

# MATH 273 - Lecture 16 (10/16/2014)

16.1

## Second derivative test

12.  $f(x,y) = 1 - \sqrt[3]{x^2+y^2} = 1 - (x^2+y^2)^{1/3}$  → find local extrema, saddle points, and other critical points, if any.

$$f_x = -\frac{1}{3}(x^2+y^2)^{-2/3} \cdot 2x = \frac{-2x}{3(x^2+y^2)^{2/3}} = 0 \quad \text{--- (1)}$$

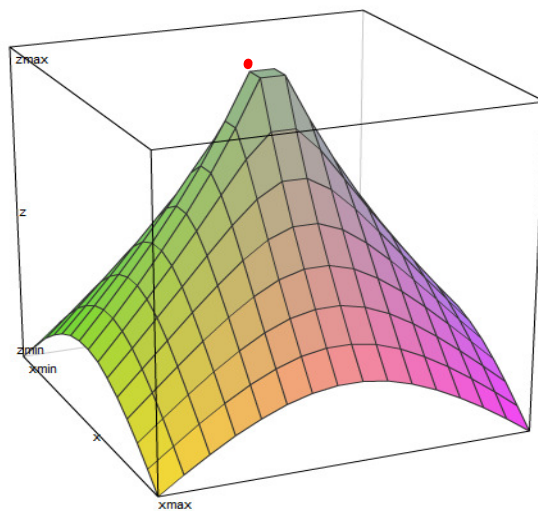
$$f_y = -\frac{1}{3}(x^2+y^2)^{-2/3} \cdot 2y = \frac{-2y}{3(x^2+y^2)^{2/3}} = 0 \quad \text{--- (2)}$$

The only possible solution could have been  $(0,0)$ , but both  $f_x$  and  $f_y$  are undefined at  $(0,0)$ . Hence  $(0,0)$  is the only critical point (as  $f_x, f_y$  both do not exist at  $(0,0)$ ).

There is no point in finding the second derivatives here (as they also will be undefined at  $(0,0)$ ).

Notice that  $f(0,0) = 1$ , and  $f(x,y) = 1 - \sqrt[3]{x^2+y^2} \leq 1$  for any  $(x,y)$ . Hence  $(0,0)$  is a local maximum.

Here is the surface  
 $z = 1 - \sqrt[3]{x^2 + y^2}$ . Indeed,  
 the point  $(0,0,1)$  is a  
 local maximum. In fact,  
 it is the global maximum!



$$(15) \quad f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$$

$$f_x = 12x - 6x^2 + 6y = 0 \quad \text{--- (1)}$$

$$f_y = 6y + 6x = 0 \quad \text{--- (2)}$$

$$(2) \Rightarrow y = -x. \text{ So (1) } \Rightarrow 12x - 6x^2 - 6x = 0$$

$$\text{i.e., } 6x - 6x^2 = 0 \quad 6x(1-x) = 0 \Rightarrow x=0, 1$$

$$\text{Hence } y=0, -1$$

The critical points are  $(0,0)$  and  $(1,-1)$ .

$$f_{xx} = 12 - 12x, \quad f_{yy} = 6, \quad f_{xy} = f_{yx} = 6$$

$$H = f_{xx}f_{yy} - f_{xy}^2$$

(0,0)

$$f_{xx} = 12 > 0$$
$$H = 12 \cdot 6 - 6^2 = 36 > 0$$

Hence (0,0) is a local minimum  
 $f(0,0) = 0$ .

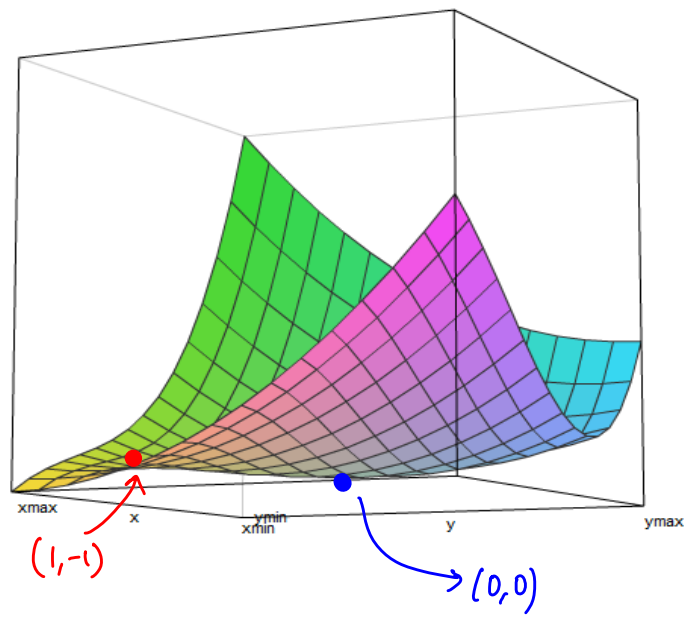
(1,-1)

$$f_{xx} = 12 - 12x = 0$$

$$H = 0 \cdot 6 - 6^2 = -36 < 0$$

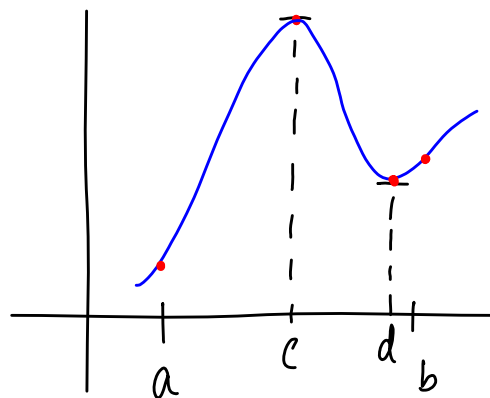
$f_{xx} \cdot f_{yy} - f_{xy}^2$

So (1,-1), and hence  
 $f(1,-1) = 1$  is a saddle point.



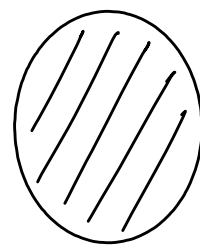
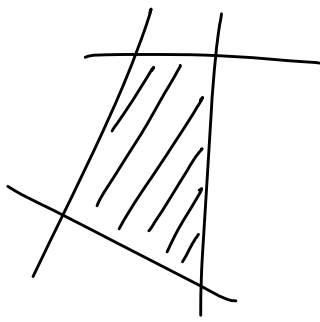
## Finding absolute maxima/minima in a region

In 1D, to find absolute extrema of  $f(x)$  in interval  $[a, b]$ , we first find local extrema in  $[a, b]$ , and then compare  $f(\cdot)$  at these critical points with  $f(a)$  and  $f(b)$ .



We extend this approach to 2D. In place of interval  $[a, b]$ , we get a region, typically a closed set in 2D (defined by lines/curves that form its boundaries).

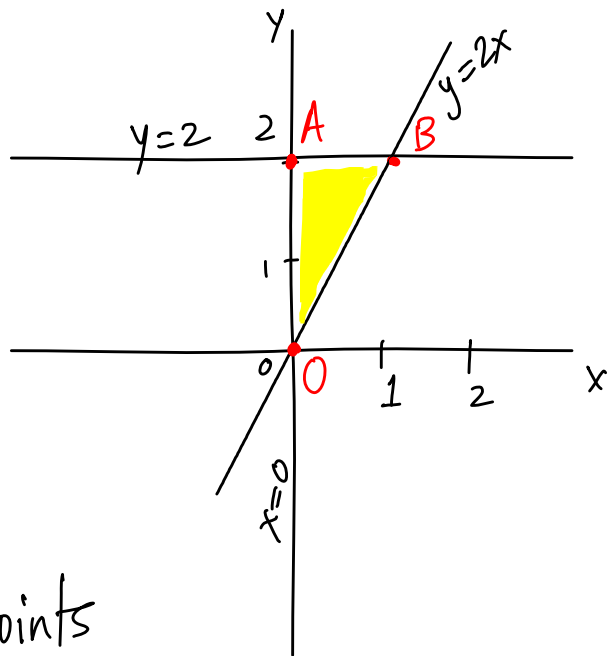
e.g.



, etc.

31. Find the absolute maximum and minimum of  $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$  on the closed triangular plate bounded by the lines  $x=0$ ,  $y=2$ , and  $y=2x$  in the first quadrant.

The region  $R$  is the triangle  $OAB$ , where  $O(0,0)$ ,  $A(0,2)$ ,  $B(1,2)$ .



Step(i) Find any critical points that are interior to  $R$ .

$$\left. \begin{aligned} f_x &= 4x - 4 = 0 & \text{--- (1)} \\ f_y &= 2y - 4 = 0 & \text{--- (2)} \end{aligned} \right\}$$

$x=1, y=2$ . But  $(1,2)$  is point  $B$ , which is not an interior point.

So, we do not consider the second derivative test at  $B$ .

Step(ii) Investigate the behavior of the function on each boundary segment—OA, OB, AB.

OA  $x=0$  on OA. Hence

$$f(0,y) = y^2 - 4y + 1$$

Hence,  $f'(0,y) = 2y - 4 = 0$ , gives  $y=2$ , which corresponds to  $A(0,2)$ , an end point.

As we did in 1D, we will investigate the function on the boundaries here. But now, we look for critical points on each boundary segment, and also look at the end points. We will finish this problem in the next lecture.