

Computational Topology (Spring 2024): Homework 4

- You **must email your submission** as a **PDF file** to kbala@wsu.edu. You are welcome to write answers by hand, and scan or take photos of the writings. Put all the images on a PDF file, though.
- Your main file's name should identify you in this manner. If you are Gregory Of Yardale, you should name your submission GregoryYardale_Hw4.pdf. **Please start your name in this format. If you want to add details to the title, you could name it GregoryYardale_Math529_Hw4.pdf, for instance. Please avoid white spaces in the file name :-).**
- If you want to **send additional files**, e.g., containing your code, then you should **include all files in a compressed folder**, e.g., zipped or tar-gzipped, named in the same manner as the main PDF file, i.e., **GregoryYardale_Hw4.zip**. You could include the PDF file inside this zipped folder, along with the other files.
- **WSU Email will not allow attachments of .py files, even inside Zipped folders!** You can save such files as .txt files, or include corresponding Jupyter notebooks (.ipynb files) instead.
- Begin the SUBJECT of your email submission with the same **FirstnameLastname**, expression, e.g., “**GregoryYardale Hw4 submission**”.
- **This homework is due by 11:59 PM on Thursday, March 7.**

1. (25) Let the simplicial complex K consist of a d -simplex σ and its faces. Justify your answers to both the following questions.

- (a) How many d -simplices does $Sd K$, the barycentric subdivision of K , have?
- (b) What is the Euler characteristic $\chi(K)$?

2. (20) Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of points in \mathbb{R}^d . The *furthest point Voronoi cell* of a point $\mathbf{v}_j \in S$ is defined as

$$F_{\mathbf{v}_j} = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x} - \mathbf{v}_j\| \geq \|\mathbf{x} - \mathbf{v}_i\| \forall \mathbf{v}_i \in S\}.$$

The collection of $F_{\mathbf{v}_j}$ for all $\mathbf{v}_j \in S$ is called the *furthest point Voronoi diagram* of S . Draw the furthest point Voronoi diagram of a set S with at least *eight* points in \mathbb{R}^2 . On the same diagram, also draw the nerve of this furthest point Voronoi diagram, i.e., the furthest point Delaunay triangulation of S .

3. (30) Let S be a finite set of points in \mathbb{R}^d , and let $\text{Alpha}(r)$, $\check{\text{Cech}}(r)$, and Del denote the alpha, Čech, and Delaunay complexes of S , with the first two complexes defined for radius r of balls centered at each $\mathbf{v}_j \in S$. Either prove each of the following two subset relationships, or give counterexamples violating them.

- (a) $\text{Alpha}(r) \subseteq \check{\text{Cech}}(r) \cap \text{Del}$.
- (b) $\check{\text{Cech}}(r) \cap \text{Del} \subseteq \text{Alpha}(r)$.

(more problems on the next page...)

4. (40) Create a dataset S of *at least* ten points in \mathbb{R}^2 or \mathbb{R}^3 . You should be able to perform *both* the following computations on the chosen dataset.
- Pick an appropriate radius r at which $\check{\text{C}}\text{ech}_S(r) \subset \text{VR}_S(r)$, i.e., the Čech complex is a strict subcomplex of the Vietoris-Rips complex. Produce a visualization showing both complexes **simultaneously**. One option is to use TetView, and set a different *attribute* for those extra triangles that are present only in $\text{VR}_S(r)$ (the attribute values are given in the last (fifth column) in a face file).
 - Choose an appropriate radius at which at least *two* tetrahedra are included in $\text{VR}_S(r)$. Produce visualization(s) of both the Čech and Vietoris-Rips complexes at this radius, using TetView or another software package.
5. (50) This exercise asks you to create a filtration of a Delaunay complex. You could use Matlab (or Python) to do most of the computations. The visualizations could be generated in *TetView* (or in Matlab itself).
- Let the set S contain the eight vertices of the unit cube in \mathbb{R}^3 sitting in the nonnegative orthant, along with ten (10) extra points lying in the interior of the cube (you could choose these points in any way you want—just describe how you locate them). Find a Delaunay tetrahedralization of S using the function `delaunay` in Matlab, or another similar program (*TetGen*, for instance). Denote this complex by Del_S .
 - Rather than finding alpha complexes at various radii, we will use pairwise distance cutoffs similar to those used in defining Vietoris-Rips complexes. For a given radius r , define the Beta complex of S as a subcomplex of its Delaunay complex as follows.

$$\text{Beta}_S(r) = \{\sigma \in \text{Del}_S \mid \text{diam}(\sigma) \leq 2r\}.$$

Write some scripts in Matlab (or another package/language) which can run through the list of simplices in Del_S to identify $\text{Beta}_S(r)$. To carry out these computations efficiently, notice that you need to compute the lengths of all edges in Del_S *only once*, and do comparisons with $2r$ as needed.

- Generate $\text{Beta}_S(r)$ for $r = 1/2, 1/\sqrt{2}$, and $\sqrt{3}/2$. Create visualizations showing the three complexes using *TetView* or a similar program. You are welcome to try Matlab for this task, but TetView might prove easier. Notice that there is an option to display just the outline of a 2- or 3-complex in TetView.

Describe each $\text{Beta}_S(r)$ complex, and include snapshot(s) of each complex.