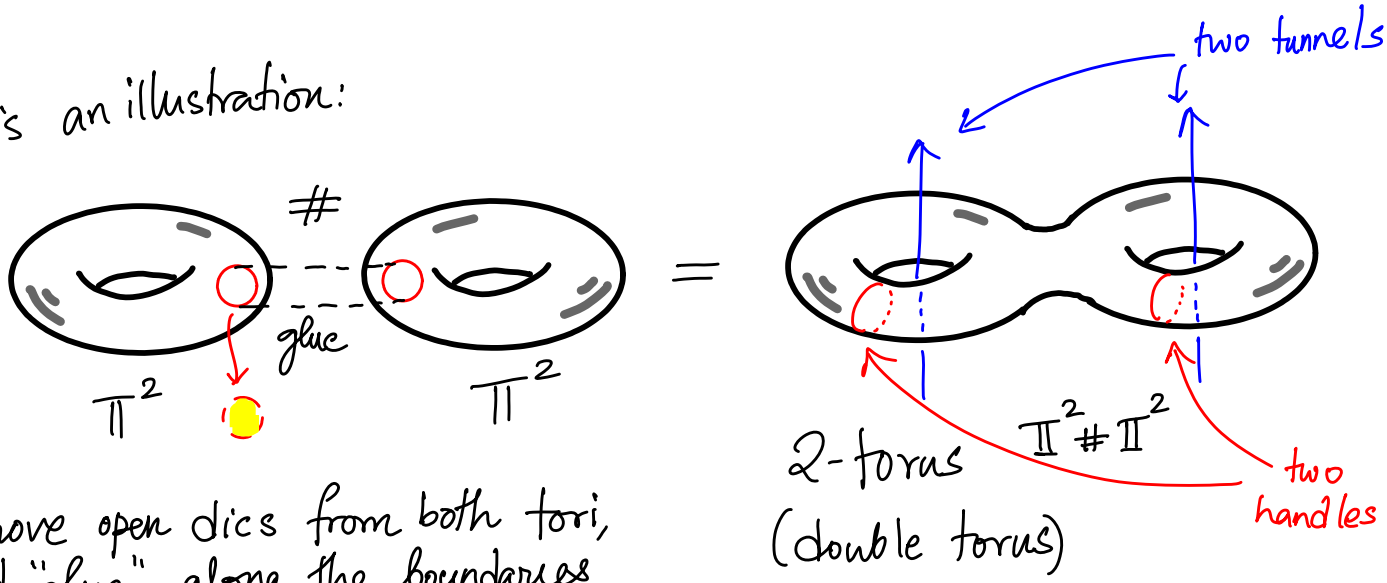
Def (Connected sum). Let $\mathbb{M}_1, \mathbb{M}_2$ be d-manifolds. The **connected sum** of these d-manifolds is another d-manifold defined as follows.

$$\mathbb{M}_1 \# \mathbb{M}_2 = (\mathbb{M}_1 - D_1^d) \bigcup_{\partial \bar{D}_1^d \cong \partial \bar{D}_2^d} (\mathbb{M}_2 - D_2^d)$$

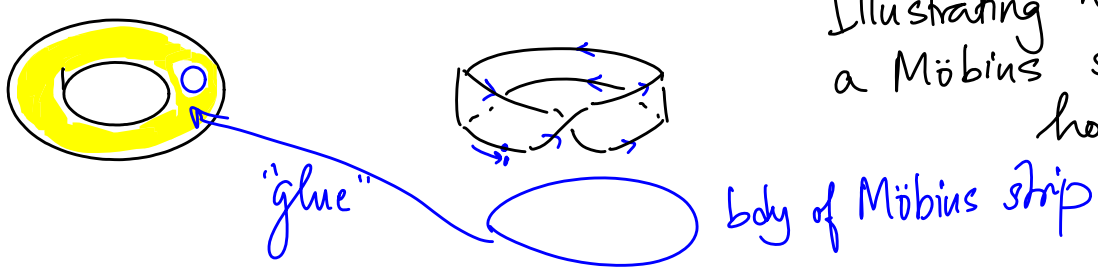
identified by homeomorphism

D_1^d, D_2^d are d-dimensional open discs in $\mathbb{M}_1, \mathbb{M}_2$, respectively.

Here is an illustration:



Remove open dics from both tori, and "glue" along the boundaries of these circular holes.



Illustrating how to glue a Möbius strip to a hole in a torus.