MATH 529 – Lecture 16 (02/29/2024)

Today: * Betti numbers
* Euler–Poincaré theorem
* boundary matrix

Recall: $B_p \subseteq Z_p \subseteq C_p$, $H_p = Z_p/B_p \ldots$

Order and Rank

Recall that the cardinality of a group is called its **order**.

$\text{ord } C_p = |C_p|$. Since we are working over $\mathbb{Z}_2$, if there are
order
n p-simplices in K, there are 2^n p-chains.

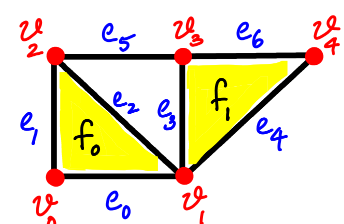
So, $\text{ord } C_p = 2^n$. Each p-simplex is either present or absent in a p-chain. (coefficient of 1 or 0, resp.)

Also, the **rank** of the chain group is $\text{rank } C_p = n$ here.
C_p is isomorphic to $\mathbb{Z}_2^n$, group of binary n-vectors, with addition satisfying $1+1=0$ (componentwise).

$\mathbb{Z}_2^n$ is an n-dimensional vector space. For instance, think of the n unit vectors, and the space generated by them (with $\mathbb{Z}_2$ addition).
$\text{rank } C_p$ is the dimension of this vector space.

Z_p and B_p also show the same vecter space structure, but typically have smaller orders and ranks.

K:



e.g., K has 7 edges $(e_0, \ldots, e_6)$.
$|C_1| = 2^7$ and $\text{rank } C_1 = 7$.

All 1-chains including cycles and boundaries, can be represented by 7-vectors.

$\text{ord } H_p = \text{ord } Z_p / \text{ord } B_p$ can add some $\bar{b} \in B_p$ to $\bar{z} \in H_p$ to get a single cycle in the same homology class.

Equivalently, $\text{rank } H_p = \text{rank } Z_p - \text{rank } B_p$.

We study these ranks in particular. We define

$\text{rank } H_p = \beta_p$, the **p-th Betti number**.

β_p's have intuitive interpretations for $p = 0, 1, 2$.

$\beta_0 = \#$ connected components.

$\beta_1 = \#$ holes

$\beta_2 = \#$ enclosed spaces or voids

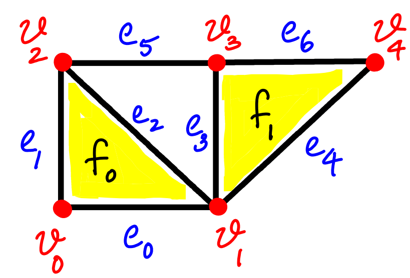
e.g.,

$\beta_0(K) = 1$ there is a single connected component

$\beta_1(K) = 1$ there is a single hole

$\beta_2(K) = 0$. There are no enclosed voids.

K:

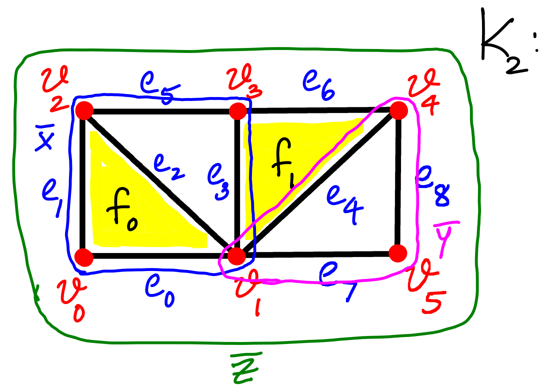


Intuitively, β_p captures the topology of K at dimension p upto homology.

Examples of Homology groups

1. To describe of $H_1(K_2)$, we could first choose one cycle around each hole.

K_2:

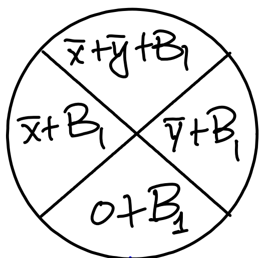


$\bar{x} = \{e_0, e_3, e_5, e_1\}$

$\bar{y} = \{e_4, e_7, e_8\}$

Here is a coset decomposition of $H_1(K_2)$:

H_1:



empty cycle $+B_1$

Thus, $\bar{z} = \{e_0, e_1, e_5, e_6, e_8, e_7\} \in \bar{x}+\bar{y}+B_1$, as

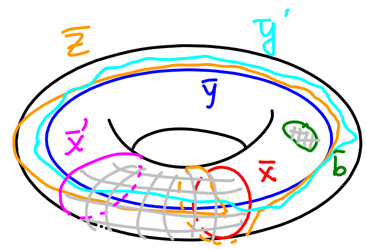
$\bar{z} = \bar{x}+\bar{y}+\{\bar{e}_3, \bar{e}_4, \bar{e}_6\} \rightarrow \partial \bar{f}_1 \in B_1$

$H_1(K_2) \overset{\text{isomorphic}}{\cong} \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $\beta_1(K_2) = 2$. Also,

$\beta_0(K_2) = 1$ and $\beta_p(K_2) = 0 \;\; \forall p \geq 2$.

Intuitively, K_2 has 1 component, 2 holes, and no voids (or other higher dimensional "holes").

2. Torus ($\mathbb{T}^2$)



$\bar{y}$: tunnel loop
$\bar{x}$: handle loop

$\bar{x} \sim \bar{x}'$, $\{\bar{x}, \bar{x}'\}$ form the boundary of the solid 2D patch in between.

$\bar{b}$ is also a cycle, but is in fact a boundary, and hence is not a member of H_1.

H_1 is generated by $\bar{x}$ and $\bar{y}$. Here $H_1 \cong \underset{\bar{x}}{\mathbb{Z}_2} \oplus \underset{\bar{y}}{\mathbb{Z}_2}$, and

$\underbrace{\beta_0 = 1}_{\text{one component}}$, $\beta_1 = 2$, and $\beta_2 = 1 \longrightarrow$ one enclosed void

$\bar{z} \in \bar{x} + \bar{y} + B_1$ here.

Similarly, $\bar{y}' \sim \bar{y}$ here (Both are tunnel loops).

To be exact, we should consider a triangulation of the torus, and $\bar{x}, \bar{y}, \bar{z}, \bar{b}$ are 1-cycles in the triangulation. Alternatively, we could talk about general curves instead of simplicial chains — this setting leads to singular homology, which is equivalent to simplicial homology. We will concentrate on simplicial homology, as it is more amenable to computation.

But as we will see soon, homology groups are invariants of the underlying space (up to ranks), and do not depend on the specific complex used.

3. <u>p-ball</u> $\quad \mathbb{B}_p = \{\bar{x} \in \mathbb{R}^d \mid \|\bar{x}\| \le 1\}$ $(d \geq p)$.

For instance, when $p=3$, $\mathbb{B}_3 \simeq |K|$ where K has a single solid tetrahedron and all its faces.

For general p, we get that $H_0 \cong \mathbb{Z}_2$ (recall, all 0-chains are also 0-cycles, and half of them are 0-boundaries), and hence $\beta_0 = 1$.
one connected component

Further, H_p is trivial for $p \geq 1$, and $\beta_p = 0$ for $p \geq 1$.
Intuitively, there are no holes, no pockets, etc.

How to compute homology?

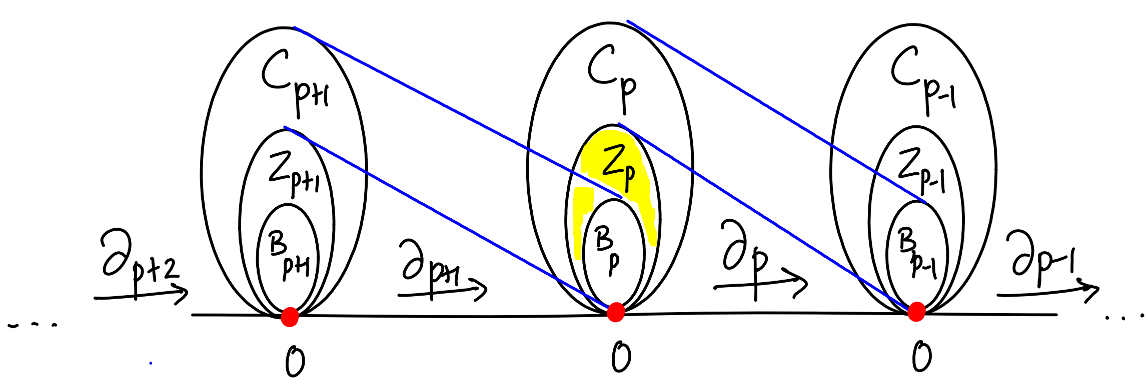
calculate β_p's, and find representative cycles for each homology class

We first give a result which shows that β_p's are invariants of the underlying space. Recall the Euler characteristic:

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i, \quad \text{where } s_i = \#\text{ } i\text{-simplices in } K.$$

Let us write $z_p = \operatorname{rank} Z_p$, $b_p = \operatorname{rank} B_p$, and write $s_p = \operatorname{rank} C_p$ (# p-simplices). We get that

$$s_p = z_p + b_{p-1}.$$



This result could be seen as the implication of a general result on linear transformations (LTs). Let $f: \mathcal{U} \to V$ be an LT from vector space $\mathcal{U}$ to vector space V. Then

$$\dim(\mathcal{U}) = \dim(\ker f) + \dim(\operatorname{im} f).$$

Applying this result to $\partial_p : C_p \to C_{p-1}$ gives $s_p = z_p + b_{p-1}$.

Alternatively, we could think about $Z_p = C_p / B_{p-1}$, and hence $\operatorname{rank} Z_p = \operatorname{rank} C_p - \operatorname{rank} B_{p-1}$, i.e., $z_p = s_p - b_{p-1}$. Intuitively, if a p-chain is not a p-cycle, it will generate a (p-1)-boundary.

By definition, $\beta_p = z_p - b_p$ $(\operatorname{rank} Z_p - \operatorname{rank} B_p)$.

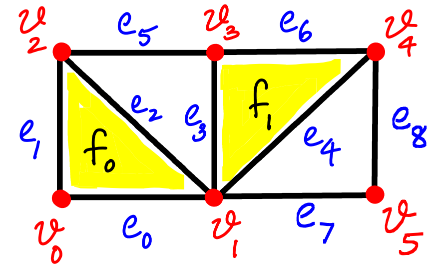
$$\begin{aligned} \Rightarrow \quad \chi &= \sum_{p \geq 0} (-1)^p s_p \\ &= \sum_{p \geq 0} (-1)^p (z_p + b_{p-1}) \\ &= \sum_{p \geq 0} (-1)^p (z_p - b_p) = \sum (-1)^p \beta_p . \end{aligned}$$

Euler-Poincaré theorem: $\boldsymbol{\chi(K) = \sum_{p \geq 0} (-1)^p \beta_p}$.

Significance: Homology groups do not depend on the particular triangulation chosen. χ is an invariant of $|K|$, and so are β_p's.

Illustration

K_2:



$\chi(K_2) = 6 - 9 + 2 = -1.$

$\beta_p = 0 \quad \forall p \geq 2.$

$\beta_0 = 1$ (one connected component).

$\Rightarrow \chi = -1 = \beta_0 - \beta_1 = 1 - \beta_1 \Rightarrow \beta_1 = 2$, which agrees with the intuition that K_2 has two holes.

Boundary Matrices

To compute, i.e., find z_p, b_p, β_p and bases for Z_p, B_p, H_p, we combine the information about cycles and boundaries into a single matrix, the **boundary matrix**.

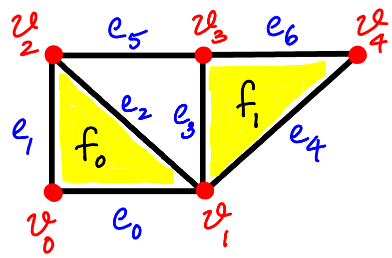
$[\partial_p(K)]$ is the p-th boundary matrix of K, $0 \leq p \leq \dim K$. It is an $m \times n$ matrix when K has m $(p-1)$-simplices $\tau_1, \ldots, \tau_m$, and n p-simplices $\sigma_1, \ldots, \sigma_n$.

$$[\partial_p]_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ } (p-1)\text{-simplex is a face of the } j^{th} \text{ } p\text{-simplex, i.e., } \tau_i \leq \sigma_j. \\ 0, & \text{otherwise} \end{cases}$$

This is the definition over $\mathbb{Z}_2$. We will introduce the same matrices over $\mathbb{Z}$, where the nonzero entries are ± 1 depending on orientations.

Example

K:



$$[\partial_0] = \begin{array}{c} \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} v_0 & v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

default, as there are no (−1)-dimensional simplices

$$[\partial_1] = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & & & & & \\ 1 & & 1 & 1 & 1 & & \\ & 1 & 1 & & & 1 & \\ & & & 1 & & 1 & 1 \\ & & & & 1 & & 1 \end{bmatrix} \end{array}$$

(entries not listed are zeros).

$$[\partial_2] = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array}\!\!\begin{array}{c} \begin{array}{cc} f_0 & f_1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

A collection of columns represents a p-chain, and a collection of rows represents a $(p-1)$-chain. Given a p-chain $\bar{c} = \sum_{j=1}^{n} a_j \sigma_j$, $a_j \in \{0,1\}$, its p-boundary $\partial_p \bar{c}$ is given by the matrix-vector product $[\partial_p]\bar{c}$, i.e., the sum of the corresponding columns with weights a_j gives its p-boundary.

Similarly a collection of rows represents a $(p-1)$-chain, and their sum gives the $(p-1)$-coboundary. Coboundary and cohomology is a dual concept to boundary and homology.

Since every p-chain can be written as $\bar{c} = \sum_{j=1}^{n} a_j \sigma_j$, the columns of $[\partial_p]$ generate C_p. Similarly, rows of $[\partial_p]$ generate C_{p-1}.

To compute z_p, b_p, β_p for all p, we will use operations similar to Gaussian elimination in linear algebra. Here we perform both elementary row operations (EROs) as well as elementary column operations (ECOs).