MATH 401 : Lecture 20 (10/23/2025)

Today: * compact v/s totally bounded
* open cover property (OCP)

Recall Prop 3.5.12 : compact $\Rightarrow$ totally bounded.

What about the converse? Does total boundedness with some extra structure imply compactness?

Recall Corollary 3.5.5: Closed and bounded $\Longleftrightarrow$ compact in $\mathbb{R}$, but equivalence does not hold for all metric spaces.

need completeness also, with closed & totally bounded.

Theorem 3.5.13 A subset $A \subseteq (X,d)$ of a complete metric space (X,d) is compact iff A is closed and totally bounded.

See LSIRA for proof.

What is the relation between total boundedness and boundedness?

LSIRA Problem 9, Pg 68 Show that a totally bounded subset of (X,d) is always bounded. Find a bounded set in some (X,d) that is not totally bounded.

Let $A \subseteq (X,d)$ be totally bounded.

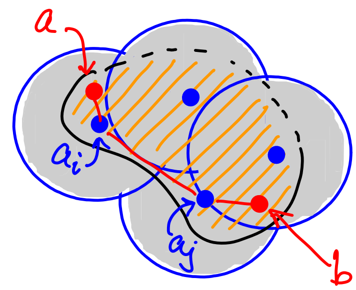
$\Rightarrow$ For $\epsilon = 1$, say, $\exists$ points $a_1, \dots, a_n \in A$ s.t. any finite ϵ will do.

$$\bigcup_{i=1}^{n} B(a_i, 1) \supseteq A.$$

Want to show

$\exists M$ s.t. $d(a,b) \le M \;\forall\, a,b \in A$.

WLOG, let $a \in B(a_i,1)$ and $b \in B(a_j,1)$, $i \neq j$.



$$d(a,b) \le d(a,a_i) + d(a_i,a_j) + d(a_j,b) \qquad (\triangle\text{le ineq})$$

$$< 1 + \underbrace{\max_{1\le k<l\le n} d(a_k,a_l)}_{\text{finite}} + 1 = M \text{ works!}$$

$d(a_i,a_j) \le \max\limits_{\substack{1\le k,l\le n \\ k\neq l}} \{d(a_k,a_l)\}$ is finite, as it is the largest of $\binom{n}{2}$ pairwise distances (of centers).

Take any infinite set A in (X,d) where d is the discrete metric.

$d(a,b) \le 1 \;\forall\, a,b \in A \Rightarrow A$ is bounded.

A cannot be totally bounded since for $0<\epsilon<1$, $B(a_i,\epsilon) = \{a_i\}$, so we need infinitely many a_i to have ϵ-balls that cover A.

the only values for d are 0 and 1, and $d(a_i,x)=1$ whenever $x \neq a_i$.

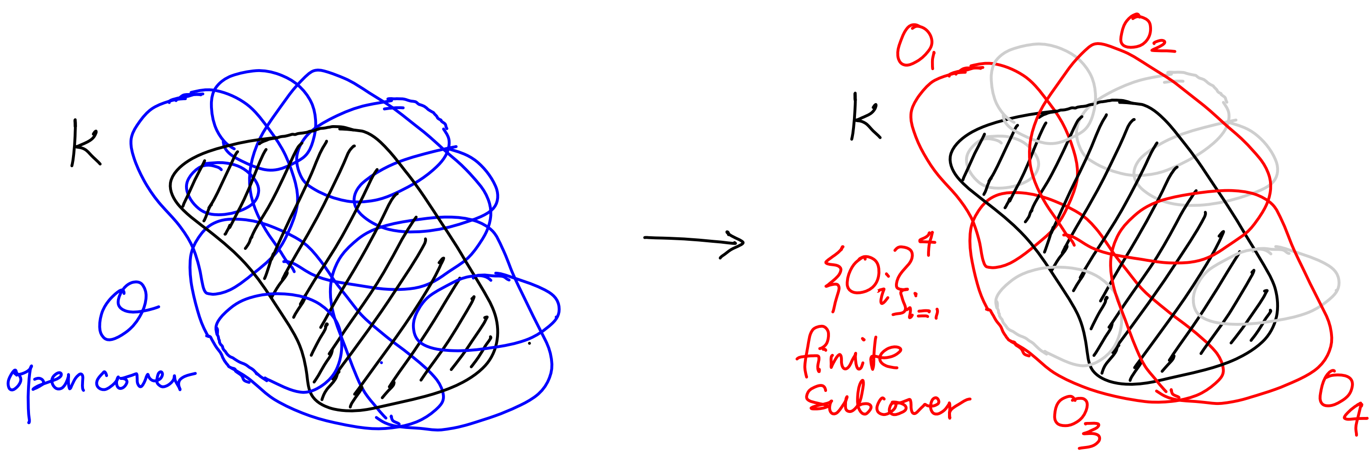
LSIRA 3.6 Compactness using Finite Covers

Recall how we extended the notion of continuous functions mapping open balls to (subsets of) open balls to that of mapping open sets to open sets. We can d ine the concept of total boundedness (with finite # centers a_i) more generally for open sets.

Def (open cover) An **open cover** of $K \subseteq (X,d)$ is a collection $\mathcal{O}$ (finite or infinite) of open sets, i.e., $\mathcal{O} = \{O_i\}$ s.t. $K \subseteq \bigcup_{O_i \in \mathcal{O}} O_i$.

Def (open cover property) Let $K \subseteq (X,d)$. If for every open cover of K $\mathcal{O} = \{O_i\}$, $\exists$ a finite # elements $O_1, O_2, \ldots, O_n$, $O_i \in \mathcal{O}$ s.t. $K \subseteq \bigcup_{i=1}^{n} O_i$, then K has the **open cover property**.

In words, every open cover has a finite subcover.



Note: Each '$\bigcirc$' is supposed to be an open set, even though it's draw with a solid line as opposed to a dashed one...

We get one direction of the equivalence between OCP and compactness readily.

Proposition 3.6.2 If $K \subseteq (X,d)$ has the O.C.P., then it is compact.

Proof Show K not compact $\Rightarrow$ K does not have OCP. → contrapositive argument

K not compact $\Rightarrow \exists \{x_n\}$ without a convergent subsequence in K.

$\Rightarrow \exists\, x \in K$ and $B(x, \overset{>0}{r_x})$ that contains only finitely many terms of $\{x_n\}$. → as no subseq. converges to x

Note that $\mathcal{O} = \{B(x,r_x)\}_{x\in K}$ is an open cover of K.

But $\mathcal{O}$ cannot have a finite subcover, as any $\{B(x_i, r_{x_i})\}_{i=1}^{n}$ for $n<\infty$ (finite subcollection of the balls) can have only finitely many terms of $\{x_n\}$. So K does not have the OCP. □

→ each ball has only finitely many terms.

What about the converse result? We need a lemma first.

Lemma 3.6.3 Let $\mathcal{O}$ be an open cover of $A \subseteq (X,d)$. Let $f: A \to \mathbb{R}$ be defined as

$$f(x) = \sup\{r\in\mathbb{R} \mid r<1 \text{ and } B(x,r)\subseteq O \text{ for some } O\in\mathcal{O}\}.$$

Then f is continuous and is strictly positive.

→ upper bound on the radius of an open ball at x that sits entirely inside a single cover element O of $\mathcal{O}$.

Proof (Strictly positive) $\mathcal{O}$ is an open cover of A. → follows from definition!

$\Rightarrow \exists\, O \in \mathcal{O}$ s.t. $x \in O$ for any $x \in A$.

O is open (by definition) $\Rightarrow \exists\, r>0$ s.t. $B(x;r) \subseteq O$.

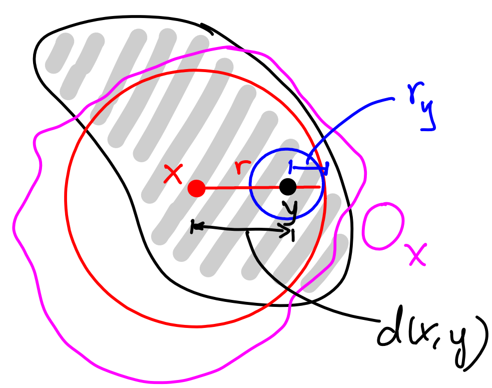
Can also take $r<1$ here. → helps to keep $f(\cdot)$ bounded.

Why continuity? We want to use EVT to argue that $f(\cdot)$ has a minimum!

(Continuity) We show $|f(x)-f(y)| \le d(x,y)$ by choosing $\delta=\epsilon$ in the definition of continuity

If $f(x), f(y) \le d(x,y)$, the result follows directly.

So assume, WLOG, $f(x) > d(x,y)$, and $f(y) < d(x,y)$.



$\Rightarrow \exists\, r > d(x,y)$ s.t $B(x,r) \subseteq O_x$.

$\Rightarrow$ with $r_y = r - d(x,y)$, we have

$B(y,r_y) \subseteq O_x$ as $B(y,r_y) \subseteq B(x,r_x)$.

$\Rightarrow f(y) \ge r_y = r - d(x,y)$

Since this inequality holds for all such r, it holds for its supremum as well, and hence we get

$$f(y) \ge f(x) - d(x,y).$$

We assumed $f(x)$ is larger $\Rightarrow$

$f(x) - f(y) \le d(x,y) \implies |f(x)-f(y)| \le d(x,y)$ as desired.

We can consider $f(y) > d(x,y) \ge f(x)$, or $f(x), f(y) \ge d(x,y)$ in a similar fashion.

$\square$

We are now ready to present the main theorem, which specifies the equivalence of compactness and O.C.P. This theorem is called the Heine-Borel theorem, but some other books/authors refer to the corresponding result in $\mathbb{R}$ (or $\mathbb{R}^m$) as the Heine-Borel theorem. See Problem 1 in the next page.

Theorem 3.6.4 $K \subseteq (X,d)$ is compact iff it has the O.C.P.

We will present the proof in the next lecture...