

MATH 273 - Lecture 16 (10/16/2014)

Second derivative test

$$12. f(x,y) = 1 - \sqrt[3]{x^2+y^2} = 1 - (x^2+y^2)^{\frac{1}{3}} \quad \text{→ find local extrema, saddle points, and other critical points, if any.}$$

$$f_x = -\frac{1}{3}(x^2+y^2)^{-\frac{2}{3}} \cdot 2x = \frac{-2x}{3(x^2+y^2)^{\frac{2}{3}}} = 0 \quad \text{--- (1)}$$

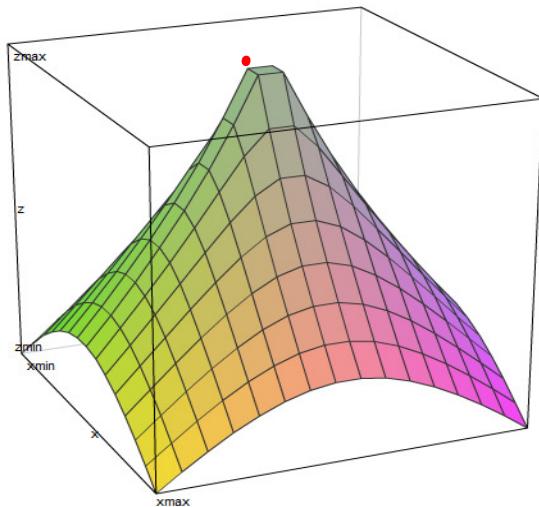
$$f_y = -\frac{1}{3}(x^2+y^2)^{-\frac{2}{3}} \cdot 2y = \frac{-2y}{3(x^2+y^2)^{\frac{2}{3}}} = 0 \quad \text{--- (2)}$$

The only possible solution could have been $(0,0)$, but both f_x and f_y are undefined at $(0,0)$. Hence $(0,0)$ is the only critical point (as f_x, f_y both do not exist at $(0,0)$).

There is no point in finding the second derivatives here (as they also will be undefined at $(0,0)$).

Notice that $f(0,0) = 1$, and $f(x,y) = 1 - \sqrt[3]{x^2+y^2} \leq 1$ for any (x,y) . Hence $(0,0)$ is a local maximum.

Here is the surface $z = 1 - \sqrt[3]{x^2 + y^2}$. Indeed, the point $(0, 0, 1)$ is a local maximum. In fact, it is the global maximum!



$$\textcircled{15} \quad f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$$

$$f_x = 12x - 6x^2 + 6y = 0 \quad \text{--- (1)}$$

$$f_y = 6y + 6x = 0 \quad \text{--- (2)}$$

$$(2) \Rightarrow y = -x. \text{ So (1)} \Rightarrow 12x - 6x^2 - 6x = 0$$

$$\text{i.e., } 6x - 6x^2 = 0 \quad 6x(1-x) = 0 \Rightarrow x=0, 1$$

Hence $y=0, -1$

The critical points are $(0, 0)$ and $(1, -1)$.

$$f_{xx} = 12 - 12x, \quad f_{yy} = 6, \quad f_{xy} = f_{yx} = 6$$

$$H = f_{xx}f_{yy} - f_{xy}^2$$

(0, 0)

$$f_{xx} = 12 > 0$$

$$H = 12 \cdot 6 - 6^2 = 36 > 0$$

Hence (0, 0) is a local minimum

$$f(0, 0) = 0.$$

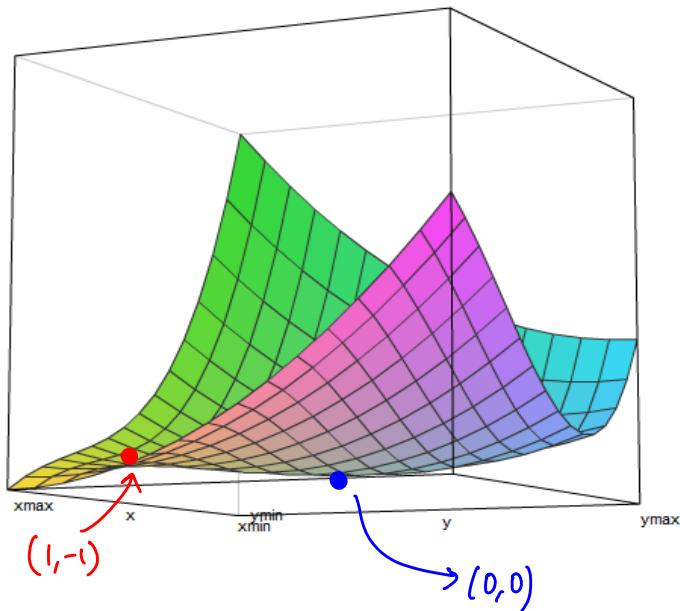
(1, -1)

$$f_{xx} = 12 - 12 \cdot 1 = 0$$

$$H = 0 \cdot 6 - 6^2 = -36 < 0$$

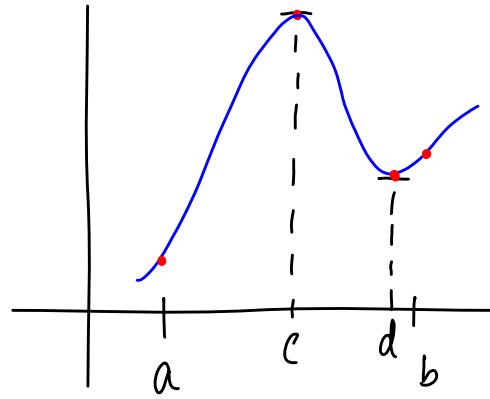
So (1, -1), and hence

$f(1, -1) = 1$ is a saddle point.

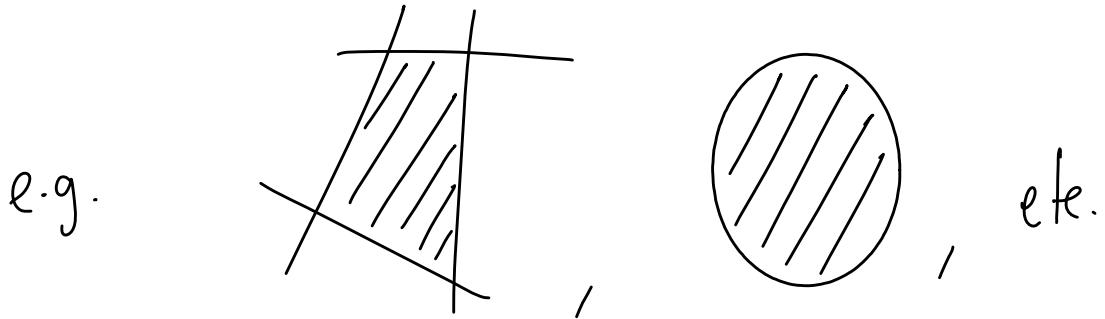


Finding absolute maxima/minima in a region

In 1D, to find absolute extrema of $f(x)$ in interval $[a,b]$, we first find local extrema in (a,b) , and then compare $f(\cdot)$ at these critical points with $f(a)$ and $f(b)$.

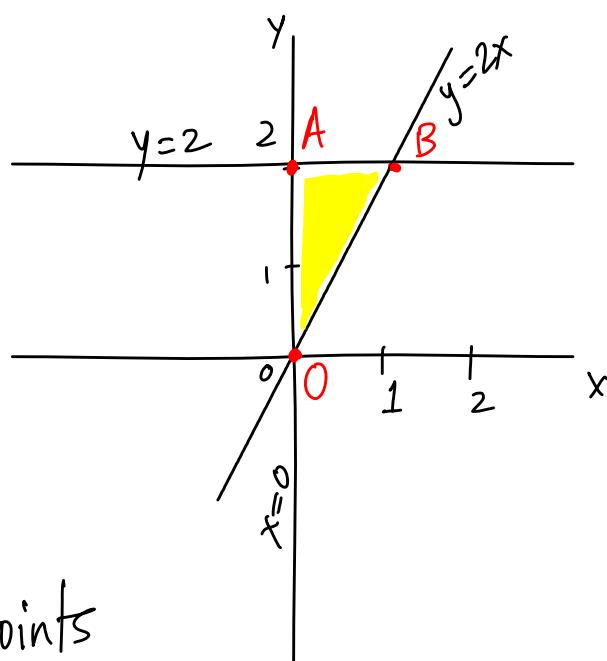


We extend this approach to 2D. In place of interval $[a,b]$, we get a region, typically a closed set in 2D (defined by lines/curves that form its boundaries).



31. find the absolute maximum and minimum of

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x=0$, $y=2$, and $y=2x$ in the first quadrant.



The region R is the triangle OAB , where $O(0,0)$, $A(0,2)$, $B(1,2)$.

Step(i) find any critical points
that are interior to R .

$$\begin{aligned} f_x &= 4x - 4 = 0 \quad \text{--- (1)} \\ f_y &= 2y - 4 = 0 \quad \text{--- (2)} \end{aligned} \quad \left. \begin{array}{l} x=1, y=2. \text{ But } (1,2) \text{ is} \\ \text{point } B, \text{ which is not an} \\ \text{interior point.} \end{array} \right\}$$

So, we do not consider the second derivative test at B .

Step(ii) Investigate the behavior of the function
on each boundary segment—OA, OB, AB.

OA $x=0$ on OA. Hence

$$f(0,y) = y^2 - 4y + 1$$

Hence, $f'(0,y) = 2y - 4 = 0$, gives $y=2$, which
corresponds to A(0,2), an end point.

As we did in 1D, we will investigate the function on
the boundaries here. But now, we look for critical
points on each boundary segment, and also look at
the end points. We will finish this problem in the
next lecture.