MATH 567: Lecture 6 (01/28/2025)

Today: * comparing formulations
* sharp/ideal formulation

Recall Farkas' lemma. We present one more version now.

(3) $\exists\, \bar{x} : A\bar{x} = \bar{b} \iff \not\exists\, \bar{u} \neq \bar{0} : \bar{u}^T A = \bar{0}^T,\ \bar{u}^T \bar{b} = -1$ can be any nonzero #

$A\bar{x} = \bar{b}$: $[A|\bar{b}] \xrightarrow{\text{EROs}} [\quad]$ echelon form

If the echelon form has a row of the form $[0\,0 \cdots 0 | \blacksquare]$, $\neq 0$
the system $A\bar{x} = \bar{b}$ is inconsistent.

Naturally, we can prove the other versions if we assume one version of Farkas' lemma (i.e., they are equivalent).

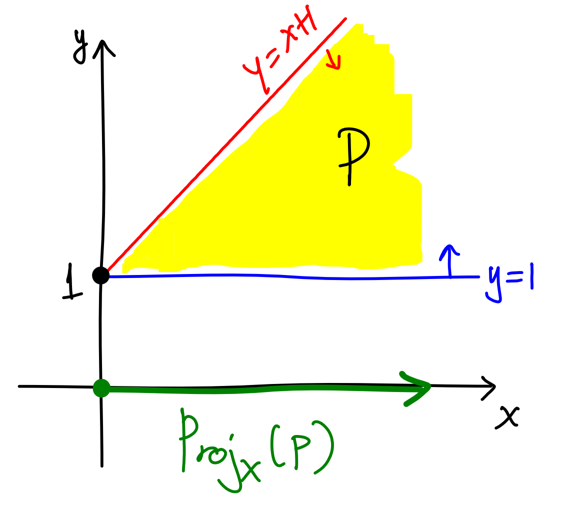
Def If $P = \{(\bar{x}, \bar{y}) \mid A\bar{x} + B\bar{y} \leq \bar{b}\}$, then the **projection of P** onto the space of $\bar{x}$ variables is

$$\text{Proj}_{\bar{x}}(P) = \{\bar{x} \mid \exists\, \bar{y} : (\bar{x}, \bar{y}) \in P\}.$$

$\Downarrow$ "push down" P onto the x-axis

e.g., $P = \{(x,y) \mid y \geq 1,\ y \leq x+1\}$

$\text{Proj}_x(P) = \{x \mid x \geq 0\}.$



Theorem 4 $\text{Proj}_{\bar{x}}(P) = \underbrace{\{\bar{x} \mid \bar{v}^T A\bar{x} \leq \bar{v}^T\bar{b} \ \forall \bar{v} \geq \bar{0},\ \bar{v}^T B = \bar{0}^T\}}_{\text{(RHS)}}$.

In words, all nonnegative linear combinations of $A\bar{x} + B\bar{y} \leq \bar{b}$ that eliminate the "unwanted" $\bar{y}$ variables.

Proof '$\subseteq$': $\bar{x} \in \text{Proj}_{\bar{x}}(P) \Rightarrow \exists \bar{y} \mid A\bar{x} + B\bar{y} \leq \bar{b}$

$\Rightarrow \bar{v}^T A\bar{x} \leq \bar{v}^T\bar{b}$ holds $\forall \bar{v} \geq 0$, $\bar{v}^T B = \bar{0}^T$.

'$\supseteq$': Show that if $\bar{x} \notin \text{Proj}_{\bar{x}}(P)$, then $\bar{x} \notin$ (RHS).

Can use Farkas' lemma!

$\nexists \bar{y}: B\bar{y} \leq \bar{b} - A\bar{x}$, i.e., the system $B\bar{y} \leq \bar{b} - A\bar{x}$ has no solutions (in $\bar{y}$).

$\Rightarrow \exists \bar{v} \geq \bar{0},\ \bar{v}^T B = \bar{0}^T$ and $\bar{v}^T(\bar{b} - A\bar{x}) < 0$.

$\Rightarrow \exists \bar{v} \geq \bar{0},\ \bar{v}^T B = \bar{0}^T$ for which $\bar{v}^T A\bar{x} > \bar{v}^T\bar{b}$.

$\Rightarrow \bar{x} \notin$ (RHS). $\square$

Back to 2D example:

$$\begin{array}{l} y \geq 1 \\ y \leq x+1 \end{array} \Rightarrow \left.\begin{array}{l} -y \leq -1 \\ y \leq x+1 \end{array}\right\} \xrightarrow{\text{ADD}} x \geq 0, \text{ which is } \text{Proj}_x(P).$$

Equivalently, $\left\{\begin{array}{rl} -y & \leq -1 \\ -x+y & \leq 1 \end{array}\right\} \equiv \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{B} y \leq \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\bar{b}}$.

What $\bar{v} \geq \bar{0}$ with $\bar{v}^T B = 0$ can we take to eliminate y?

$\bar{v} = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$, $\lambda \geq 0$ works! Or, $\bar{v} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda \geq 0$ are all the multipliers!

Could we generalize this result illustrated in the example, i.e., could we describe $\text{Proj}_{\bar{x}}(P)$ using only a finite # $\bar{v}^i$'s?

Theorem 5 If $\{\bar{v} \geq \bar{0},\ \bar{v}^T B = \bar{0}^T\} = \underbrace{\textbf{cone}\{\bar{v}^1, \dots, \bar{v}^k\}}$ the projection cone is finitely generated

$\overset{\text{def}}{\equiv} \left\{ \sum_{i=1}^{k} \lambda_i \bar{v}^i \,\middle|\, \lambda_i \geq 0 \right\}$, then

$$\text{Proj}_{\bar{x}}(P) = \{\bar{x} \mid (\bar{v}^i)^T A \bar{x} \leq (\bar{v}^i)^T \bar{b},\ i = 1, \dots, k\}.$$

Def The set $\{\bar{v} \mid \bar{v} \geq \bar{0},\ \bar{v}^T B = \bar{0}^T\}$ is called the **projection cone** of $\text{Proj}_{\bar{x}}(P)$.

Example (continued): The projection cone of $\text{Proj}_x(P)$ is

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \,\middle|\, \begin{array}{l} v_1, v_2 \geq 0 \\ -v_1 + v_2 = 0 \end{array} \right\} = \left\{ \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \,\middle|\, \lambda \geq 0 \right\}.$$

Definition of Comparison

Let $S \subseteq \mathbb{Z}^n \times \mathbb{R}^m$ have two formulations

$$P_1 = \{(\bar{x}, \bar{y}, \bar{u}^1, \bar{v}^1) \in \mathbb{R}^{n+m+p_1+q_1} \mid A_1\bar{x} + B_1\bar{y} + C_1\bar{u}^1 + D_1\bar{v}^1 \leq \bar{b}^1\}$$

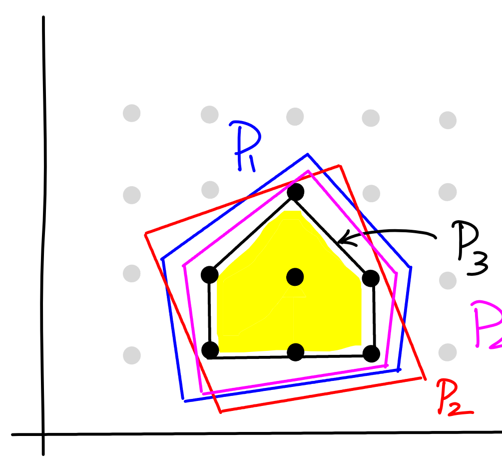
and

$$P_2 = \{(\bar{x}, \bar{y}, \bar{u}^2, \bar{v}^2) \in \mathbb{R}^{n+m+p_2+q_2} \mid A_2\bar{x} + B_2\bar{y} + C_2\bar{u}^2 + D_2\bar{v}^2 \leq \bar{b}^2\}$$

where $p_1 \neq p_2$, $q_1 \neq q_2$ and $p_1+q_1 \neq p_2+q_2$.

Then P_1 is a **better (stronger, tighter)** formulation than P_2 if $\mathrm{Proj}_{(\bar{x},\bar{y})}(P_1) \subset \mathrm{Proj}_{(\bar{x},\bar{y})}(P_2)$.

P_j, $j=1,2,3,4$ are formulations of S.



Here, $P_3 \subset P_j$, $j=1,2,4$.

So, P_3 is stronger than P_j.

Similarly, $P_4 \subset P_1$, so is stronger than P_1.

Example

$$S = \{ \bar{x} \mid \bar{x} \in \{0,1\}^n, (x_1 = 1) \Rightarrow (x_2=1) \wedge \cdots \wedge (x_n = 1) \}.$$

$$P_1 = \{ \bar{x} \mid x_1 \le x_2, \ldots, x_1 \le x_n, \quad 0 \le x_i \le 1, \quad i = 1, \ldots, n \}$$ and

$$P_2 = \{ \bar{x} \mid (n-1)x_1 \le x_2 + \cdots + x_n, \quad 0 \le x_i \le 1, \quad i = 1, \ldots, n \}$$

are formulations for S. ($P_i \cap \mathbb{Z}^n$ gives S for $i = 1,2$).

P_1 is the **disaggregated** formulation, while P_2 is an **aggregated** formulation.

Claim $P_1 \subset P_2$.

$P_1 \subseteq P_2$ is trivial — just add up $x_1 \le x_i, i=2,\ldots,n,$ to get $(n-1)x_1 \le x_1 + \cdots + x_2$.

To show $P_1 \subset P_2$, identify one point in P_2 / P_1.

$\left(\frac{1}{n-1}, 1, 0, \ldots, 0\right) \in P_2 / P_1$. For instance, $x_3 \ge x_1$ is violated here.

In fact, P_1 is the strongest formulation for S here!

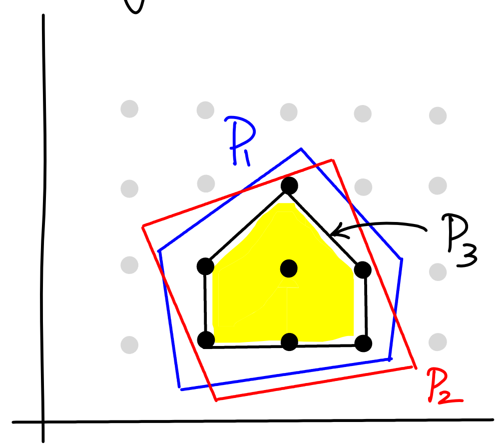
Def Given $S \in \mathbb{Z}^n \times \mathbb{R}^m$, $P \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is a **sharp** or **ideal formulation** for S if

(1) $\forall \begin{bmatrix} \bar{c} \\ \bar{d} \end{bmatrix} \in \mathbb{R}^{n+m}$ such that $\max \left\{ [\bar{c}^T \bar{d}^T] \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \,\middle|\, \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \in P \right\}$ is finite, the optimum is obtained for some element of S.

(2) An extended formulation of S (using extra variables) is sharp if its projection to $(\bar{x}, \bar{y})$-space is sharp in the sense of (1) above.

Intuitively, all corner points of P are integral.

P_3 is the sharp formulation of S here:



More generally, P is the convex hull of S.

Def For $X \subseteq \mathbb{R}^n$, the **convex hull** is defined as

$$\text{conv}(X) = \left\{ \bar{x} \in \mathbb{R}^n \,\middle|\, \bar{x} = \sum_{i=1}^{k} \lambda_i \bar{x}^i,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1, \text{ for all finite subsets } \{\bar{x}^1, \dots, \bar{x}^k\} \text{ of } X \right\}.$$

To show a formulation P is sharp for a set S, we can show every corner point of P is integral, i.e., all their entries are integers.

In 2D, any two non-parallel lines representing equations from P **could** intersect at a corner point, assuming it is feasible.

In general, in $\mathbb{R}^n$, we get a corner point from n linearly independent (LI) equations that define P, assuming their intersection is feasible.

We saw that $(\frac{1}{n-1}, 1, 0, 0 \ldots, 0) \in P_2$ (aggregated formulation). In fact, this point is a corner point of P_2, defined by the n LI constraints.

→ more on this and other details in the next lecture...