MATH 524: Lecture 1 (08/19/2025)

This is Algebraic Topology.
I'm Bala Krishnamoorthy (Call me Bala).

Today:
* syllabus, logistics
* neighborhoods, continuous functions
* topology using neighborhoods
* homeomorphism

I will be teaching computational topology (Math 529) next semester. The two classes – Math 524 and Math 529 will be kept independent. In particular, we will spend nearly no focus on computational aspects in Math 524.

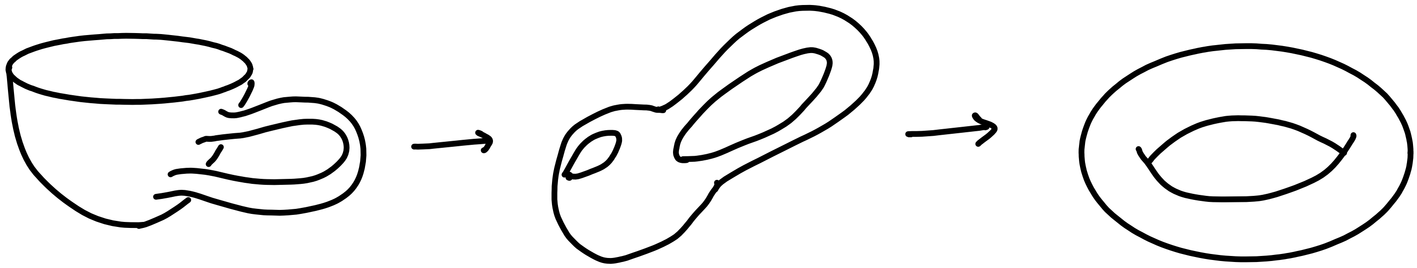
Check the course web page at

https://bala-krishnamoorthy.github.io/Math524.html

All documents, important updates, homework assignments etc. will be posted there. Check the class page frequently.

More about the video assignment to come soon. But you're encouraged to start looking for topics that you might want to make the video on as we proceed in the course.

Topology studies how "space is connected". You might have heard the (true!) joke that the topologist cannot distinguish between a coffee cup and a donut! Indeed, they both are connected the same way.



In algebraic topology, we cast problems on how space is connected as equivalent problems on algebraic objects — groups, rings, etc., and maps between them (homomorphisms).

As a subfield of mathematics, algebraic topology started in late 19th and early 20th century. Poincaré introduced the fundamental group first. Later Betti introduced homology groups, which are much easier to compute (both by hand as well as algorithmically) than the former.

We will spend a lot of time talking about homology groups, and the dual concept of cohomology. We will not be spending much attention on the fundamental group. There are several (equivariant) ways to define homology groups. Perhaps the "nicest" way to do so is using simplicial complexes. We will spend a fair bit of time studying simplicial homology.

We will introduce/refresh background concepts as needed. First, we will talk about continuous functions and topological spaces, defined in terms of neighborhoods.

Continuous functions

We first give the classical ε-δ definition in Euclean spaces.

Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$. f is continuous at $\bar{x} \in \mathbb{R}^n$ if there exists $\delta > 0$ for every $\varepsilon > 0$ such that $\|f(\bar{y}) - f(\bar{x})\| < \varepsilon$ whenever $\|\bar{y} - \bar{x}\| < \delta$ for $\bar{y} \in \mathbb{R}^n$. f is continuous (in all of $\mathbb{R}^n$) if it is so at every $\bar{x} \in \mathbb{R}^n$.

my notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\mu}$, etc., are all vectors — lower case letters with a bar.

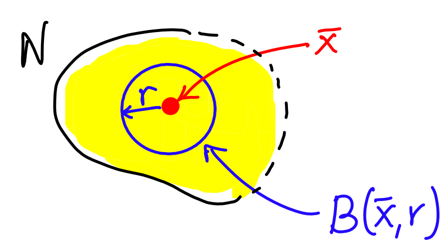
We give an equivalent definition based on neighborhoods.

Def A subset N of $\mathbb{R}^n$ is a **neighborhood** of $\bar{x} \in \mathbb{R}^n$ if for some $r > 0$, the closed ball $B(\bar{x}, r)$ centered at $\bar{x}$ is contained entirely within N.

Notice that neighborhood N can be open or closed.

$B(\bar{x}, r) = \{\bar{y} \in \mathbb{R}^n \mid \|\bar{x} - \bar{y}\| \le r\}$

closed Ball of radius r centered at $\bar{x}$



Def $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous if given any $\bar{x} \in \mathbb{R}^n$ and a neighborhood N of $f(\bar{x})$ in $\mathbb{R}^m$, $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in $\mathbb{R}^n$.

Now we define what a topological space (or topology) is. We give the definition in terms of neighborhoods first. In most textbooks, you will see the definition given in terms of open sets. Later, we will see that both definitions are equivalent.

Topological space (or topology)

more notation: Upper case letters, e.g., A, B, X, Y, etc, denote sets or matrices.

Def I We are given a set X and a nonempty collection of subsets of X for each $\bar{x} \in X$ called the neighborhoods of $\bar{x}$. This is a topological space if it satisfies the following axioms.

(a) $\bar{x}$ lies in each of its neighborhood.

(b) Intersection of two neighborhoods of $\bar{x}$ is itself a neighborhood of $\bar{x}$.

(c) If N is a neighborhood of $\bar{x}$, and $\mathcal{U} \subseteq X$ contains N, then $\mathcal{U}$ is a neighborhood of $\bar{x}$.

(d) If N is a neighborhood of $\bar{x}$, $\mathring{N}$, the **interior** of N is also a neighborhood of $\bar{x}$.

The interior of N is $\mathring{N} = \{\bar{y} \in N \mid N \text{ is a neighborhood of } \bar{y}\}$.

Intuitively, every point of N not on its boundary is in its interior.

We can extend the definition of continuous functions to functions defined between topological spaces.

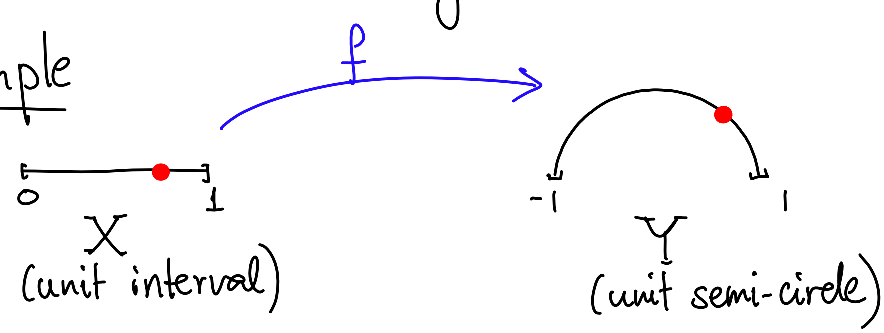
Def Let X, Y be topological spaces. $f: X \to Y$ is continuous if $\forall\, \bar{x} \in X$ and for every neighborhood N of $f(\bar{x})$ in Y, the set $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in X.

We are interested in studying when two topological spaces are similar. There are a few different notions of topological similarity, and the strongest notion is that of homeomorphism. For two spaces to be homeomorphic, we need a function between them that is "nicer" than just a continuous function.

Def A function $f: X \to Y$ is a **homeomorphism** if it is one-to-one, onto, continuous, and has a continuous inverse.

When such a function exists between two spaces X and Y, we say they are **homeomorphic**, or are topologically equivalent. We denote this fact by $X \approx Y$.

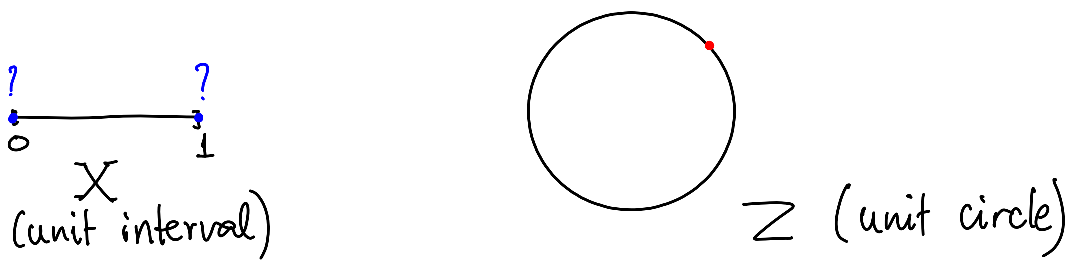
Example



$X \approx Y$. Can you define the function f?

Think of X & Y as subsets of $\mathbb{R}^2$, and write down the form of f, as well as f^{-1}. You can show f satisfies all requirements for being a homeomorphism.

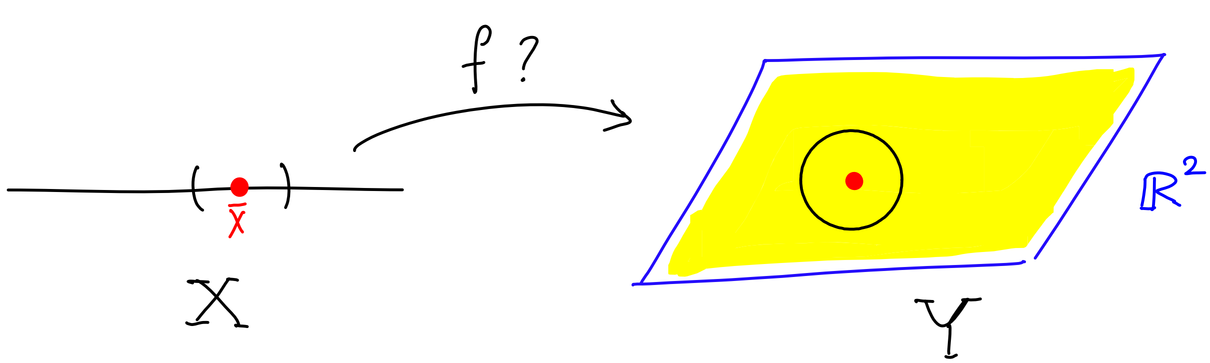
Showing two spaces are **not** homeomorphic could be harder — we need to show that no such function exists between X and Y.



Here, $X \not\approx Z$. Where do things breakdown?

Intuitively, one can notice the two end points of X behave distinctly from any point in Y.

Here is another example. Perhaps the simplest example of a topological space is $\mathbb{R}^d$ under the usual definition of neighborhoods, which specifies that any set $N \subseteq \mathbb{R}^d$ big enough to contain $B(\bar{x}, r)$ for some $r > 0$ is a neighborhood of $\bar{x} \in \mathbb{R}^d$. But notice that $\mathbb{R}^1 \not\approx \mathbb{R}^2$, for instance. It is not straightforward to prove this fact rigorously. But, how would one "argue" for it?



One method is to appeal to how the two spaces are connected. Recall that topologically similar spaces are "connected" the same way. Here, if we remove one point from both $X = \mathbb{R}^1$ and $Y = \mathbb{R}^2$, we can see that it affects the connectivity differently. Removing one point leaves X disconnected (into two pieces). But removing a point from Y still leaves it connected – its just like poking a hole in the "sheet" that is $\mathbb{R}^2$, which remains connected.

More formally, we could try to define a homeomorphism from X to Y. But we can observe that neighborhoods in X are 1-dimensional, while those in Y are 2D. Hence we cannot define a bijection between them.

We will talk about open sets in the next lecture, and define a topology using open sets. That definition is equivalent to the one introduced earlier today, i.e., Def I.

MATH 524 : Lecture 2 (08/21/2025)

Today: * open sets, topology using open sets
* simplices, properties of simplices

We now consider topology defined in terms of open sets. This is the default approach taken in most textbooks. We first define open sets using the concept of neighborhoods.

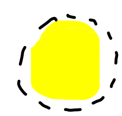
Def $\mathcal{O} \subseteq X$ is **open** if it is a neighborhood of each of its points. By (c) of **Def I**, union of any collection of open sets is also open. Also, by (b) of **Def I**, the intersection of any <u>finite</u> number of open sets is open.

We mention unions and finite intersections of open sets as they are both required to be open in a topology. See below.

Notice, $\mathring{N}$ (interior of neighborhood N) is always open.

Alternatively, we can start by defining open sets directly.

Def A set $A \subseteq \mathbb{R}^n$ is **open** if each $\bar{x} \in A$ can be surrounded by a ball of positive radius that lies entirely inside the set.



We can also define open sets more generally, starting with collections of subsets of some set X.

We could define neighborhoods in terms of open sets.

Def A subset $N \subseteq X$ is a neighborhood of $\bar{x}$ if there exists an open set $\mathcal{O}$ s.t. $\bar{x} \in \mathcal{O} \subseteq N$.

We now formally state the definition of topology in terms of open sets. This definition sees more use than the one using neighborhoods.

Def II A **topology** on a set X is a collection of open sets of X such that any union and finite intersection of open sets is open, and ϕ (empty set) and X are open.
The set X along with the topology is called a **topological space**.

We can define continuous functions also in terms of open sets.

Def $f: X \to Y$ is **continuous** iff (if and only if) the inverse image of each open set of Y is open in X.

We now start the discussion of homology, which is a less strict version of topological similarity than homeomorphism. We study in detail simplicial homology, where the spaces are made of "gluing" "nice" objects called simplices together, and are hence are very "regular".

As we will see, it is also much easier to algebraize questions about homology (than those about homeomorphism).

There is a "continuous" version of homology defined on spaces not composed to regular pieces (simplices), termed singular homology. It turns out singular homology is equivalent to simplicial homology.

We start by defining simplices, which are the building blocks.

Simplices

We define simplices in the usual geometric setting first, and then define them abstractly. We need some concepts from geometry first.

Def The set $\{\bar{a}_0, \ldots, \bar{a}_n\}$ of points in $\mathbb{R}^d$ is **geometrically independent** (**GI**) if for any scalars $t_i \in \mathbb{R}$, the equations
$$\sum_{i=0}^{n} t_i = 0, \quad \sum_{i=0}^{n} t_i \bar{a}_i = \bar{0} \quad \text{imply that } t_0 = t_1 = \cdots = t_n = 0.$$

Here are some observations about GI sets.

* $\{\bar{a}_i\}$ is GI $\forall i$. (singleton sets)

* $\{\bar{a}_0, \ldots, \bar{a}_n\}$ is GI $\Longleftrightarrow$ ← if and only if
  $\{\bar{a}_1 - \bar{a}_0, \bar{a}_2 - \bar{a}_0, \ldots, \bar{a}_n - \bar{a}_0\}$ is linearly independent (LI).

$\bar{a}_0$ is chosen as the "origin", so to speak. But any $\bar{a}_i$ could play the role of $\bar{a}_0$ here.

IDEA: $\displaystyle\sum_{i=1}^{n} t_i(\bar{a}_i - \bar{a}_0) = \bar{0} \implies t_i = 0 \;\forall i$ (LI)

$$\sum_{i=1}^{n} t_i \bar{a}_i + \underbrace{\left(-\sum_{i=1}^{n} t_i\right)}_{t_0} \bar{a}_0 = \bar{0}$$

$\Bigg\}$ $\displaystyle\sum_{i=0}^{n} t_i \bar{a}_i = 0$ & $\displaystyle\sum_{i=0}^{n} t_i = 0 \implies t_i = 0 \;\forall i$

* 2 distinct points in $\mathbb{R}^d$ are GI,
  3 non-collinear points are GI,
  4 non-coplanar points are GI, and so on.

Notice the relationship/correspondence to LI vectors. For instance, $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ is GI, but of course the set is not LI.

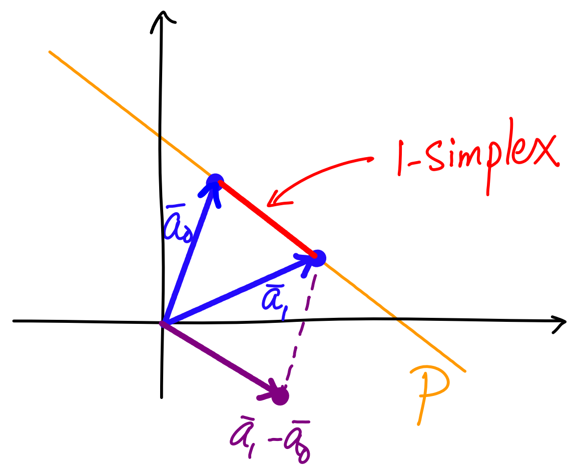
Def Given GI set $\{\bar{a}_0, \ldots, \bar{a}_n\}$, the **n-plane** P spanned by these points consists of all $\bar{x}$ such that $\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i$ for scalars t_i with $\sum_{i=0}^{n} t_i = 1$.

The scalars t_i are uniquely determined by $\bar{x}$.

Notice that t_i could be ≥ 0 or ≤ 0 here.

P can also be described as the set of $\bar{x}$ such that

$$\bar{x} = \bar{a}_0 + \sum_{i=1}^{n} t_i (\bar{a}_i - \bar{a}_0).$$



Hence P is the plane through $\bar{a}_0$ parallel to the vectors $\bar{a}_i - \bar{a}_0$.

Going back to the previous example with $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$, the plane P is the line generated by one of the two vectors.

Q. What is the set described by $\bar{x} = \sum_{i=0}^{n} t_i a_i$, $\sum t_i = 0$?

e.g., consider $n=1$: $\bar{x} = t_0 \bar{a}_0 + t_1 \bar{a}_1$ with $t_0 + t_1 = 0 \Rightarrow t_0 = -t_1$.

$\Rightarrow \bar{x} = t_0(\bar{a}_0 - \bar{a}_1)$, i.e., it's the line generated by $\bar{a}_0 - \bar{a}_1$.

We now define a simplex as the set "spanned" by a set of GI points.

Def Let $\{\bar{a}_0, \dots, \bar{a}_n\}$ be a GI set in $\mathbb{R}^d$. The **n-simplex** σ spanned by $\bar{a}_0, \dots, \bar{a}_n$ is the set of points $\bar{x} \in \mathbb{R}^d$ s.t. $\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i$ with $\sum_{i=0}^{n} t_i = 1$, $t_i \geq 0 \; \forall i$.

The t_i are uniquely determined by $\bar{x}$, and are called the **barycentric coordinates** of $\bar{x}$ (in σ) w.r.t. $\bar{a}_0, \dots, \bar{a}_n$.

we will later extend definition of t_i to $\bar{x} \notin \sigma$. the

0-simplex : a point

1-simplex : line segment

2-simplex

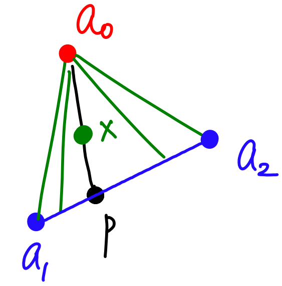
$\bar{x} = \bar{a}_0$ is trivial to consider.

Assume $\bar{x} \neq \bar{a}_0$, i.e., $t_0 \neq 1$. Now consider

$$\bar{x} = \sum_{i=0}^{2} t_i a_i = t_0 \bar{a}_0 + (1-t_0) \underbrace{\left[\frac{t_1}{1-t_0} \bar{a}_1 + \frac{t_2}{1-t_0} \bar{a}_2 \right]}_{\bar{p}}$$

Since $\sum_{i=0}^{2} t_i = 1$, $1 - t_0 = t_1 + t_2$. Hence $\left(\frac{t_1}{1-t_0}\right)\bar{a}_1 + \left(\frac{t_2}{1-t_0}\right)\bar{a}_2$ is a point $\bar{p}$ on the line segment $\overline{\bar{a}_1\bar{a}_2}$, and $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{p}$ is a point on the line segment $\overline{\bar{a}_0 \bar{p}}$.

Hence the 2-simplex is the union of such line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\overline{a_1 a_2}$, i.e., the triangle $a_0 a_1 a_2$ ($\triangle a_0 a_1 a_2$).



This result extends to higher order simplices. For instance, a tetrahedron is the union of all line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\triangle a_1 a_2 a_3$.

Properties of Simplices

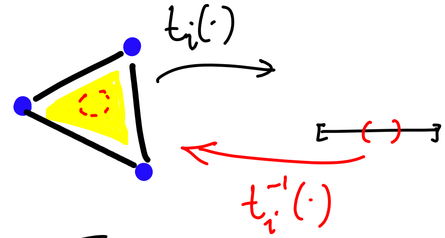
(1) $t_i(\bar{x})$ are continuous functions of $\bar{x}$.

IDEA: $t_i : \mathbb{R}^d \to \mathbb{R}$

convex hull $\{\bar{x} \mid \bar{x} = \sum_{i=0}^{n} t_i a_i, t_i \geq 0, \sum t_i = 1\}$

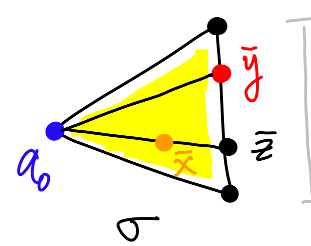
domain $\to$ $\mathrm{Dom}(t_i) = \mathrm{conv}(\{\bar{a}_0, \dots, \bar{a}_n\})$

$\mathrm{Range}(t_i) = [0, 1]$



Prove that $t_i^{-1}(\text{open set in } [0,1])$ is open in σ.

(2) σ is the union of all line segments joining $\bar{a}_0$ to points of the simplex spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$.
Two such line segments intersect only at $\bar{a}_0$.
$\to$ proof?



Assume two such ($\bar{y} \neq \bar{z}$) line segments from $\bar{a}_0$ to $\bar{y}, \bar{z} \in \tau$, the simplex spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$, meet at $\bar{x} \neq \bar{a}_0$.

Then $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{y} = s_0 \bar{a}_0 + (1-s_0)\bar{z}$, for $t_0, s_0 \in [0,1]$,
where $t_0 \neq s_0$ by assumption (else $\bar{y} = \bar{z}$!).

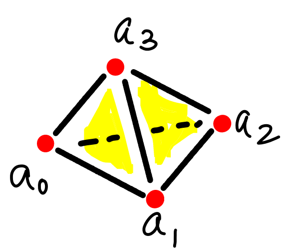
$\Rightarrow \bar{a}_0 = u\bar{y} + v\bar{z}$, where $u, v \in \mathbb{R}$ with $u+v=1$.

$\Rightarrow \bar{a}_0 \in P(\{\bar{y}, \bar{z}\}) \in P(\tau)$ $\to$ $(n-1)$-plane spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$.

which contradicts the GI of $\{\bar{a}_0, \dots, \bar{a}_n\}$.

Def The points $\bar{a}_0, \dots, \bar{a}_n$ which span σ are called its **vertices**. The **dimension** of σ is n $(\dim(\sigma) = n)$.
A simplex spanned by a non empty subset of $\{\bar{a}_0, \dots, \bar{a}_n\}$ is a **face** of σ. The face spanned by $\{\bar{a}_0, \dots, \hat{\bar{a}}_i, \dots \bar{a}_n\}$ where $\hat{\bar{a}}_i$ means $\bar{a}_i$ is not included, is the **face opposite $\bar{a}_i$**.
Faces of σ distinct from σ itself are its **proper faces**, their union is its **boundary**, $\mathrm{Bd}\,\sigma$ or $\partial\sigma$.

$\partial(\bar{a}_0) = \phi \longrightarrow$ there are no proper faces of a vertex.



tetrahedron $a_0 a_1 a_2 a_3 = \sigma$ → a 3-simplex

proper faces : $\Delta a_0 a_1 a_2, \Delta a_0 a_2 a_3, \dots$ (4)

edges → $\overline{a_0 a_1}, \overline{a_0 a_2}, \dots$ (6)

vertices → $\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3$ (4)

$\partial\sigma = \bigcup(\text{proper faces})$ (triangles, edges, vertices)

⟶ the "hollow" tetrahedron

Def The **interior** of σ, $\mathrm{Int}(\sigma)$ or $\mathring{\sigma}$, is $\mathrm{Int}(\sigma) = \sigma - \mathrm{Bd}\,\sigma$.

$\mathrm{Int}(\sigma)$ is called an open simplex.

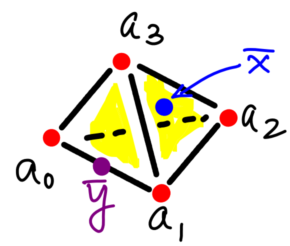
$\mathrm{Int}(\bar{a}_0) = \bar{a}_0$. $\longrightarrow$ as $\partial a_0 = \phi$.

$\mathrm{Bd}\,\sigma$ consists of all $\bar{x} \in \sigma$ with at least one $t_i(\bar{x}) = 0$.
$\mathrm{Int}\,\sigma$ consists of all $\bar{x} \in \sigma$ with $t_i(\bar{x}) > 0$ $\forall i$.

could be σ itself!

Given $\bar{x} \in \sigma$, there is exactly one face τ s.t. $\bar{x} \in \text{Int}\,\tau$.
τ is that face of σ spanned by those $\bar{a}_i$ for which $t_i(\bar{x}) > 0$.

$\bar{x}$ is interior to $\triangle a_1 a_2 a_3$
$\bar{y}$ is interior to $\overleftrightarrow{a_0 a_1}$



(3) σ is a compact, convex set in $\mathbb{R}^d$, and is the intersection of all convex sets in $\mathbb{R}^d$ containing $\bar{a}_0, \dots, \bar{a}_n$.

(4) There exists one and only one GI set of points $\{\bar{a}_0, \dots, \bar{a}_n\}$ spanning σ.

(5) $\text{Int}\,\sigma$ is convex, and is open in P, and $Cl(\text{Int}\,\sigma) = \sigma$. $\text{Int}\,\sigma$ is the union of all "open line segments" joining $\bar{a}_0$ with points in $\text{Int}\,\tau$, where τ is the face opposite $\bar{a}_0$.

closure

MATH 524: Lecture 3 (08/26/2025)

Today: * simplicial complexes
* underlying space

One more property of simplices first...

Def Unit ball: $\mathbb{B}^n = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| \leq 1\}$

Unit sphere: $\mathbb{S}^{n-1} = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| = 1\}$

We will use these definitions later on

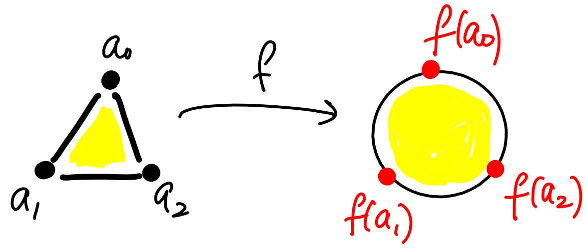
upper/lower hemisphere: $\mathbb{E}^{n-1}_+ / \mathbb{E}^{n-1}_- =$

$\{\bar{x} \in \mathbb{S}^{n-1} \mid x_n \geq 0 / x_n \leq 0\}$.

e.g., $\mathbb{B}^0 = \{\bar{0}\}$, $\mathbb{B}^1 = [-1, 1]$, $\underbrace{\mathbb{S}^0 = \{-1, 1\}}_{\text{two points}}$.

(6) There is a homeomorphism of $\underset{n\text{-simplex}}{\sigma}$ with $\mathbb{B}^n$ that carries $\partial\sigma$ to $\mathbb{S}^{n-1}$.

(proof in Munkres [M] EAT)



See [M] (Munkres - Elements of Algebraic Topology) for the proof.

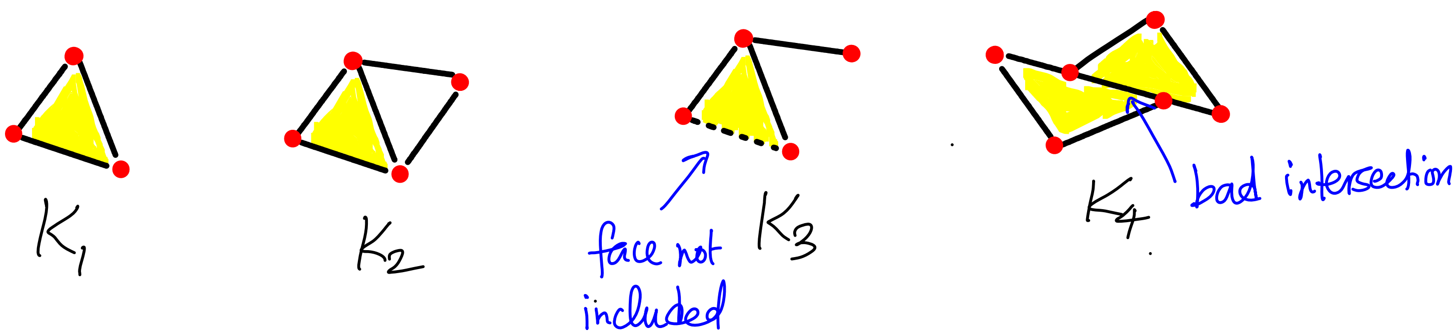
In summary, simplices are "nice" elementary objects that can be used as building blocks to build larger spaces or objects. We will now introduce these larger objects, which are quite general, but are still "nice" since we "glue" simplices together nicely to build them.

Simplicial Complexes

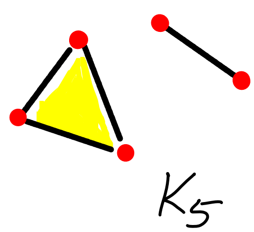
Def A simplicial complex K in $\mathbb{R}^d$ is a collection of simplices in $\mathbb{R}^d$ such that

(1) every face of a simplex in K is in K, and

(2) the intersection of any two simplices of K, when non-empty is a face of each of them.



K_1, K_2 are simplicial complexes, while K_3, K_4 are not.



K_5 is a simplicial complex – in particular, a simplicial complex need not be a single connected component.

Here is another equivalent definition:

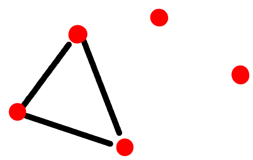
Lemma 2.1 [M] A collection of simplices K is a simplicial complex iff

(1) every face of a simplex in K is in K; and

(2) every pair of distinct simplices in K have disjoint interiors.

A simplex σ and all its proper faces together is a simplicial complex.

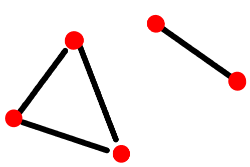
Def If L is a subcollection of K that contains all faces of its elements, then it is a simplicial complex on its own, called a **subcomplex** of K.



L_5. A subcomplex of K_5

Def The subcomplex of K that is the collection of all simplices in K of dimension at most p is the **p-skeleton** of K, denoted $K^{(p)}$.

$K^{(0)}$ are the vertices of K.

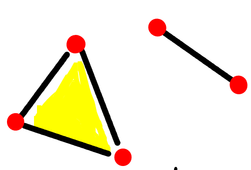


$K_5^{(1)}$ (the 1-skeleton of K_5).

Def The **dimension** of a simplicial complex K is the largest dimension of any simplex in K.

$$\dim(K) = \max_{\sigma \in K} \{\dim(\sigma)\}.$$

e.g.,



K_5

$\dim(K_5) = 2$, also referred to as a 2-complex.

A p-dimensional simplicial complex is referred to, in short, as a p-complex.

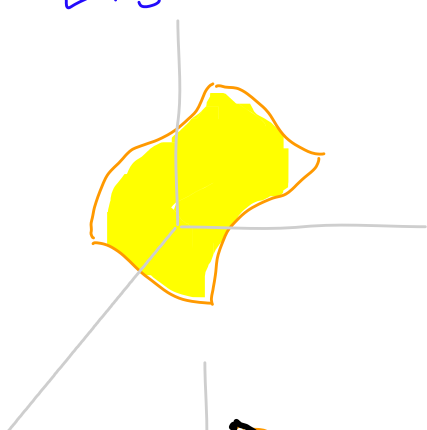
Q. What is $\dim\left(K^{(p)}\right)$? → p-skeleton of K

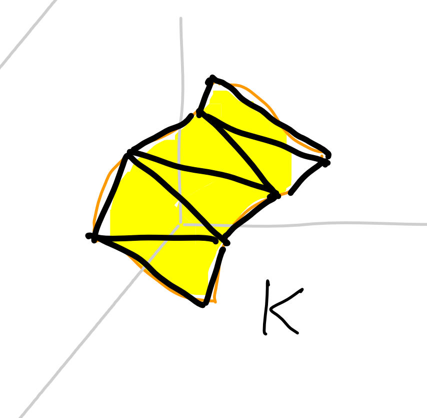
One can immediately conclude $\dim\left(K^{(p)}\right) \leq p$. But notice that $\dim\left(K^{(p)}\right)$ need not always be $= p$. For instance, $\dim\left(K_5^{(3)}\right) = 2$, since $K_5^{(3)} = K_5$ itself. But if we avoid this somewhat trivial case, $\dim\left(K^{(p)}\right) = p$, typically.

Or, more generally, $\dim\left(K^{(p)}\right) = \min\left(p, \dim(K)\right)$.

Recall that we want to use simplicial complexes as a "nice" structured way to model spaces. We now outline the somewhat subtle distinction between the simplicial complex and the (sub)space that it models.

Let's start with an illustration.





K

Consider a subspace of, say, $\mathbb{R}^3$ modeled by a sheet of paper. We could capture this space by a simplicial complex K consisting of six triangles.

Complementarily, if we start with K, we could talk about the subspace of $\mathbb{R}^3$ that it captures. We can specify the usual topology on this subspace (as inherited from $\mathbb{R}^3$).

Def Let $|K|$ be the subset of $\mathbb{R}^d$ which is the union of all simplices in K. Give each simplex its natural topology as a subspace of $\mathbb{R}^d$. Then we can topologize $|K|$ by declaring a subset A of $|K|$ is closed in $|K|$ if $A \cap \sigma$ is closed in K $\forall \sigma \in K$. $|K|$ is called the **underlying space** of K, or the **polytope** of K.

also referred to as "polyhedron"

Some people use the word polytope only when K is finite, i.e., it has a finite number of simplices, while using the word polyhedron more generally, i.e., even for the case where K is not finite.

In convex geometry, $P = \{\bar{x} \in \mathbb{R}^d \mid A\bar{x} \leq \bar{b}\}$ is a polyhedron, and a closed polyhedron is referred to as a polytope.

The two topologies – one as a subspace of $\mathbb{R}^d$, and the other defined using the simplices as above – need not be identical in all cases. But if K is finite, they usually concide. In fact, typical examples where they differ come from infinite simplicial complexes K.

$|K|$ topologized in two different ways: here is an example where the two topologies are different.

Example $K = \{\bigcup [m, m+1] \ \forall m \in \mathbb{Z} \setminus \{0\}\} \cup$ notice that $[-1,0]$ is included

$\{[\frac{1}{n+1}, \frac{1}{n}] \ \forall n \in \mathbb{Z}_{>0}\}$ and all faces.

K is an infinite 1-complex. infinitely many simplices

$|K| = \mathbb{R}$ as a set, but not as a topological space. Indeed, $A = \{\frac{1}{n}, n \in \mathbb{Z}_{>0}\}$ is closed in $|K|$, but not in $\mathbb{R}$.
A does not include 0.

But if K is finite, the topologies are the same.

Properties of $|K|$

Munkres – Elements of Algebraic Topology

Lemma 2.2 [M] If $L \subseteq K$ is a subcomplex, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.
$|\sigma|$ to be precise, but notice σ and $|\sigma|$ are identical!

Lemma 2.3 [M] A map $f: |K| \to X$ is continuous iff $f|_\sigma$ is continuous for each $\sigma \in K$.
f restricted to σ

Recall the barycentric coordinates of $\bar{x} \in \sigma$ ($t_{\bar{a}_i}(\bar{x})$ for vertices $\bar{a}_i$).
We can naturally extend the barycentric coordinates to $\bar{x} \notin \sigma$.

Def If $\bar{x} \in |K|$, then $\bar{x}$ is interior to precisely one simplex in K, whose vertices are, say, $\bar{a}_0, \dots, \bar{a}_n$. Then

$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i, \quad \text{where} \quad t_i > 0 \ \forall i, \ \sum_{i=0}^{n} t_i = 1.$$

If $\bar{v}$ is an arbitrary vertex of K, then the barycentric coordinate of $\bar{x}$ w.r.t $\bar{v}$, $t_{\bar{v}}(\bar{x})$, is defined as $t_{\bar{v}}(\bar{x}) = 0$ if $\bar{v} \notin \{\bar{a}_0, \dots, \bar{a}_n\}$, and $t_{\bar{v}}(\bar{x}) = t_i$ if $\bar{v} = \bar{a}_i$.

Notice that $t_{\bar{v}}(\bar{x})$ is continuous on $|K|$, as $t_{\bar{a}_i}(\bar{x})$ are continuous, as we noted in the last lecture, and then by Lemma 2.3.

Lemma 2.4[M] $|K|$ is Hausdorff.

A space X is Hausdorff if every pair of distinct points $\bar{x}, \bar{y} \in X$ can be surrounded by open sets $\mathcal{U}, \mathcal{V} \subseteq X$ s.t. $\bar{x} \in \mathcal{U}$, $\bar{y} \in \mathcal{V}$, $\mathcal{U} \cap \mathcal{V} = \phi$.

Proof For $\bar{x}_i \neq \bar{x}_j$ in $|K|$, by definition, there exists at least one $\bar{v}$ (vertex) s.t. $t_{\bar{v}}(\bar{x}_i) \neq t_{\bar{v}}(\bar{x}_j)$. Choose r in between $t_{\bar{v}}(\bar{x}_i)$ and $t_{\bar{v}}(\bar{x}_j)$ and define $\mathcal{U} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) < r\}$ and $\mathcal{V} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) > r\}$ as the required open sets.

We now study some important subspaces of $|K|$.

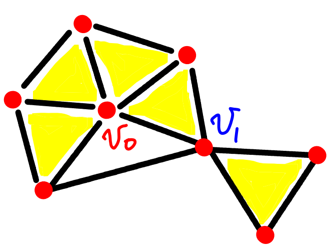
MATH 524: Lecture 4 (08/28/2025)

Today:
- * star, closed star, link
- * simplicial maps
- * abstract simplicial complexes

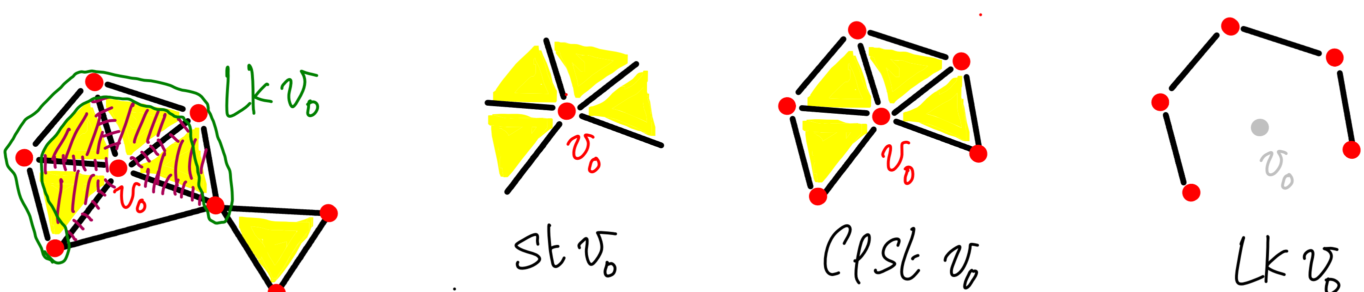
We now study some important subspaces of $|K|$.

Three Subspaces of $|K|$

Def If $\bar{v}$ is a vertex of K, then the **star** of $\bar{v}$ in K, denoted $\operatorname{St}\bar{v}$ (or $\operatorname{St}(\bar{v}, K)$) is the union of the interiors of all simplices in K that contain $\bar{v}$ as a vertex. The closure of $\operatorname{St}\bar{v}$, denoted $\overline{\operatorname{St}\bar{v}}$ or $\operatorname{Cl}\operatorname{St}\bar{v}$, is the **closed star** of $\bar{v}$. It is the union of all simplices of K which have $\bar{v}$ as a vertex. $\operatorname{Cl}\operatorname{St}\bar{v}$ is a polytope of a subcomplex of K. $\operatorname{Cl}\operatorname{St}\bar{v} - \operatorname{St}\bar{v}$ is called the **link** of $\bar{v}$, denoted $\operatorname{Lk}\bar{v}$.

K_6 

We illustrate these subcomplexes on K_6 for vertices v_0 and v_1. Note that the unshaded triangle below v_0 is not part of K_6.

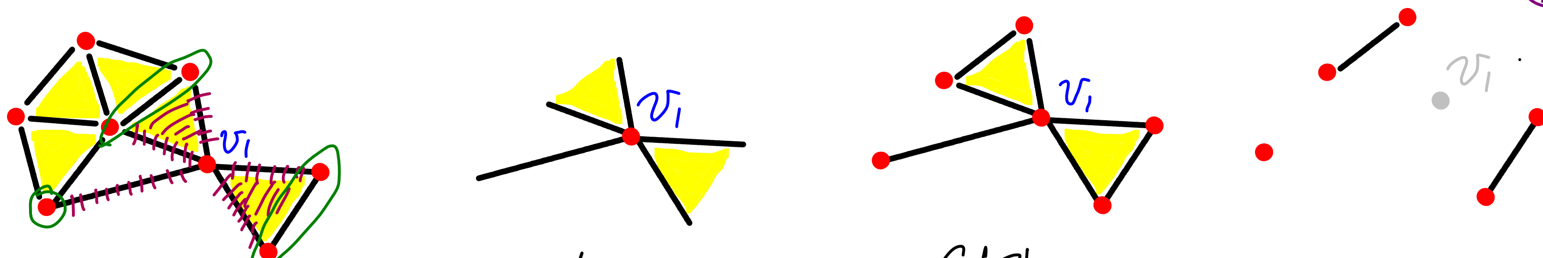


St v_0 St v_0 Cl St v_0 Lk v_0

add to get Cl St v_0

Note that $\operatorname{Lk} v_0 = \operatorname{Cl}\operatorname{St} v_0 - \operatorname{St} v_0$.

Also note that $v_0 \in \operatorname{St} v_0$ (indeed, $\operatorname{Int} v_0 = v_0$, and v_0 is a simplex that contains v_0 as a vertex, trivially).



St v_1 St v_1 ClSt v_1 Lk v_1

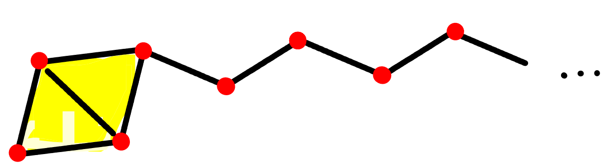
Properties of star, closed star, link

* St $\bar{v}$ is open in $|K|$. → We could use $t_{\bar{v}}(\cdot)$ to prove.
* The complement of St $\bar{v}$ is the union of all simplices that do not contain $\bar{v}$ as a vertex, and hence it is the polytope of a subcomplex of K.
* Lk $\bar{v}$ is the polytope of a subcomplex of K.
* $\text{Lk}\,\bar{v} = \text{Cl St}\,\bar{v} \cap (\text{complement of St}\,\bar{v})$.
* St $\bar{v}$ and Cl St $\bar{v}$ are both path-connected.
  X is path-connected if $\forall \bar{u}, \bar{v} \in X, \bar{u} \neq \bar{v}$, $\exists$ a path connecting $\bar{u}$ and $\bar{v}$ in X.
* Lk $\bar{v}$ need not be connected.

Def A simplicial complex K is **locally finite** if each vertex of K belongs to only finitely many simplices of K. Equivalently, K is locally finite iff each closed star is the polytope of a finite subcomplex of K.

Note: A locally finite simplicial complex could be infinite, e.g., K_7.

K_7



(the edges continue forever)

Simplicial Maps

We study maps between simplicial complexes as a first step toward developing the tools to compare spaces modeled by the simplicial complexes.

Def Let K, L be simplicial complexes. A function $f: |K| \to |L|$ is a (linear) **simplicial map** if it takes simplices of K linearly onto onto simplices of L. In other words, if $\sigma \in K$, then $f(\sigma) \in L$.

linearly: If $\sigma = \text{conv}\{\bar{v}_0, \dots, \bar{v}_n\}$ and $\bar{x} = \sum_{i=0}^{n} t_i \bar{v}_i$, $t_i \geq 0$, $\sum_{i=0}^{n} t_i = 1$, then $f(\bar{x}) = \sum_{i=0}^{n} t_i f(\bar{v}_i)$.

Note that $\{f(\bar{v}_0), \dots, f(\bar{v}_n)\}$ span a simplex τ of L, which could be of a lower dimension than σ.

Munkres takes a slightly different approach in defining simplicial maps.

[M]: Starts with $f: K^{(0)} \to L^{(0)}$, then insist that when

$\{\bar{v}_0, \dots, \bar{v}_n\}$ span $\sigma \in K$, $\{f(\bar{v}_0), \dots f(\bar{v}_n)\}$ span $\tau \in L$.

f is a continuous map of σ onto τ, and hence as a map of σ onto $|L|$. Then by Lemma 2.3, it is a continuous map from $|K|$ to $|L|$.

If $g: |K| \to |L|$ and $h: |L| \to |M|$ are simplicial maps, then $f = h \circ g$ is a simplicial map from $|K|$ to $|M|$.

If we further insist that $f: K^{(0)} \to L^{(0)}$ is a **bijective** correspondence such that vertices $\bar{v}_0, \dots, \bar{v}_n$ of K span a simplex of K iff $f(\bar{v}_0), \dots, f(\bar{v}_n)$ span a simplex of L, then the induced simplicial map $g: |K| \to |L|$ is a homeomorphism. We call this map an **isomorphism** of K with L (or a simplicial homeomorphism).

Abstract Simplicial Complexes (ASCs)

Def An abstract simplicial complex (ASC) is a collection $\mathcal{S}$ of finite nonempty sets such that if $A \in \mathcal{S}$, then so is every nonempty subset of A.

Note: $\mathcal{S}$ itself could be infinite, but each $A \in \mathcal{S}$ is finite.

Example: $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ is an ASC.

We specify several more definitions related to ASCs.

Def A (any element of $\mathcal{S}$) is a **simplex** of $\mathcal{S}$. Its **dimension** is given as $\dim(A) = |A| - 1$.

↳ # elements in A, or size of A

The **dimension of the ASC** is defined as follows.

$\dim(\mathcal{S})$ = largest dimension of any simplex in $\mathcal{S}$, or ∞ if no such largest dimension exists.

The **vertex set** V of $\mathcal{S}$ (or $V(\mathcal{S})$) is the union of all singleton elements (simplices) of $\mathcal{S}$. We do not distinguish between the individual vertices and the singleton sets they represent.

$$v_0 \text{ (vertex)} \equiv \{v_0\} \quad \text{0-simplex of } \mathcal{S}.$$

A subcollection of $\mathcal{S}$ that is a simplicial complex by itself is a **subcomplex** of $\mathcal{S}$.

We can now talk about when two ASCs are "similar".

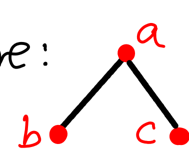
Def Two ASCs $\mathcal{S}$ and $\mathcal{T}$ are **isomorphic** if there exists a bijective correspondence f mapping $V(\mathcal{S})$ to $V(\mathcal{T})$ such that $\{a_0, \ldots, a_n\} \in \mathcal{S}$ iff $\{f(a_0), \ldots, f(a_n)\} \in \mathcal{T}$.

e.g., With $\mathcal{T} = \{\{d\}, \{e\}, \{f\}, \{d,e\}, \{d,f\}\}$, $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.

It turns out the previous notion of simplicial complexes (in $\mathbb{R}^d$) and ASC are directly related.

Def Let K be a (geometric) simplicial complex. Let V be its vertex set. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, \ldots, a_n\}$ of V such that $\bar{a}_0, \ldots, \bar{a}_n$ span a simplex of K. Then $\mathcal{K}$ is an ASC called the **vertex scheme** of K. Symmetrically, we call K a **geometric realization** of $\mathcal{K}$.

e.g., (continued) $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ has a geometric realization S as shown here:



This complex could be sitting in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Theorem 3.1 [M] (a) Every ASC $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K.

A version of this result is given as the **geometric realization theorem** which states that every abstract d-complex has a geometric realization in $\mathbb{R}^{2d+1}$.

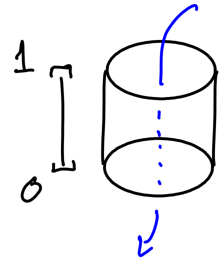
IDEA: If $\dim(\mathcal{S}) = d$ then let $f: V(\mathcal{S}) \to \mathbb{R}^{2d+1}$ be an injective function whose image is a set of GI points in $\mathbb{R}^{2d+1}$. Specify that for each abstract simplex $\{a_0, \ldots, a_n\} \in \mathcal{S}$, $\{f(a_0), \ldots, f(a_n)\} \in K$. Then $\mathcal{S}$ is isomorphic to the vertex scheme of K.

MATH 524: Lecture 5 (09/02/2025)

Today: * Examples of ASCs
* Review of algebra

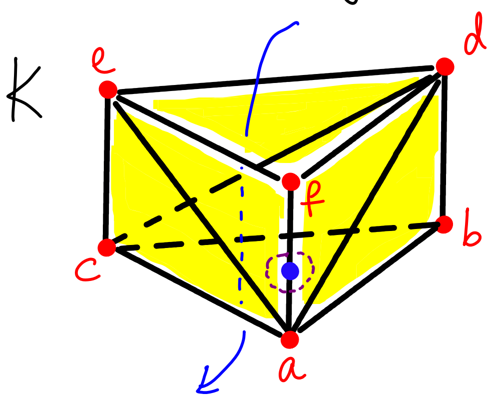
Examples of ASCs

1. Cylinder $\qquad \underbrace{S^1}_{\text{circle}} \times \underbrace{I}_{[0,1]}$



We want to describe a simplicial complex K such that $|K|$ is homeomorphic to the cylinder.

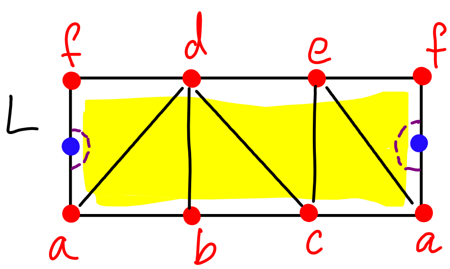
We first describe a geometric simplicial complex K, which could be sitting in $\mathbb{R}^3$, for instance.



K comprises of the six triangles adf, abd, bcd, cde, ace, and aef.

Indeed, $|K| \approx$ cylinder.

But we now specify an abstract simplicial complex whose underlying space is homeomorphic to the cylinder. We start with a rectangle L, and then assign labels to specific vertices in L. Thus, L along with the labels is the ASC.



Notice that both the left and right border edges of L are labeled af going from bottom to top.

We can describe the required map between K and L as follows.

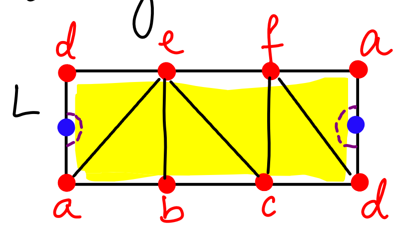
Let $f: K^{(0)} \to L^{(0)}$ is the vertex map that assigns vertices in K the labels in L. We can extend f to a simplicial map $g: |K| \to |L|$. This map g is a "pasting map", or a quotient map.

indeed, we are starting with the rectangular strip (of paper, say) L, and pasting its end edges together (af).

Notice how we can visualize a neighborhood of a point on edge af in K and correspondingly on L.

2. Möbius strip

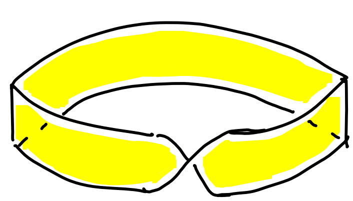
We now start with the rectangular space L and a specific vertex labeling as shown here.



The ASC $\mathcal{S}$ here has 6 triangles ade, abe, bce, cef, cdf, adf, as well as their faces.

We're again gluing the end edges, but now with a "twist."

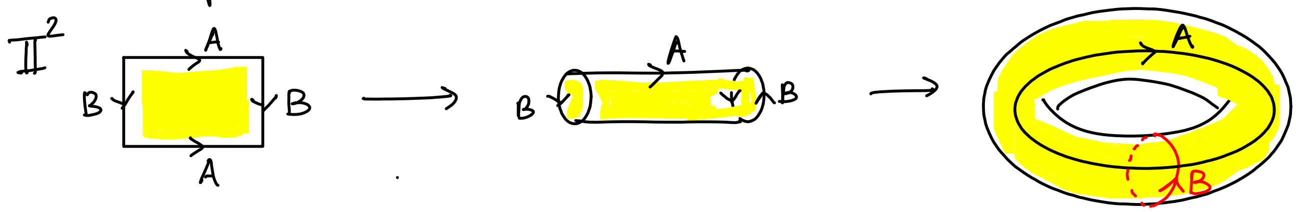
Let K be a geometric realization of $\mathcal{S}$. We can consider a simplicial map $g: |L| \to |K|$, which maps vertices in L to vertices in K. Again, g is a quotient (or "pasting") map that maps the left edge of |L| to the right edge, but with a "twist."

$L \xrightarrow{g}$ 

Notice that we do want a homeomorphism from $|L|$ to K, and just a vertex map is not enough. But of course, the vertex map is naturally (linearly) extended to the desired map from $|L|$ to $|K|$.

\mathbb(T)^2 in LaTeX!

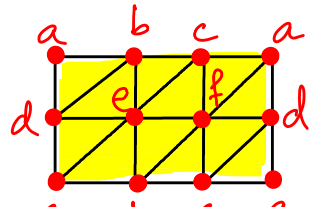
3. Torus ($\mathbb{T}^2$) The quotient space obtained by making identifications on the sides of a rectangle as follows.



Notice that this is an example of a quotient map defined on a general space, and not on an ASC.
This is the surface of a "donut", and not the solid donut itself.

Now, let us find an ASC K such that $|K| \approx \mathbb{T}^2$.
Let's start with a rectangular space as before, and assign labels that could work. Here is a first try.

K_1:

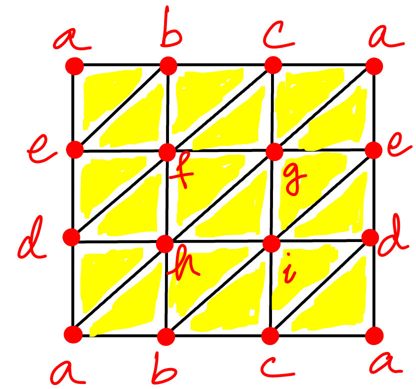


Is $|K_1| \approx \mathbb{T}^2$? No!
We are doing too much gluing!

Notice that $\overline{ad}$ is part of 4 triangles ade, adb, adc, adf, for instance.
The gluings specified above glue only two edges together at a time.
With this gluing, edge ad is part of four triangles, i.e., we get a "fan" of four "flaps" meeting at ad. But notice that there are no such 4-way junctions in the torus.

We need to "spread out" more!

K_2:



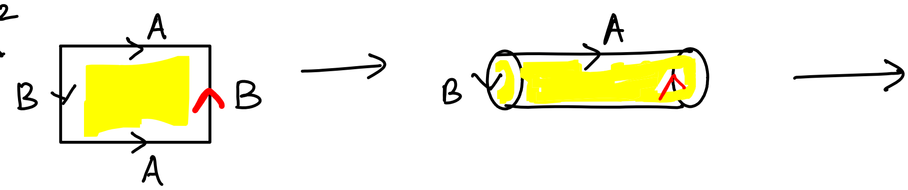
$|K_2| \approx \mathbb{T}^2$

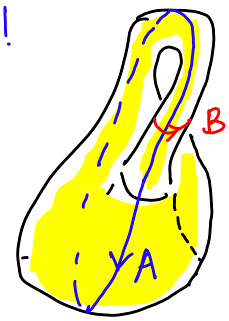
We can show that $|K_2| \approx \mathbb{T}^2$. See [M] for details, but on a complex similar to K_2.

Every edge is face of exactly two triangles.

4. Klein bottle ($\mathbb{K}^2$) → \mathbb{K}^2 in LaTeX!

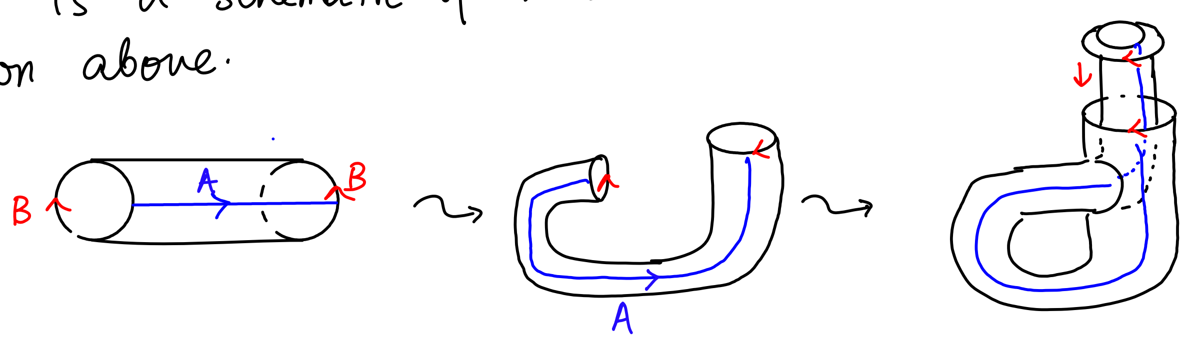
$\mathbb{K}^2$





Here, we identify the opposite pairs of edges – one pair with a twist as in the Möbius strip (B here) and the other without (A here; similar to torus or cylinder). The Klein bottle does not have an embedding in $\mathbb{R}^3$, but has in $\mathbb{R}^4$. We must go to the higher dimension to avoid self intersections.

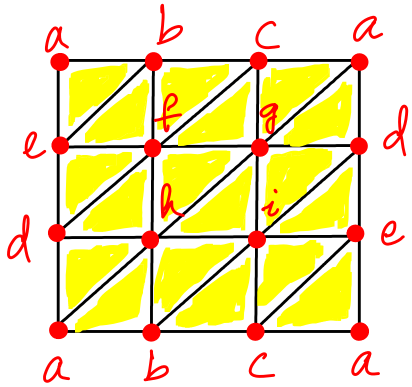
We do get an **immersion** in $\mathbb{R}^3$, which allows self intersection. Here is a schematic of how one arrives at the immersion shown above.



This instance illustrates the difficulty faced when working with geometric embeddings. We could instead work with the abstract space along with the quotient map!

We now construct an ASC for $\mathbb{K}^2$.

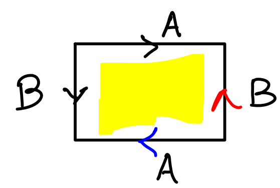
K_3



$|K_3| \approx \mathbb{K}^2 \rightsquigarrow$ one can check to make sure we are not gluing more than two edges anywhere.

Of course, $|K_3| \not\approx |K_2|$, and in general, $\mathbb{K}^2 \not\approx \mathbb{T}^2$.

Notice that we could start with the rectangular space (for L) and identify pairs of edges in several ways. For instance, when we glue both pairs of opposite edges with twists, we get the real projective plane ($\mathbb{RP}^2$).



$\mathbb{RP}^2$

The related question now is how to identify homeomorphic simplicial complexes K for any such quotient space. In particular, when do we get "nice" labelings (or gluings)?

See Lemma 3.2 in [M] for a condition given in terms of closed stars of vertices in K. This result is left as a **candidate for video tutorial**.

Review of Abelian Groups

We now review several properties and results from groups and homomorphisms between groups. The idea is to cast questions about similarity of topological spaces as corresponding questions on homomorphisms between groups defined on simplicial complexes that are homeomorphic to the spaces in question.

A good book – Fraleigh (First course in Abstract Algebra).

closure is assumed, i.e., $a+b \in G \;\forall a,b \in G$.

Group: set G with an operation $+$ "addition", such that

(1) there exists an **identity**, $0 \in G$, s.t.

$$a+0 = 0+a = a \quad \forall a \in G;$$

(2) $\forall a \in G$, there is an **inverse**, i.e., $-a \in G$ s.t.

$$a+(-a) = (-a)+a = 0; \text{ and}$$

(3) $a+(b+c) = (a+b)+c \quad \forall a,b,c \in G$; i.e., $+$ is **associative**.

(4) Further, if $a+b = b+a \;\forall a,b \in G$, then G is an **abelian group**.

In general, we will work with abelian groups in this class.

Notation: $\quad ng = \underbrace{g+g+\cdots+g}_{n \text{ times}}$ for $g \in G$.

<u>Homomorphisms</u> $f: G \to H$, G, H are groups is a homomorphism if $f(g_1 +_G g_2) = f(g_1) +_H f(g_2)$.

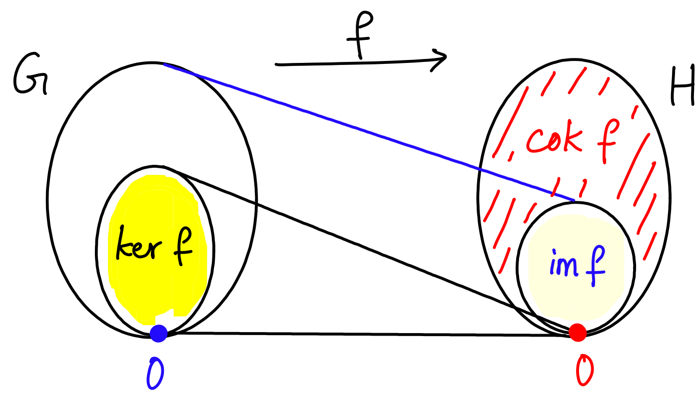
Intuitively, homomorphisms "preserve the structure" of groups.

We study subgroups specified by homomorphism f:

kernel of f: $f^{-1}(0)$, is a subgroup of G, denoted **ker f**.

image of f: $f(G)$, is a subgroup of H, denoted **im f**.

cokernel of f: quotient group of H given as $H/f(G)$, denoted **cok f**.



f is a monomorphism (injection) iff $\ker f = 0$.

f is an epimorphism (surjection) iff $\operatorname{cok} f = 0$, and in this case, f defines an isomorphism $G/_{\ker f} \simeq H$.

An abelian group G is **free** if it has a basis $\{g_\alpha\}$ of elements in G such that $\forall g \in G$, $g = \sum n_\alpha g_\alpha$ is a unique finite sum, for $n_\alpha \in \mathbb{Z}$.

This <u>uniqueness</u> (for the free abelian group) implies that each basis element g_α generates an infinite cyclic group $H = \{ng_\alpha \mid n \in \mathbb{Z}\}$.

Note: $\mathbb{Z}/n$ (or $\mathbb{Z}_n$) has elements $\{0, 1, \ldots, n-1\}$ with addition mod n.
→ notation used in [M]

More generally, if each $g \in G$ can be written as $\sum_i n_\alpha g_\alpha$, but not necessarily uniquely, then $\{g_\alpha\}$ **generates** G.
If $\{g_\alpha\}$ is finite, we say that G is **finitely generated**.

We will work mostly with finitely generated abelian groups

Def If G is free, and has a basis of n elements, then every basis of G has n elements. The number of elements in a basis of G is its **rank**, denoted $rk(G)$.
→ or $rank(G)$

The **order** of G is the # elements in G, denoted $|G|$.

<u>A crucial property</u>: If $\{g_\alpha\}$ is a basis of G, any function f from $\{g_\alpha\}$ to abelian group H extends uniquely to a homomorphism from G to H.

→ Somewhat similar in flavor to a vertex map extending to the corresponding simplicial map

MATH 524: Lecture 6 (09/04/2025)

Today: * two results on abelian groups
* orientation of simplices

More results on groups...

Let G be an abelian group. $g \in G$ has **finite order** if $ng = 0$ for some $n \in \mathbb{Z}_{>0}$. The set of all elements of finite order in G is a subgroup T of G, called the **torsion subgroup**. If T vanishes, we say G is **torsion-free**.

Notice that $0 \in G$ is a trivial case in this context, as $n0 = 0$ for any $n \in \mathbb{Z}$.

We now consider how to "combine" (abelian) groups to form bigger (abelian) groups. The intuition is similar to combining multiple individual dimensions to form a higher dimensional space.

[M] defines internal direct sums, direct products, and external direct sums. We discuss them all for the sake of completeness.

Internal direct sums

Let G be an abelian group, and let $\{G_\alpha\}_{\alpha \in J}$ be a collection of subgroups of G indexed bijectively by the index set J. If each $g \in G$ can be written uniquely as finite sum $g = \sum_\alpha g_\alpha$, where $g_\alpha \in G_\alpha$ for each $\alpha \in J$, then G is the **internal direct sum** of the groups G_α, and is written
$$G = \bigoplus_{\alpha \in J} G_\alpha .$$

If $J = \{1, 2, \ldots, n\}$ for finite n, say, we also write

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \quad \text{or} \quad G = \bigoplus_{\alpha=1}^{n} G_\alpha$$

There is a similar distinction here to a basis vs generating set of a group.

If each $g \in G$ can be written as a finite sum $g = \sum_\alpha g_\alpha$, but not necessarily uniquely, then G is simply the **sum** of groups $\{G_\alpha\}$.

We write $G = \sum_\alpha G_\alpha$, or $G = G_1 + \cdots + G_n$ (if finite).

internal sum, to be precise

Here, we say $\{G_\alpha\}$ **generates** G.

Notice that if G is free abelian with basis $\{g_\alpha\}$, then G is the direct sum of subgroups $\{G_\alpha\}$, where G_α is the infinite cyclic group generated by g_α.

The converse is also true here, i.e., if G is the direct sum of $\{G_\alpha\}$ where G_α is the infinite cyclic group generated by g_α, then G is free abelian with basis $\{g_\alpha\}$.

Direct Products and External direct sums

Def Let $\{G_\alpha\}_{\alpha \in J}$ be an indexed family of abelian groups. The **direct product** $\prod_{\alpha \in J} G_\alpha$ is the group whose set is the cartesian product of sets G_α, and the operation is component-wise addition.

J can be infinite here; you could assume it is finite, though, to get the intuition. There is technical work required to extend the results and definitions to the infinite case — but it's not critical for us.

The **external direct sum** G is the subgroup of the direct product $\prod_{\alpha \in J} G_\alpha$ consisting of all tuples

$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{\alpha_i} \\ \vdots \end{pmatrix}$ such that $g_{\alpha_i} = 0_{\alpha_i}$ for all but finitely many values of α_i.

Examples

1. $G_1 = \mathbb{Z} \times \mathbb{Z}$ $\quad G_1$ has rank 2; basis is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. $rk(G_1) = 2$.

   operation is componentwise addition.

2. $G_2 = \mathbb{Z}/_2 \times \mathbb{Z}/_3$ $\quad$ (or $\mathbb{Z}_2 \times \mathbb{Z}_3$)

   componentwise addition mod 2 and mod 3.

   $1 +_2 1 = 0$ $\qquad$ $1 +_3 1 = 2$, $\; 1 +_3 2 = 0$

   G_2 is a cyclic group, $\underbrace{|G_2|}_{\text{order}} = 6$,

   $G_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.

   $rk(G_2) = 1$, as $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is a basis.

   $1 \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\quad 2 \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\quad 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   $4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\quad 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\quad 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Theorem The group $\prod_{i=1} \mathbb{Z}/t_i$ for $t_i \in \mathbb{Z}_{>0}$ is cyclic and is isomorphic to $\mathbb{Z}_{t_1 t_2 \cdots t_n}$ iff $\gcd(t_i, t_j) = 1 \; \forall i, j$.

t_i and t_j are relatively prime

Back to example 2: $\mathbb{Z}/2 \times \mathbb{Z}/3 \simeq \mathbb{Z}_6$.

If $n = (p_1)^{n_1}(p_2)^{n_2} \cdots (p_r)^{n_r}$ for primes $p_1, \dots, p_r$, then

$$\mathbb{Z}_n \simeq \mathbb{Z}/(p_1)^{n_1} \times \mathbb{Z}/(p_2)^{n_2} \times \cdots \times \mathbb{Z}/(p_r)^{n_r}.$$

Structure of finitely generated abelian groups

Two main results that we will use in characterizing the structure of homology groups on simplicial complexes.

Theorem 4.2 [M] Let F be a free abelian group. If R is a subgroup of F, then R is a free abelian group. If $\operatorname{rank}(F) = n$, then $\operatorname{rank}(R) = r \leq n$. Furthermore, there is a basis $e_1, \dots, e_n$ of F and numbers $t_1, \dots, t_k$ $(t_i \in \mathbb{Z}_{>0})$ such that

(1) $t_1 e_1, \dots, t_k e_k, e_{k+1}, \dots, e_r$ is a basis for R, and

(2) $t_1 | t_2 | \cdots | t_k$, i.e., t_i divides t_{i+1} $\forall i \geq 1$. $(i \leq k-1)$.

The t_i's are uniquely determined by F and R.

Intuitively, the subgroup inherits the structure of the original group...

Theorem 4.3 [M] (Fundamental theorem of finitely generated abelian groups).

Let G be a finitely generated abelian group, and let T be its torsion subgroup. The following results hold.

(a) There is a free abelian subgroup H of G such that $G = H \oplus T$. The rank of H $rk(H) = \beta$, a finite number.

(b) There exist finite cyclic groups $T_1, \ldots, T_k$ with $|T_i| = t_i > 1$, and $t_1 | t_2 | \cdots | t_k$ such that

$$T = T_1 \oplus \cdots \oplus T_k.$$

(c) The numbers β and $t_1, \ldots, t_k$ are uniquely determined by G.

β is the **betti number** of G, and $t_1, \ldots, t_k$ are the **torsion coefficients** of G.

"torsion" meaning "twistedness" or "cyclic nature"; as opposed to the free part.

A quick example on torsion...

Example What is the torsion subgroup of the multiplicative group $\mathbb{R}^*$ of all nonzero real numbers?

$G = \mathbb{R}/\{0\}$, operation is $*$ (multiplication), identity is 1, $g^{-1} = \frac{1}{g}$ $\forall g \in G$.

The answer is $\{1, -1\}$.

Here is the main consequence of the previous theorem:

Any finitely generated abelian group G can be written as (is isomorphic to) a direct sum of cyclic groups, i.e.,

$$G \simeq \underbrace{(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})}_{\beta} \oplus \mathbb{Z}/t_1 \oplus \mathbb{Z}/t_2 \oplus \cdots \oplus \mathbb{Z}/t_k$$

where $\beta \geq 1$, $t_i > 1$, and $t_i \mid t_{i+1}$ $\forall i$. This is a canonical form, called the **invariant factor decomposition** of G.

We can also get the **primary decomposition**, which is another canonical form:

$$G \simeq \underbrace{(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})}_{\beta} \oplus \mathbb{Z}/(p_1)^{n_1} \oplus \cdots \oplus \mathbb{Z}/(p_r)^{n_r}$$ for primes $p_1, \ldots, p_r$.

Examples

1. What are the beth number and torsion coefficients of

$$G = \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z} \ ?$$

$$G \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3,$$ so $\beta = 2$.

Notice that $4 \times 3 = (2)^2 \times 3 = 2 \times 6$, and $2 \mid 6$. Hence $t_1 = 2, t_2 = 6$ are the torsion coefficients.

2. Find the primary and invariant factor decompositions of $\mathbb{Z}/4 \times \mathbb{Z}/12 \times \mathbb{Z}/18$.

We do not get $\mathbb{Z}/4 \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, as 2 and 2 are not coprime.

$$\mathbb{Z}/4 \to \cancel{(2)^2} \qquad \mathbb{Z}/12 \to \mathbb{Z}/3 \times \mathbb{Z}/4 \qquad \mathbb{Z}/18 \to \mathbb{Z}/2 \times \mathbb{Z}/9$$

Primary decomposition: $\mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/9$.

invariant factor decomposition:

$$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/36$$

$$\begin{array}{ccc} 2 & 4 & 4 \\ & \times 3 & \times 9 \\ \hline 2 & 12 & 36 \end{array}$$

Notice that $2 \mid 12 \mid 36$.

A standard "trick" is to write the factors for each prime in a line in a right justified fashion. Then multiply the numbers in each column to get the torsion coefficients.

Homology Groups

We now study groups and homomorphisms defined on simplicial complexes. Questions about topological similarity are posed as equivalent questions on corresponding groups' structure.

We need a few foundational concepts.

Orientation of a simplex

Let σ be a simplex (geometric or abstract). We define two orderings of its vertex set to be equivalent if they differ by an even permutation, i.e., you can go from one ordering to the other using an even number of pairwise swaps.

If $\dim(\sigma) > 0$, the orderings fall into two equivalence classes. Each class is an **orientation** of σ.

If $\dim(\sigma) = 0$, it has only one orientation.

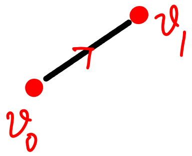
An **oriented simplex** is a simplex σ together with an orientation of σ.

Notation: Let $v_0, \ldots, v_p$ be independent. Then $\sigma = v_0 v_1 \cdots v_p$ is the simplex spanned by $v_0, \ldots, v_p$, and $[v_0, \ldots, v_p]$ denotes the oriented simplex σ with the orientation $(v_0, \ldots, v_p)$.

GI if $\bar{v}_0, \ldots, v_p \in \mathbb{R}^d$ and distinct if $v_0, \ldots, v_p$ are (just) labels in the abstract setting.

When it is clear from the context, we will use σ to denote both the simplex as well as its orientation (or the oriented simplex).

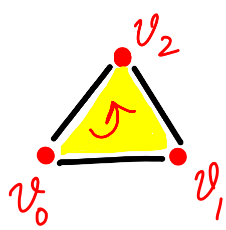
1-simplex $[v_0, v_1]$, $[v_1, v_0] \to$ opposite orientation



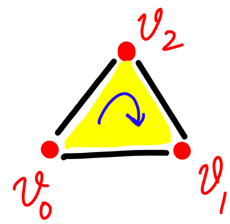
equivalent to orienting the edge from v_0 to v_1.
$[v_1 v_0] \to$ draw the arrow the other way.

2-simplex Notice that $[v_0, v_1, v_2]$ is the same as $[v_1, v_2, v_0]$.

$(v_0 v_1 v_2) \to (v_1 v_0 v_2) \to (v_1 v_2 v_0)$ two pairwise swaps

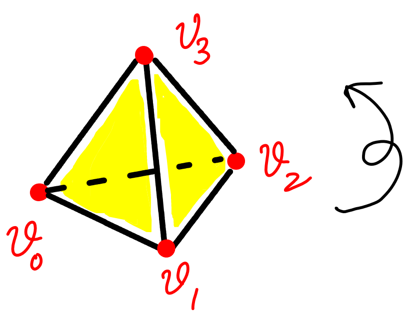


$[v_0 v_1 v_2] \to$ can be the counterclockwise orientation



$[v_0 v_2 v_1] \to$ is the clockwise orientation

3-simplex $[v_0 v_1 v_2 v_3]$



We could imagine orienting the tetrahedron as per the right-hand thumb rule – $v_0 \to v_1 \to v_2$ as the fingers of your right hand curl around, and $v_2 \to v_3$ points up along your thumb.

Notice that $[v_0 v_2 v_1 v_3]$, the opposite orientation, then corresponds to the left-hand thumb rule.