MATH401: Lecture 21 (10/28/2025)

Today: * compact $\Leftrightarrow$ OCP
* modes of continuity

Recall OCP: $\mathcal{O} = \{O_i\}_{i\in I}$, $K \subseteq \bigcup_{i\in I} O_i$, $\exists \{O_i\}_{i=1}^n$ s.t. $K \subseteq \bigcup_{i=1}^n O_i$.

Prop 3.6.2: OCP $\Rightarrow$ compact.

Theorem 3.6.4 compact iff OCP.

Proof Need to prove: If K is compact, and $\mathcal{O}$ is an open cover, then $\mathcal{O}$ has a finite subcover.
$\to$ We already showed the reverse implication in Prop. 3.6.2.

By the Extreme Value Theorem (EVT, Theorem 3.5.10), $f(x)$ defined in Lemma 3.6.3 taken over K has a minimum value r over K. We can conclude that $r>0$, as $f(x)>0$ $\forall x \in K$.

$\Rightarrow$ $B(x, \frac{r}{2}) \subset O_x \in \mathcal{O}$ $\forall x \in K$. $\to$ $\exists\, O_x \in \mathcal{O}$ s.t. $B(x, r/2) \subset O_x$ $\forall x \in K$.

Now, K is compact $\Rightarrow$ K is totally bounded.
$\to$ By Proposition 3.5.12.

$\Rightarrow$ $\exists$ a finite collection of balls $\{B(x_i, \frac{r}{2})\}_{i=1}^n$ that cover K.

And each such $B(x_i, r/2) \subset O_i \in \mathcal{O}$.

$\Rightarrow$ $\{O_i\}_{i=1}^n$ is a finite subcover of $\mathcal{O}$. $\square$

sometimes referred to as the

Problem 1, LSIRA pg 71 (Heine-Borel theorem) Let $\mathcal{I}$ be a collection of open intervals in $\mathbb{R}$ s.t. $[0,1] \subseteq \bigcup_{I \in \mathcal{I}} I$. Show that there is a finite collection $\{I_i\}_{i=1}^{n}$ of intervals from $\mathcal{I}$ s.t. $[0,1] \subseteq \bigcup_{i=1}^{n} I_i$.

Proof $[0,1]$ is a closed and bounded set in $\mathbb{R}$.

$\Rightarrow$ it is compact $\Rightarrow$ it has the open cover property.

$\Rightarrow$ We can find a finite subset $\{I_i\}_{i=1}^{n}$, $I_i \in \mathcal{I}$ s.t. $[0,1] \subseteq \bigcup_{i=1}^{n} I_i$.

One more problem on open cover property and compactness.

Problem 4, LSIRA pg 71 Let $K_1, \dots, K_n$ be compact subsets of (X,d). Use the O.C.P to show that $\bigcup_{i=1}^{n} K_i$ is compact.

Need to show: Any open cover $\mathcal{O} = \{O_i\}$ of $\bigcup_{i=1}^{n} K_i$ has a finite subcover.

Each K_j, $j=1,\dots,n$, is compact. For any K_j, as $K_j \subset \bigcup_{i=1}^{n} K_i$, $\mathcal{O}$ is an open cover for K_j as well.

as an open cover of K_j

K_j is compact $\Rightarrow \exists$ finite subcover of $\mathcal{O}$, say, $K_j \subseteq \{O_r^j\}_{r=1}^{n_j}$ with $O_r^j \in \mathcal{O}$, and $n_j < \infty$, for $j=1,\dots,n$.

Now consider $\mathcal{F} = \bigcup_{j=1}^{n} \{O_r^j\}_{r=1}^{n_j}$, or $\mathcal{F} = \bigcup_{j=1}^{n} \bigcup_{r=1}^{n_j} O_r^j$.

$\mathcal{F}$ is a finite collection of open sets, and $K_j \subseteq \mathcal{F}$ $\forall j$.

open sets $\leq \sum_{j=1}^{n} n_j$.

$\Rightarrow \bigcup_{j=1}^{n} K_j \subseteq \mathcal{F}$.

$\Rightarrow \bigcup_{j=1}^{n} K_j$ has the O.C.P, and hence it is compact. $\square$

The result may not hold if we have an infinite collection $\{K_j\}_{j\in J}$, $|J|$ not finite, of compact sets.

What about (in)finite intersections of compact sets?

4. Spaces of Continuous Functions

Chapter 4 in LSIRA

We generalize notions of continuity and convergence now to spaces of functions — where elements of the space are functions.

LSIRA 4.1 Modes of Continuity

we first generalize notions of continuity of functions

Recall Definition of continuity: $f: X \to Y$ is continuous at $a \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x,a) < \delta$.

$\delta \to \delta(\epsilon, a)$

Q. Can we use the same δ for all $a \in X$?

This is the first generalization we consider...

metric spaces

Def 4.1.1 A function $f: X \to Y$ is **uniformly continuous** if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in X$ with $d_X(x,y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

$\delta \to \delta(\epsilon) \to$ independent of $x, y \in X$ (x is used in place of a)

Example 1 Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = 3x + 2$. Then f is uniformly continuous.

$\forall \epsilon > 0$, we can choose $\delta = \frac{\epsilon}{3}$, as $\forall x, y \in \mathbb{R}$ with $|x - y| < \frac{\epsilon}{3}$ we get $|f(x) - f(y)| = |3x - 3y| = 3|x - y| < 3 \cdot \frac{\epsilon}{3} = \epsilon$.

Note that the choice of $\delta \left(= \frac{\epsilon}{3}\right)$, while independent of $x, y \in X$, depends on the function. If $f(x)$ were $5x + 2$, we would have chosen $\delta = \frac{\epsilon}{5}$.

A function that is continuous at all points but not uniformly continuous is called **pointwise continuous**.

Example 2 $X = (0,2) \subset \mathbb{R}$, $f(x): X \to \mathbb{R}$ is $f(x) = x^2$.

Show $f(x)$ is uniformly continuous over X.

$\forall \epsilon > 0$, let $\delta = \frac{\epsilon}{4}$.

$\forall x, y \in (0,2)$ and $|x-y| < \delta$, we get

as $x, y \in (0,2)$

$$|f(x) - f(y)| = |x^2 - y^2| = |(x+y)(x-y)| < (2+2)|x-y| < 4 \cdot \frac{\epsilon}{4} = \epsilon.$$

Note that we get the same result for $X = [0,2]$ here.

Problem 1, LSIRA pg 80 Show $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$.

Proof by contradiction Assume f is uniformly continuous. Thus, for any $\epsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \epsilon$ whenever $|x-y| < \delta$. Then find an x, y and a δ (for a given ϵ) that violates this condition (to get a contradiction).

Goal: Start with x, y s.t. $|x-y| < \delta$ and "solve" $|f(x) - f(y)| \geq \epsilon$, and use $|f(x) - f(y)| = |x^2 - y^2| = |(x+y)(x-y)|$. hint given in LSIRA

let $y = x + \frac{\delta}{2} \Rightarrow |x-y| = |-\delta/2| < \delta$.

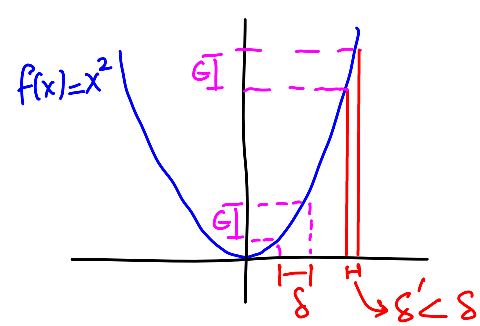
$$\Rightarrow |f(x) - f(y)| = |x^2 - y^2| = |(x+y)(x-y)| = \left|2x + \frac{\delta}{2}\right| \left|\frac{\delta}{2}\right|$$

$$= \left|x\delta + \frac{\delta^2}{4}\right| \geq \epsilon \quad \text{want!}$$

Can choose $x = \frac{\epsilon}{\delta} \Rightarrow \left|x\delta + \frac{\delta^2}{4}\right| = \left|\frac{\epsilon}{\delta}\delta + \frac{\delta^2}{4}\right| = \epsilon + \underbrace{\frac{\delta^2}{4}}_{>0} > \epsilon$.

Contradiction!

From the graph of $f(x) = x^2$, we can see that we have to choose smaller and smaller δ's to get the same ϵ bound as we go higher.



How are compactness and uniform continuity related?

Proposition 4.1.2 Let X, Y be metric spaces. If X is compact, then all continuous functions $f: X \to Y$ are uniformly continuous.

See LSIRA for proof.