MATH 524: Lecture 26 (11/18/2025)

Today: * elementary cochains
* computing coboundaries, cohomology

Recall: Elementary cochain: σ_α^* : 1 on σ_α, 0 o.w.

p-cochain $\phi^p = \sum g_\alpha \sigma_\alpha^*$

$$\delta\phi^p = \sum g_\alpha (\delta \sigma_\alpha^*) \qquad (*)$$

Let's verify $(*)$: Let τ be a $(p+1)$-simplex, and suppose $\partial\tau = \sum_{i=0}^{p+1} \epsilon_i \sigma_{\alpha_i}$, $\epsilon_i = \pm 1 \; \forall i$.

Then $\langle \delta\phi^p, \tau \rangle = \langle \phi^p, \partial\tau \rangle = \sum_{i=0}^{p+1} \epsilon_i \langle \phi^p, \sigma_{\alpha_i} \rangle$

$= \sum_{i=0}^{p+1} \epsilon_i g_{\alpha_i}$, where g_{α_i} = value of ϕ^p on σ_{α_i}.

Also, $\langle g_\alpha (\delta\sigma_\alpha^*), \tau \rangle = g_\alpha \langle \delta\sigma_\alpha^*, \tau \rangle = g_\alpha \langle \sigma_\alpha^*, \partial\tau \rangle$

$$= \begin{cases} \epsilon_i g_{\alpha'} & \text{if } \alpha = \alpha_i,\ i = 0, \dots, p+1; \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

So, $(*)$ does hold.

By $(*)$, to compute $\delta\phi^p$, it suffices to compute $\delta\sigma^*$ for each oriented p-simplex σ. But

$$\delta\sigma^* = \sum \epsilon_j \tau_j^*$$

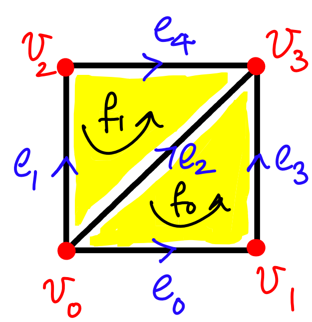
where the sum extends over all $(p+1)$-simplices τ_j that are cofaces of σ, i.e., $\tau_j > \sigma$ (or, τ_j has σ as a face), and $\epsilon_j = \pm 1$ is the sign with which σ appears in the expression for $\partial\tau_j$.

So, we can compute cohomology using elementary cochains.
We now explore several examples.

Examples

1. Vertices $\{v_i\}$
   edges $\{e_i\}$
   faces $\{f_i\}$

K



Let's evaluate some cochains, and their coboundaries.

$\delta e_2^* = f_1^* - f_0^*$ → notice $\bar{e}_2$ has +1 in $\partial \bar{f}_1$ and −1 in $\partial \bar{f}_0$

$\delta v_3^* = e_2^* + e_3^* + e_4^*.$

Cocycles and coboundaries

Both f_0^* and f_1^* are trivial 2-cocycles (as K has no 3-simplices, so $\delta f_0^* = \delta f_1^* = 0$).
Also, both f_0^* and f_1^* are coboundaries, since

$\delta e_0^* = f_0^*$ and $\delta e_1^* = -f_1^*$.

Also, $\delta e_3^* = f_0^*$ and $\delta e_4^* = -f_1^*$.

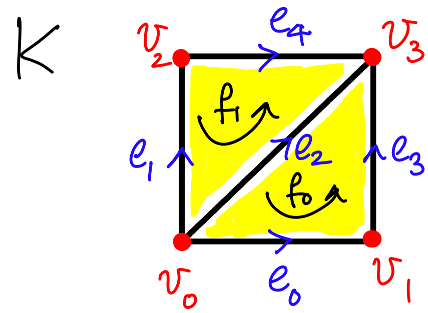
The 1-cochain $\phi^1 = e_0^* + e_2^* + e_4^*$ is a 1-cocycle, as

$\delta \phi^1 = f_0^* + (f_1^* - f_0^*) + -f_1^* = 0.$

It is also a 1-coboundary, as $\delta(v_1^* + v_3^*) = \phi^1.$

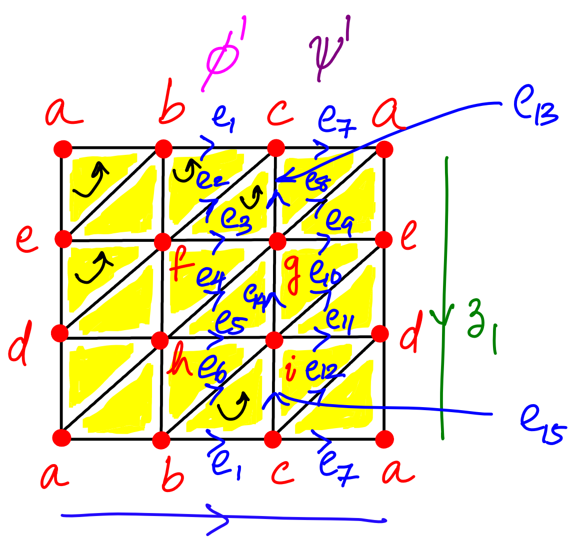
Here are all the 0-coboundaries:

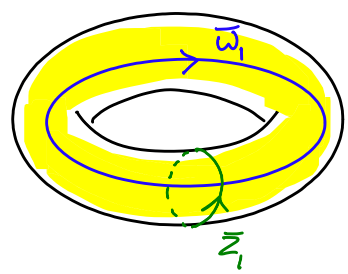
$$\delta v_0^* = -e_0^* - e_1^* - e_2^*$$
$$\delta v_1^* = e_0^* - e_3^*$$
$$\delta v_2^* = e_1^* - e_4^*$$
$$\delta v_3^* = e_2^* + e_3^* + e_4^*$$

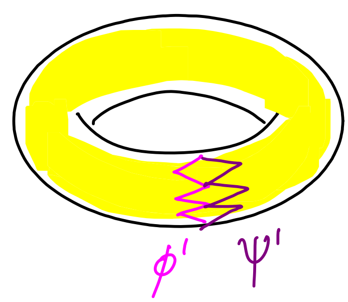


Hence the 0-cochain $\phi^0 = v_0^* + v_1^* + v_2^* + v_3^*$ is a 0-cocycle (as $\delta\phi^0 = 0$). It cannot be a coboundary, as there are no cochains of dimension -1.

2. Torus







Consider the 1-cochain $\phi^1 = e_1^* + \dots + e_6^*$. It is a 1-cocycle! Each triangle in the middle patch appears with a +1 and −1 in the expressions for δe_i^*.

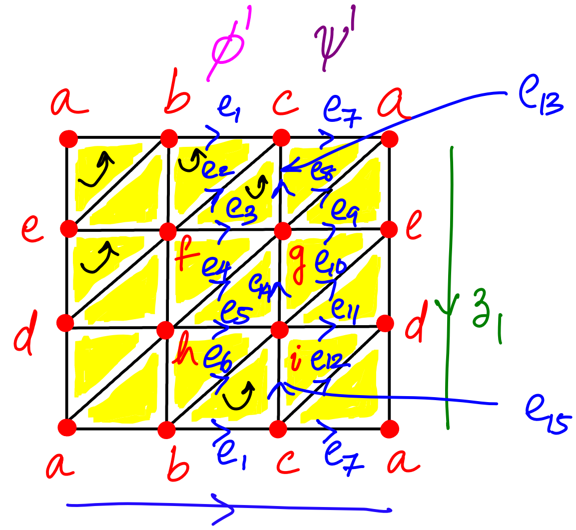
Similarly, $\psi' = e_7^* + \cdots + e_{12}^*$ is also a 1-cocycle, as $\delta\psi' = 0$.

ϕ' and ψ' are **cohomologous**, as $\phi' - \psi' = \delta^0(c^* + g^* + i^*)$.

$$\delta i^* = e_5^* + e_6^* + \cancel{e_{15}^*} - e_{10}^* - e_{11}^* - \cancel{e_{14}^*}$$

$$\delta g^* = e_3^* + e_4^* + \cancel{e_{14}^*} - e_8^* - e_9^* - \cancel{e_{13}^*}$$

$$\delta c^* = e_1^* + e_2^* + \cancel{e_{13}^*} - e_7^* - e_{12}^* - \cancel{e_{15}^*}$$



$$\delta(c^* + g^* + i^*) = \phi' - \psi'.$$

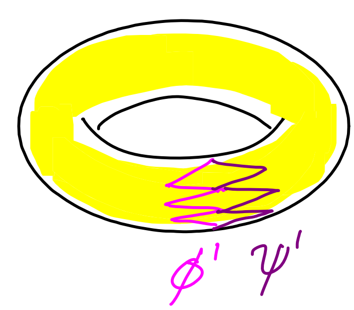
Two cocycles are cohomologous if their difference is the coboundary of a one-dim lower cochain.

We write $\psi' \underset{c}{\simeq} \phi'$ here.

$\underset{c}{\simeq}$: "cohomologous"

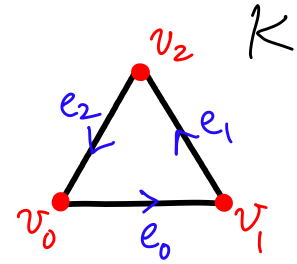
Recall, 2-chains $\bar{c}, \bar{c}'$ are homologous, $\bar{c} \sim \bar{c}'$, if $\bar{c} - \bar{c}' = \partial \bar{d}$.

We can visualize the cocycles as "picket fences". Two 1-cocycles are cohomologous if the picket fences are attached at the "right" vertices "along the middle".



Example 3

Let's compute cohomology groups of K.



Note that $H_0(K) \simeq \mathbb{Z}$ (one component), and
$H_1(K) \simeq \mathbb{Z}$ (one hole).

The general 0-cochain is $\phi^0 = n_0 v_0^* + n_1 v_1^* + n_2 v_2^*$.
We have $\delta v_0^* = e_2^* - e_0^*$, $\delta v_1^* = e_0^* - e_1^*$, and $\delta v_2^* = e_1^* - e_2^*$.

$$\Rightarrow \delta \phi^0 = \sum_{i=0}^{2} n_i (\delta v_i^*) = (n_1 - n_0) e_0^* + (n_2 - n_1) e_1^* + (n_0 - n_2) e_2^*.$$

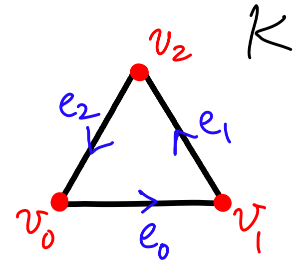
Hence ϕ^0 is a 0-cocycle if $\delta \phi^0 = 0$, i.e., when $n_0 = n_1 = n_2 = n$ (say).

Then $\phi^0 = n\left(\sum_{i=0}^{2} v_i^*\right)$. It is trivially not a coboundary as there are no (-1)-dim. cochains.

$$\Rightarrow H^0(K) \simeq \mathbb{Z}, \text{ and is generated by } \left\{\sum_{i=0}^{2} v_i^*\right\}.$$

Notice the correspondence of the argument used here to the one used to find the structure of $H_1(K)$ — they're are essentially identical!

Consider the 1-cochain $\psi^1 = \sum_{i=0}^{2} m_i e_i^*$. It is a cocycle (trivially), as there are no 2-cochains. We show that $\psi^1 \underset{c}{\sim}$ some multiple of e_0^*.



We show $e_1^* \underset{c}{\sim} e_0^*$ and $e_2^* \underset{c}{\sim} e_0^*$.

But we get these results from

$\delta v_0^* = e_2^* - e_0^*$ and $\delta v_1^* = e_0^* - e_1^*$.

$\Rightarrow \psi^1 \underset{c}{\sim} m e_0^*$ for some $m \in \mathbb{Z}$, $m \neq 0$.

$m e_0^*$ is not a coboundary unless $m=0$.

Suppose $m e_0^* = \delta\left(\sum_{i=0}^{2} n_i' v_i^*\right) = \sum_{i=0}^{2} n_i' (\delta v_i^*)$

$$= \underset{=0}{(n_1' - n_0')} e_0^* + \underbrace{(n_2' - n_1')}_{=0} e_1^* + \underbrace{(n_0' - n_2')}_{=0} e_2^* \longrightarrow \text{needed.}$$

$\Rightarrow n_0' = n_1' = n_2'$. $\Rightarrow m = 0$ if $m e_i^*$ is a coboundary.

Hence we conclude that $H^1(K) \simeq \mathbb{Z}$, and is generated by $\{e_0^*\}$, or by $\{e_1^*\}$ or $\{e_2^*\}$.

Here $H^i(K) \simeq H_i(K)$ $\forall i$ (they are both trivial for $i \geqslant 2$).

But in general, $H^i(K) \not\simeq H_i(K)$.

Here, $H^1(K) \simeq H_0(K)$ and $H^0(K) \simeq H_1(K)$, actually.