MATH 220 - Lecture 21 (10/29/2013)

Basis for a subspace H

A linearly independent set in H that spans H is a basis of H.

Set of all unit vectors, $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is the standard basis for $\mathbb{R}^n$.

(bases is the plural of basis).

Bases for Nul A can be found from the parametric vector form of the solutions to $A\bar{x} = \bar{0}$.

The collection of vectors that are scaled by each parameter (or free variable) gives a basis for Nul A.

Basis for Col A: Pivot columns in A (in the original matrix, and **not** in the echelon form).

It is important to remember that the columns in any basis for A should be chosen from the original matrix A.

Dimension of a subspace H: The number of vectors in any basis of H. (denoted by dim H)

Any basis for H has the same number of vectors. This number is its dimension.

Problem

Find bases for Col A and Nul A, and their dimensions.

row equivalent

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

$\left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\}$ is a basis of Col A.

Notice that the vectors in the basis for Col A come from A, and not from its echelon form.

dim Col A = 2, as there are 2 pivot columns.

$$\begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \times \frac{1}{4}} \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 6R_2} \begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2 x_4 are free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} t, \quad s, t \in \mathbb{R}.$$

A basis for Nul A is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} \right\}$.

dim Nul A is 2, as there are 2 free variables.

$\dim(\{\bar{0}\}) = ?$ By following the definition, we get $\dim(\{\bar{0}\}) = 0$.

Since dim H is the number of vectors in any basis of H, and $\{\bar{0}\}$ has no basis. Notice that $\{\bar{0}\}$ is LD, and hence has no basis.

Another Problem

row equivalent

pivot columns

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find bases and dimensions of Col A and Nul A.

Columns 1, 2, and 4 are pivot columns. So, a basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$\dim(\text{Col}A) = 3$, as there are 3 pivot columns.

$\dim(\text{Nul }A) = 2$, as there are two free variables.

$$\begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 3 & 5 & -10 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 5R_3}$$

$$\begin{array}{c} \begin{matrix} & & x_3 & & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

are free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} x_5,$$

x_3, x_5 are real numbers.

A basis for Nul A is $\left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$

Problem

$A_{3\times 5}$ has 3 pivot columns.

Is $\text{Col}\, A = \mathbb{R}^3$? Yes! As there are 3 pivots, there is a pivot in every row. So columns of A span $\mathbb{R}^3$.

Is $\text{Nul}\, A = \mathbb{R}^2$? No! Every solution to $A\bar{x}=\bar{0}$ sits in $\mathbb{R}^5$. But, $\dim(\text{Nul}\, A) = 2$ here.

Rank of a matrix A (A is m×n)

Def The rank of an m×n matrix A, rank(A) or rank A, is the dimension of Col A. So

$$\text{rank}(A) = \#\text{ pivot columns in } A.$$

Rank Theorem:

$$\boxed{\underbrace{\text{rank}(A)}_{\#\text{ pivot columns}} + \underbrace{\dim(\text{Nul}\, A)}_{\substack{\#\text{ free variables} \\ \text{(or, nonpivot columns)}}} = \underbrace{n}_{\text{total }\#\text{ columns}}}$$

$n = \#$ columns in A.

Problem

What is rank(A) when A is 4×5 and Nul A is 3-dimensional?

$\text{rank}(A) + \dim(\text{Nul}\, A) = 5.$ So $\text{rank}(A) = 5-3 = 2.$

Problem

Create a 3×4 matrix A with $\dim(\text{Nul } A) = 2$ and $\dim \text{Col } A = 2$.

So, A has 2 pivot columns and 2 free variables

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ will work.

Basis Theorem: Let H be a p-dimensional subspace of $\mathbb{R}^n$. Then any LI set of exactly p elements (vectors) in H is a basis for H. Also, any set of p elements in H that spans H is a basis.

e.g., any 3 LI vectors in $\mathbb{R}^3$ will be a basis for $\mathbb{R}^3$.

We have 3 qualifications, or properties, that a (potential) basis B of H has to satisfy.

1. B spans H.
2. B is LI.
3. # vectors in B = $\dim H$.

If B satisfies any two of these three properties, it is automatically a basis, i.e., the third property is satisfied in this case.

Invertible Matrix Theorem (IMT) $A_{n \times n}$

Recall, (a) A is invertible.

We add equivalent statements related to Col A, Nul A, and their dimensions now.

(m) Columns of A form a basis for $\mathbb{R}^n$.

(n) $\operatorname{Col} A = \mathbb{R}^n$.

(o) $\dim \operatorname{Col} A = n$.

(p) $\operatorname{rank} A = n$.

(q) $\operatorname{Nul} A = \{\bar{0}\}$.

(r) $\dim \operatorname{Nul} A = 0$.