MATH 567: Lecture 10 (02/11/2025)

Today: * Definitions on polyhedra
* Integral polyhedra

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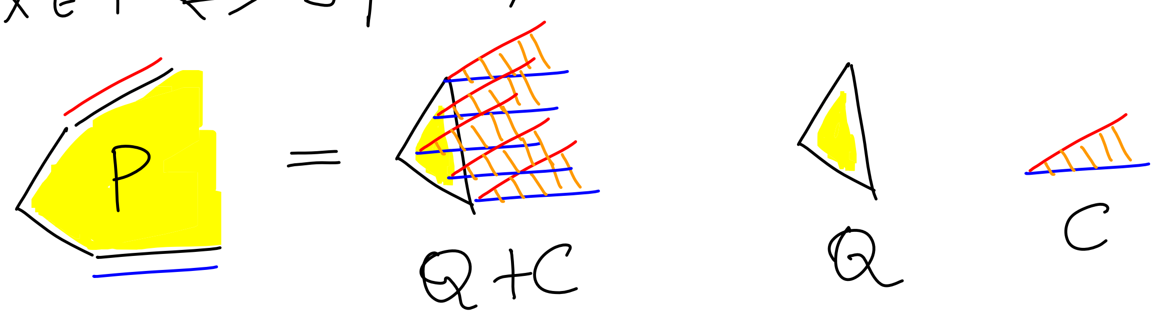
* $P \subseteq \mathbb{R}^n$ is a (convex) polyhedron if $P = \{\bar{x} \mid A\bar{x} \leq \bar{b}\}$.
P is the intersection of finitely many affine half-spaces
$\{\bar{x} \mid \bar{a}^T\bar{x} \leq \beta\}$, $\bar{a} \neq \bar{0}$, $\beta \neq 0$.
some entries in $\bar{b}$ are $\neq 0$ (not necessary to have all entries $\neq 0$)

* $P \subseteq \mathbb{R}^n$ is a (convex) polytope if it is the convex hull of finitely many vectors.
$P = \text{conv}(\bar{v}^1, \ldots, \bar{v}^k) = \{\bar{x} \mid \bar{x} = \sum_{i=1}^{k} \lambda_i \bar{v}^i,\ 0 \leq \lambda_i \leq 1,\ \sum_{i=1}^{k} \lambda_i = 1\}$.

P is a bounded polyhedron.

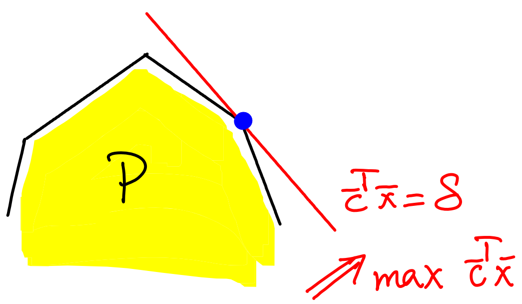
* Motzkin's decomposition theorem: P is a polyhedron iff $P = Q + C$ for some polytope Q and convex cone C.
$\bar{x} \in P \iff \exists\, \bar{y} \in Q, \bar{z} \in C$, s.t. $\bar{x} = \bar{y} + \bar{z}$.



Polytopes and convex cones both have several nice structural properties that might not always hold for general polyhedra. But because of this decomposition theorem, we could present results in terms of polytopes and convex cones.

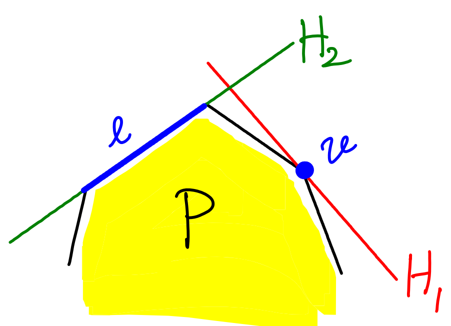
* Farkas' lemma: $\exists\, \bar{x} \mid A\bar{x} \le \bar{b} \iff$
$\nexists\, \bar{u} \ge \bar{0},\quad \bar{u}^T A = \bar{0}^T,\quad \bar{u}^T\bar{b} < 0.$

* ① Let $P = \{\bar{x} \mid A\bar{x} \le \bar{b}\}$, $\delta = \max\{\bar{c}^T\bar{x} \mid \bar{x} \in P\}$, $\bar{c} \ne \bar{0}$.
Then the affine hyperplane $\{\bar{x} \mid \bar{c}^T\bar{x} = \delta\}$ is a **supporting hyperplane** of P.



The line $\bar{c}^T\bar{x} = \delta$ "supports" the polyhedron here.

② $F \subseteq P$ is called a **face** of P if $F = P$ or if $F = P \cap H$ for some supporting hyperplane H of P.



Vertex v and line segment ℓ are faces of P here

We give an alternative definition for a face of P.

③ If $\bar{a}^T\bar{x} \le \beta$ is a valid inequality for P, and $F = \{\bar{x} \in P \mid \bar{a}^T\bar{x} = \beta\}$, then F is a face of P.

(4) F is a **proper face** of P if $F \neq \phi$, $F \neq P$, and F is a face of P.

$\bar{a}^T\bar{x} \leq \beta$: $[\bar{a}, \beta]$ represents the face defined by $\bar{a}^T\bar{x} = \beta$.

valid inequality

compact notation

Also $[\bar{a}, \beta]$ supports P.

* Alternatively, F is a face of $P \iff F = \{\bar{x} \mid \bar{x} \in P, A'\bar{x} = \bar{b}'\}$ where $A'x \leq \bar{b}'$ is a subsystem of $A\bar{x} \leq \bar{b}$.

(i) P has only finitely many faces;

(ii) each face is a nonempty polyhedron; and

(iii) if F is a face of P, then $F' \subseteq F$ is a face of $P \iff F'$ is a face of F.

* Active (tight) constraint : A constraint $\bar{a}^T\bar{x} \leq \beta$ from $A\bar{x} \leq \bar{b}$ is **tight** or **active** in a face F if $\bar{a}^T\bar{x} = \beta \quad \forall \bar{x} \in F$.

also, binding

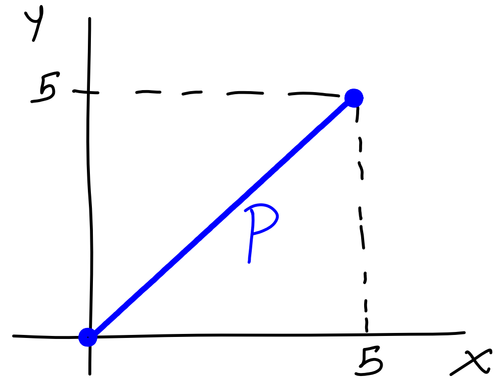
* An inequality $\bar{a}^T\bar{x} \leq \beta$ from $A\bar{x} \leq \bar{b}$ is an **implicit equality** if $A\bar{x} \leq \bar{b} \implies \bar{a}^T\bar{x} = \beta$.

* Let $A'\bar{x} \leq \bar{b}'$ be the subsystem of implicit equalities in $A\bar{x} \leq \bar{b}$. Then the **dimension** of P is

$$\mathbf{dim(P) = n - rank(A')}.$$

Example

$y \geq x$
$y \leq x$
$x \leq 5$
$x \geq 0$



Both $y \geq x$ and $y \leq x$ are implicit equalities here. We get $x - y \leq 0$ and $-x + y \leq 0$, to give $A' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, which has $\operatorname{rank}(A') = 1$. Thus, $\dim(P) = 2 - 1 = 1$, which agrees with our intuition.

* P is **full-dimensional** if $\dim(P) = n$, i.e., it has no implicit equalities.

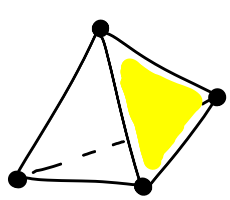
* $\dim\{\bar{v}\} = 0$. (one point/vertex)

* by convention, $\dim(\phi) = -1$.

* The **affine hull** of P is $\operatorname{affhull}(P) = \{\bar{x} \mid A'\bar{x} = \bar{b}'\}$.

* **facet**: inclusionwise minimal face F of P with $F \neq P$.

* If F is a facet of P, then $\dim(F) = \dim(P) - 1$.



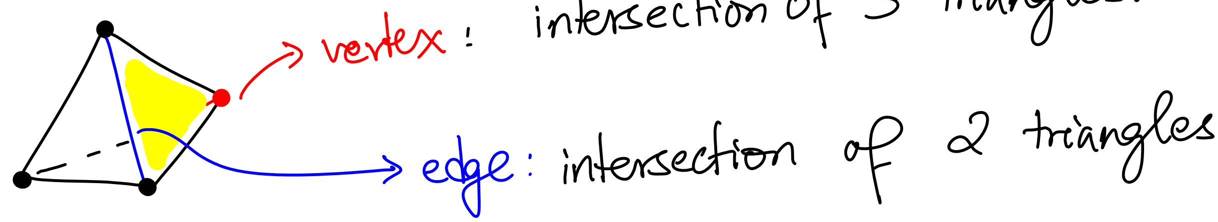
P is a solid tetrahedron.
Each triangle is a facet.
Each vertex and edge is a face, but not a facet.

Let $A'\bar{x} \le \bar{b}'$ be the subsystem of implicit equalities in $A\bar{x} \le \bar{b}$, and $A^+\bar{x} \le \bar{b}^+$ be the remaining inequalities.

If no inequality in $A^+\bar{x} \le \bar{b}^+$ is redundant in $A\bar{x} \le \bar{b}$, then for any facet F of P,
$F = \{\bar{x} \in P \mid \bar{a}^{+T}\bar{x} = \beta^+\}$ for an inequality $\bar{a}^{+T}\bar{x} \le \beta^+$ from $A^+\bar{x} \le \bar{b}^+$.

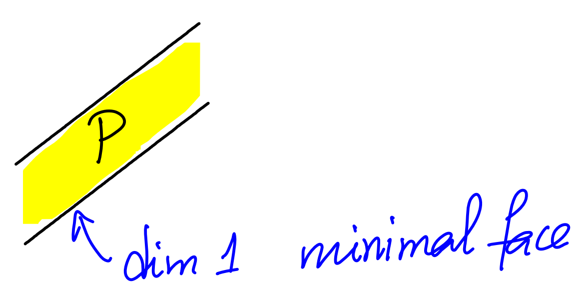
In this case, $\bar{a}^{+T}\bar{x} = \beta^+$ **determines** the facet F.

* Each face of P, except P itself, is the intersection of facets of P.



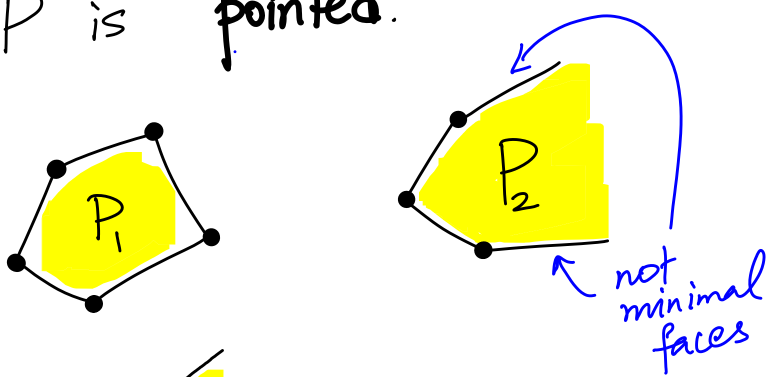
* A **minimal face** of P is a face of P not containing any other face of P.

For the tetrahedron, vertices are minimal faces.

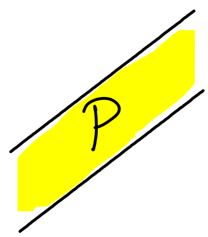


* A vertex of P is $\bar{z} \in P$ such that $\{\bar{z}\}$ is a minimal face of dimension zero.

* If each minimal face of P has dimension zero, then P is **pointed**.



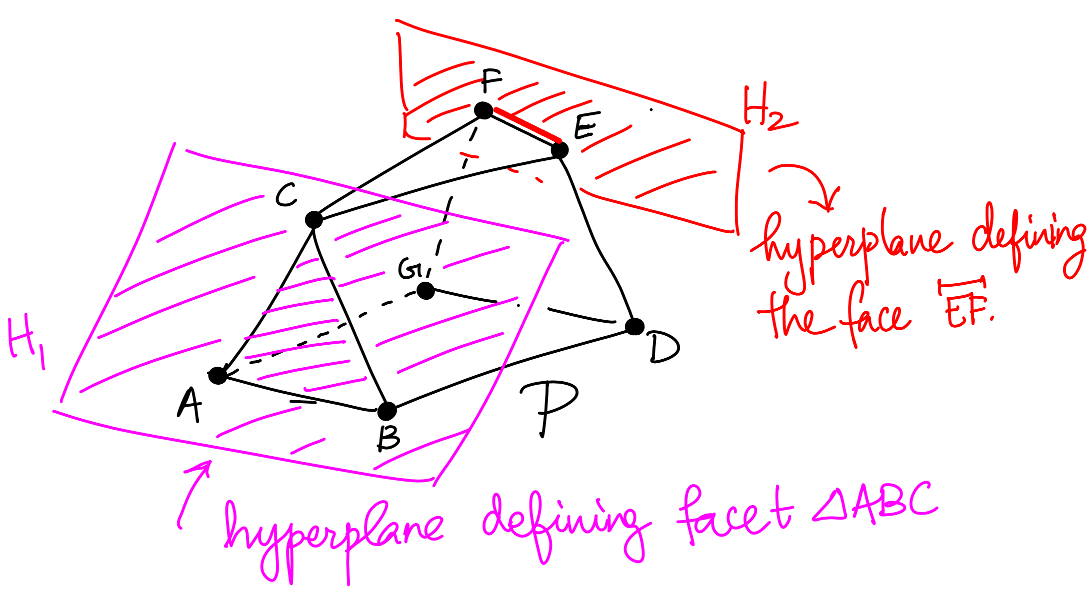
P_1 and P_2 are pointed.



P_3 is not pointed

Here is another example illustrating several of these definitions

P is the solid object in $\mathbb{R}^3$.



$\dim(P) = 3$, and it has no implicit equalities.

All the vertices, edges (line segments), triangles, and quardilaterals are all faces of P.

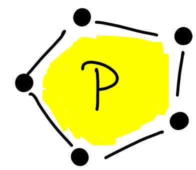
The triangles and quardilaterals are facets of P, and their dimension is 2 each.

The vertices are minimal faces of P. Also, $\dim(\text{edge}) = 1$.

H_1 is the supporting hyperplane defining face $\triangle ABC$, and H_2 defines the face which is edge $\overline{EF}$.

Special Cases: Well-Solved IPs

Recall: A polytope is the convex hull of its vertices.



We study problems of the form

$$\max\{\bar{c}^T\bar{x} \mid \bar{x} \in X\} = \max\{\bar{c}^T\bar{x} \mid \mathrm{conv}(X)\},$$

where conv(X) is "efficiently" described, i.e., using a polynomial # inequalities in a polynomial # variables. Then we can solve as an LP efficiently (in polynomial time), and get integrality for free.

We restrict our attention to rational polyhedra, i.e., $P = \{\bar{x} \mid A\bar{x} \leq \bar{b}\}$ where entries in A, $\bar{b}$ are rational.

The subtour formulation with **all** subtour constraints added will describe the convex hull, but there are exponentially many constraints.

Integral Polyhedra

Def A rational polyhedron is called **integral** if every non-empty face contains an integer vector.

We need to consider only minimal faces.

A pointed rational polyhedron is integral iff **every vertex is integral**.

A polyhedron could be integral even if it is not pointed, e.g., the infinite band as shown here.