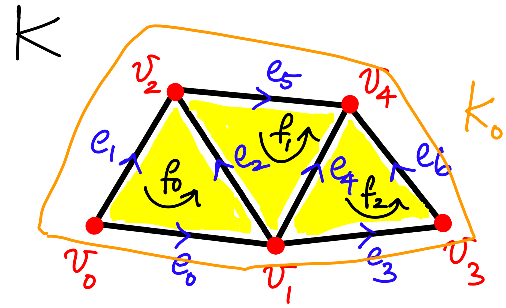
MATH 524 – Lecture 18 (10/19/2023)

Today: * exact sequences
* chain complex

Exact Sequences

What are the relationships between $H_p(K,K_0)$, $H_p(K)$, and $H_p(K_0)$?



Example (same as example 3 in lecture 9)

Here, $H_2(K,K_0) \simeq \mathbb{Z}$. $\bar{r} = \sum_{i=0}^{2} \bar{f}_i$ is a generator.

Also, $H_1(K_0) \simeq \mathbb{Z}$, $\{\bar{z}\}$ is a basis, where $\bar{z} = \bar{e}_0 + \bar{e}_3 + \bar{e}_6 - \bar{e}_5 - \bar{e}_1$.

So $H_2(K,K_0) \simeq H_1(K_0)$.

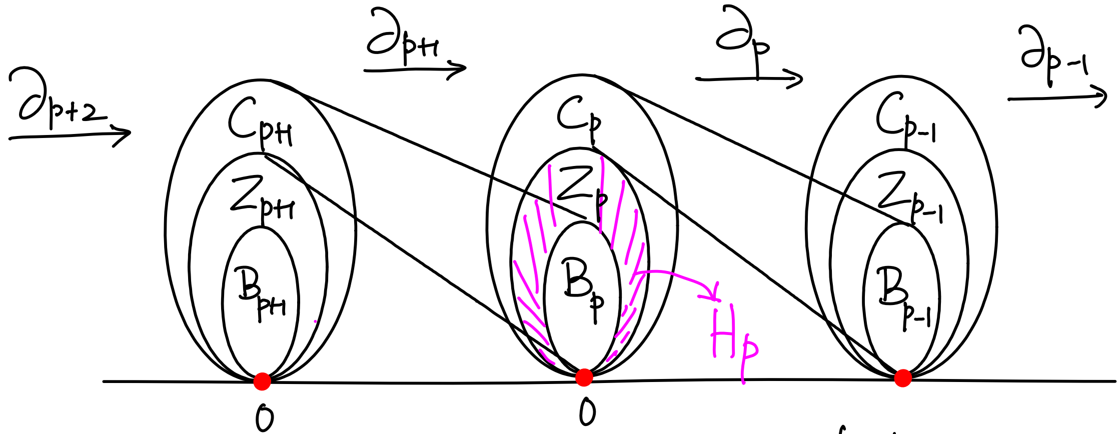
It turns out that $H_2(K,K_0) \simeq H_1(K_0)$ here is not a mere coincidence. To present the general result, we first need to introduce the algebraic machinery of exact sequences – of objects (think groups, rings, etc.) and maps (homomorphisms) between them.

Def Consider a sequence (finite or infinite) of groups and homomorphisms

$$\cdots \xrightarrow{\phi_{i-2}} A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \to \cdots$$

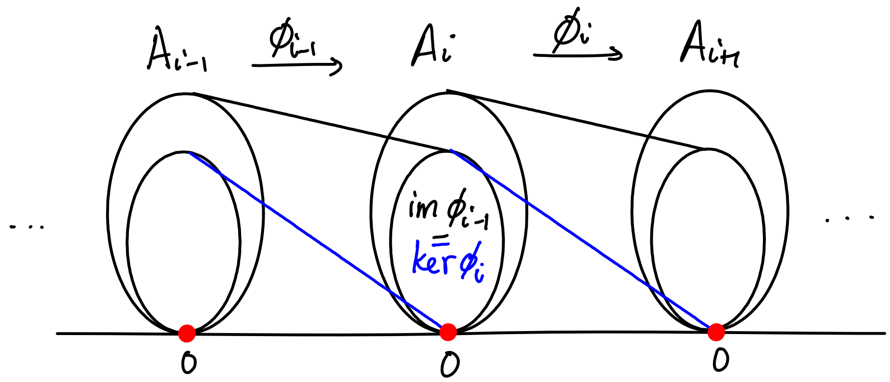
This sequence is **exact** at A_i if image ϕ_{i-1} = kernel ϕ_i. If it is exact everywhere, it is an **exact sequence**. Exactness is not defined at the first and last group of the sequence, if they exist.

The sequence we have seen already, of chain groups and boundary homomorphisms, is **not** exact!



The indices are decreasing left to right here, but that is not an issue. Indeed, notice that $\operatorname{im} \partial_{p+1} = B_p \neq \ker \partial_p = Z_p$.

Here is the picture of exact sequences that we want.

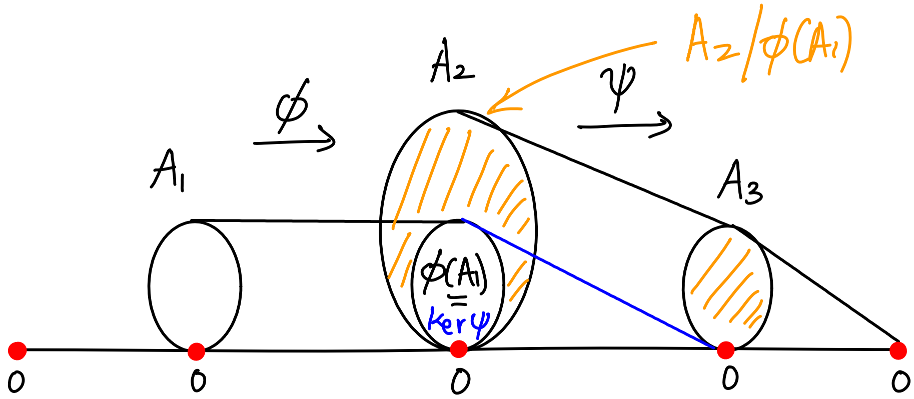


Several results on exact sequences (with abelian groups)

$0 \to$ denotes the trivial group

1. $A_1 \xrightarrow{\phi} A_2 \to 0$ is exact iff ϕ is an epimorphism (surjective/onto).
2. $0 \to A_1 \xrightarrow{\phi} A_2$ is exact iff ϕ is a monomorphism (injective/1-to-1).

3. Suppose the sequence $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ is exact. Such a sequence is called a **short exact sequence** (SES).



Then $A_2/\phi(A_1) = \operatorname{cok}\psi$ is isomorphic to A_3; this isomorphism is induced by ψ. Conversely, if $\psi : A \to B$ is an epimorphism with $\ker\psi = K$, then the sequence

$$0 \to K \xrightarrow{i} A \xrightarrow{\psi} B \to 0$$

is exact, where i is inclusion.

4. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\phi} A_3 \xrightarrow{\beta} A_4$ is exact. Then the following statements are equivalent.
   (i) α is an epimorphism.
   (ii) β is a monomorphism.
   (iii) ϕ is the zero homomorphism.

5. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \to A_3 \to A_4 \xrightarrow{\beta} A_5$ is exact. Then so is the induced sequence $0 \to \operatorname{cok}\alpha \to A_3 \to \ker\beta \to 0$.

It will be instructive to draw diagrams similar to the base case for each of these results!

Def Consider two sequences of groups and homomorphisms having the same index set.

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] & A_{i+1} \arrow[r, "\phi_{i+1}"] \arrow[d, "\alpha_{i+1}"] & \cdots \\
\cdots \arrow[r] & B_i \arrow[r, "\psi_i"] & B_{i+1} \arrow[r, "\psi_{i+1}"] & \cdots
\end{tikzcd}$$

A **homomorphism of the first sequence into the second** is a family of homomorphisms $\alpha_i : A_i \to B_i$ such that each square of maps

$$\begin{tikzcd}
A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] \arrow[dr, phantom, "\square"] & A_{i+1} \arrow[d, "\alpha_{i+1}"] \\
B_i \arrow[r, "\psi_i"] & B_{i+1}
\end{tikzcd}$$

commutes, i.e., $\alpha_{i+1} \circ \phi_i = \psi_i \circ \alpha_i$.

It is an isomorphism of sequences if each α_i is an isomorphism.

We had studied simplicial maps (from K to L), and associated homomorphisms between the chain groups (and its subgroups) in both complexes. Indeed, that set up illustrates the above definition. At the same time, it turns out we could study such collections of groups and homomorphisms in a much more general setting — and not necessarily on a simplicial complex.

Example: chain maps between chain complexes, which we define now.

Def A **chain complex** $\mathcal{C}$ is a family $\{C_p, \partial_p\}$ of abelian groups C_p and homomorphisms ∂_p
$\partial_p : C_p \to C_{p-1}$ such that $\partial_p \circ \partial_{p+1} = 0 \ \forall p$.

The group $H_p(\mathcal{C}) = \ker \partial_p / \operatorname{im} \partial_{p+1}$ is the p-th **homology group** of the chain complex $\mathcal{C}$.

Notice that the chain, cycle, boundary and homology groups, along with boundary homomorphisms does indeed fit this framework — and hence the overloading of notation! At the same time, chain complexes could be much more general! We do need $\partial_p \circ \partial_{p+1} = 0 \ \forall p$ in the general setting.

Now consider two chain complexes $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C_p', \partial_p'\}$. We can define a family of homomorphisms from C_p to C_p' with additional requirements on "connecting" them to ∂_p and ∂_p' as follows. We define $\phi_p : C_p \to C_p'$ to be the homomorphism from the p^{th} abelian group of $\mathcal{C}$ to the p^{th} abelian group of $\mathcal{C}'$, for each p.

ϕ_p should be such that each "square" in the diagram commutes.

ϕ satisfies $\partial_p' \circ \phi_p = \phi_{p-1} \circ \partial_p \ \forall p$.

$$\begin{tikzcd}
\mathcal{C} & \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[d, "\phi_{p+1}"] & C_p \arrow[r, "\partial_p"] \arrow[d, "\phi_p"'] \arrow[dr, phantom, "\square"] & C_{p-1} \arrow[r] \arrow[d, "\phi_{p-1}"] & \cdots \\
\mathcal{C}' & \arrow[r] & C'_{p+1} \arrow[r, "\partial'_{p+1}"] & C'_p \arrow[r, "\partial'_p"] & C'_{p-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p$$

Then entire family of homomorphisms, $\phi_p \; \forall p$, is referred to as a **chain map** from $\mathcal{C}$ to $\mathcal{C}'$ — $\phi : \mathcal{C} \to \mathcal{C}'$.

Recall how we talked about simplicial maps inducing homomorphisms at the homology level. We get the same result in the general setting as well.

A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ induces a homomorphism

$$(\phi_*)_p : H_p(\mathcal{C}) \longrightarrow H_p(\mathcal{C}').$$

We now introduce one more concept related to short exact sequences.

Def Consider a short exact sequence

$$0 \longrightarrow A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \longrightarrow 0.$$

This sequence is said to **split** if the group $\phi(A_1)$ is a direct summand in A_2. So the sequence becomes

$$0 \longrightarrow A_1 \xrightarrow{\phi} \phi(A_1) \oplus B \xrightarrow{\psi} A_3 \longrightarrow 0.$$

where ϕ defines an isomorphism of A_1 with $\phi(A_1)$, and ψ defines an isomorphism of B with A_3.

We end by stating two results on short exact sequences that split. See [M] for details and proofs.

Theorem 23.1 [M] Let $0 \longrightarrow A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \longrightarrow 0$ be exact. Then the following statements are equivalent.

(1) The sequence splits.

(2) There exists a map $p : A_2 \longrightarrow A_1$ such that $p \circ \phi = i_{A_1}$.
identity in A_1

(3) There exists a map $j : A_3 \longrightarrow A_2$ such that $\psi \circ j = i_{A_3}$.
identity in A_3

$$
\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, shift left, "\phi"] & A_2 \arrow[l, shift left, "p"] \arrow[r, shift left, "\psi"] & A_3 \arrow[l, shift left, "j"] \arrow[r] & 0
\end{tikzcd}
$$

Corollary 23.2 [M] Let $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ be exact. If A_3 is free abelian, the sequence splits.