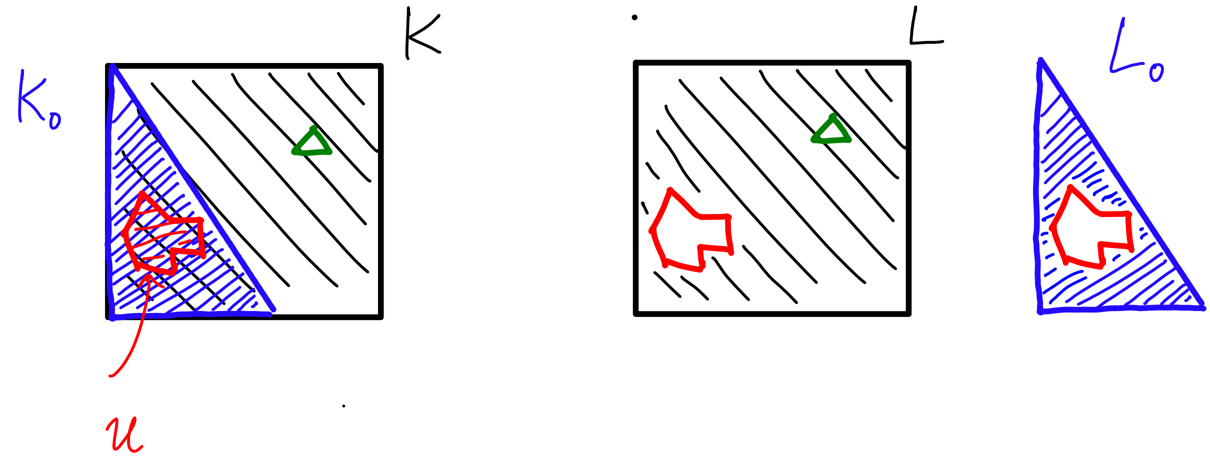
MATH 524 – Lecture 13 (10/03/2023)

Today: * Proof of excision theorem
* simplicial maps and homomorphisms

Excision Theorem $H_p(L, L_0) \simeq H_p(K, K_0)$



Proof idea: Consider the composite map ϕ

$$C_p(L) \longrightarrow C_p(K) \longrightarrow C_p(K)/C_p(K_0)$$

defined as inclusion followed by projection. "project out" parts in K_0

a p-chain in L is extended to a p-chain in K by setting the weights on p-simplices in K/L to zero.

ϕ is surjective, as $C_p(K)/C_p(K_0)$ has as basis all cosets $\{\sigma_i\}$ for p-simplices σ_i in K not in K_0, and all such $\sigma_i \in L$. Also, $\ker \phi$ is $C_p(L_0)$.

So, ϕ induces an isomorphism $C_p(L)/C_p(L_0) \simeq C_p(K)/C_p(K_0) \;\forall p$.

The p-simplex σ is mapped to empty (i.e., to zero) if it is in L_0 by the projection part of ϕ.

And ∂ is preserved under this isomorphism.

Hence, $H_p(L, L_0) \simeq H_p(K, K_0)$. $\square$

We now turn to simplicial maps, and how the groups we have studied – chains, cycles, boundaries, and homology groups – behave under them. We introduce several useful algebraic tools in this process.

Homomorphisms induced by Simplicial Maps

§12 in [M]

Recall Simplicial map: Given simplicial complexes K and L, $f: K \to L$ is a simplicial map if f is a continuous map of $|K|$ to $|L|$ that maps each simplex of K linearly onto a simplex of L.

We could start with the corresponding vertex map, and extend the same linearly to the simplicial map.

Note that a simplex in K could be mapped to a lower dimensional simplex in L by f. We define a homomorphism from f by "staying in the same dimension".
$(\dim(f(\sigma)) \leq \dim \sigma)$.

Def Let $f: K \to L$ be a simplicial map. If $(v_0, \dots, v_p)$ is a simplex of K, then $f(v_0), \dots, f(v_p)$ span a simplex of L. We define a homomorphism $f_\#: C_p(K) \to C_p(L)$ by defining it on oriented p-simplices as follows.

$$f_\#([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_i) \text{ are distinct,} \\ 0, & \text{otherwise.} \end{cases}$$

This map is indeed well-defined, i.e., $f_\#(-\sigma) = -f_\#(\sigma)$.
If we swap v_i and v_j in $[v_0, \dots, v_p]$, the sign of the right-hand side expression is changed.
The family of homomorphisms $\{f_\#\}$, one in each dimension, is called the **chain map induced by the simplicial map f.**

<u>Notation</u>: Ideally, we should write $(f_\#)_p : C_p(K) \to C_p(L)$, for each dimension p. But we avoid writing p, when the dimension is evident. We will work with $f_\#$ and ∂ in both K and L.
We use ∂ to denote the boundary in K and in L.
We could be particular, and write ∂_K and ∂_L, when necessary.

Notice that $f_\#$ is a homomorphism for C_p groups. What about the homology groups H_p? It turns out that $f_\#$ induces a homomorphism f_* from $H_p(K)$ to $H_p(L)$.

Lemma 12.1 [M] The homomorphism $f_\#$ commutes with ∂; therefore $f_\#$ induces a homomorphism $f_* : H_p(K) \to H_p(L)$.

$$\begin{tikzcd} C_p(K) \arrow[r, "(f_\#)_p"] \arrow[d, "\partial_K"'] & C_p(L) \arrow[d, "\partial_L"] \\ C_{p-1}(K) \arrow[r, "(f_\#)_{p-1}"] & C_{p-1}(L) \end{tikzcd}$$

composing the maps along either path gives the same result

To be exact, we say $\partial_L \circ (f_\#)_p = (f_\#)_{p-1} \circ \partial_K$, or just briefly, $\partial f_\# = f_\# \partial$.

Proof We first show that

$$\partial f_{\#}([v_0,\dots,v_p]) = f_{\#}(\overbrace{\partial[v_0,\dots,v_p]}^{\sum_{i=0}^{p}(-1)^i [v_0 \dots \hat{v}_i \dots v_p]}) \quad \text{———} \quad (*).$$

Let τ be the simplex spanned by $f(v_0),\dots,f(v_p)$. We consider three cases, based on the dimension of τ.

Case 1. $\dim(\tau) = p$. Here, $f(v_0),\dots,f(v_p)$ are distinct, and hence the result follows directly since $f_{\#}$ and ∂ are homomorphisms.

Case 2. $\dim(\tau) \le p-2$. → three or more vertices are mapped to one vertex, or two or more pairs are identified.

LHS of $(*)$ vanishes, as $f(v_i)$ are not all distinct.
RHS of $(*)$ is also 0, as $\forall i$, two or more terms out of $f(v_0),\dots,f(v_{i-1}), f(v_{i+1}),\dots, f(v_p)$ are the same.

Case 3 $\dim(\tau) = p-1$. → exactly one pair of v_i's are mapped to the same vertex in L

WLOG, assume vertices are ordered such that

$f(v_0) = f(v_1) \neq f(v_2) \neq \dots \neq f(v_p)$. → $f(v_0) = f(v_1)$, while the remaining $f(v_i)$ are all distinct, and also distinct from $f(v_0)$

Again, LHS of $(*)$ vanishes by definion.
RHS of $(*)$ has only two nonzero terms:
$[f(v_1), f(v_2),\dots,f(v_p)]$ and $-[f(v_0), f(v_2),\dots,f(v_p)]$.

As $f(v_0) = f(v_1)$, these two terms cancel.

Further, $f_\#$ carries cycles to cycles and boundaries to boundaries.

Let $\bar{z} \in Z_p(K)$. Then $\partial(\bar{z}) = 0$. By Lemma,
$\partial f_\#(\bar{z}) = f_\# \partial(\bar{z}) = 0$. So $f_\#(\bar{z})$ is a cycle, i.e., $f_\#(\bar{z}) \in Z_p(L)$.

Similarly, if $\bar{b} \in B_p(K)$, then $\bar{b} = \partial_{p+1}\bar{d}$ for $\bar{d} \in C_{p+1}(K)$.
But $\partial f_\#(\bar{d}) = f_\# \partial \bar{d} = f_\#(\bar{b})$, and hence $f_\#(\bar{b}) \in B_p(L)$.

Thus, $f_\#$ induces a homomorphism of the homology groups, $f_* : H_p(K) \to H_p(L)$. $\square$

We can naturally combine the homomorphisms induced by multiple simplicial maps, as the following theorem describes.

Theorem 12.2 [M] (a) Let $i : K \to K$ be the identity simplicial map. Then $i_* : H_p(K) \to H_p(K)$ is the identity homomorphism.
(b) Let $f : K \to L$ and $g : L \to M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$, i.e., the following diagram commutes.

$$\begin{tikzcd} H_p(K) \arrow[rr, "(g\circ f)_*"] \arrow[dr, "f_*"'] & & H_p(M) \\ & H_p(L) \arrow[ur, "g_*"'] & \end{tikzcd}$$

This theorem presents the **functorial** property of the induced homomorphism. Think of H_p as an operator that assigns to each simplicial complex an abelian group, and $*$ as another operator that assigns to each simplicial map of one complex to another, a homomorphism between the corresponding abelian groups.

We say that $(H_p, *)$ is a "functor" from the "category" of simplicial complexes and simplicial maps to the "category" of abelian groups and homomorphisms.

Intuitively, a "category" consists of a collection of sets (or "objects") along with maps between them. A functor assigns pairs of such structure, i.e., (object, map) pairs, in one category to those in the other category such that it "preserves the structure" of the category.

The ideas of commuting diagrams in particular, and functoriality in general, are used widely in algebraic topology. We will see more of these concepts in the upcoming lectures.

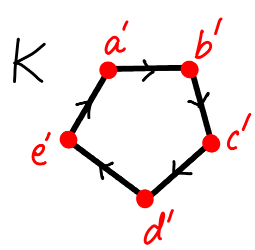
We study further the homomorphisms induced by simplicial maps. In particular, we talk about when distinct simplicial maps induce equal homomorphisms on homology groups.

Lemma 12.3 [M] $f_\#$ preserves the augmentation map ϵ; therefore it induces a homomorphism f_* of reduced homology groups.

Proof Let $f: K \to L$ be a simplicial map. Then $\epsilon f_\#(v) = 1$, and $\epsilon(v) = 1$ $\forall v \in K^{(0)}$. Hence $\epsilon \circ f_\# = \epsilon$. Thus $f_\#$ carries the kernel of $\epsilon_K : C_0(K) \to \mathbb{Z}$ into the kernel of $\epsilon_L : C_0(L) \to \mathbb{Z}$, and so it induces a homomorphism $f_* : \tilde{H}_0(K) \to \tilde{H}_0(L)$.

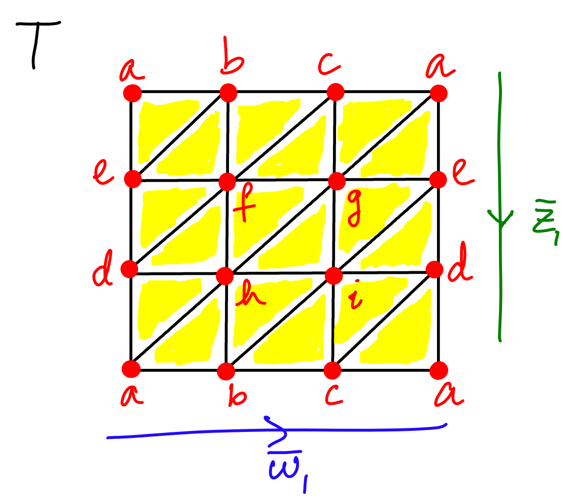
Example

Consider K that is a loop, and T, which represents $\mathbb{T}^2$.



$\bar{z} : a'b'c'd'e'$

$H_1(K) \simeq \mathbb{Z}$, $\{\bar{z}\}$ generates this group



As described previously, with $\bar{w}_1 = [a,b]+[b,c]+[c,a]$ and $\bar{z}_1 = [a,e]+[e,d]+[d,a]$, $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and $\{\bar{w}_1, \bar{z}_1\}$ is a basis.

We consider three different simplicial maps $f, g, h : K \to T$, described by the maps for each vertex in K.

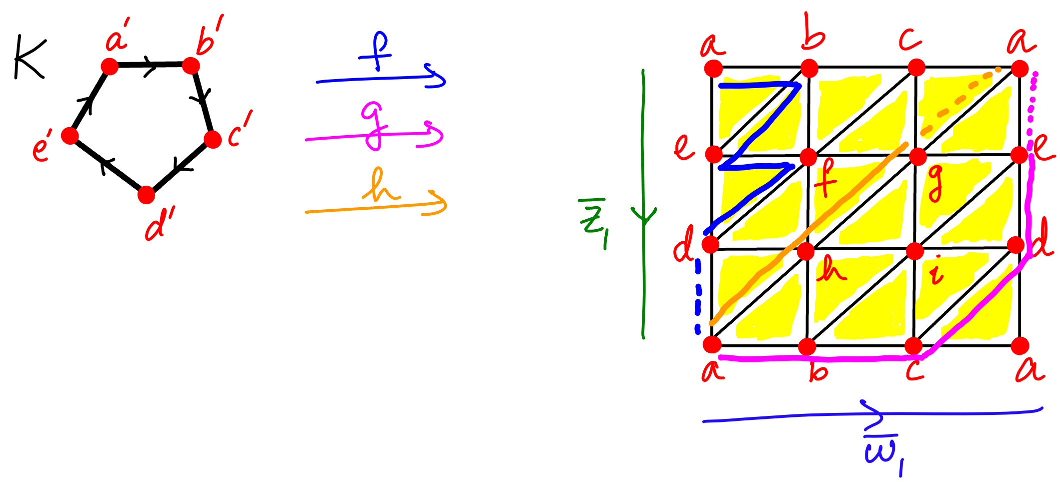
3rd one added after lecture; but we'll discuss the example again in the next lecture...

$f: a' \to a$, $b' \to b$, $c' \to e$, $d' \to f$, $e' \to d$

$g: a' \to a$, $b' \to b$, $c' \to c$, $d' \to d$, $e' \to e$

$h: a' \to a$, $b' \to h$, $c' \to h$, $d' \to g$, $e' \to g$

We can visualize the three maps as follows.



More on these maps in the next lecture...