MATH 524 – Lecture 8 (09/14/2023)

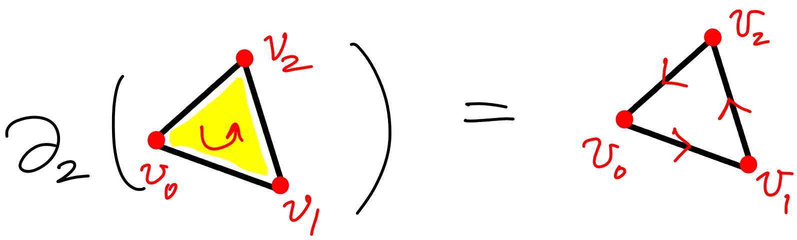
Today: * boundary homomorphism
* cycles, boundaries, homology group

Now that we have defined the chain groups $C_p(K)$ for each p, we now talk about how to connect/relate the $C_p(K)$ for various p. In particular, how are $C_p(K)$ and $C_{p-1}(K)$ related?

Called the "p-boundary"

We define a homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ called the **boundary operator** (or boundary homomorphism).

Intuitively, the boundary of a triangle is made of its three edges. But now we take the orientation also into account.



Def We define the homomorphism
$$\partial_p : C_p(K) \to C_{p-1}(K)$$ called the **boundary operator** as follows. If $\sigma = [v_0, \ldots, v_p]$, $p > 0$, then

$$\partial_p \sigma = \partial_p [v_0, \ldots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_p] \qquad (1)$$

where $\hat{v}_i$ means vertex v_i is deleted from $[v_0, \ldots, v_p]$.

As $C_p(K)$ is trivial for $p < 0$, ∂_p is the trivial homomorphism for $p \leq 0$.

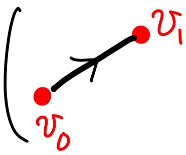
Since ∂_p is a homomorphism, we naturally extend the definition of boundary from p-simplices to p-chains. If $c = \sum n_i \sigma_i$ is a p-chain, then $\partial_p c = \partial_p\left(\sum n_i \sigma_i\right) = \sum n_i (\partial_p \sigma_i)$.

Examples

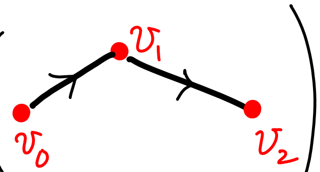
1-simplex

$\partial_1 [v_0 v_1] = v_1 - v_0$

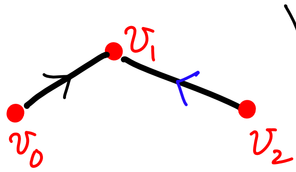
Notice that $\partial_1 [v_1 v_0] = v_0 - v_1$;

$\partial_1 \Bigg($  $\Bigg) = v_1 - v_0$

head − tail, if you think of the oriented edge as an "arrow".

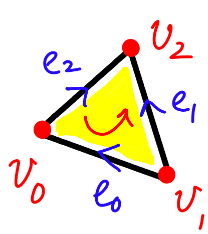
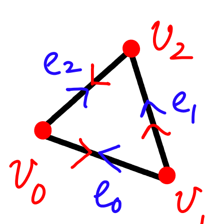
$\partial_1 \Bigg($  $\Bigg) = v_1 - v_0 + v_2 - v_1 = v_2 - v_0$

Notice that the computations are sensitive to the choice of orientations.

$\partial_1 \Bigg($  $\Bigg) = 2v_1 - v_0 - v_2$

2-simplex

$$\partial_2 [v_0 v_1 v_2] = (-1)^0 [v_1 v_2] + (-1)^1 [v_0 v_2] + (-1)^2 [v_0 v_1] = [v_1 v_2] - [v_0 v_2] + [v_0 v_1].$$

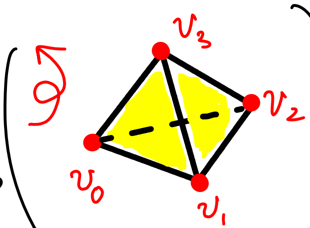
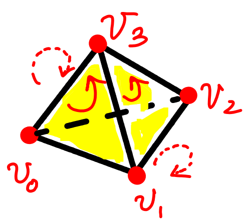
$\partial_2 \Bigg($  $\Bigg) =$ 

The 1-boundary is $-e_0 + e_1 - e_2$

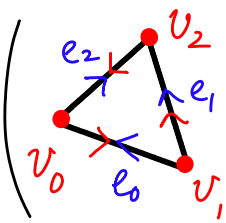
Notice that the orientation induced from the 2-simplex onto its faces (1-simplices) by the boundary operation could be distinct from the individual orientations of the 1-simplices themselves.

3-simplex

$$\partial_3 [v_0 v_1 v_2 v_3] = [v_1 v_2 v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 v_1 v_2]$$

$\partial_3 \Bigg($  $\Bigg) =$ 

We observe that $\partial_1 (\partial_2 [v_0 v_1 v_2]) = 0$. (both algebraically and geometrically)

$\partial_1 \Bigg($  $\Bigg) = \partial_1 (-e_0 + e_1 - e_2) = -(v_0 - v_1) + (v_2 - v_1) - (v_2 - v_0) = 0.$

A similar observation can be made for the tetrahedron:

$$\partial_2 (\partial_3 [v_0 v_1 v_2 v_3]) = \partial_2 \big([v_1 v_2 v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 v_1 v_2] \big) = 0$$

$+[v_1 v_2]$ $\qquad\qquad$ $-[v_1 v_2]$

every edge cancels in pairs.

Indeed, this result holds in general — $\partial_p \partial_{p+1} \sigma = 0$. And we can prove it using the definition of ∂_p.

Before that, let's make sure ∂_p is well-defined. In particular, we need to check that $\partial_p(-\sigma) = -\partial_p(\sigma)$.

We check what happens in Sum (1) when we swap v_j & v_{j+1}.

Consider $\sum_{i=0}^{p-1}(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_p]$ and $\sum_{i=0}^{p-1}(-1)^i(-[v_0,\dots,\hat{v}_i,\dots v_p])$. If $i\neq j, j+1$, the corresponding terms do differ by a sign. When $i=j$, compare terms in

$$\partial_p[v_0,\dots,v_{j-1},v_j,v_{j+1},v_{j+2},\dots,v_p] \quad \text{———} \quad (1a)$$

and swapped

$$\partial_p[v_0,\dots,v_{j-1},v_{j+1},v_j,v_{j+2},\dots,v_p]. \quad \text{———} \quad (1b)$$

before we leave out one vertex at a time...

We have $(-1)^j[v_0,\dots v_{j-1},\hat{v}_j,v_{j+1},v_{j+2},\dots,v_p]$ in (1a), and

$(-1)^{j+1}[v_0,\dots,v_{j-1},v_{j+1},\hat{v}_j,v_{j+2},\dots,v_p]$ in (1b)

These two terms do differ by a sign: $(-1)^j$ and $(-1)^{j+1}$. Argument for $i=j+1$ is similar.

We now prove the general result on taking the boundary of a boundary. Indeed, we will use this result to define homology groups as subgroups of $C_p(K)$. Hence this result is called the fundamental lemma of homology.

Lemma 5.3 [M] $\quad \partial_{p-1}\circ\partial_p=0. \longrightarrow$ Fundamendal lemma of homology

Proof

$$\partial_{p-1}\partial_p[v_0,\dots,v_p]$$

$$=\sum_{i=0}^{p}(-1)^i\partial_{p-1}[v_0,\dots,\hat{v}_i,\dots v_p]$$

$$=\sum_{j<i}(-1)^i(-1)^j[\dots\hat{v}_j,\dots,\hat{v}_i,\dots]+\sum_{j>i}(-1)^i(-1)^{j-1}[\dots\hat{v}_i,\dots\hat{v}_j\dots]$$

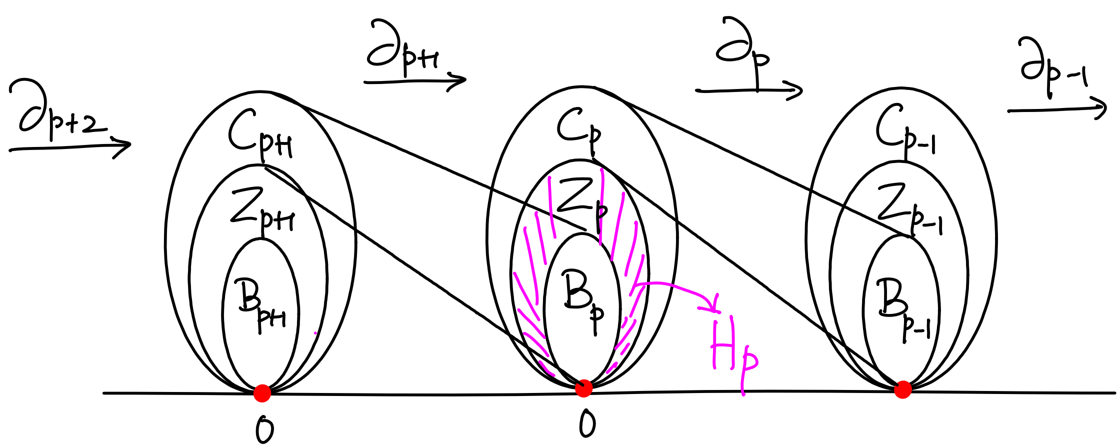
$=0$, as the terms cancel in pairs! $\square$

Def The kernel of $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$ is the group of **p-cycles**, denoted $Z_p(K)$. The image of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is the group of **p-boundaries**, denoted $B_p(K)$.

Since $\partial_p \partial_{p+1} = 0$ by the above lemma, each boundary of a $(p+1)$-chain is automatically a p-cycle. Hence, $B_p(K) \subset Z_p(K) \subset C_p(K)$.

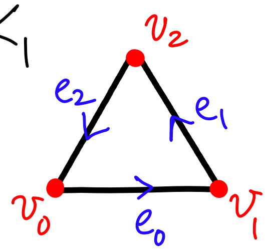
We now define $H_p(K) = Z_p(K)/B_p(K)$, and call it the **p-th homology group** of K.

The various groups and ∂ homomorphisms have the following structure:



Examples

1. K_1



$C_1(K_1)$ is free abelian, generated by, e.g., $\{e_0, e_1, e_2\}$.

We use vector notation for chains

Any 1-chain in K can be given as $\bar{c} = n_0 e_0 + n_1 e_1 + n_2 e_2$ for $n_i \in \mathbb{Z}$.

ideally, $\bar{c} = n_0\bar{e}_0 + n_1\bar{e}_1 + n_2\bar{e}_2$

When is $\bar{c}$ a cycle?

$$\partial_1 \bar{c} = n_0(v_1 - v_0) + n_1(v_2 - v_1) + n_2(v_0 - v_2)$$
$$= (n_2 - n_0)v_0 + (n_0 - n_1)v_1 + (n_1 - n_2)v_2.$$

So, $\partial_1 \bar{c} = 0$ iff $n_0 = n_1 = n_2$. Thus $\bar{c}$ is a 1-cycle iff $n_0 = n_1 = n_2$.

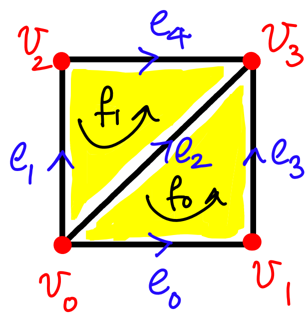
We can see that $Z_1(K_1)$ is infinite cycle, generated by $\bar{e}_0 + \bar{e}_1 + \bar{e}_2$. In other words, we can pick an integer, and that number tells us how many times we go around the cycle. If $n_0 = n_1 = n_2 = -3$, for instance, we go around in the opposite direction (ie, clockwise) 3 times.

There are no 2-simplices, so $B_1(K_1)$ is trivial. In other words, there are no 1-boundaries. Hence $H_1(K_1) = Z_1(K_1) \simeq \mathbb{Z}$.

Also, $\beta_1(K_1) = rk(H_1(K_1)) = 1$.

Example 2

K_2



This is the same example 2 in [M], but with different choices of orientations.

$|K_2|$ is a square.

A general 1-chain in K_2 is $\bar{c} = \sum_{i=0}^{4} n_i \bar{e}_i$. Then $\partial_1 \bar{c} = \sum n_i \partial_1(\bar{e}_i)$.

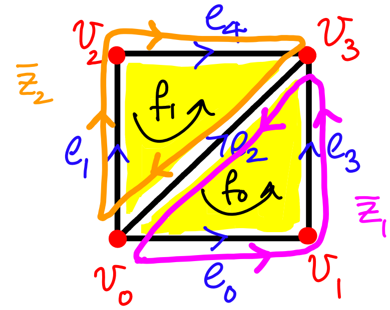
When is $\bar{c}$ as 1-cycle?

We need $n_0 = n_3$ (at v_1), and $n_1 = n_4$ (at v_2).

Similarly, $n_2 = -(n_0 + n_1)$ at v_0, and $n_2 = -(n_3 + n_4)$ at v_3.

Hence we can choose n_0, n_1 arbitrarily, and the other n_i's are fixed. So $Z_1(K_2)$ is free abelian with rank 2. One basis is $\{ \underbrace{\bar{e}_0 + \bar{e}_3 - \bar{e}_2}_{n_0 = 1,\, n_1 = 0}, \underbrace{\bar{e}_1 + \bar{e}_4 - \bar{e}_2}_{n_0 = 0,\, n_1 = 1} \}$.

Let's call these cycles as $\{\bar{z}_1, \bar{z}_2\}$.

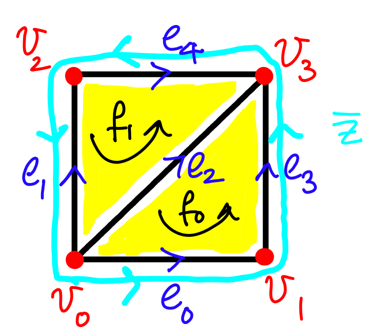


Indeed, any other cycle in K_1 can be written as a sum of $\bar{z}_1$ and $\bar{z}_2$. For instance, let $\bar{z}$ be the 1-cycle $e_0 + e_3 - e_4 - e_1$

Indeed, $\bar{z}$ can be written as $\bar{z}_1 - \bar{z}_2$

$= (\bar{e}_0 + \bar{e}_3 - \bar{e}_2) - (\bar{e}_1 + \bar{e}_4 - \bar{e}_2)$.

that the e_2 portions from $\bar{z}_1$ and $\bar{z}_2$ cancel.



Now let's characterize $B_1(K_2)$. Notice that both 1-cycles $\bar{z}_1$ and $\bar{z}_2$ are also 1-boundaries. Indeed, we have

$\partial_2 \bar{f}_0 = \bar{e}_0 + \bar{e}_3 - \bar{e}_2 = \bar{z}_1$ and $\partial_2 \bar{f}_1 = \bar{e}_2 - \bar{e}_4 - \bar{e}_1 = -\bar{z}_2$.

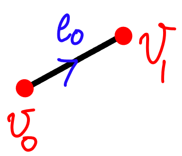
So $B_1(K_2) = Z_1(K_2)$. Hence $H_1(K_2) = Z_1(K_2)/B_1(K_2) = 0$.

(i.e., the first homology group is trivial; there are no 1-cycles that are not 1-boundaries).

Likewise, $H_2(K_2) = 0$. The general 2-chain is $\bar{d} = m_0 \bar{f}_0 + m_1 \bar{f}_1$. And $\partial_2 \bar{d} = 0$ iff $m_0 = m_1 = 0$. There are no 2-cycles. And $H_p(K_2) = 0$ for $p \geq 3$ trivially.

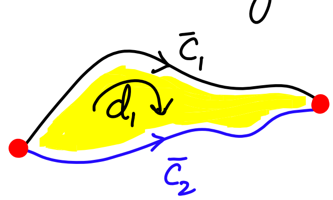
We now present some definitions we will use subsequently.

Def A chain $\bar{c}$ is **carried by a subcomplex** L if $\bar{c}(\sigma) = 0 \;\forall\, \sigma \notin L$. Two p-chains $\bar{c}_1, \bar{c}_2$ are **homologous** if $\bar{c}_1 - \bar{c}_2 = \partial_{p+1} \bar{d}$ for some $(p+1)$-chain $\bar{d}$. → we write $\bar{c}_1 \sim \bar{c}_2$. In particular, if $\bar{c} = \partial_{p+1} \bar{d}$, then $\bar{c}$ is homologous to zero, or we say that $\bar{c}$ **bounds**, i.e., $\bar{c}$ is a **boundary**.

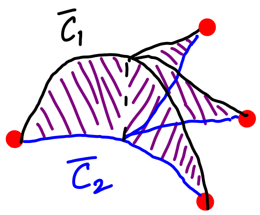


Here, the 2 0-chains v_0 and v_1 are homologous, since $v_1 - v_0 = \partial_1 e_0$.

Consider two 1-chains $\bar{c}_1$, $\bar{c}_2$ representing two 1D curves starting and ending at the same pair of vertices as shown.

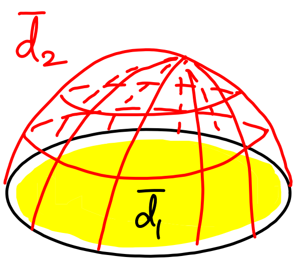


Then $\bar{c}_1$ and $\bar{c}_2$ are homologous here, as $\bar{c}_1 - \bar{c}_2 = \partial_2 \bar{d}_1$, where $\bar{d}_1$ is the 2-chain representing the 2D patch in between $\bar{c}_1$ & $\bar{c}_2$.



Notice that $\bar{c}_1$ and $\bar{c}_2$ need not be just simple open curves. Here, $\bar{c}_1$ and $\bar{c}_2$ both represent Y-shaped 1D curves. Again, $\bar{c}_1 - \bar{c}_2$ is the boundary of the 2D patch in between the two Y-shaped curves.

Consider two 2-chains $\bar{d}_1$ and $\bar{d}_2$, one representing a disc, and another representing the upper hemispherical surface that has the same boundary as the disc.



$\bar{d}_1$ and $\bar{d}_2$ are homologous, as $\bar{d}_1 - \bar{d}_2$ represents the boundary of the 3D solid hemisphere bounded by the two surfaces.