STABLE COMPARISON OF TIME SERIES USING TOPOLOGY

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Washington State University

arXiv: 2501.02817

WADEPS

COMPARING TIME SERIES

→ stock price v/s volume of trading

(image: www)

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→ stock price v/s volume of trading

(image: www)

→ rainfall amount v/s river levels

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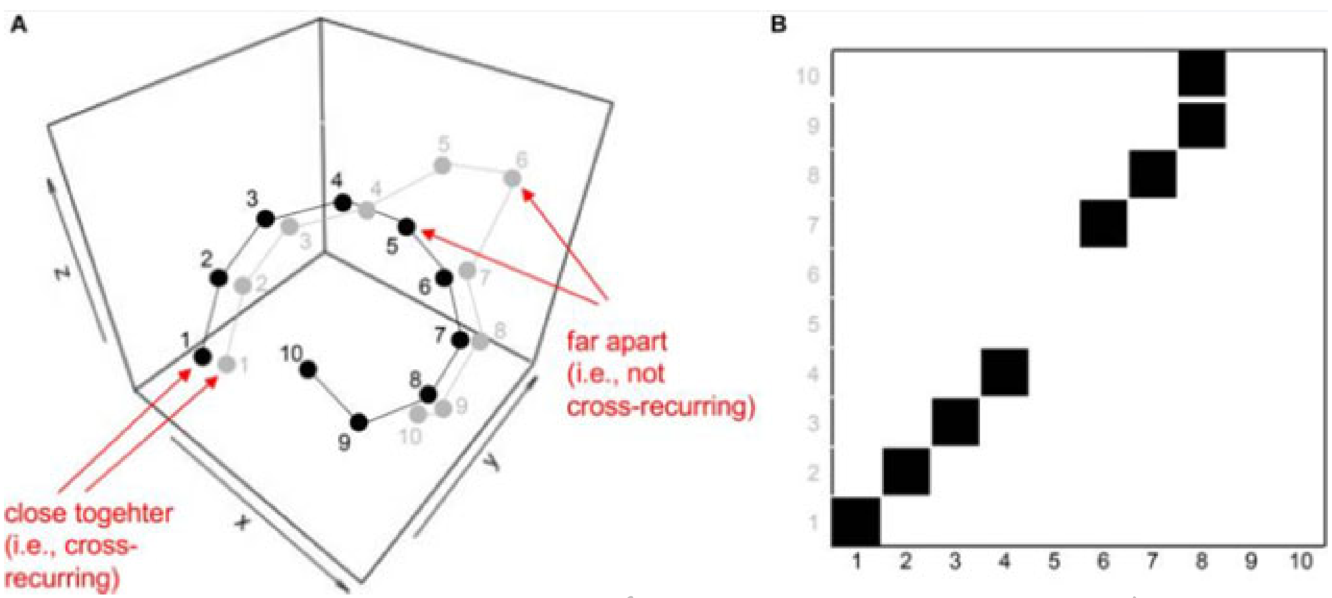
? How to compare?

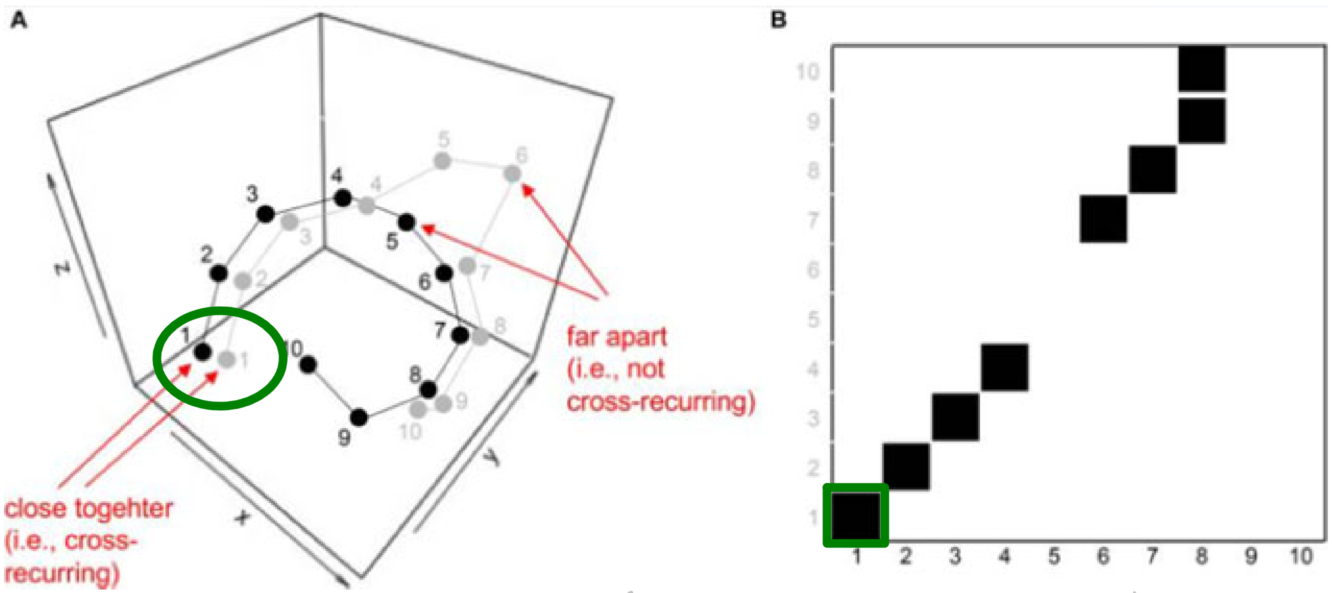
CROSS-RECURRENCE (%DET)



(Wallot & Leonardi, 2018)

Cross-recurrence matrix $\mathcal{C}$

Cross-Recurrence (%DET)



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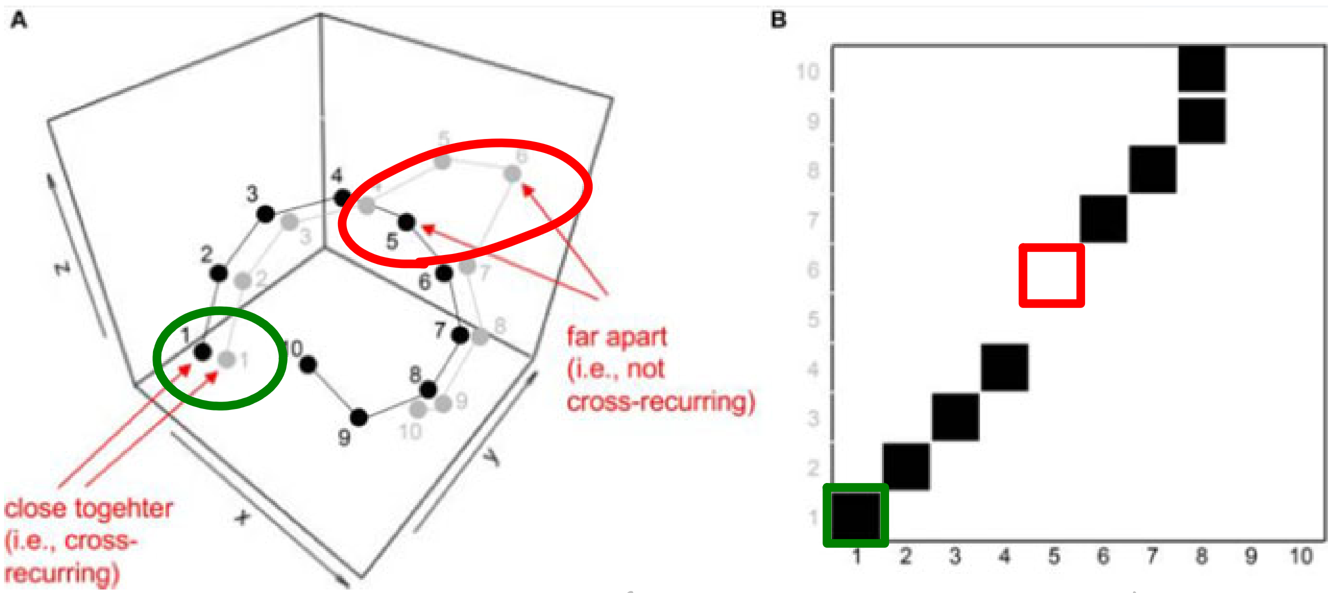

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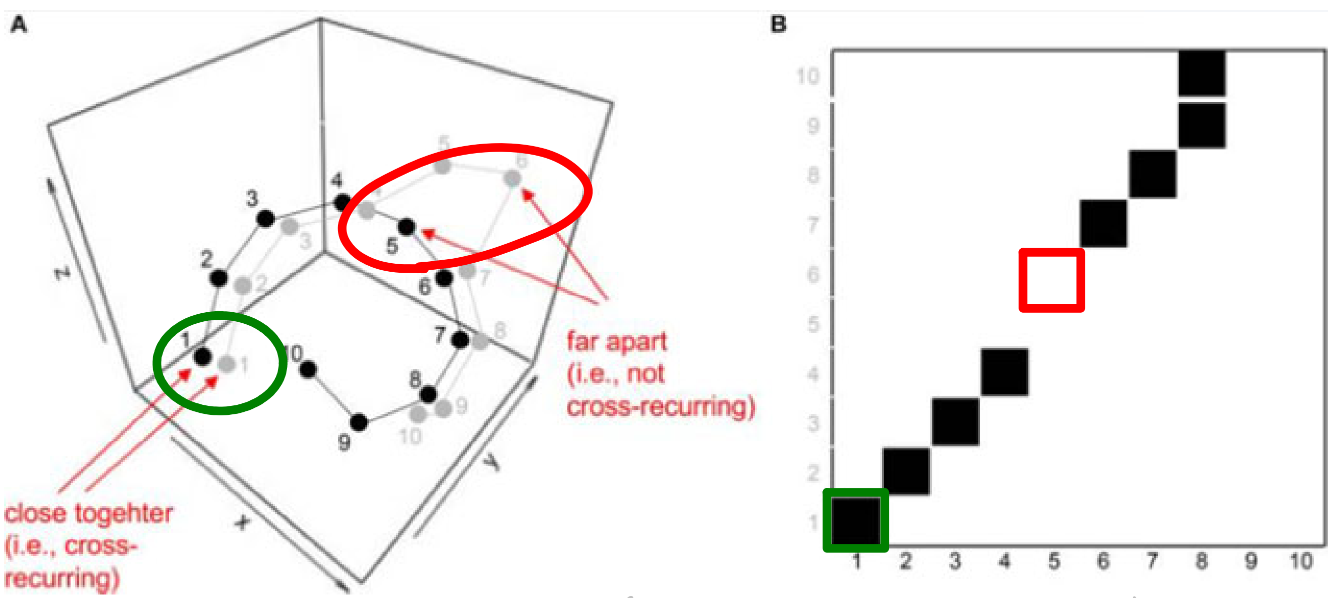

Cross-recurrence matrix $\mathcal{C}$

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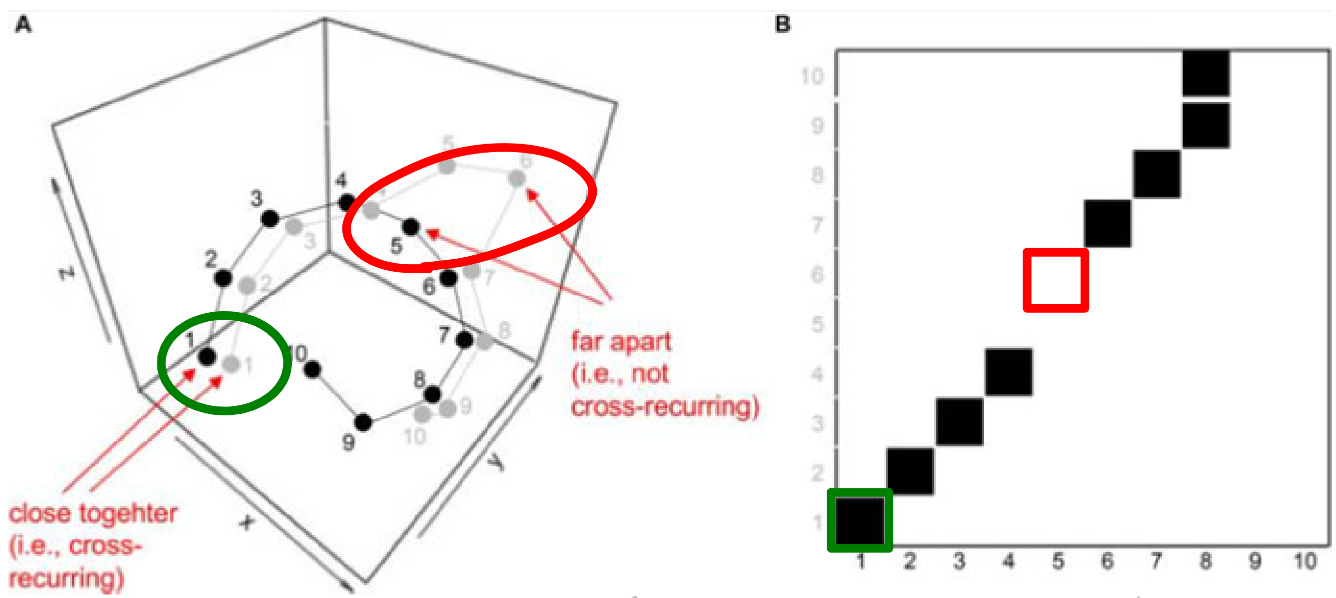
→ % Determinism (%DET)

– % of 1's in diagonal strips of $\mathcal{C}$

CROSS-RECURRENCE (%DET)



Cross-recurrence matrix $\mathcal{C}$

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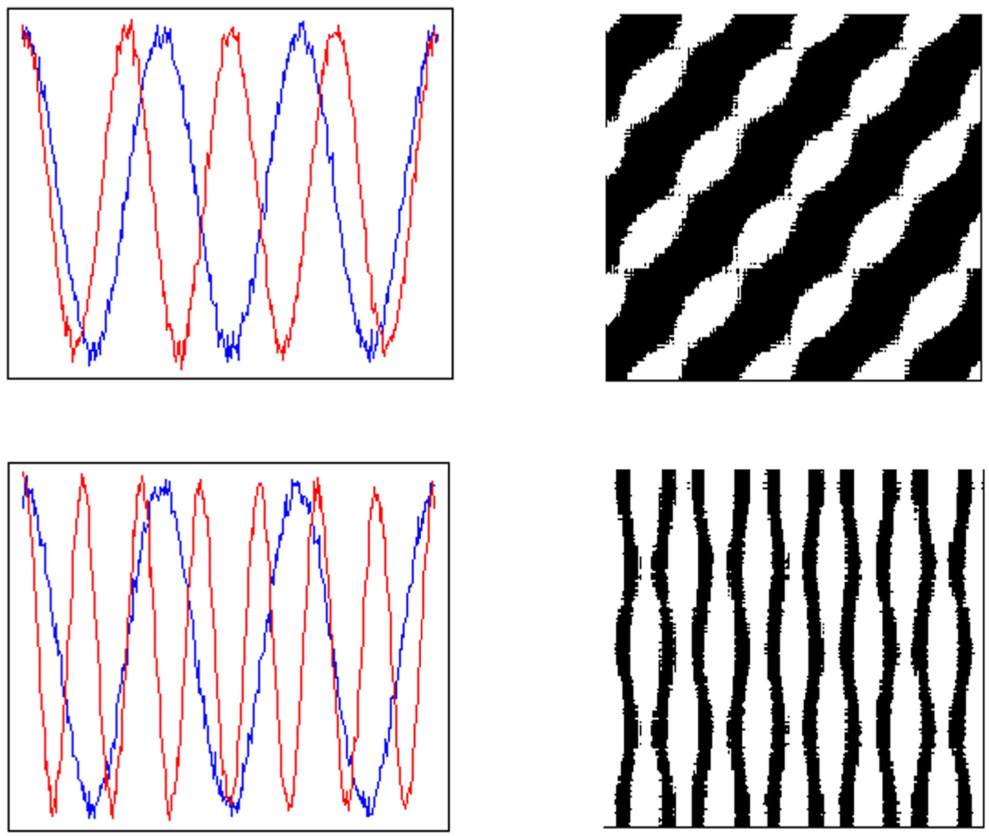
→ % Determinism (%DET)

– % of 1's in diagonal strips of $\mathcal{C}$

→ four parameters: time lag τ, embed. dimension, dist. threshold (tol), min # 1's in a diagonal (minDL)

%DET: INSTABILITY

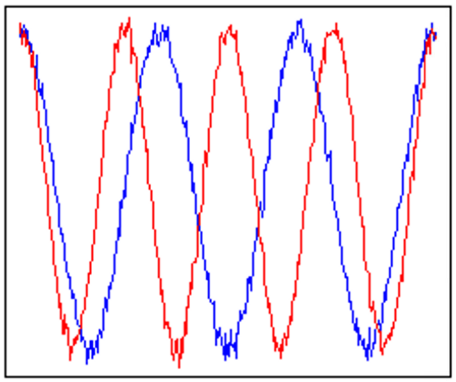


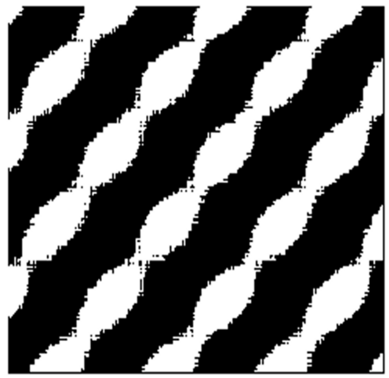
%DET = 95

minDL = 7

%DET = 83

%DET: INSTABILITY





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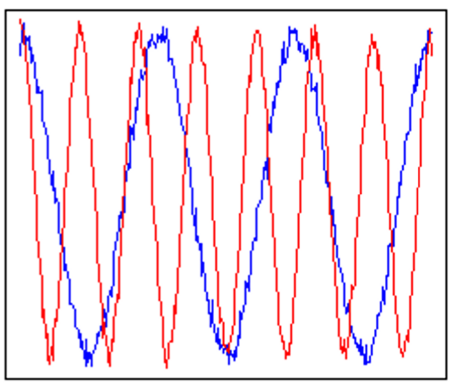

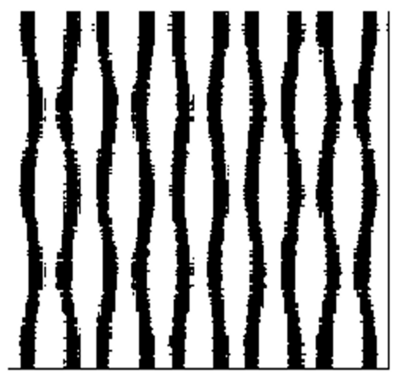

%DET = 8~~3~~ 97!

STABLE MEASURE?

? Define a measure of similarity that is provably stable?

- small change in input $\Rightarrow$ small change in measure

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? Does it work well in practice?

RESULTS

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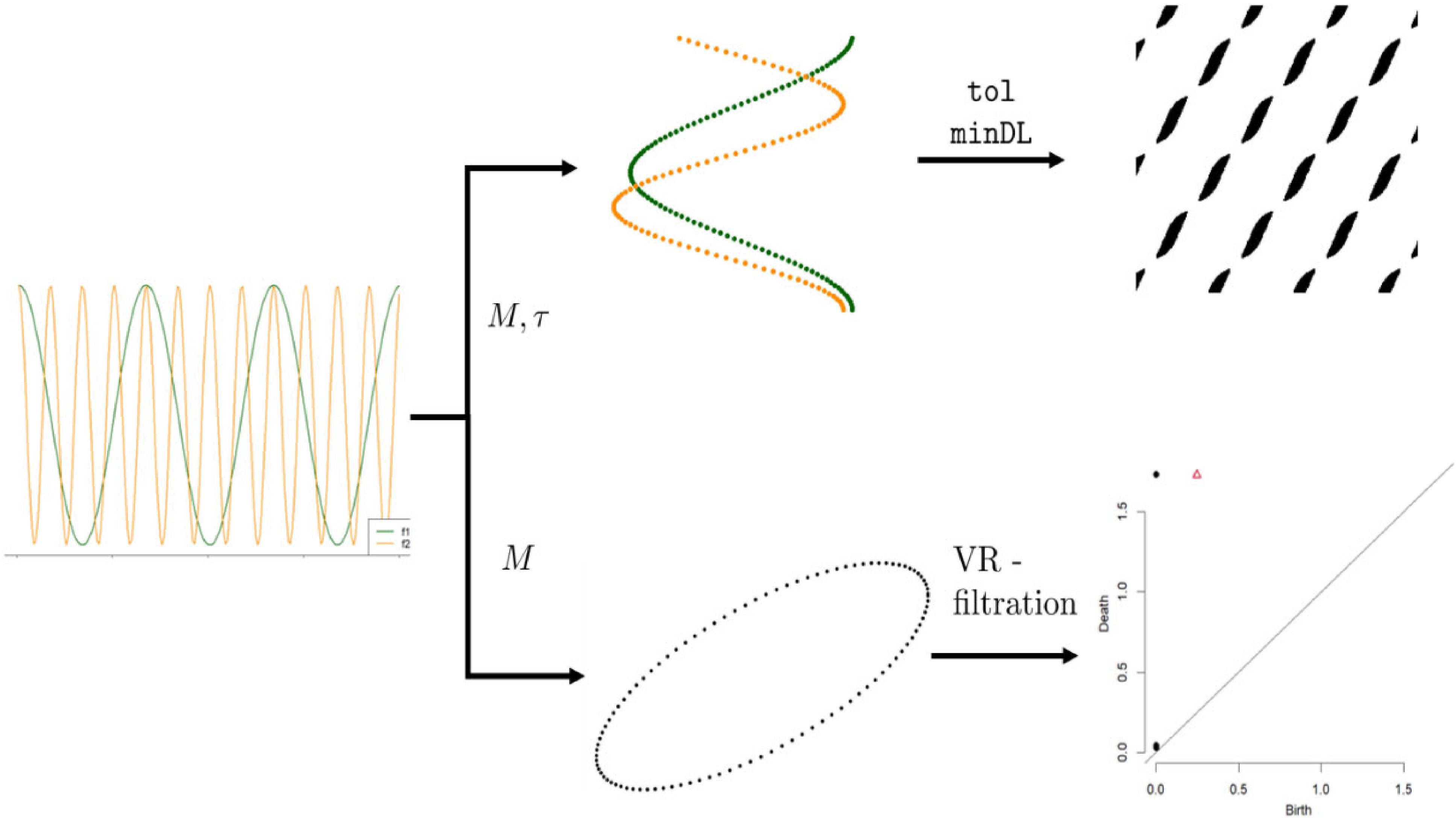
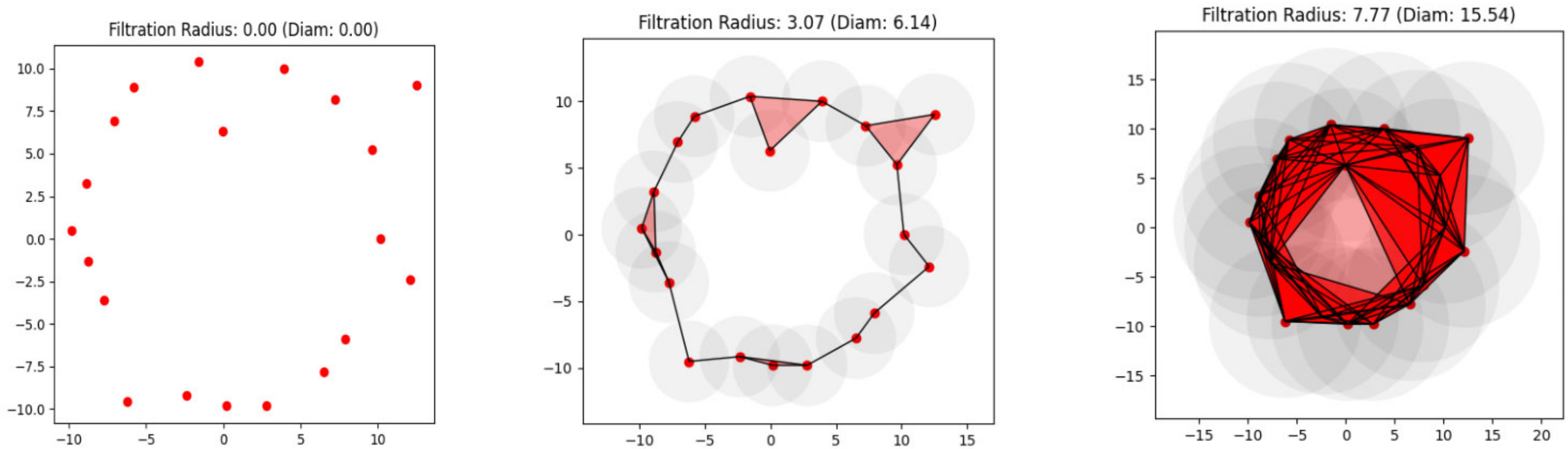
- ✓ computational evidence

SCORE($f_1|f_2$) vs %DET



Vietoris-Rips (VR) Persistence

Edelsbrunner et al., 2002



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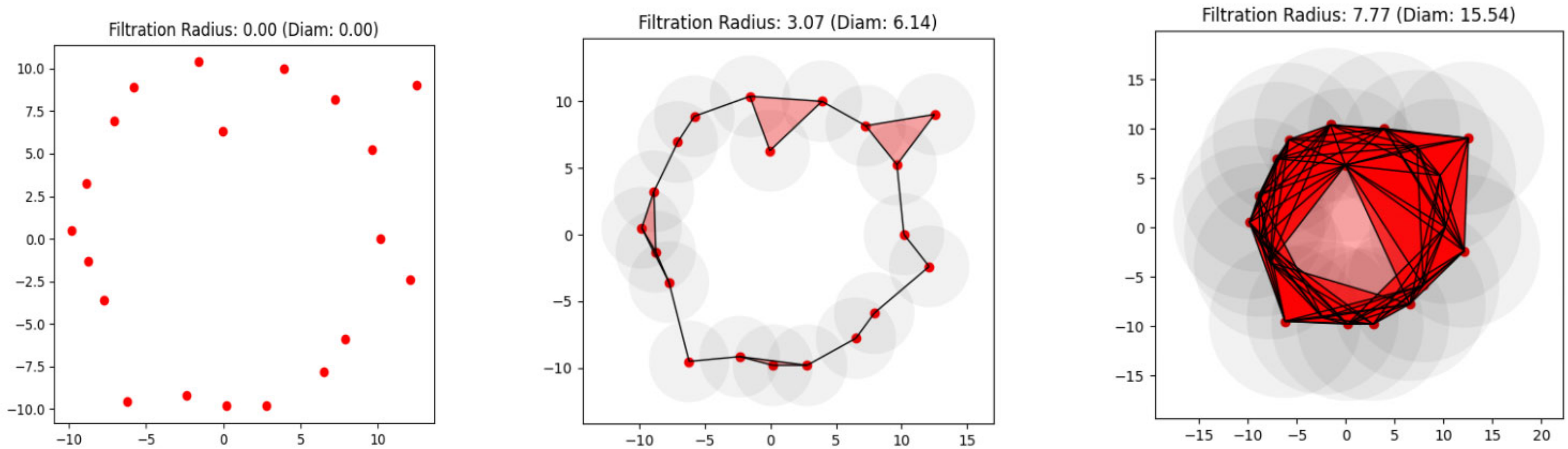
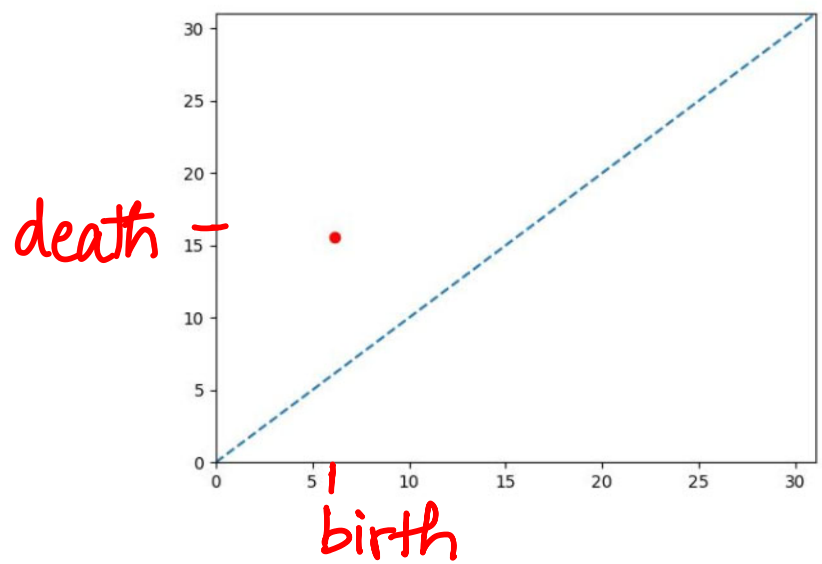


persistence diagram (PD):

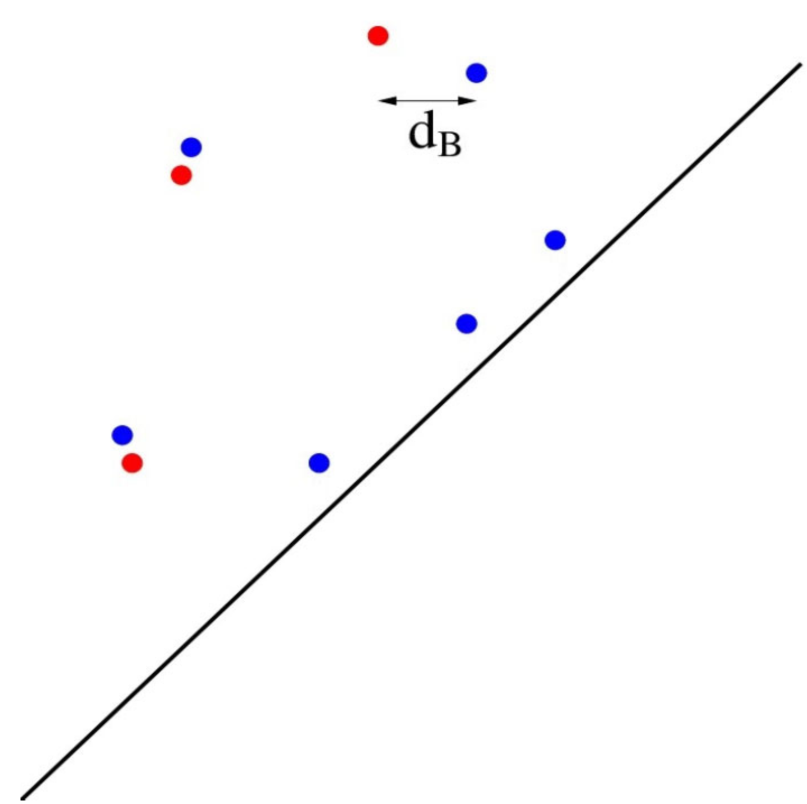


Bottleneck Distance

for two PDs X and Y

$$d_B(X,Y) = \min_{\gamma: X \to Y} \max_{x \in X} \|x - \gamma(x)\|_\infty$$

bijection



Persistence Stability

Chazal, de Silva, Oudot (2014)
Chazal, de Silva, Glisse, Oudot (2016)

$$d_B\big(\mathrm{dgm}(VR(X)), \mathrm{dgm}(VR(Y))\big) \leq 2\, d_{GH}(X,Y) \leq 2\, d_H(X,Y)$$

Gromov-Hausdorff

Hausdorff

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X, Y in common metric space

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X, Y in common metric space

PH: VR Persistent Homology

PH on Time Series

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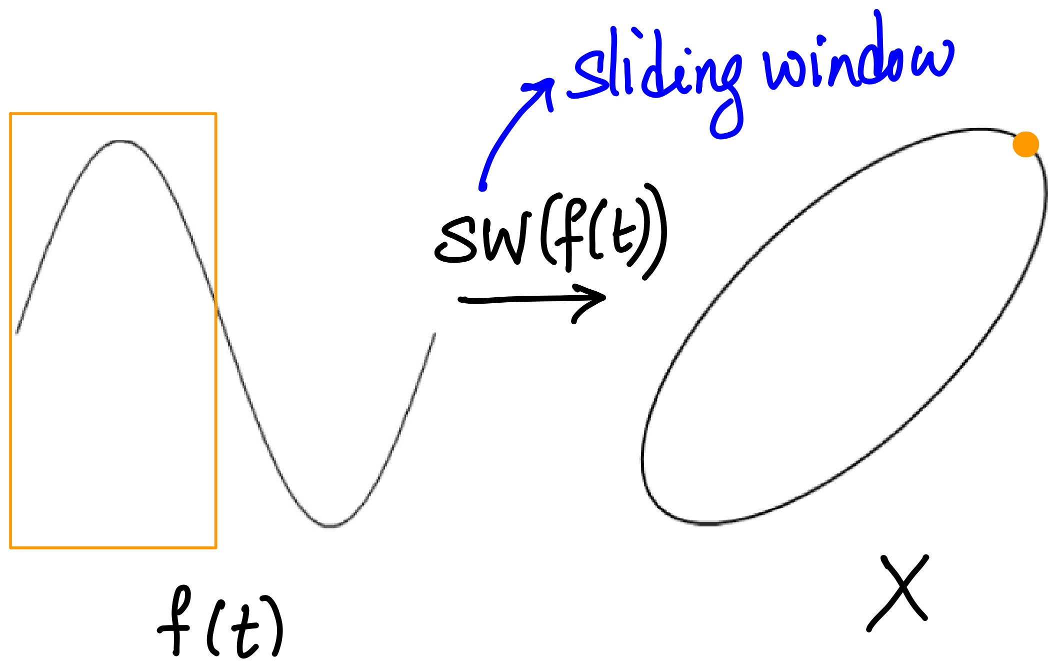

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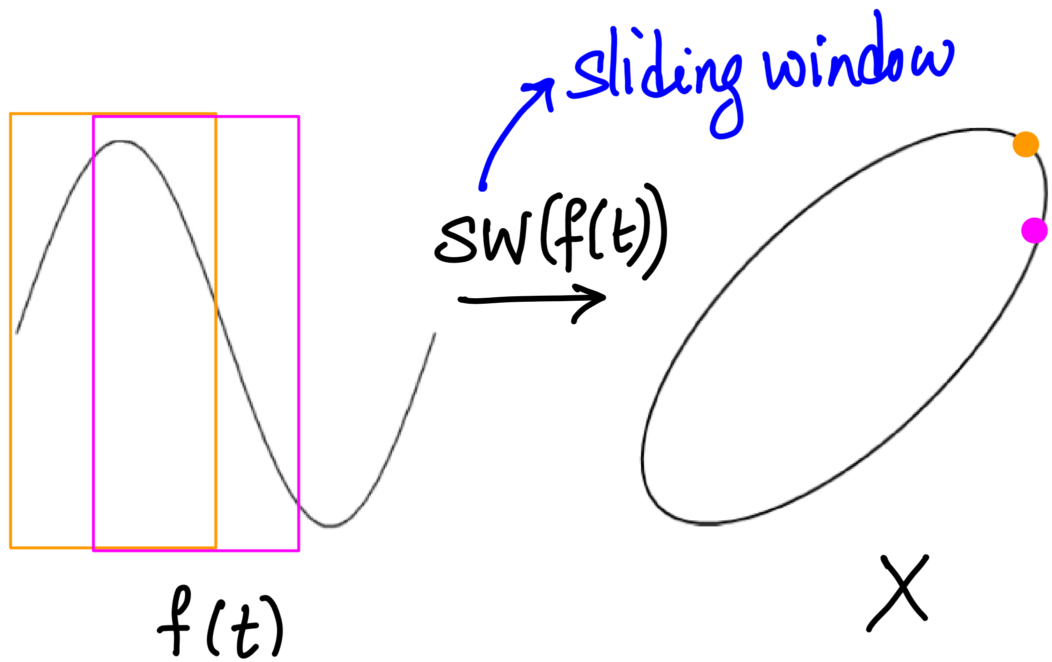

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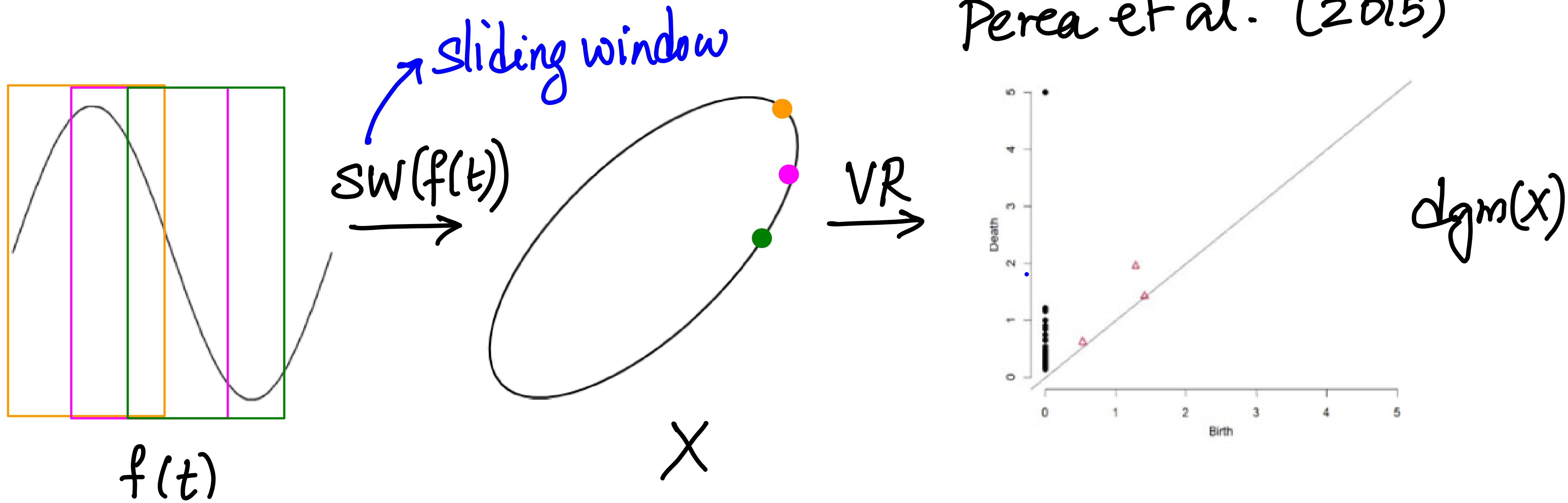

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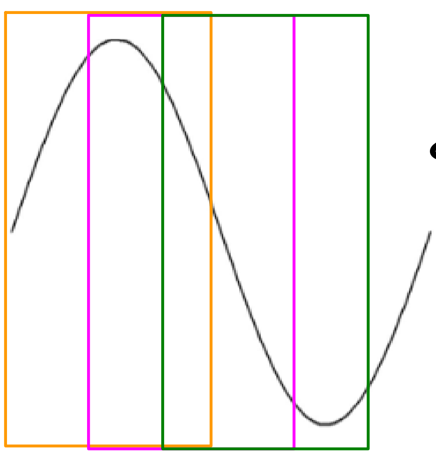
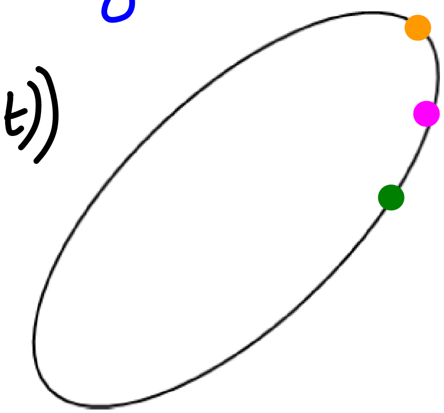


$f(t)$

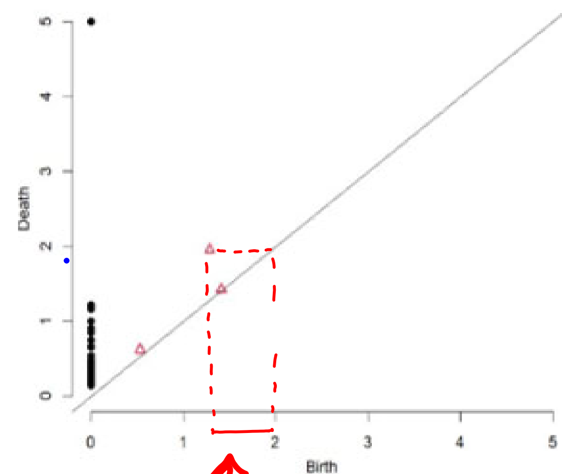
sliding window

$\xrightarrow{SW(f(t))}$



X

$\xrightarrow{VR}$



$dgm(X)$

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max 1-persistence

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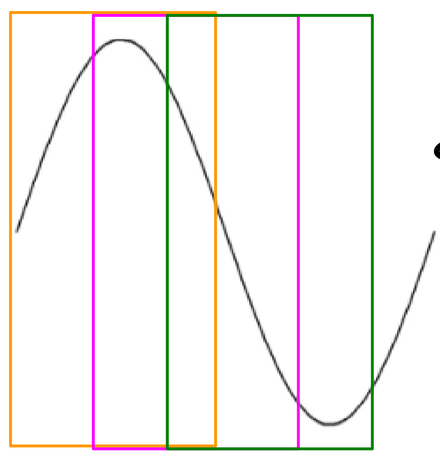
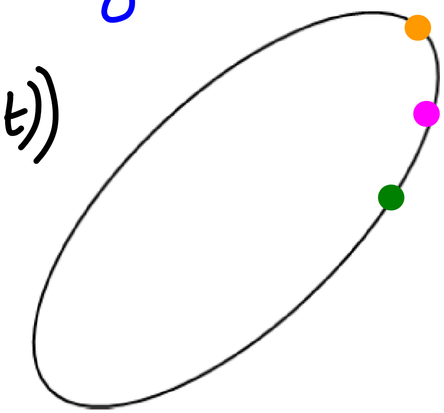


$f(t)$

$\xrightarrow{SW(f(t))}$ (sliding window)



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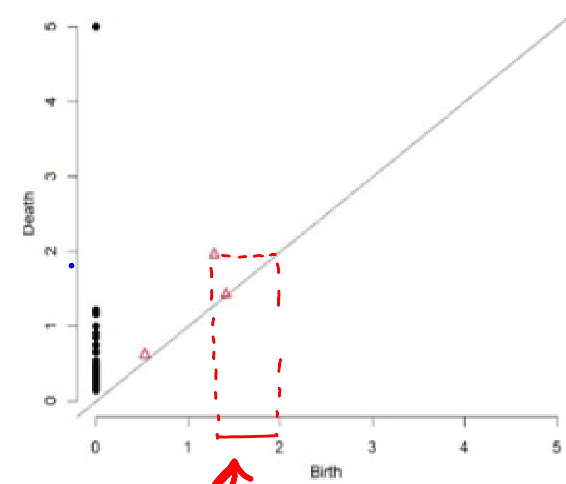

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$$\rightarrow \text{score}(f) = \frac{mp(dgm(X))}{\sqrt{3}}$$

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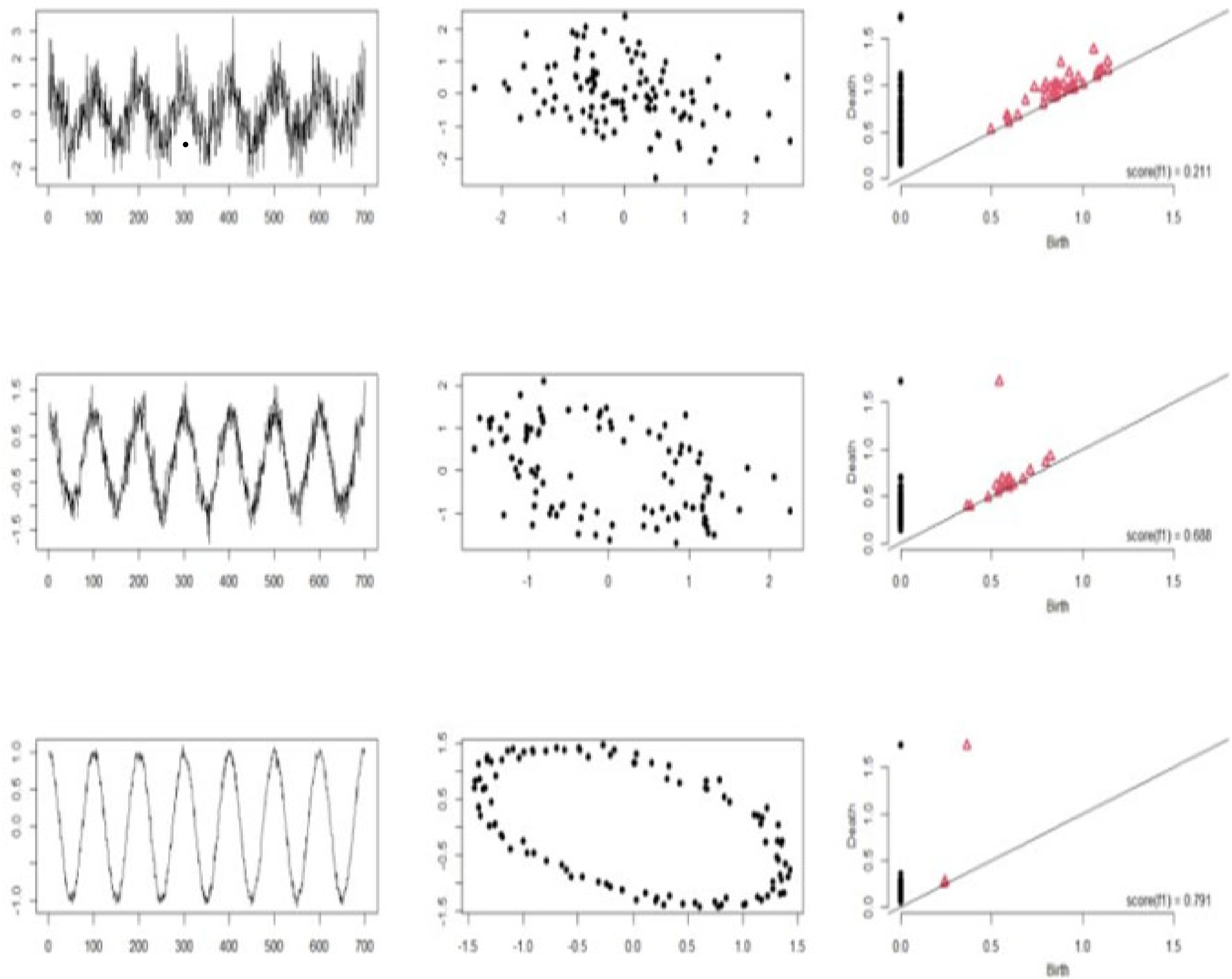

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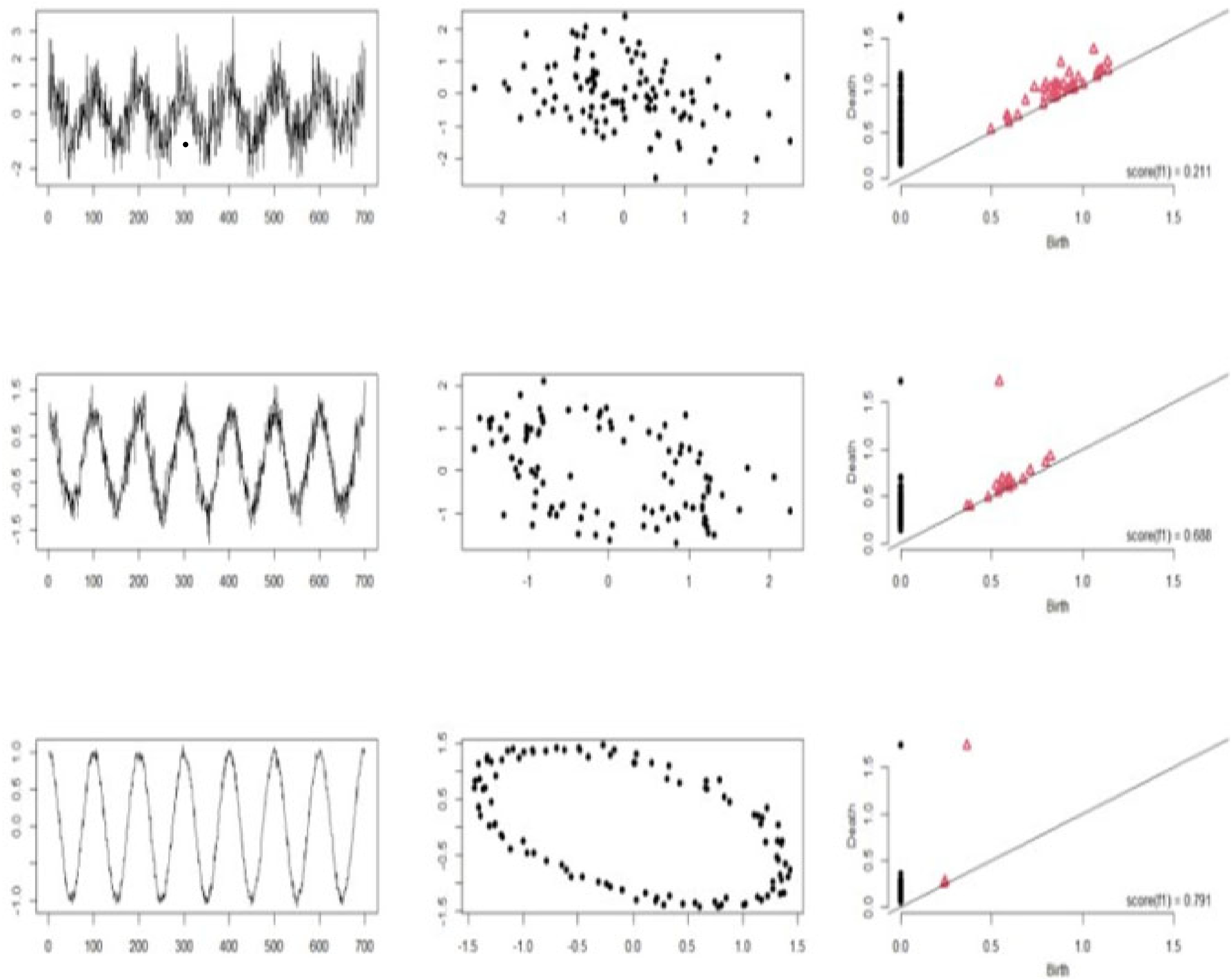

→ stability : closeness of $SW(f)$ and $SW(S_N f)$

N-truncated Fourier series of f

CONDITIONAL PERIODICTY SCORE

Def $f_1, f_2 : [0, 2\pi] \to \mathbb{R}$ continuous, periodic time series

f_2 more-periodic than f_1 $\left(\frac{2\pi}{\omega_2} \leq \frac{2\pi}{\omega_1} \right)$

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Conditional SW Embedding of f_1 given f_2:

$$SW_{M,\tau} f_{1|2}(t) = \left(f_1(t), \ldots, f_1(t + M\tau)\right)^T \text{ for } \tau = \frac{2\pi}{\omega_2(M+1)}.$$

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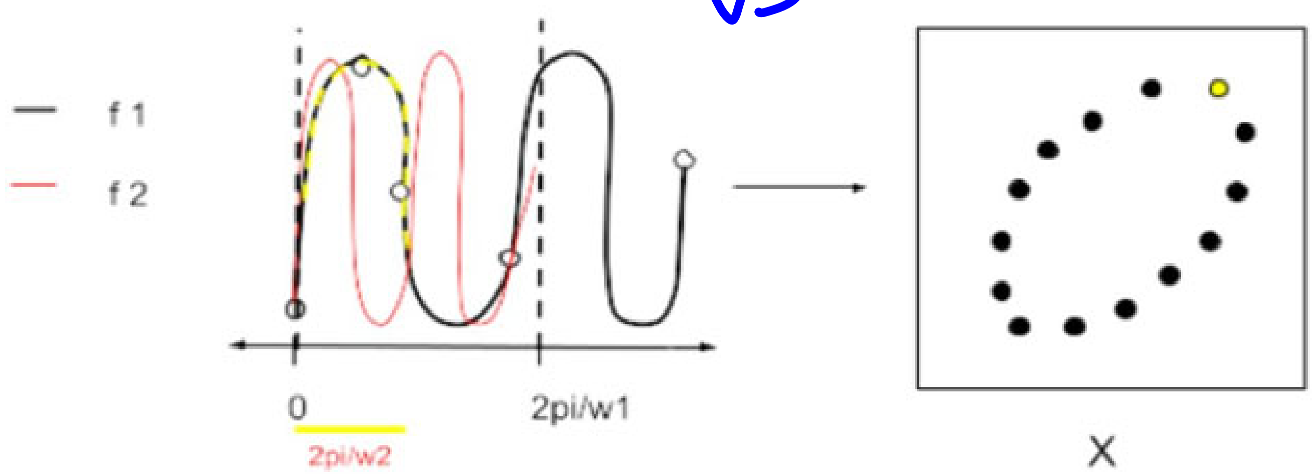

REDUCTION TO PERIODICTY SCORE

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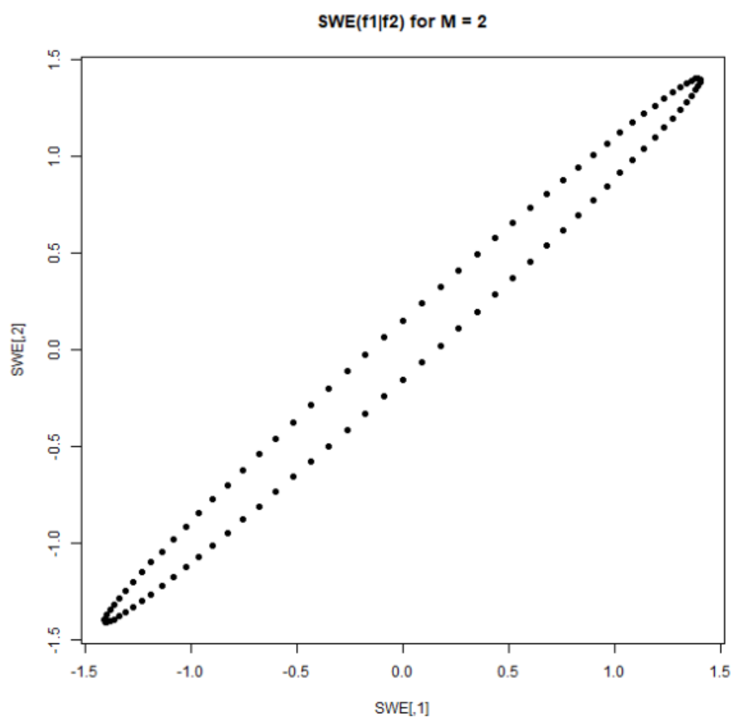

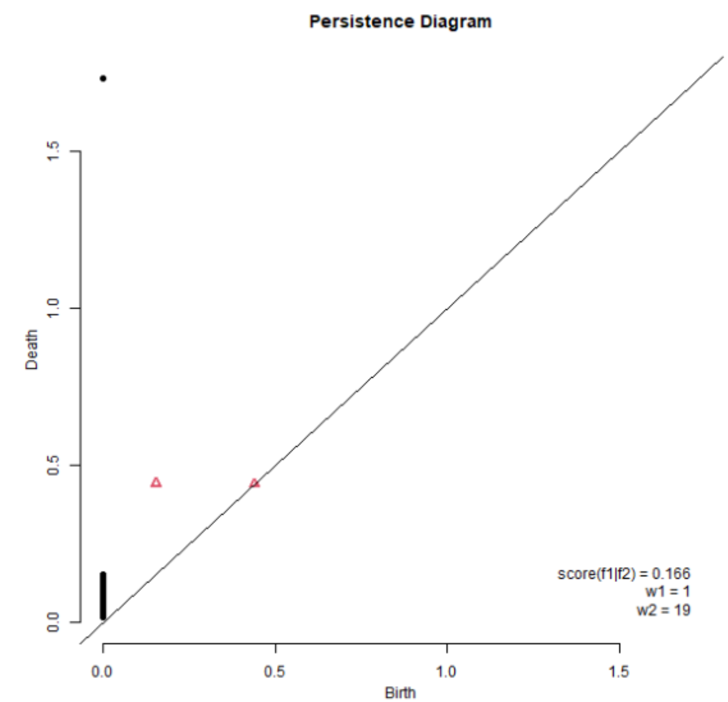

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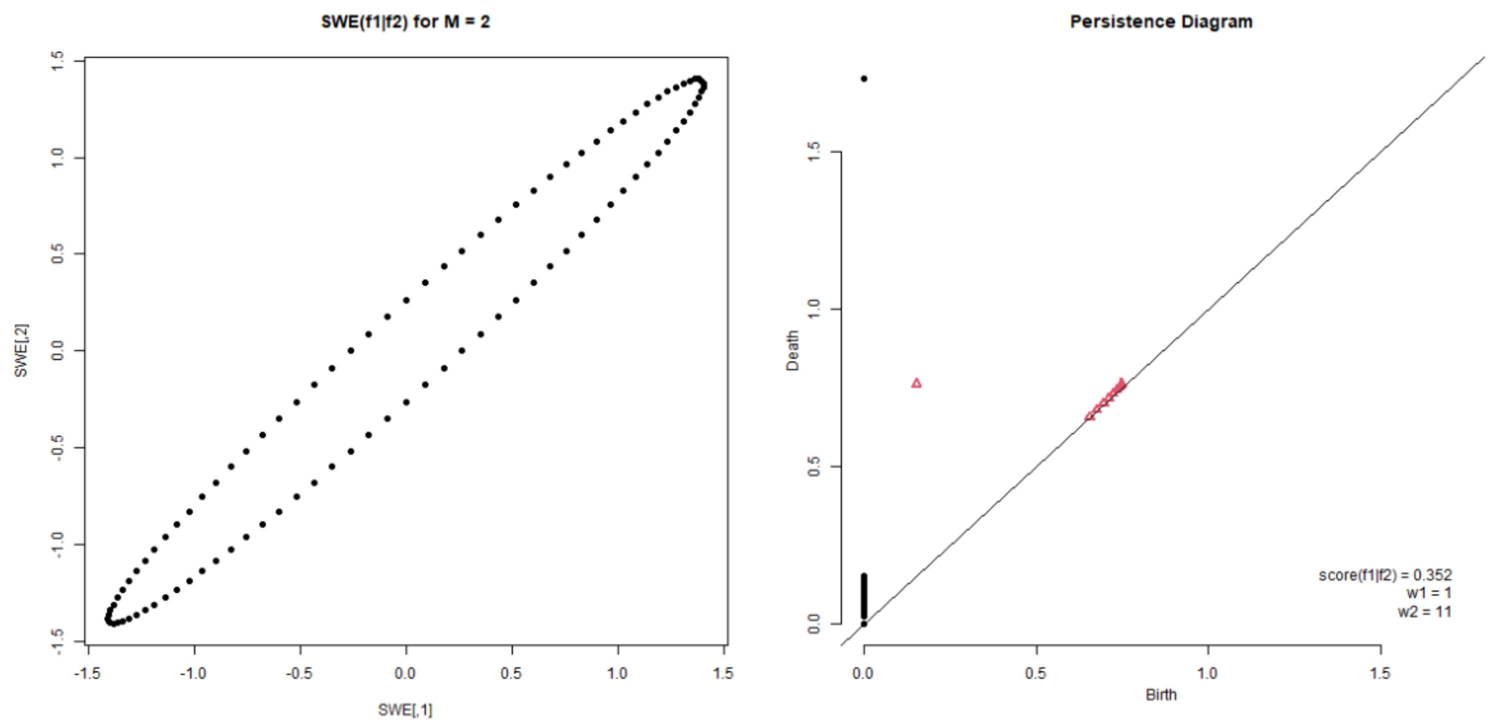

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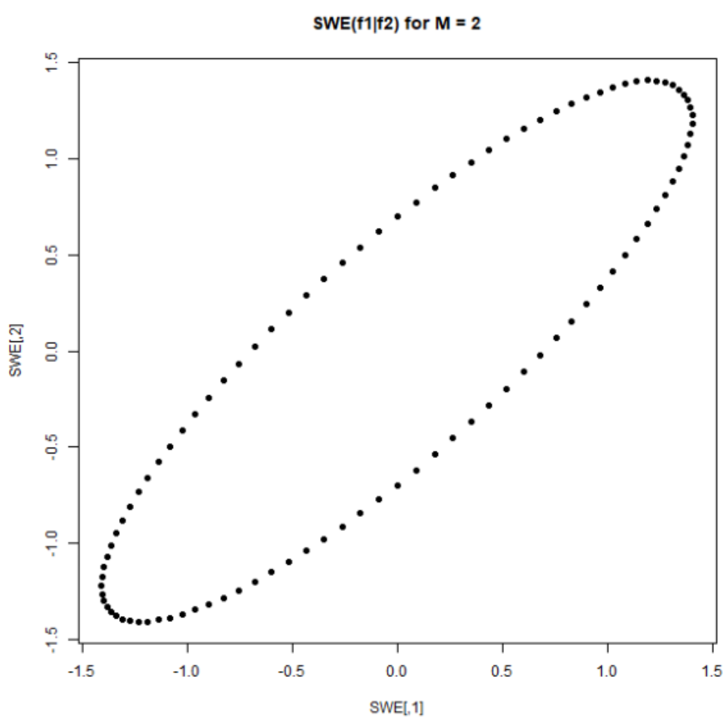

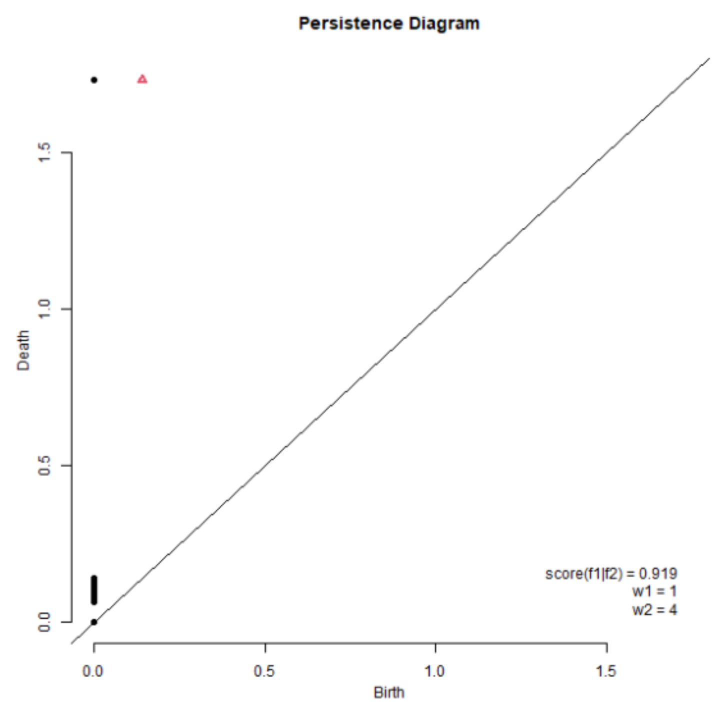

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$X_1 = SW_{M,\tau_1} f_{1|21}(T)$, $X_2 = SW_{M,\tau_2} f_{1|22}(T)$

$$d_H(X_1, X_2) \leq \sqrt{M+1} \left| \frac{2\pi}{\omega_{21}} - \frac{2\pi}{\omega_{22}} \right| \sqrt{\sum_i |f_1'(c_i)|^2}$$

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$$\left|\text{score}(f_1|f_{21}) - \text{score}(f_1|f_{22})\right| \leq 4\sqrt{\frac{M+1}{3}}\left|\frac{2\pi}{\omega_{21}} - \frac{2\pi}{\omega_{22}}\right| \sqrt{\sum_i |f_1'(c_i)|^2}$$

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<u>Lemma 3</u> $f_1^\sigma(t) = f_1(t) + \epsilon_t$ for $\epsilon_t \sim N(0,\sigma^2)$

For $\delta \in (0,1)$, it is at least $(1-\delta)\cdot 100\%$ likely that

$$\left|\text{score}(f_1|f_2) - \text{Score}_\sigma(f_1|f_2)\right| \leq 4\sigma\sqrt{\frac{M+1}{3\delta}}$$

STABILITY RESULTS

✓ Using top K principal components of PCA projection of SW $f_{1|2}(t)$ preserves score$(f_1|f_2)$

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✓ Using top K principal components of PCA projection of SW $f_{1|2}(t)$ preserves score$(f_1|f_2)$

<u>Theorem 4</u> $\phi : \mathbb{R}^{M+1} \to \mathbb{R}^K$: PCA projection with eigen(vectors/values) $\{c_k, \lambda_k\}_{k=1}^{N}$.

$$\left|\text{score}(f_1|f_2) - \text{Score}_\phi(f_1|f_2)\right| \le \sqrt{\frac{8}{3}}\,\sqrt[4]{\sum_{i=K+1}^{N} \lambda_i^2}$$

"unused" (N−K) eigenvalues

MIN. EMBEDDING DIMENSION

- ✓ Embedding dim. M above which score ($F_1 | F_2$) does not change much with dimension

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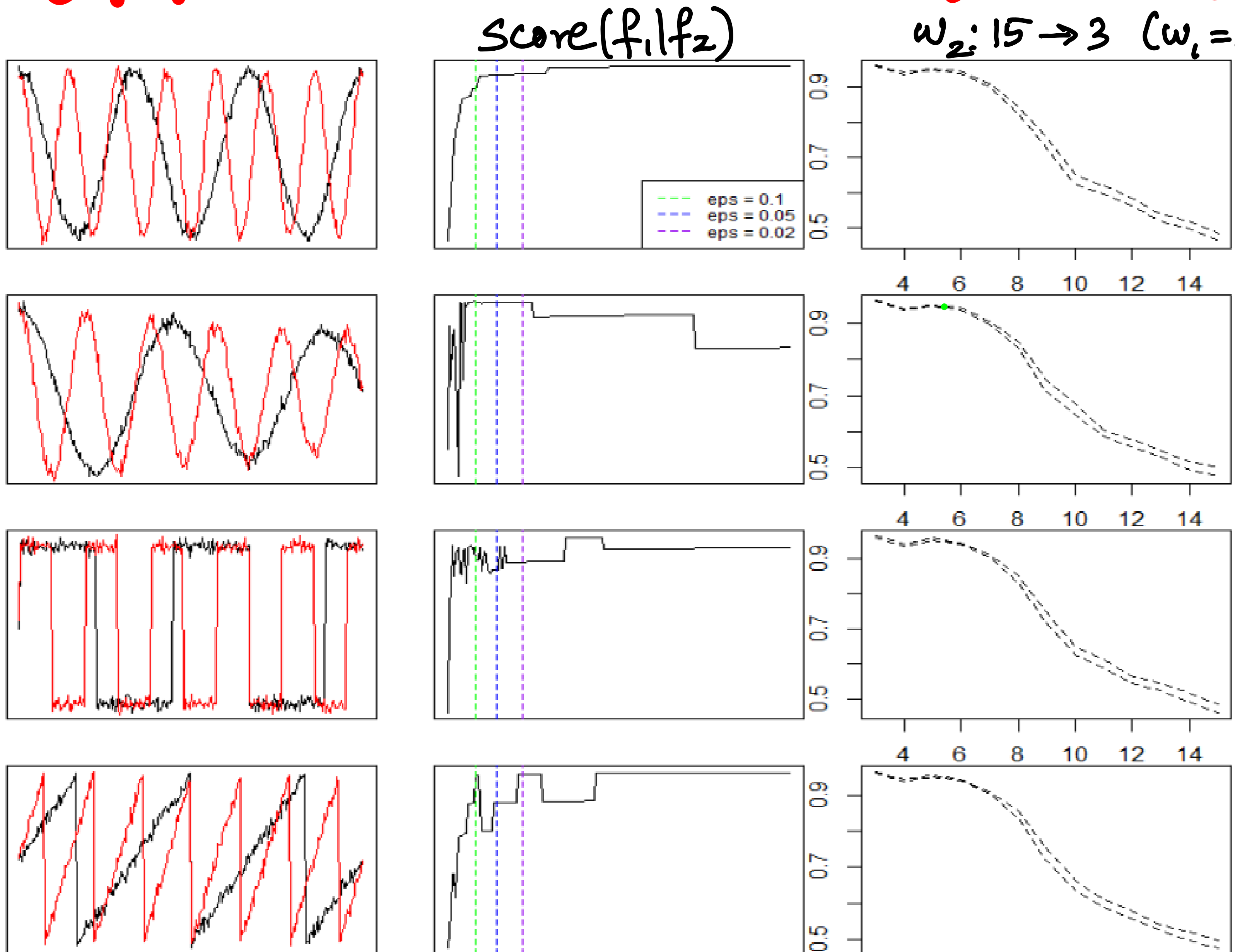
✓ Embedding dim. $\mathcal{M}$ above which $\text{Score}(f_1|f_2)$ does not change much with dimension

Theorem 5 For $\epsilon > 0$, with $\mathcal{M} = \left\lceil \frac{2\pi}{\omega_2 \epsilon} \right\rceil$, for any $M_2 > M_1 \geq \mathcal{M}$

$$\left| \text{score}_{M_1}(f_1|f_2) - \text{Score}_{M_2}(f_1|f_2) \right| \leq \epsilon \cdot g(M_1, f_1) + h(M_1, M_2, f_1)$$

constants w.r.t. ϵ

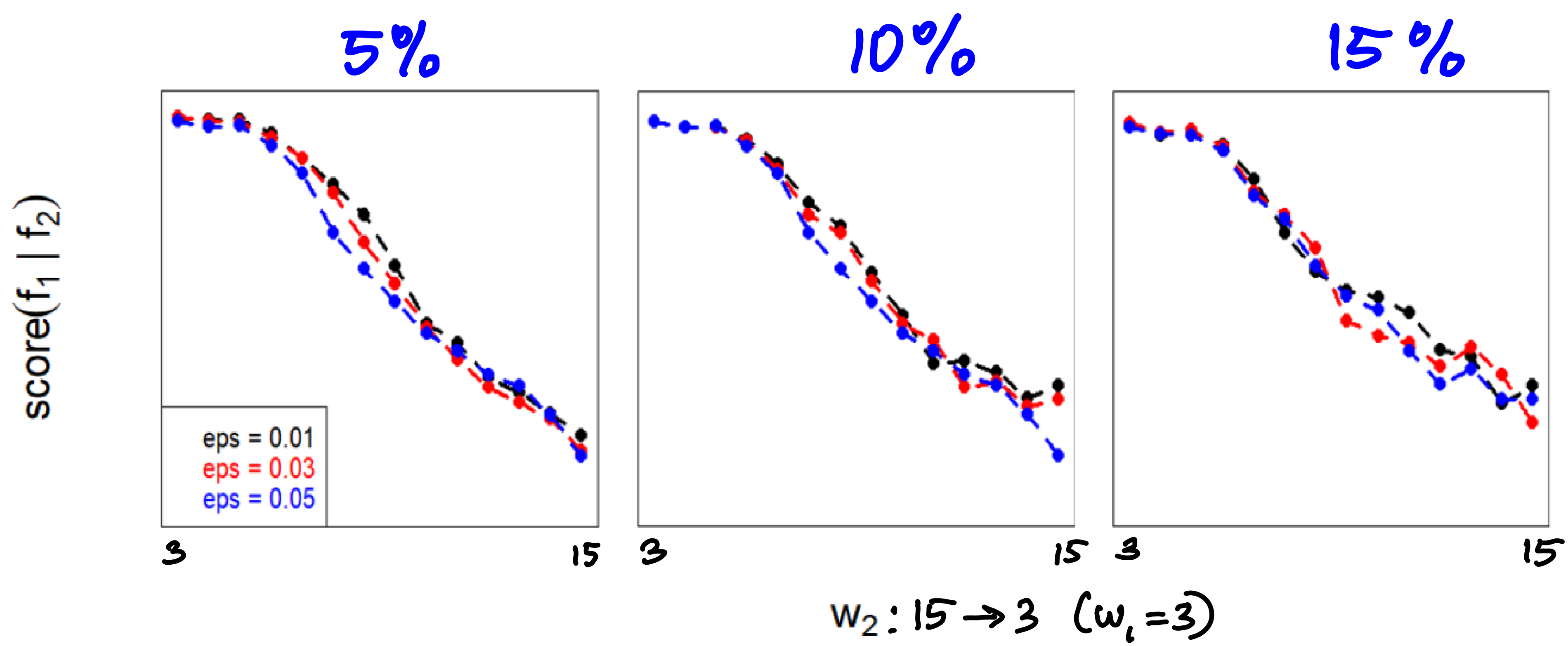
Computational Results



$\mathcal{M}$ for $\epsilon = 0.1, 0.05, 0.03$

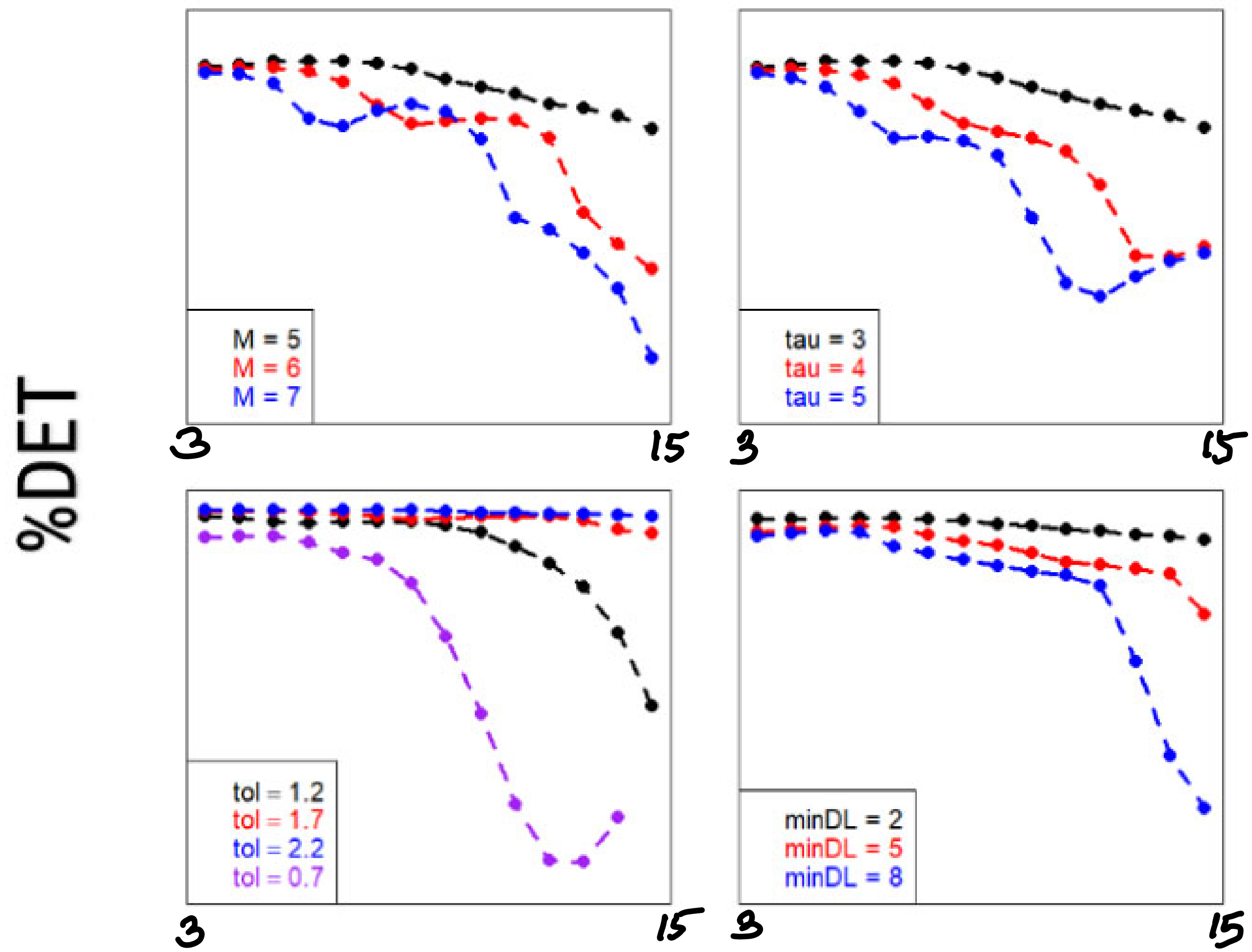
Computational Results

$\mathrm{score}(f_1|f_2)$ under Gaussian noise



Computational Results

%DET under 10% Gaussian noise



$W_2 : 15 \to 3$ $(w_1 = 3)$

OPEN QUESTIONS

? Testing on real data?

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Thank You!