Introduction to Analysis I (Fall 2025)
Midterm Examination

- There are **six** problems in this exam, all presented in the next page.
- The total points (given in parentheses) add to 100.
- This is a **CLOSED RESOURCES** exam. You are not supposed to use any external resources—checking textbooks, notes, cheat/summary sheets, AI/LLM tools, phone and internet resources, or communicating with other people about the exam are all **not permitted**.
- You **must start your exam** by writing down the following statement word-by-word, and signing under the same.

  I promise that I will not use any external resources while working on this exam. I will not search the internet for any hints on the problems in the exam, and I will not look at any textbook, notes, handouts, or use online search and online resources, including any AI/LLM tools. I will also not communicate with any one else about this exam while working on the same.
  —*Signature*

- You **must end your exam** by writing down the following **second** statement word-by-word, and **again signing** under the same.

  As promised, I did not use any external resources while working on this exam.
  —*Signature*

- You **must email your submission as a SINGLE PDF file** to kbala@wsu.edu. You are welcome to write answers by hand, and scan the writings.
- Your **file name should identify you** in the usual manner. If you are Uncle Tricky, you should name your submission UncleTricky_Midterm.pdf (and **NOT** Uncle_Tricky or "Uncle Tricky" or ...). You could add anything more to your filename *after* these terms, e.g., UncleTricky_Midterm_Math401.pdf. Please avoid white spaces in the file name :-).
- **Begin the SUBJECT of your email submission with the same `FirstnameLastname`, e.g., "UncleTricky Midterm submission".**
- This exam must be emailed to me **before 10:00 PM on Tuesday, October 7**.

---

1. (16) Prove the following statements for sets A, B, C and for the family of sets $\mathcal{A}$. The *symmetric difference* of two sets A and B is defined as $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$.

$$(A \,\Delta\, B) \cap C = (A \cap C) \,\Delta\, (B \cap C) \tag{1}$$

$$\left(\bigcup_{A_i \in \mathcal{A}} A_i \right)^c = \bigcap_{A_i \in \mathcal{A}} A_i^c \tag{2}$$

2. (16) Consider the relation R defined as follows.

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}. \tag{3}$$

Check if each of the three properties required for a relation to be an equivalence relation holds for R or not. Based on your conclusions, decide if R is an equivalence relation or not.

3. (16) Let S be the set of all points lying on a straight line in $\mathbb{R}^2$. Is the set S countable? You need to clearly justify your Yes/No response.

4. Recall that, informally, a sequence is Cauchy if we can go "far out enough" such that the difference between *any* two terms that come after is arbitrarily small.

   (a) (10) Consider the sequence $\{1/n\}$ over the set $X = (0, 1] \cap \mathbb{Q}$. Show that the sequence is Cauchy. Is the sequence convergent? Justify your answer.

   (b) (9) Let $\{a_n\}$ be Cauchy sequence in $\mathbb{R}$. Show that this sequence has a subsequence $\{a_{n_k}\}$ such that
   $$|a_{n_{k+1}} - a_{n_k}| < 1/2^k \quad \text{for all } k \in \mathbb{N}\,.$$

5. (16) Use the definition of continuity (using $\epsilon, \delta > 0$) of a function ***directly*** to show that if $f, g, h \to \mathbb{R}$ are all continuous functions at $x = a$ then the function $fg + h$ is also continuous at $x = a$. You **cannot** use results we saw in class or homework about continuity of fg or of sums of continuous functions as part of your proof.

6. (17) Assume that the sequence $\{a_n\}$ is nonnegative and converges to a. Also assume that for the sequence $\{b_n\}$, $\limsup_{n\to\infty} b_n = b < \infty$ is positive. Show that $\limsup_{n\to\infty} a_n b_n = ab$.