MATH 401: Lecture 12 (09/25/2025)

Today: * Examples of metric spaces
* isometry

Recall (X,d): Metric space (i) positivity: $d(x,y) \geq 0 \;\forall x,y \in X$, and $d(x,y) = 0 \iff x = y$.

(ii) symmetry: $d(x,y) = d(y,x) \;\forall x,y \in X$

(iii) triangle inequality: $d(x,y) \leq d(x,z) + d(z,y) \;\forall x,y,z \in X$

One more requirement: $d(x,y) < \infty \;\forall x,y \in X$. (finiteness)

The finiteness requirement is usually satisfied. But you should use your judgement to decide in which cases this property needs to be proved.

LSIRA 3.1 Example 4 (Problem 1)

Let X be the space of messages, where each message is a vector (k is fixed)

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \text{ with } x_i \in \underbrace{S}_{\text{alphabet}} = \{\underbrace{s_1, \ldots, s_N}_{\text{symbols in the alphabet}}\}.$$

Let $d(\bar{x},\bar{y}) = \#$ indices i where $x_i \neq y_i$. Show (X,d) is a metric space.

Note that shorter messages could be padded up with empty cells, or a "dummy" symbol (equivalent to 0 when using numbers). And larger messages can be chopped up into pieces of length k each (with padding up if needed for the last piece).

An example: Let $S = \{R, G, B\}$ for colors red, green, blue. Consider vectorizing each image by assigning to each pixel its predominant color, and stacking these color symbols into a vector. For instance, a 12×12 pixel image is represented by a 144-vector of color values from $\{R, G, B\}$.

(i) $d(\bar{x},\bar{y}) \geq 0$, and $d(\bar{x},\bar{y}) = 0 \iff \bar{x} = \bar{y}$ as messages.

↳ $d(\bar{x},\bar{y})$ is the # places (or indices) where the messages differ, and hence is ≥ 0.

$d(\bar{x},\bar{y}) = 0 \iff \bar{x}$ and $\bar{y}$ are identical in all entries, i.e., they do not differ at all. Hence $\bar{x} = \bar{y}$.

(ii) symmetry ✓

$$\begin{aligned} d(\bar{x},\bar{y}) &= \# \text{ indices where } x_i \neq y_i \\ &= \# \text{ indices where } y_i \neq x_i \\ &= d(\bar{y},\bar{x}) \end{aligned}$$

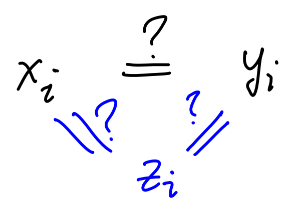
(iii) triangle inequality.

$d(\bar{x},\bar{y})$ counts # indices i where $x_i \neq y_i$

$x_i \neq y_i \implies$ cannot have $x_i = z_i$ and $z_i = y_i$.

Combining with z_i, here are the possibilities:

1. $x_i \neq z_i$, $z_i = y_i$
2. $x_i = z_i$, $z_i \neq y_i$
3. $x_i \neq z_i$, $z_i \neq y_i$



$\implies d(x,y) \leq d(x,z) + d(z,y)$, as there are three possible cases for each index contributing 1 to the right-hand side sum corresponding to the one case possibly contributing 1 to the left-hand side distance.

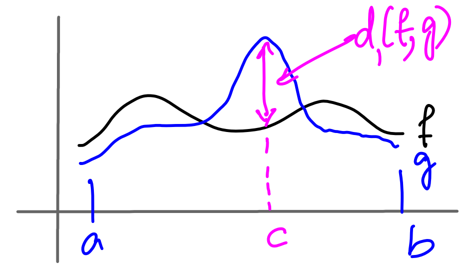
□

LSIRA Pg47, Problem 2 **Distance between Functions.**

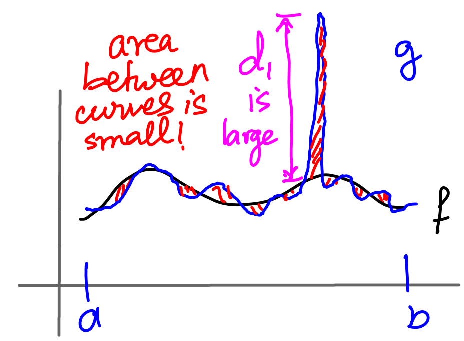
Let X = set of all continuous functions from $[a,b] \to \mathbb{R}$, and let

$$d_1(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\}.$$

Show that d_1 is a metric over X.



Measuring distance between functions is a widely studied problem in analysis as well as applications. We illustrate use of d_1 here (top right). But d_1 may not be the best choice in all cases — in the second instance, f and g are quite close to each other except at one point $x=c$, where g shoots up a lot. $d_1(f,g)$ will be quite large here, even though f and g are near equal. Measuring distance using the area between f and g may be better here.



Proof: We first show $d_1(f,g)$ is finite for any $f, g \in X$. $\to$ finitness is not obvious in this case!

f, g are continuous over $[a,b]$

$\Rightarrow f-g$ is continuous over $[a,b]$.

By the Extreme Value Theorem (Theorem 2.3.4), $h = f-g$ has a maximum and minimum over $[a,b]$.

$\Rightarrow \sup\{|h(x)| : x \in [a,b]\}$ is finite.

$\Rightarrow d_1(f,g)$ is finite.

(i) (positivity) $d_1(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\} \geq 0$, as the supremum of a set of ≥ 0 values is ≥ 0.

Need to also show $d_1(f,g) = 0 \iff f = g$ over $[a,b]$

$(\Rightarrow)$ $f(x) = g(x)\ \forall x \in [a,b]$

$\Rightarrow |f(x)-g(x)| = 0\ \forall x \in [a,b]$

$\Rightarrow \sup\{|f(x)-g(x)| : x \in [a,b]\} = 0 \Rightarrow d_1(f,g) = 0.$

$(\Leftarrow)$ $d_1(f,g) = 0 \Rightarrow \sup\{|f(x)-g(x)| : x \in [a,b]\} = 0$

The supremum of a set of ≥ 0 is zero $\Rightarrow$ each element $= 0$!

$\Rightarrow f(x) = g(x)\ \forall x \in [a,b]$.

(ii) (symmetry) $|f(x)-g(x)| = |g(x)-f(x)|$

$\Rightarrow d(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\} =$

$\sup\{|g(x)-h(x)| : x \in [a,b]\} = d_1(g,f)$.

(iii) <u>Triangle inequality</u>

$|f(x)-g(x)| = |f(x)-h(x)+h(x)-g(x)|$

$\leq |f(x)-h(x)| + |h(x)-g(x)|$ by standard triangle inequality over $\mathbb{R}$.

$d_1(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\}$

$\leq \sup\{|f(x)-h(x)| + |h(x)-g(x)| : x \in [a,b]\}$

$\leq \sup\{|f(x)-h(x)| : x \in [a,b]\} +$ as $\sup\{a+b\} \leq \sup\{a\} + \sup\{b\}$

$\sup\{|h(x)-g(x)| : x \in [a,b]\}$

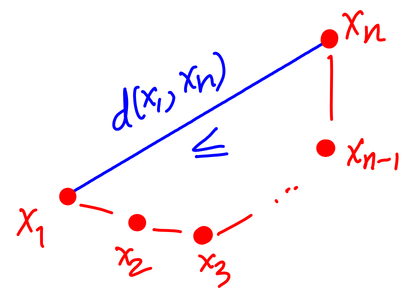
$= d_1(f,h) + d_1(h,g)\quad \forall f,g,h \in X$.

□

LSIRA Pg 48, Problem 7

Let (X, d) be a metric space, and $x_i \in X$, $i = 1, \ldots, n$.

Show $$d(x_1, x_n) \leq \underbrace{\sum_{i=1}^{n-1} d(x_i, x_{i+1})}_{d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n)}.$$



Can use induction

$n = 2$ (base case)

$d(x_1, x_2) \leq d(x_1, x_2)$ holds, as both sides are the same.

$n = 3$ case could be considered as the base case as well:

$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ follows from triangle inequality for (X, d).

Assume result is true for $n = k$, i.e.,

$$d(x_1, x_k) \leq \sum_{i=1}^{k-1} d(x_i, x_{i+1}) \qquad (*)$$

For $n = k+1$,

$$\begin{aligned} d(x_1, x_{k+1}) &\leq d(x_1, x_k) + d(x_k, x_{k+1}) && \text{by triangle inequality in } (X, d) \\ &\leq \sum_{i=1}^{k-1} d(x_i, x_{i+1}) + d(x_k, x_{k+1}) && \text{by } (*) \\ &= \sum_{i=1}^{k} d(x_i, x_{i+1}) && \checkmark \end{aligned}$$

Hence the result holds for all n.

$\square$

We now talk about comparing two metric spaces, and functions between metric spaces. When are two metric spaces "the same"? As metric spaces are about pairwise distances between points, we want these distances to be preserved.

Def 3.1.2 Let (X, d_X) and (Y, d_Y) are metric spaces.

An **isometry** between the spaces is a bijection $i : X \to Y$ such that $d_X(x,y) = d_Y(i(x), i(y))$ $\forall x, y \in X$.

The two spaces are isometric if an isometry exists between them.

Since i is a bijection, its inverse exists, and i^{-1} is an isometry from (Y, d_Y) to (X, d_X). Hence we can just say isometry between the spaces.

LSIRA Pg48, Problem 11 For $a \in \mathbb{R}$, let $f(x) = x + a$. Show f is an isometry from $\mathbb{R}$ to $\mathbb{R}$.

$$d(x,y) = |x - y| \quad \forall x, y \in \mathbb{R}.$$

To show f is an isometry, we need to show

1. f is a bijection; and
2. $d(f(x), f(y)) = d(x,y)$ $\forall x, y \in \mathbb{R}$

1\. $f(x) = x + a$ is a bijection as $\overbrace{x_1 \neq x_2 \Rightarrow f(x_1) = x_1 + a \neq x_2 + a = f(x_2)}^{\text{injection}}$; and $\underbrace{\forall y \in \mathbb{R}, \exists x = f^{-1}(y) = y - a}_{\text{surjection}}$.

2\. $$d(f(x), f(y)) = |f(x) - f(y)| = |x + a - (y + a)| = |x - y| = d(x,y) \quad \forall x, y \in \mathbb{R}.$$

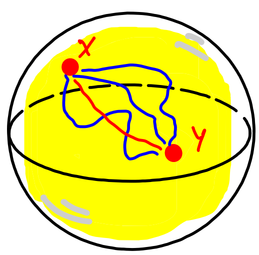
$\square$

The requirement of i being a bijection is too strict in some settings. There may be spaces that look otherwise quite similar, even if they are not isometric.

If $i: X \to Y$ in Definition 3.1.2 is only an injection (and not a bijection), we call i an <u>embedding</u>.

You may have heard about embeddings in other geometric settings. for instance, think of a sphere (surface of a ball) in 3D space. We can work with the sphere as a metric space — the distance between any two points is the length of the shortest curve connecting the points that lies entirely on the surface (of the sphere). This is called the shortest geodesic distance. We can prove it is a metric.

S^2 (2-sphere) surface of a ball



in $\mathbb{R}^3$

There are many curves between x and y that lie on the surface of the sphere. Length of a shortest geodesic curve defines the distance between x and y.

The embedding here is literally the positioning of the sphere in $\mathbb{R}^3$.