

MATH 529 – Lecture 17 (03/05/2024)

Today: * Smith normal form of $[a_p]$
* read off z_p, b_p , and maintain bases

Recall $[\partial_p]$: $m \times n$ $\{0, 1\}$ -matrix (over \mathbb{Z}_2) when K has m $(p-1)$ -simplices and n p -simplices.
 $[\partial_p]_{ij} = \begin{cases} 1 & \text{if } \tau_i \leq \sigma_j, 0, \text{ otherwise} \end{cases}$

EROS on $A_{m \times n}$: $R_i \rightleftharpoons R_j$ (swap rows i and j)
 over $\mathbb{R}, c \in \mathbb{R}$, and $R_i \leftarrow R_i + cR_j$ ($R_i + cR_j$, replacement)
 over $\mathbb{Z}, c \in \mathbb{Z}$. $R_i \leftarrow cR_i, c \neq 0$ (cR_i , scaling)

Over \mathbb{Z}_2 , $c \in \{0, 1\}$, and only swap and replacement need to be considered. We write $R_i \leq R_j$, $R_i + R_j$, in short.

ECOs are defined similarly: $C_i \geq C_j$, $C_i + C_j$ (over \mathbb{Z}).

ECOs can be represented by multiplication on the right by the corresponding elementary matrix.

e.g.,

	i		j	
	B			
	$m \times n$			

$\xrightarrow{C_i + C_j}$

	$i+j$		j	
	B'			
	$m \times n$			

$$B' = BV, \text{ where } V = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n \times n}$$

Similarly, EROs can be represented by multiplication the left by an elementary matrix- $\begin{bmatrix} & \\ i & j \end{bmatrix}$

A diagram showing the effect of row interchange on a matrix. On the left, a matrix B of size $m \times n$ is shown with its rows labeled i and j . An arrow labeled $R_i \Leftrightarrow R_j$ points to the right, indicating the interchange of rows i and j . On the right, the resulting matrix B' of size $m \times n$ is shown with its rows labeled j and i .

can be written as $B' = UB$ for $U = \begin{pmatrix} 0 & 1 & \dots & 1 \\ j & 0 & \dots & 0 \\ & & \ddots & \\ & & & 0 \end{pmatrix}$.

Smith Normal Form (SNF)

This is the analogue of reduced row echelon form (RREF) of a matrix, from linear algebra. SNF is defined over \mathbb{Z} , in general, and accounts for both EROs and ECOS.

Def A matrix $B \in \mathbb{Z}^{m \times n}$ is in **Smith normal form** if

$$B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} d_1 d_2 \dots d_l & 0 \\ \hline 0 & 0 \end{array} \right] \text{ where}$$

Each '0' is
a submatrix
of all zeros.

$$d_i \in \mathbb{Z}, d_i \geq 1 \text{ and } d_1 | d_2 | d_3 | \dots | d_l.$$

↳ "divides": $a|b$ means a divides b .

When working over \mathbb{Z}_2 , we will get

$$\text{SNF}([\partial_p]) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right].$$

In more detail, we will have the following structure for $\text{SNF}([\partial_p])$.

$$\text{SNF}([\partial_p]) = \left[\begin{array}{c|c} 1 & \dots \\ \hline 0 & \end{array} \right]$$

$s_p = \text{rank } C_p$
 $\text{rank } Z_p = s_p$
 $b_{p-1} = \text{rank } B_{p-1}$
 $\text{rank } C_{p-1} = s_{p-1}$

We get Z_p and B_{p-1} from $\text{SNF}([\partial_p])$. To compute β_p , we need to know also b_p , which is obtained from $\text{SNF}([\partial_{p+1}])$.

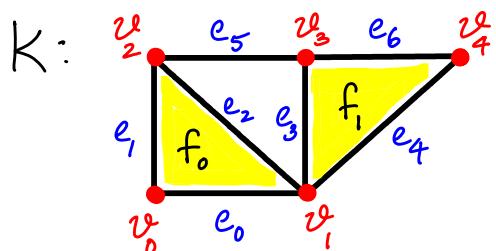
Overall, we can write $\text{SNF}([\partial_p]) = U_{p-1} [\partial_p] V_p$, where U_{p-1} captures all EROs performed, and V_p captures all ECOS performed. We also get information about bases of Z_p and B_{p-1} from U_{p-1} and V_p .

More specifically, a basis for the cycle group Z_p is encoded in the last 3_p columns of V_p . Similarly, a basis for the boundary group B_{p-1} is encoded in U_{p-1} — in the first b_{p-1} columns of U_{p-1}^{-1} , to be exact.

Thus we can reduce each $[\partial_p]$ to SNF, and in that process, compute all β_p , and also identify bases for Z_p and B_p . In the following illustration, we keep track of the bases as we perform the corresponding EROs or ECOS on $[\partial_p]$. Later on, we describe an algorithm that does all the operations in a unified manner.

Example

We consider our favorite example and reduce each $[\partial_p]$ to SNF for $p=0, 1, 2$.



$$[\partial_2] = \begin{bmatrix} f_0 & f_1 \\ e_0 & 1 \\ e_1 & 1 \\ e_2 & 1 \\ e_3 & 0 \\ e_4 & 0 \\ e_5 & 0 \\ e_6 & 0 \end{bmatrix}$$

$$[\partial_0] = \begin{bmatrix} v_0 & v_1 & v_2 & v_3 & v_4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

↳ default, as there are no (-1)-dimensional simplices

$$[\partial_1] = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ v_0 & 1 & & & & & \\ v_1 & & 1 & 1 & & & \\ v_2 & & & 1 & 1 & & \\ v_3 & & & & 1 & 1 & \\ v_4 & & & & & 1 & 1 \end{bmatrix}$$

(entries not listed are zeros).

We reduce each $[\partial_p]$ to SNF.

We consider $[\partial_0]$ and $[\partial_2]$ first, since they are simpler compared to $[\partial_1]$ (we consider that the last).

$$[\partial_0] = 1 \begin{bmatrix} v_0 & v_1 & v_2 & v_3 & v_4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[\text{for } j=2, \dots, 5]{C_j + C_1} 1 \begin{bmatrix} v_0 & v_1+v_0 & v_2+v_0 & v_3+v_0 & v_4+v_0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad z_0 = 4$$

But $s_0 = 5 = z_0$!

We had noticed previously that every 0-chain is also a 0-cycle, as vertices have empty boundaries. So, we should have got $s_0 = z_0 = 5$ here! We will address this discrepancy in $p=0$ soon. For now, work with $z_0 = s_0 = 5$.

A basis for $B_0(K) = \{v_j + v_0 \mid j=1, 2, 3, 4\}$. A collection of even pairs of vertices, such that any even set of vertices can be written as a combination of these pairs.

Recall that every 0-chain is also a 0-cycle since it has no boundary (when we work over \mathbb{Z}_2). And every 0-chain with an even number of vertices is a 0-boundary (assuming K is connected).

Adding any one vertex (by itself) to the above basis will give another basis for \mathbb{Z}_0 (and hence C_0), e.g., $\{v_j + v_0\}_{j=1}^4, v_0\}$. Recall that $\{v_j\}_{j=0}^4$ gives the default (elementary chain) basis for $\mathbb{Z}_0(C_0)$.

Notation: $\ell_{ijk} \equiv \ell_i + \ell_j + \ell_k$

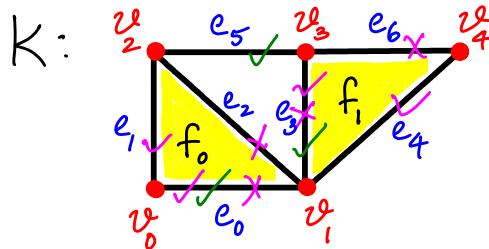
$$[\partial_2] = \begin{bmatrix} e_0 & f_0 \\ e_1 & f_1 \\ e_2 & 0 \\ e_3 & 0 \\ e_4 & 0 \\ e_5 & 0 \\ e_6 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 + R_1 \\ R_3 + R_1}} \begin{bmatrix} e_0 & f_0 \\ e_1 & f_1 \\ e_{01} & 0 \\ e_{02} & 0 \\ e_{03} & 0 \\ e_{34} & 0 \\ e_{35} & 0 \\ e_{36} & 0 \end{bmatrix} \xrightarrow{\substack{R_5 + R_4 \\ R_7 + R_4}}$$

$$\begin{bmatrix} e_0 & f_0 \\ e_1 & f_1 \\ e_{01} & 0 \\ e_{02} & 0 \\ e_{03} & 0 \\ e_{34} & 0 \\ e_{35} & 0 \\ e_{36} & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4}$$

$$\begin{bmatrix} e_0 & f_0 \\ e_3 & 0 \\ e_{02} & 0 \\ e_{01} & 0 \\ e_{34} & 0 \\ e_5 & 0 \\ e_{36} & 0 \end{bmatrix}$$

$e_2 = 0, b_1 = 2$

Basis for $C_2(K) = \{f_0, f_1\}$.



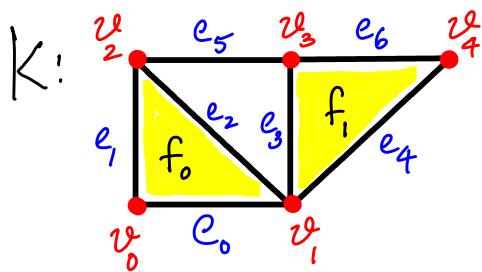
A basis for $C_1(K) = \{e_0, e_3, e_5, e_0 + e_1, e_0 + e_2, e_3 + e_4, e_3 + e_6\}$.

Notice that we have found also a basis for $B_1(K)$ - consisting of the boundaries of f_0 and f_1 .

We are numbering the rows and columns starting from 1. As we proceed further into the reduction (to SNF), the labels for the rows/columns could become more complicated.

Similar notation for column labels: $C_{ijk} \equiv C_i + C_j + C_k$.

Let's look at $[\partial_1]$ now.



$$[\partial_1] = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ v_0 & 1 & 1 & 1 & 1 & 1 & 1 \\ v_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ v_2 & 1 & 1 & 1 & 1 & 1 & 1 \\ v_3 & 1 & 1 & 1 & 1 & 1 & 1 \\ v_4 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1}$$

and then
 $C_2 + C_1$

$$\begin{bmatrix} e_0 & e_{01} & e_2 & e_3 & e_4 & e_5 & e_6 \\ v_0 & 1 & 0 & & & & \\ v_{01} & 0 & 1 & 1 & 1 & 1 & \\ v_2 & 1 & 1 & 1 & & & \\ v_3 & & 1 & 1 & 1 & 1 & \\ v_4 & & & 1 & 1 & 1 & \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2}$$

then
 $C_j + C_2$
 $j = 3, 4, 5$

$$\begin{bmatrix} e_0 & e_{01} & e_{012} & e_{013} & e_{014} & e_5 & e_6 \\ v_0 & 1 & & & & & \\ v_{01} & & 1 & 0 & 0 & 0 & \\ v_{012} & 0 & 0 & 1 & 1 & 1 & \\ v_3 & & & 1 & 1 & 1 & \\ v_4 & & & & 1 & 1 & \end{bmatrix}$$

$$\xrightarrow{C_3 \Leftrightarrow C_4}$$

$$\begin{bmatrix} e_0 & e_{01} & e_{013} & e_{012} & e_{014} & e_5 & e_6 \\ v_0 & 1 & & & & & \\ v_{01} & & 1 & 0 & 0 & 0 & \\ v_{012} & 0 & 1 & 0 & 1 & 1 & \\ v_3 & & & 1 & 1 & 1 & \\ v_4 & & & & 1 & 1 & \end{bmatrix}$$

$$\xrightarrow{R_4 + R_3}$$

then
 $C_5 + C_3$
 $C_6 + C_3$

$$\begin{bmatrix} e_0 & e_{01} & e_{013} & e_{012} & e_{34} & e_{0135} & e_6 \\ v_0 & 1 & & & & & \\ v_{01} & & 1 & 0 & 0 & 0 & \\ v_{012} & 0 & 1 & 0 & 0 & 0 & \\ v_{0123} & & & 1 & 0 & 1 & \\ v_4 & & & & 1 & 1 & \end{bmatrix}$$

$$\xrightarrow{C_5 \Leftrightarrow C_4}$$

then
 $R_5 + R_4$
then
 $C_7 + C_4$

$$\begin{bmatrix} e_0 & e_{01} & e_{013} & e_{34} & e_{012} & e_{0135} & e_{346} \\ v_0 & 1 & & & 0 & & \\ v_{01} & & 1 & 0 & 0 & 0 & \\ v_{012} & 0 & 1 & 0 & 0 & 0 & \\ v_{0123} & & 0 & 1 & 0 & 0 & \\ & & & & 0 & 0 & \\ v_{01234} & & & & & 0 & \end{bmatrix}$$

$$= \text{SNF } ([\partial_1]).$$

$$z_1 = 3, b_0 = 4$$

Thus we get $\beta_1 = z_1 - b_1 = 3 - 2 = 1$, and $\beta_0 = z_0 - b_0 = 5 - 4 = 1$, and both of these numbers agree with intuition (1 hole and 1 component).

$$\beta_p = 0 \quad \forall p \geq 2.$$