MATH 524 — Lecture 20 (10/26/2023)

Today: * exact sequences of chain complexes
* zig-zag lemma, "diagram chasing"

Recall: chain complexes and chain maps

We had introduced the (far more) general concept of chain complexes and chain maps between them. A chain complex $\mathcal{C}$ consists of a set of objects (groups, for instance) with maps (homomorphisms) between them that satisfy the condition that composition of consecutive maps is trivial (i.e., zero).

We have $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$, with $\partial'_p \circ \partial'_{p+1} = 0$. A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ is a family of homomorphisms that commutes with ∂_p, ∂'_p, i.e.,

$$\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p \quad \forall p.$$

Each "square" commutes:

$$\begin{tikzcd} \cdot \arrow[r, "\partial_p"] \arrow[d, "\phi_p"'] & \cdot \arrow[d, "\phi_{p-1}"] \\ \cdot \arrow[r, "\partial'_p"'] & \cdot \end{tikzcd}$$

So, cycle (boundaries) in $\mathcal{C}$ get mapped to cycles (boundaries) in $\mathcal{C}'$, and ϕ induces a homomorphism of the homology groups

$$(\phi_*)_p : H_p(\mathcal{C}) \longrightarrow H_p(\mathcal{C}').$$

Notice that we can define $Z'_p = \ker \partial'_p$, $B'_p = \operatorname{im} \partial'_{p+1}$, and $H'_p = Z'_p / B'_p$ for $\mathcal{C}'$.

We present the result on existence of long exact sequences given a family of short exact sequences in the general setting of chain complexes.

Notation $\mathcal{C}, \mathcal{D}, \mathcal{E}$: chain complexes

$$\mathcal{C} = \{C_p, \partial_C\}, \quad \mathcal{D} = \{D_p, \partial_D\}, \quad \mathcal{E} = \{E_p, \partial_E\}$$

groups in the chain complexes

homomorphisms for each chain complex

We will supress listings of subscripts to avoid clutter.

Def Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be chain complexes and 0 denote the trivial chain complex whose groups vanish in each dimension. Let $\phi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{E}$ be chain maps. We say the sequence $\mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E}$ is **exact** at $\mathcal{D}$ if $\ker \psi_p = \operatorname{im} \phi_p \ \forall p$, i.e., if the sequence $C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p$ is exact $\forall p$.

We say the sequence $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$ is a **short exact sequence of chain complexes** if in each dimension p, the sequence

$$0 \to C_p \xrightarrow{\phi} D_p \xrightarrow{\psi} E_p \to 0$$

is an exact sequence of groups.

Example Let $K_0 \subseteq K$ be a subcomplex of simplicial complex K.
Then the sequence

$$0 \longrightarrow \mathcal{C}(K_0) \xrightarrow{i} \mathcal{C}(K) \xrightarrow{\pi} \mathcal{C}(K, K_0) \longrightarrow 0$$

is exact, as $C_p(K, K_0) = C_p(K)/C_p(K_0)$ by definition.

We have $\ker \pi_p = \operatorname{im} i_p \;\; \forall p$.

Here $\mathcal{C}(K) = \{C_p(K), \partial_p\}$, $\mathcal{C}(K_0) = \{C_p(K_0), \partial_p\}$, and so on. Notice that we directly get the following results:

$\mathcal{C}(K_0) \longrightarrow \mathcal{C}(K)$ is injective and
$\mathcal{C}(K) \longrightarrow \mathcal{C}(K, K_0)$ is surjective.

We can construct/define connecting homomorphisms using which we can connect such short exact sequences of chain complexes to build long exact sequences of chain complexes. Recall the result from the previous lecture about long exact sequences for homology groups of a pair (K, K_0) — we will see that this result follows as a direct instance of the more general result specified on chain complexes and chain maps. We first state the general result, and come back to the above example to illustrate the same.

Lemma 24.1 [M] (The **zig-zag lemma**) or (Snake lemma).
Suppose one is given chain complexes $\mathcal{C} = \{C_p, \partial_C\}$, $\mathcal{D} = \{D_p, \partial_D\}$, and $\mathcal{E} = \{E_p, \partial_E\}$, and chain maps ϕ, ψ such that the sequence $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$ is exact. Then there is a **long exact homology sequence**

$$\cdots H_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \xrightarrow{\phi_*} H_{p-1}(\mathcal{D}) \to \cdots$$

where ∂_* is the **connecting homomorphism** and is induced by the boundary operator in $\mathcal{D}$ (∂_D).

<u>Back to the example</u> on long exact sequence of homology.

We just saw that the sequence

$$0 \longrightarrow \mathcal{C}(K_0) \xrightarrow{i} \mathcal{C}(K) \xrightarrow{\pi} \mathcal{C}(K, K_0) \longrightarrow 0$$

is exact. The exactness in the middle follows from the fact that a chain of K is carried by K_0 iff it is zero in $C(K, K_0)$.

So Lemma 24.1 implies the existence of a long exact homology sequence of pair (K, K_0):

$$\cdots \to H_p(K_0) \longrightarrow H_p(K) \longrightarrow H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

Proof (Sketch).

Main step: define connecting homomorphism ∂_*. We illustrate the technique of "diagram chasing" here – it's applied in more general settings (and not just to simplicial complexes).

$$
\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] & D_{p+1} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\square_2"] & E_{p+1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_p \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\square_3"] & D_p \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\square_0"] & E_p \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\square_1"] & D_{p-1} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] & E_{p-1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "\phi"] & D_{p-2} \arrow[r, "\psi"] & E_{p-2} \arrow[r] & 0
\end{tikzcd}
$$

d_p $\quad$ e_p $\quad$ c_{p-1} $\quad$ $\partial_D d_p$ $\quad$ $\partial_C c_{p-1}$

$\psi\partial_D = \partial_E\psi$

$\square_i$: squares are indexed in the order in which they're used in the proof.

<u>Step 1</u>: (defining ∂_*). Given a cycle e_p in E_p, since ψ is surjective, we can choose $d_p \in D_p$ such that $\psi(d_p) = e_p$. Since $\square_0$ commutes, the element $\partial_D d_p$ of D_p lies in $\ker \psi$, as

$$\psi(\partial_D d_p) = \partial_E(\psi(d_p)) = \partial_E(\overset{\text{cycle}}{e_p}) = 0.$$

Therefore, there exists $c_{p-1} \in C_{p-1}$ such that $\phi(c_{p-1}) = \partial_D d_p$ as $\ker \psi = \operatorname{im} \phi$. Since ϕ is injective, c_{p-1} is unique here. Further, c_{p-1} is a cycle here, since

$$\phi(\partial_C c_{p-1}) = \partial_D \phi(c_{p-1}) = \partial_D(\partial_D d_p) = 0,$$

as $\square_1$ commutes. Again, since ϕ is injective, $\partial_C c_{p-1} = 0$.

We define $\partial_* \{e_p\} = \{c_{p-1}\}$, where $\{\cdot\}$ means "homology class of."

<u>Step 2</u> Show ∂_* is well defined – independent of the choices $e_p \in \ker \partial_E$ and choice of c_{p-1} from $\{c_{p-1}\}$.

Recall that we defined ∂_* on homology classes – $\partial_*\{e_p\} = \{c_{p-1}\}$ for cycle $e_p \in E_p$ and corresponding cycle $c_{p-1} \in C_{p-1}$. We want to now show that this definition is independent of the choice of e_p and c_{p-1}. To this end, we start with cycles e_p, e_p' in E_p $(e_p, e_p' \in \ker \partial_E : E_p \to E_{p-1})$. We assume that $e_p \sim e_p'$ (homologous), and then argue that $c_{p-1} \sim c_{p-1}'$.

Given $e_p \sim e_p'$, we can find $e_{p+1} \in E_{p+1}$ such that $e_p - e_p' = \partial_E e_{p+1}$ (by definition of homology). Using the upper portion of the diagram, we argue that we can find $c_p \in C_p$ such that $c_{p-1} - c_{p-1}' = \partial_C c_p$.

$$
\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial_C"] & D_{p+1}\ {\color{blue}d_{p+1}} \arrow[r, "\psi"] \arrow[d, "\partial_D"] \arrow[dr, phantom, "\square_2"] & E_{p+1}\ {\color{blue}e_{p+1}} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & {\color{blue}c_p}\ C_p \arrow[r, "\phi"] \arrow[d, "\partial_C"] \arrow[dr, phantom, "\square_3"] & D_p\ {\color{blue}d_p}, {\color{green}d_p'} \arrow[r, "\psi"] \arrow[d, "\partial_D"] \arrow[dr, phantom, "\square_0"] & E_p\ {\color{blue}e_p} \sim {\color{green}e_p'} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & {\color{green}c_{p-1}'} \sim {\color{blue}c_{p-1}}\ C_{p-1} \arrow[r, "\phi"] \arrow[d, "\partial_C"] \arrow[dr, phantom, "\square_1"] & D_{p-1}\ {\color{blue}\partial_D d_p} \arrow[r, "\psi"] \arrow[d, "\partial_D"] & E_{p-1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "\phi"] & D_{p-2} \arrow[r, "\psi"] & E_{p-2} \arrow[r] & 0
\end{tikzcd}
$$

ψ is surjective. So choose d_p, d_p' such that $\psi(d_p) = e_p$ and $\psi(d_p') = e_p'$. Using the same arguments in Step 1, choose c_{p-1} and c_{p-1}' such that $\phi(c_{p-1}) = \partial_D d_p$ and $\phi(c_{p-1}') = \partial_D d_p'$.

Suppose $e_p - e_p' = \partial_E e_{p+1}$. $\overbrace{\text{Choose } d_{p+1} \in D_{p+1}}^{\text{recall that } \psi \text{ is surjective}}$ such that $\psi(d_{p+1}) = e_{p+1}$. Notice that

as $\square_2$ commutes, $\psi\partial_D = \partial_E\psi$

$$
\begin{aligned}
\psi(d_p - d_p' - \partial_D d_{p+1}) &= e_p - e_p' - \partial_E \psi(d_{p+1}) \\
&= e_p - e_p' - \partial_E e_{p+1} = 0.
\end{aligned}
$$

So $d_p - d_p' - \partial_D d_{p+1} \in \ker \psi : D_p \to E_p$. By exactness, it should also be in $\operatorname{im} \phi : C_p \to D_p$.

So we can choose $c_p \in C_p$ such that $\phi(c_p) = d_p - d_p' - \partial_D d_{p+1}$.

So $\quad \phi(\partial_C c_p) = \partial_D \phi(c_p)$ as $\square_3$ commutes, $\phi \partial_C = \partial_D \phi$

$$= \partial_D (d_p - d_p' - \partial_D d_{p+1}) = \phi(c_{p-1} - c_{p-1}').$$

But ϕ is injective, so $\partial_C c_p = c_{p-1} - c_{p-1}'$. So $c_{p-1} \sim c_{p-1}'$.

We need to provide some more arguments to finish the proof. We will do that in the next lecture...