MATH 524 – Lecture 22 (11/02/2023)

Today: * Proof of Mayer-Vietoris Sequence (MVS)
* Application of MVS

Recall: Mayer-Vietoris Sequence (MVS): $K', K'' \subseteq K$, $K' \cup K'' = K$, $K' \cap K'' = A$.

$$\cdots H_p(A) \longrightarrow H_p(K') \oplus H_p(K'') \longrightarrow H_p(K) \xrightarrow{\partial} H_{p-1}(A) \longrightarrow \cdots$$

Proof $\quad 0 \longrightarrow \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K') \oplus \mathcal{C}(K'') \xrightarrow{\psi} \mathcal{C}(K) \longrightarrow 0$

$\mathcal{C}(K') \oplus \mathcal{C}(K'')$: chain groups are $C_p(K') \oplus C_p(K'')$

$$\partial(\bar{c}', \bar{c}'') = (\partial' \bar{c}', \partial'' \bar{c}'')$$

Define ϕ, ψ

$$\begin{tikzcd} & K' \arrow[dr, "j'"] & \\ A \arrow[ur, "i'"] \arrow[rr, "k"] \arrow[dr, "i''"'] & & K \\ & K'' \arrow[ur, "j''"'] & \end{tikzcd}$$

$$\phi(\bar{c}) = (i'_{\#}(\bar{c}), -i''_{\#}(\bar{c})), \text{ and}$$

$$\psi(\bar{c}', \bar{c}'') = (j'_{\#}(\bar{c}') + j''_{\#}(\bar{c}''))$$

We can verify that ϕ and ψ are indeed chain maps.

Check for exactness:

ϕ is injective, as both $i'_{\#}$ and $i''_{\#}$ are just inclusions of chains.
Also, ψ is surjective. Given $\bar{c} \in C_p(K)$, let $\bar{c}'$ be its part carried by K', and then $\bar{c} - \bar{c}'$ carried by K'', and we get $\psi(\bar{c}', \bar{c} - \bar{c}') = \bar{c}$ $(= \bar{c}' + \bar{c} - \bar{c}')$.

To confirm exactness at the middle term, note that

$$\psi\phi(\bar{c}) = k_{\#}(\bar{c}) - k_{\#}(\bar{c}) = 0$$

recall the "−" in the definition of ϕ!

Conversely, if $\psi(\bar{c}', \bar{c}'') = 0$, then $\bar{c}' = -\bar{c}''$ as chains of K. Since $\bar{c}' \in K'$ and $\bar{c}'' \in K''$, they must be carried by $A = K' \cap K''$ (as $\bar{c}' = -\bar{c}''$). Hence $(\bar{c}', \bar{c}'') = (\bar{c}', -\bar{c}') = \phi(\bar{c}')$ as needed.

The homology for the middle chain complex in dimension p is

$$\frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}} = \frac{\ker \partial'_p \oplus \ker \partial''_p}{\operatorname{im} \partial'_{p+1} \oplus \operatorname{im} \partial''_{p+1}} \simeq H_p(K') \oplus H_p(K'').$$

The Meyer-Vietoris (MV) sequence now follows from the zig-zag lemma. A similar argument can be used to get the Meyer-Vietoris sequence in reduced homology groups (when $A \neq \emptyset$).

More details on structure of $H_p(K)$ $\forall p$:

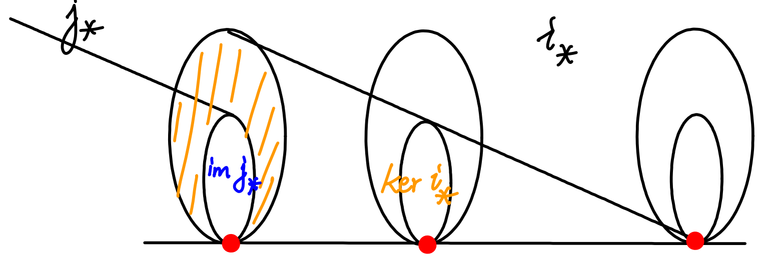
$$\cdots \longrightarrow H_p(A) \xrightarrow{i_\#} H_p(K') \oplus H_p(K'') \xrightarrow{j_\#} H_p(K) \xrightarrow{\partial_*}$$

$$\hookrightarrow H_{p-1}(A) \xrightarrow{i_\#} H_{p-1}(K') \oplus H_{p-1}(K'') \xrightarrow{j_\#} H_{p-1}(K) \to$$

We write the part of the sequence for each dimension in one level, or "floor". We will come back to this representation later...

Consider the connecting maps now.

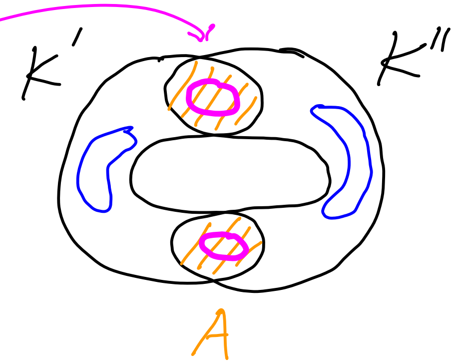
$$\longrightarrow H_p(K') \oplus H_p(K'') \xrightarrow{j_\#} H_p(K) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_\#} H_{p-1}(K') \oplus H_{p-1}(K'') \longrightarrow \cdots$$



Exactness of the Meyer-Vietoris sequence at $H_p(K)$ tells us that this group is a direct sum of the image of $j_*: H_p(K') \oplus H_p(K'') \longrightarrow H_p(K)$ with the kernel of $i_*: H_{p-1}(A) \longrightarrow H_{p-1}(K') \oplus H_{p-1}(K'')$.

We use exactness at $H_{p-1}(A)$ here.

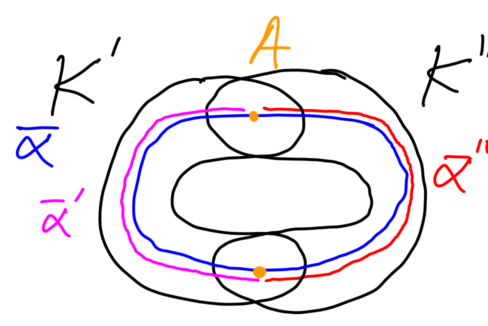
Hence we can distinguish two types of homology classes in K — one class in $\mathrm{im}\, j_*$ that lives in K' or K'', and the other one lives in both, as illustrated here.



A class in $\ker i_* \equiv (p-1)$-cycle $\bar{r}_{p-1} \in A$ that bounds both in K' and K''. If we write

$$\bar{r}_{p-1} = \partial \bar{\alpha}_p' = -\partial \bar{\alpha}_p'' \quad \text{where } \bar{\alpha}_p' \in C_p(K') \text{ and } \bar{\alpha}_p'' \in C_p(K''),$$

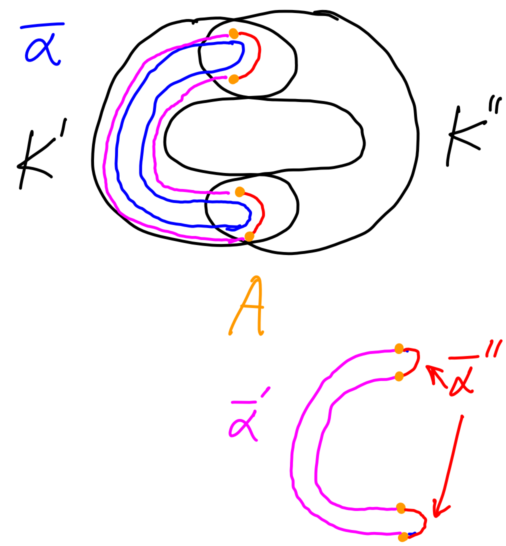
then $\bar{\alpha}_p = \bar{\alpha}_p' + \bar{\alpha}_p''$ is a cycle in K which represents the second type of the class.



Here is another example. The 1-cycle $\bar{\alpha}$ decomposes into $\bar{\alpha}'$ and $\bar{\alpha}''$. Their boundaries (∂' and ∂'' in K' and K'', respectively) is the 0-chain made of 2 points (with signs reversed) which is a reduced 0-cycle in A.

↳ between K' and K''

What about this 1-cycle $\bar{\alpha}$? This cycle also represents a homology class of the <u>second</u> type, with one possible decomposition of $\bar{\alpha}$ into $\bar{\alpha}'$ and $\bar{\alpha}''$ illustrated below.



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The connecting homo'm ∂_* can be explicitly defined as follows. Consider a cycle $\bar{z} \in K$. We can choose $\bar{c}' \in K'$ and $\bar{c}'' \in K''$ s.t. $\bar{z} = \bar{c}' + \bar{c}''$. $\bar{c}'$ and $\bar{c}''$ need not be cycles themselves, but it must hold that

$\partial \bar{c}' = -\partial \bar{c}''$, as $\partial \bar{z} = \partial(\bar{c}' + \bar{c}'') = 0$.

Also, $\partial \bar{c}'$ and $\partial \bar{c}''$ must both be carried by $A = K' \cap K''$.

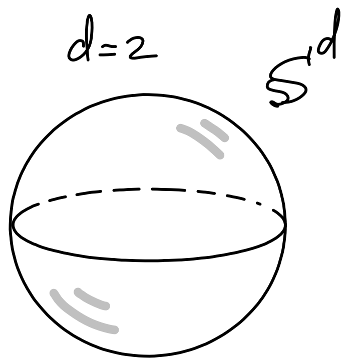
We define $\partial_* \{\bar{z}\} = \{\partial \bar{c}'\}$, or $\{-\partial \bar{c}''\}$, equivalently.

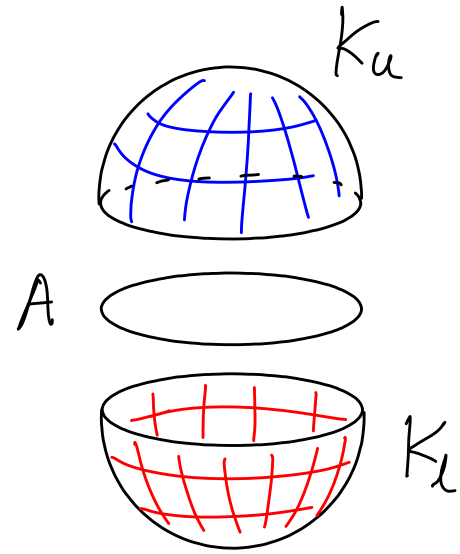
Example 1 Homology of $\mathbb{S}^d$ (d-sphere): We want to show:

$$\tilde{H}_p(\mathbb{S}^d) \simeq \mathbb{Z} \text{ if } p=d, \text{ and}$$

$$\tilde{H}_p(\mathbb{S}^d) = 0 \text{ if } p \neq d.$$

We set $\mathbb{S}^d = K_u \cup K_\ell$, where K_u, K_ℓ are the upper and lower hemispheres, respectively. And $A = K_u \cap K_\ell$ is the equator.





Note that $K_u, K_\ell \approx \mathbb{B}^d$ (d-disc or d-ball) and $A \approx \mathbb{S}^{d-1}$! We compute $\tilde{H}_p(\mathbb{S}^d)$ inductively using the reduced homology MVS.

$$\cdots \underset{A}{\tilde{H}_p(\mathbb{S}^{d-1})} \longrightarrow \underset{0}{\tilde{H}_p(K_u)} \oplus \underset{0}{\tilde{H}_p(K_\ell)} \longrightarrow \underset{K}{\tilde{H}_p(\mathbb{S}^d)} \xrightarrow{\partial_*} \underset{A}{\tilde{H}_{p-1}(\mathbb{S}^{d-1})} \rightarrow \cdots$$