MATH 524 – Lecture 19 (10/24/2023)

Today:
* connecting homom
* long exact sequence
* exact homology sequence of a pair

Exact homology sequence of a pair K, K_0

Goal: Connect $H_p(K,K_0)$, $H_p(K)$, $H_p(K_0)$

We first need to define a homomorphism connecting $H_p(K,K_0)$ and $H_{p-1}(K_0)$

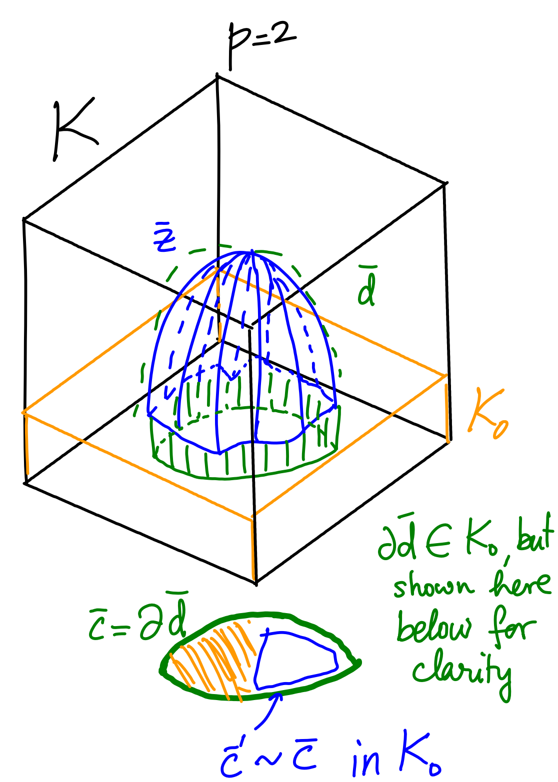
$$\partial_* : H_p(K,K_0) \longrightarrow H_{p-1}(K_0)$$

We call this homomorphism the **homology boundary homomorphism** or the **connecting homomorphism**.

Consider a cycle $\bar{z} \in C_p(K,K_0)$.

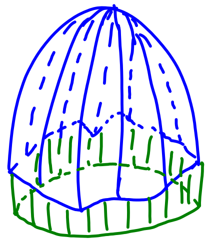
We consider the class $\{\bar{z}\}$ as the coset modulo $C_p(K_0)$ of a p-chain $\bar{d}$ of K such that $\partial \bar{d}$ is carried by K_0. Notice that $\partial \bar{d}$ is automatically a $(p-1)$-cycle of K_0. We define

$$\partial_* \{\bar{z}\} = \{\partial \bar{d}\}$$

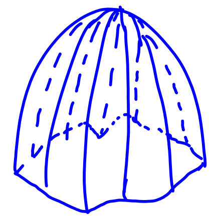


We detail the algebraic construction/definition in this fashion.

$\bar{d}$



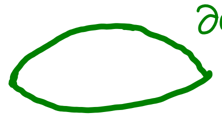
$\bar{z}$



$$\begin{tikzcd} & C_p(K) \arrow[r, "\pi_\#"] \arrow[d, "\partial"] & C_p(K, K_0) \\ C_{p-1}(K_0) \arrow[r, "i_\#"] & C_{p-1}(K) & \end{tikzcd}$$

$\pi_\#(\bar{d}) = \bar{z}$

$\bar{c}$
$(p-1)$-cycle



$\partial \bar{d} \in K_0$

$\{\bar{c}\} =: \partial_* \{\bar{z}\}$

$i : K_0 \to K$ and $\pi : (K, \phi) \to (K, K_0)$ are inclusions.

$i_\#$ is an inclusion, $\pi_\#$ is projection of $C_p(K)$ onto $C_p(K)/C_p(K_0)$.

So we define $\partial_* \{\bar{z}\}$ by a "zig-zag" process.

Def A **long exact sequence** is an exact sequence whose index set is $\mathbb{Z}$. So the sequence is infinite in both directions. It could begin or end with an infinite string of trivial groups.

Theorem 23.3 [M] (**The exact homology sequence of a pair**)

Let K_0 is a subcomplex of K. There is a long exact sequence

$$\cdots \to H_p(K_0) \xrightarrow{i_\#} H_p(K) \xrightarrow{\pi_\#} H_p(K,K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

where $i: K_0 \to K$ and $\pi: (K,\phi) \to (K,K_0)$ are inclusions and ∂_* is the connecting homomorphism. There exists a similar long exact sequence in reduced homology.

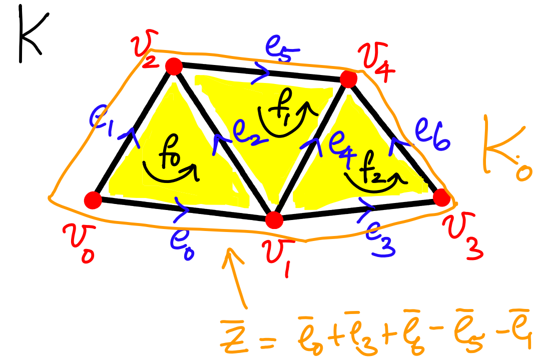
$$\cdots \to \tilde{H}_p(K_0) \xrightarrow{i_\#} \tilde{H}_p(K) \xrightarrow{\pi_\#} H_p(K,K_0) \xrightarrow{\partial_*} \tilde{H}_{p-1}(K_0) \longrightarrow \cdots$$

It turns out $\tilde{H}_p(K,K_0) = H_p(K,K_0)$ as long as $K_0 \neq \phi$. Essentially, relative homology groups are already reduced.

One direct use of the above result is in figuring out the structure of $H_p(K,K_0)$ when the structures of $H_p(K)$ and $H_p(K_0)$ are known. In many cases, the latter homology groups could be characterized more easily, and hence could be used in conjunction with this exact homology sequence to identify $H_p(K,K_0)$.

We apply this result to a few examples.

1. We had seen that ← in Lecture 12
$H_2(K,K_0) \simeq \mathbb{Z}$ with $\bar{r} = \sum_{i=0}^{2} f_i$ being a generator.



Also, $H_1(K_0) \simeq \mathbb{Z}$ with $\bar{z}$ being a generator.

Notice that $\partial \bar{r} = \bar{z}$. In this case $\partial_*: H_2(K,K_0) \longrightarrow H_1(K_0)$ is an isomorphism. We could reach the same conclusion using the exact sequence result. A portion of the long exact sequence is

$$\underbrace{H_2(K)}_{=0} \longrightarrow H_2(K,K_0) \xrightarrow{\partial_*} H_1(K_0) \longrightarrow \underbrace{H_1(K)}_{=0}.$$

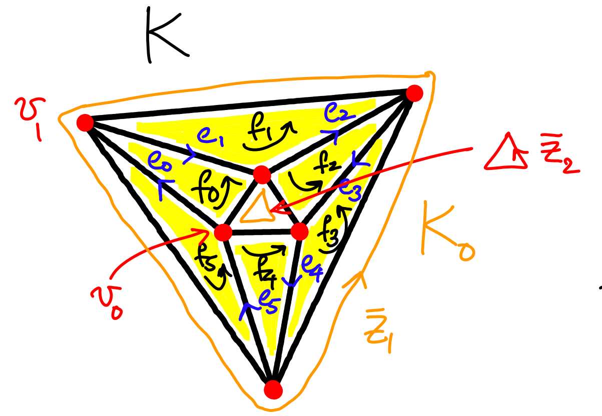
$H_2(K)$ and $H_1(K)$ are both trivial, and hence ∂_* is both a monomorphism and an epimorphism, i.e., it's an isomorphism.

There are no 2-cycles to start with. → Notice that any 1-cycle in K is also a 1-boundary. More intuitively, K has no holes.

→ Recall results 1 and 2 from Lecture 18 on exact sequences!

2. Consider the annulus we saw in Lecture 12.

$H_2(K,K_0) = ?$ $\quad H_1(K,K_0) = ?$



Consider reduced homology (for $\tilde{H}_0(K_0)$).

Recall that with $\bar{r} = \sum_{i=0}^{5} \bar{f}_i$, $\partial \bar{r} = \bar{z}_1 - \bar{z}_2$.

Also, $\partial_1 \bar{e}_0 = v_1 - v_0$.

K_0 consists of the outer and inner perimeters, both oriented CCW.

We consider the relevant portion of the exact homology sequence:

$$H_2(K) \xrightarrow{0} H_2(K,K_0) \xrightarrow{(\partial_*)_2} H_1(K_0) \xrightarrow{(i_*)_1} H_1(K) \xrightarrow{(\pi_*)_1} H_1(K,K_0) \xrightarrow{(\partial_*)_1} \tilde{H}_0(K_0) \xrightarrow{(i_*)_0} \tilde{H}_0(K)$$

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & ? & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & ? & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \mathbb{Z} & & \{\bar{z}_1\},\{\bar{z}_2\} & & \{\bar{z}_1\} & & \mathbb{Z} & & \{v_1 - v_0\} & & \\ & & & & & & \text{or } \{\bar{z}_2\} & & & & & & \end{array}$$

If $i: K_0 \to K$ is inclusion, i_* maps both $\{\bar{z}_1\}$ and $\{\bar{z}_2\}$ to say, $\{\bar{z}_1\}$. So $(i_*)_1$ is an epimorphism, and $\ker (i_*)_1 \simeq \mathbb{Z}$, and it is generated by $\{\bar{z}_1\} - \{\bar{z}_2\}$. Hence, we get that $(\pi_*)_1$ is the zero homomorphism. Equivalently, notice that any $\bar{z} \in H_1(K)$ is homologous to $\bar{z}_1$ (or $\bar{z}_2$), so is projected out by π_* in $H_1(K,K_0)$.

So, we have

$$\xrightarrow{0} H_1(K, K_0) \xrightarrow{(\partial_*)_1} \underset{\mathbb{Z}}{\tilde{H}_0(K_0)} \xrightarrow{0} 0$$

$\Rightarrow (\partial_*)_1$ is an isomorphism, so $H_1(K, K_0) \simeq \mathbb{Z}$.

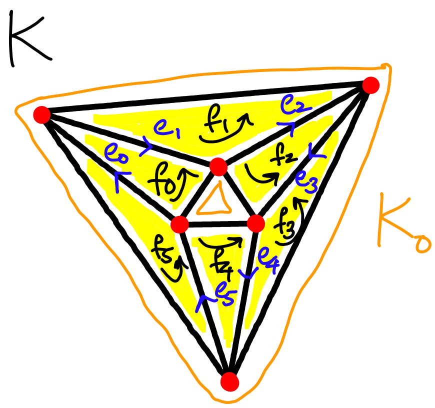
It is generated by, e.g., $\{\bar{e}_0\}$ with $\partial \bar{e}_0 = v_1 - v_0$.

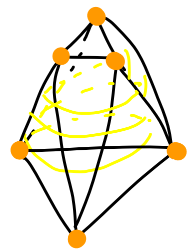
Again, by applying results 1 and 2 from Lecture 19 on exact sequences here, we notice $(\partial_*)_1$ is both an epimorphism and a monomorphism

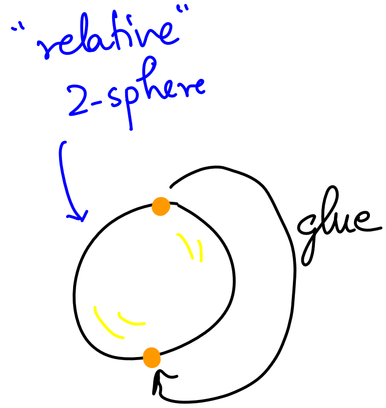
We also get $\operatorname{im}(\partial_*)_2 = \ker(i_*)_1$ and $(\partial_*)_2 : H_2(K, K_0) \longrightarrow \ker(i_*)_1$ is an isomorphism. Hence $H_2(K, K_0) \simeq \mathbb{Z}$. It is generated by $\bar{r} = \sum_{i=0}^{5} \bar{f}_i$, as $\partial_2 \bar{r} = \bar{z}_1 - \bar{z}_2$, which in turn generates $\ker(i_*)_1$, as we noted previously.

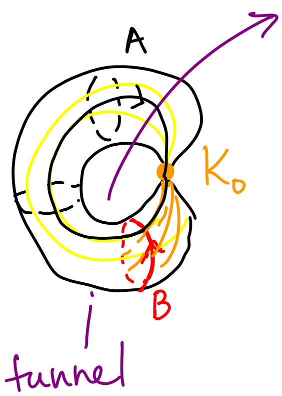
While the ~~for~~ formal method works, it also helps to think intuitively, how the complex looks after "shrinking" all of K_0 to a point.

Think about shrinking both $\bar{z}_1$ and $\bar{z}_2$ (which comprise K_0) to a point each, and then "gluing" these two points.

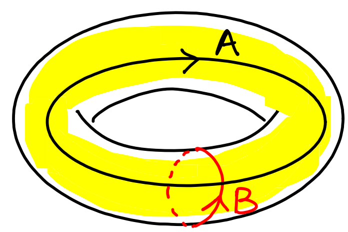








← This object is the pinched torus. Compare this object to the torus.



Notice that while the tunnel loop (A) still exists, the handle loop (B) is now a boundary - it bounds the two chain from the pinched point (representing K_0) to B (looks like a cap). Hence, $H_1(K,K_0) \simeq \mathbb{Z}$.

Also, there is still one enclosed space, or void, and hence $H_2(K,K_0) \simeq \mathbb{Z}$ as well here.