

# MATH 529 – Lecture 17 (03/05/2024)

Today: \* Smith normal form of  $[a_p]$   
\* read off  $z_p, b_p$ , and maintain bases

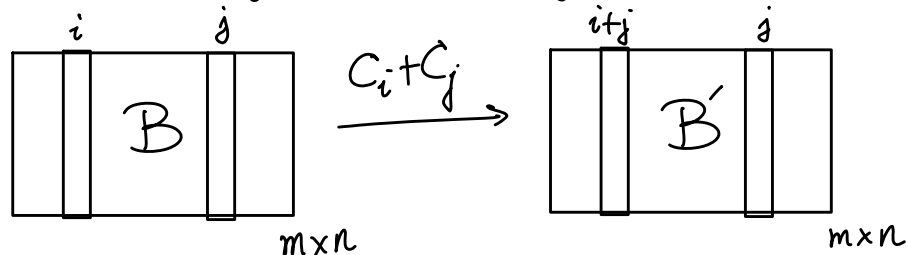
Recall  $[a_p]$ :  $m \times n$   $\{0,1\}$ -matrix (over  $\mathbb{Z}_2$ ) when  $K$  has  $m$   $(p-1)$ -simplices and  $n$   $p$ -simplices.  
 $[a_p]_{ij} = \begin{cases} 1 & \text{if } \tau_i \leq \sigma_j, 0, \text{ otherwise} \end{cases}$

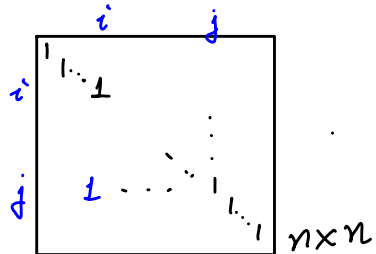
EROs on  $A_{m \times n}$ :  $R_i \rightleftharpoons R_j$  (swap rows  $i$  and  $j$ )  
over  $\mathbb{R}, c \in \mathbb{R}$ , and  $R_i \leftarrow R_i + cR_j$  ( $R_i + cR_j$ , replacement)  
over  $\mathbb{Z}, c \in \mathbb{Z}$ .  $R_i \leftarrow cR_i, c \neq 0$  ( $cR_i$ , scaling)

Over  $\mathbb{Z}_2, c \in \{0,1\}$ , and only swap and replacement need to be considered. We write  $R_i \rightleftharpoons R_j, R_i + R_j$ , in short.

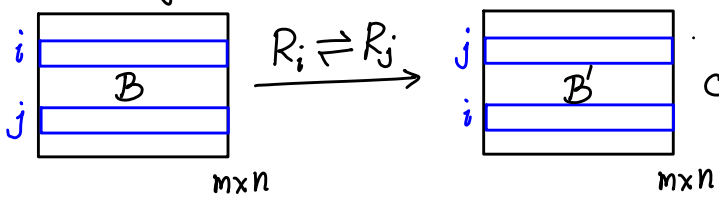
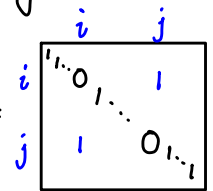
ECOs are defined similarly:  $C_i \rightleftharpoons C_j, C_i + C_j$  (over  $\mathbb{Z}_2$ ).

ECOs can be represented by multiplication on the right by the corresponding **elementary matrix**.

e.g.,  can be written as

$B' = BV$ , where  $V =$  

Similarly, EROs can be represented by multiplication the left by an elementary matrix.

 can be written as  $B' = UB$  for  $U =$  

# Smith Normal Form (SNF)

This is the analogue of reduced row echelon form (RREF) of a matrix, from linear algebra. SNF is defined over  $\mathbb{Z}$ , in general, and accounts for both EROs and ELOs.

**Def** A matrix  $B \in \mathbb{Z}^{m \times n}$  is in Smith normal form if

$$B = \begin{bmatrix} D & O \\ O & O \end{bmatrix} = \left[ \begin{array}{c|c} \begin{matrix} d_1 & & \\ & \ddots & \\ & & d_\ell \end{matrix} & O \\ \hline O & O \end{array} \right] \text{ where}$$

Each 'O' is a submatrix of all zeros.

$$d_i \in \mathbb{Z}, d_i \geq 1 \text{ and } d_1 | d_2 | d_3 | \dots | d_\ell.$$

↳ "divides":  $a|b$  means  $a$  divides  $b$ .

When working over  $\mathbb{Z}_2$ , we will get

$$\text{SNF}([a_p]) = \left[ \begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & O \\ \hline O & O \end{array} \right].$$

In more detail, we will have the following structure for  $\text{SNF}([a_p])$ .

$$\text{SNF}([a_p]) = \begin{array}{c} \begin{array}{c} \xrightarrow{\quad s_p = \text{rank } C_p \quad} \\ \xrightarrow{\quad \text{rank } Z_p = s_p \quad} \\ \begin{array}{|c|} \hline \begin{array}{c} \begin{array}{c} \begin{array}{c} 1 \quad 1 \quad \dots \quad 1 \end{array} \\ \hline \end{array} \\ \hline \end{array} \\ \xrightarrow{\quad \text{rank } C_{p-1} = s_{p-1} \quad} \end{array} \end{array}$$

We get  $z_p$  and  $b_{p-1}$  from  $\text{SNF}([a_p])$ . To compute  $\beta_p$ , we need to know also  $b_p$ , which is obtained from  $\text{SNF}([a_{p+1}])$ .

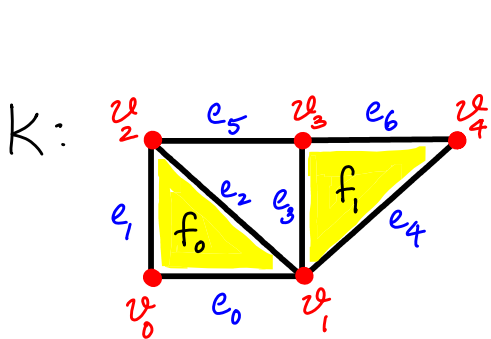
Overall, we can write  $\text{SNF}([\partial_p]) = U_{p-1} [\partial_p] V_p$ , where  $U_{p-1}$  captures all EROs performed, and  $V_p$  captures all ECOs performed. We also get information about bases of  $Z_p$  and  $B_{p-1}$  from  $U_{p-1}$  and  $V_p$ .

More specifically, a basis for the cycle group  $Z_p$  is encoded in the last  $z_p$  columns of  $V_p$ . Similarly, a basis for the boundary group  $B_{p-1}$  is encoded in  $U_{p-1}$  — in the first  $b_{p-1}$  columns of  $U_{p-1}^{-1}$ , to be exact.

Thus we can reduce each  $[\partial_p]$  to SNF, and in that process, compute all  $\beta_p$ , and also identify bases for  $Z_p$  and  $B_p$ . In the following illustration, we keep track of the bases as we perform the corresponding EROs or ECOs on  $[\partial_p]$ . Later on, we describe an algorithm that does all the operations in a unified manner.

### Example

We consider our favorite example and reduce each  $[\partial_p]$  to SNF for  $p=0,1,2$ .



$$[\partial_2] = \begin{matrix} & f_0 & f_1 \\ \begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}$$

$$[\partial_0] = \begin{matrix} & v_0 & v_1 & v_2 & v_3 & v_4 \\ 1 & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

default, as there are no (-1)-dimensional simplices

$$[\partial_1] = \begin{matrix} & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 1 & & & & & \\ & 1 & & 1 & & & \\ & & 1 & & & 1 & \\ & & & 1 & & & \\ & & & & 1 & & 1 \\ & & & & & 1 & 1 \end{bmatrix} \end{matrix}$$

(entries not listed are zeros).

We reduce each  $[2_p]$  to SNF.

We consider  $[2_0]$  and  $[2_2]$  first, since they are simpler compared to  $[2_1]$  (we consider that the last).

$$[2_0] = 1 \begin{bmatrix} v_0 & v_1 & v_2 & v_3 & v_4 \\ | & | & | & | & | \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[\text{for } j=2, \dots, 5]{C_j + 2C_1} 1 \begin{bmatrix} v_0 & v_1+v_0 & v_2+v_0 & v_3+v_0 & v_4+v_0 \\ | & | & | & | & | \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$z_0 = 4$

But  $s_0 = 5 = z_0!$

We had noticed previously that every 0-chain is also a 0-cycle, as vertices have empty boundaries. So, we should have got  $s_0 = z_0 = 5$  here! We will address this discrepancy in  $p=0$  soon. For now, work with  $z_0 = s_0 = 5$ .

A basis for  $B_0(K) = \{v_j + v_0 \mid j=1,2,3,4\}$ . A collection of even pairs of vertices, such that any even set of vertices can be written as a combination of these pairs.

Recall that every 0-chain is also a 0-cycle since it has no boundary (when we work over  $\mathbb{Z}_2$ ). And every 0-chain with an even number of vertices is a 0-boundary (assuming  $K$  is connected).

Adding any one vertex (by itself) to the above basis will give another basis for  $Z_0$  (and hence  $C_0$ ), e.g.,  $\{v_j + v_0\}_{j=1}^4, v_0\}$ . Recall that  $\{v_j\}_{j=0}^4$  gives the default (elementary chain) basis for  $Z_0(G)$ .

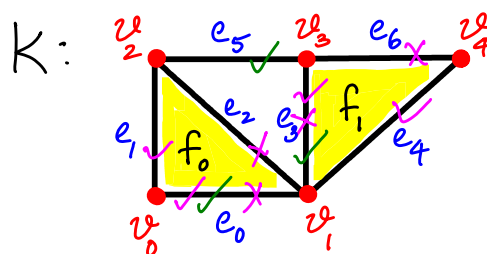
Notation:  $e_{ijk} \equiv e_i + e_j + e_k$

$$[\partial_2] = \begin{matrix} & f_0 & f_1 \\ \begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix} \xrightarrow{\begin{matrix} R_2+R_1 \\ R_3+R_1 \\ R_5+R_4 \\ R_7+R_4 \end{matrix}} \begin{matrix} & f_0 & f_1 \\ \begin{matrix} e_0 \\ e_{01} \\ e_{02} \\ e_3 \\ e_{34} \\ e_5 \\ e_{36} \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{matrix} \xrightarrow{R_2 \Leftarrow R_4} \begin{matrix} & f_0 & f_1 \\ \begin{matrix} e_0 \\ e_3 \\ e_{02} \\ e_{01} \\ e_{34} \\ e_5 \\ e_{36} \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{matrix}$$

there are no 2-cycles here!  
 $z_2 = 0, b_1 = 2$

no 0-matrix to the right!

Basis for  $C_2(K) = \{f_0, f_1\}$ .



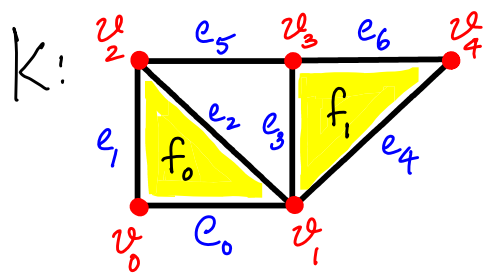
A basis for  $C_1(K) = \{e_0, e_3, e_5, e_0 + e_1, e_0 + e_2, e_3 + e_4, e_3 + e_6\}$ .

Notice that we have found also a basis for  $B_1(K)$  - consisting of the boundaries of  $f_0$  and  $f_1$ .

We are numbering the rows and columns starting from 1. As we proceed further into the reduction (to SNF), the labels for the rows/columns could become more complicated.

Similar notation for column labels:  $C_{ijk} \equiv C_i + C_j + C_k$ .

Let's look at  $[\partial_1]$  now.



$$[\partial_1] = \begin{matrix} & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} R_2 + R_1 \\ \text{and then} \\ C_2 + C_1 \end{matrix} \rightarrow \begin{matrix} & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix} \xrightarrow{\begin{matrix} R_3 + R_2 \\ \text{then} \\ C_j + C_2 \\ j=3,4,5 \end{matrix}} \begin{matrix} & e_0 & e_1 & e_{012} & e_{013} & e_{014} & e_5 & e_6 \\ \begin{matrix} v_0 \\ v_1 \\ v_{012} \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$\xrightarrow{C_3 \Rightarrow C_4} \begin{matrix} & e_0 & e_1 & e_{013} & e_{012} & e_{014} & e_5 & e_6 \\ \begin{matrix} v_0 \\ v_1 \\ v_{012} \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix} \xrightarrow{\begin{matrix} R_4 + R_3 \\ \text{then} \\ C_5 + C_3 \\ C_6 + C_3 \end{matrix}} \begin{matrix} & e_0 & e_1 & e_{013} & e_{012} & e_{34} & e_{0135} & e_6 \\ \begin{matrix} v_0 \\ v_1 \\ v_{012} \\ v_{0123} \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$\xrightarrow{\begin{matrix} C_5 \Rightarrow C_4 \\ \text{then} \\ R_5 + R_4 \\ \text{then} \\ C_7 + C_4 \end{matrix}} \begin{matrix} & e_0 & e_1 & e_{013} & e_{34} & e_{012} & e_{0135} & e_{346} \\ \begin{matrix} v_0 \\ v_1 \\ v_{012} \\ v_{0123} \\ v_{01234} \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} = \text{SNF}([\partial_1]).$$

$z_1 = 3, b_0 = 4$

Thus we get  $\beta_1 = z_1 - b_1 = 3 - 2 = 1$ , and  $\beta_0 = z_0 - b_0 = 5 - 4 = 1$ , and both of these numbers agree with intuition (1 hole and 1 component).  
 $\beta_p = 0 \quad \forall p \geq 2$ .