MATH401: Lecture 1 (08/19/2025)

This is Introduction to Analysis I
I'm Bala Krishnamoorthy (Call me Bala).

Today: * syllabus, logistics → see the course web page for details
* proof techniques
  - contrapositive proof
  - proof by contradiction
  - proof by induction

Book: Lindstrøm: Spaces–An Intro to Real Analysis (LSIRA)

LSIRA 1.1

Logical statements and notation.

If A then B (or $A \Rightarrow B$) "implies"

$A \Rightarrow B$ typically does not mean $B \Rightarrow A$.

e.g., A: p a natural number, is divisible by 6

B: p is divisible by 3.

$A \Rightarrow B$ holds, but $B \not\Rightarrow A$ (B does not imply A),

e.g., $p=9$.

But if $A \Rightarrow B$ and $B \Rightarrow A$ hold, we say A $\underbrace{\text{if and only if}}_{\text{iff}}$ B, or

$A \Leftrightarrow B$ (or A is equivalent to B).

To prove $A \Leftrightarrow B$, we often prove $A \Rightarrow B$ and $B \Rightarrow A$ ($A \Leftarrow B$) separately.

We start by reviewing certain standard techniques to construct proofs of mathematical statements.

1. Contrapositive Proof

To show $A \Rightarrow B$, equivalently show not $B \Rightarrow$ not A $(\neg B \Rightarrow \neg A)$.

"negation" or "not"

"If A happened then B happened"
This statement is equivalent to
"If B did not happen then A did not happen."

LSIRA 1.1 Prob 3. Prove the following Lemma.

Lemma 1 If n is a natural number such that n^2 is divisible by 3, then n is divisible by 3.

This is $A \Rightarrow B$ where A: $3 \mid n^2$ (n^2 is divisible by 3). or 3 divides n^2

B: $3 \mid n$ (n is divisible by 3).

Let's try to prove $A \Rightarrow B$ directly: $n^2 = 3k$ (as $n^2 \mid 3$) $\Rightarrow n = \sqrt{3}\sqrt{k}$ (taking square root on both sides)

Hard to conclude that $n \mid 3$ ☹! Would have to argue $k \mid 3$, which is not obvious!

Let's try proving $\neg B \Rightarrow \neg A$.

$\neg B$: n is not divisible by 3.

$\Rightarrow n = 3p+1$ or $n = 3q+2$, for $p, q \in \mathbb{N}$. (set of natural numbers)

Case 1. $n = 3p+1$

$$\Rightarrow n^2 = (3p+1)^2$$
$$= 9p^2 + 6p + 1$$
$$= 3(3p^2+2p)+1$$
$$= 3k+1 \text{ for } k = 3p^2+2p$$

$\Rightarrow n^2$ is not divisible by 3

Case 2. $n = 3q+2$

$$\Rightarrow n^2 = (3q+2)^2$$
$$= 9q^2 + 12q + 4$$
$$= 9q^2 + 12q + 3 + 1$$
$$= 3\underbrace{(3q^2+4q+1)}_{=k'}+1$$
$$= 3k'+1$$

$\Rightarrow n^2$ is not divisible by 3.

Hence we have proved that if n is not divisible by 3, then n^2 is not divisible by 3. Hence, by the contrapositive, we have $n^2 \mid 3 \Rightarrow n \mid 3$. $\square$

Should we always try to build a contrapositive proof?
Not necessarily! In cases where $A \Rightarrow B$ could be concluded directly, the contrapositive argument might make life harder!
It is one of the different proof approaches that you should be aware of.

2. Proof by Contradiction

Assume opposite of what you want to prove, and end up with a contradiction (or an obviously wrong statement). Hence the original assumption must be wrong, i.e., you have proved the statement.

LSIRA 1.1 Prob 3 (continued) Prove the following Theorem.

Theorem 2 $\sqrt{3}$ is irrational.

Assume $\sqrt{3}$ is rational. → the opposite of what you want to prove

$\Rightarrow \sqrt{3} = \frac{p}{q}$, $p, q \in \mathbb{N}$ with no common factors. → by definition, any positive rational number can be written in the form p/q, as specified.

Let's square both sides, and cross multiply.

$\Rightarrow 3q^2 = p^2 \Rightarrow 3 \mid p^2$ (p^2 is divisible by 3).

Hence by Lemma 1, $3 \mid p$. Let $p = 3k$. ($k \in \mathbb{N}$). Plug $p = 3k$ back in:

$\Rightarrow 3q^2 = (3k)^2 = 9k^2$ (divide both sides by 3)

$\Rightarrow q^2 = 3k^2$, i.e., $3 \mid q^2$ (q^2 is divisible by 3).

Again by Lemma 1, $3 \mid q$.

Since we started with the assumption that p and q have no common factors

Thus p and q have a common factor of 3, which is a contradiction.

Hence $\sqrt{3}$ is irrational. $\square$

3. Proof by Induction

To show a statement $P(n)$ holds for all $n \in \mathbb{N}$,

1. show $P(1)$ holds;
2. Assume $P(k)$ holds for some $k \in \mathbb{N}$.
3. Show $P(k+1)$ holds under Assumption 2.

"for all"

Example

Show that $P(n) = 3+5+\cdots+2n+1 = n(n+2)\ \forall n \in \mathbb{N}$.

1. $P(1) = 3 = 1(1+2)$ (so $P(1)$ is true).
2. Assume $P(k) = k(k+2)$ for some $k \in \mathbb{N}$.
3. $P(k+1) = P(k) + 2(k+1)+1 = P(k) + 2k+3$

$= k(k+2) + 2k+3$ by induction assumption.

$= k(k+2)+k+ k+3$

$= k(k+3) + k+3$

$= (k+1)(k+3) = n(n+2)$ for $n = k+1$.

$\Rightarrow P(n) = n(n+2)\ \forall n \in \mathbb{N}$. $\square$

MATH 401: Lecture 2 (08/21/2025)

Today: * sets and operations

Sets and Operations (LS IRA 1.2)

Set: Collection of mathematical objects.

They can be finite, e.g., $\{2,5,9,1,6\}$, or infinite, e.g., $[0,1]$, the collection of all $x \in \mathbb{R}$ with $0 \leq x \leq 1$.

"element of"   set of all real numbers

Given sets A, B we have

$A \subseteq B$: A is a subset of, or equal to, B.

$A \subset B$: A is a strict subset of B, i.e., there is at least one $x \in B$ such that $x \notin A$.

But $\forall x \in A$, $x \in B$ holds.

To prove $A=B$, we often prove $A \subseteq B$ and $A \supseteq B$ (or $B \subseteq A$).

Here are ~~some~~ standard sets we will use regularly.

ϕ: empty set.

$\mathbb{N} = \{1,2,3,\dots\}$, set of all natural numbers

$\mathbb{R}$ = set of all real numbers

$\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$, set of all integers

$\mathbb{Q}$ = set of rational numbers, $\mathbb{C}$ = set of complex numbers.

$\mathbb{R}^n$: set of all real n-tuples, or n-vectors

Notation for sets: $[-2,1] = \{x \in \mathbb{R} \mid -2 \leq x \leq 1\}$.

closed interval from -2 to 1

"such that"
could also use ":" instead of "|".

More generally, $A = \{a \in B \mid P(a)\}$.

bigger set than A   property

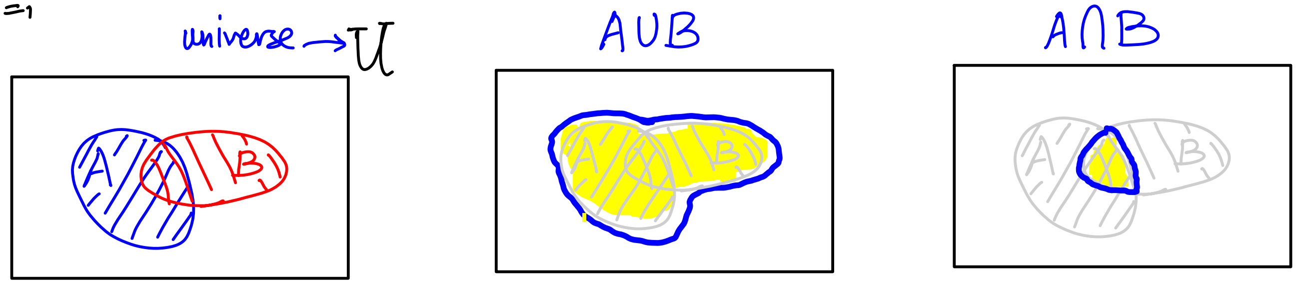
Union and Intersection

If A_i are sets for $i=1,\dots,n$, i.e., $A_1, A_2, \dots, A_n$ are sets, then

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{a \mid a \in A_i \text{ for } \underline{\text{at least one}}\ i\}$ is their union,

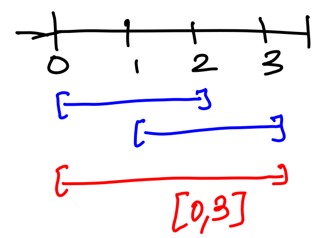
"for all"

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{a \mid a \in A_i\ \forall i\}$ is their intersection.



LSIRA 1.2 Prob1

Show $[0,2] \cup [1,3] = [0,3]$.



We show $[0,2] \cup [1,3] \subseteq [0,3]$ and
$[0,2] \cup [1,3] \supseteq [0,3]$.

The result is an obvious one. But we go through the steps of a formal proof more for practice!

($\subseteq$) Let $x \in [0,2] \cup [1,3]$

$\Rightarrow x \in [0,2]$ or $x \in [1,3]$ (definition of $\cup$).

$x \in [0,2] \Rightarrow x \in [0,3]$ (as $[0,3]$ contains $[0,2]$)

$x \in [1,3] \Rightarrow x \in [0,3]$. In either case, $x \in [0,3]$.

Hence $[0,2] \cup [1,3] \subseteq [0,3]$.

($\supseteq$) Let $x \in [0,3]$. Hence $0 \le x \le 3$. Then we get that either $x \le 2$, and hence $x \in [0,2]$, or $x \in (2,3]$.

But if $x \in (2,3]$ then $x \in [1,3]$ (as $[1,3]$ includes $(2,3]$).

$\Rightarrow x \in [0,2] \cup [1,3]$.

Hence $[0,3] \subseteq [0,2] \cup [1,3]$.

$\square$

Distributive Laws of Union and Intersection

For all sets $B, A_1, \dots, A_n$, we have

LSIRA (1.2.1) $$B \cap (A_1 \cup A_2 \cup \dots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n).$$

Using more compact notation, we can write

$$B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (B \cap A_i)$$

Proof

We will prove

$$B \cap (A_1 \cup \dots \cup A_n) \subseteq (B \cap A_1) \cup \dots \cup (B \cap A_n), \text{ and}$$

$$B \cap (A_1 \cup \dots \cup A_n) \supseteq (B \cap A_1) \cup \dots \cup (B \cap A_n).$$

('$\subseteq$') Let $x \in B \cap (A_1 \cup \dots \cup A_n)$.

$\Rightarrow x \in B$ and $x \in (A_1 \cup \dots \cup A_n)$ (definition of $\cap$)

$\Rightarrow x \in B$ and $x \in A_i$ for at least one A_i. (defn. of $\cup$)

$\Rightarrow x \in B \cap A_i$ for at least one A_i.

$\Rightarrow x \in (B \cap A_1) \cup \dots \cup (B \cap A_n)$.

('$\supseteq$') Let $x \in (B \cap A_1) \cup \dots \cup (B \cap A_n)$.

$\Rightarrow x \in (B \cap A_i)$ for at least one A_i.

$\Rightarrow x \in B$ and $\underbrace{x \in A_i \text{ for at least one } A_i}$

$\Rightarrow x \in B$ and $x \in (A_1 \cup \dots \cup A_n)$

$\Rightarrow x \in B \cap (A_1 \cup \dots \cup A_n)$. $\square$

LSIRA (1.2.2) is assigned in Homework 1.

Set Difference and Complement

Caution! $\ast$ $A \setminus B \neq B \setminus A$!

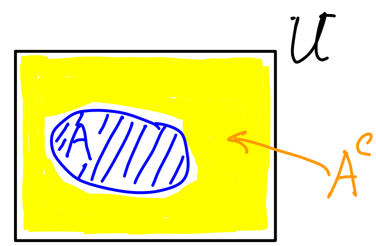
We write $A \setminus B$ or $A - B$ "setminus"

"A setminus B" is $A \setminus B = \{a \mid a \in A, a \notin B\}$.

If $\mathcal{U}$ is the universe, i.e., $A \subseteq \mathcal{U}$ for all sets A, then

$$A^c = \mathcal{U} \setminus A = \{a \in \mathcal{U} \mid a \notin A\}$$ is the

complement of A (or A-complement).



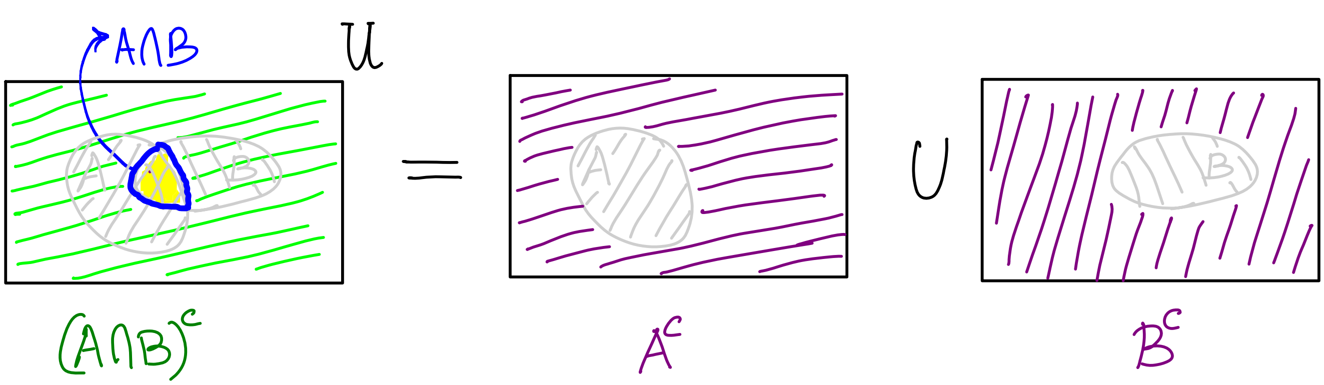
De Morgan's Laws

LSIRA (1.2.3) $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$. "complement of union = intersection of complements"

LSIRA (1.2.4) $(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c$. complement of intersection = union of complements.

See LSIRA for the proof.

Let's illustrate (1.2.4) for $n=2$, i.e., with A_1 and A_2 first.



We will prove subset inclusion in both directions.

($\subseteq$) Let $x \in (A_1 \cap \cdots \cap A_n)^c$.

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$ (definition of complement)

$\Rightarrow x \notin A_j$ for some j. (definition of $\cap$)

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \in A_1^c \cup \cdots \cup A_n^c$.

Hence $(A_1 \cap \cdots \cap A_n)^c \subseteq A_1^c \cup \cdots \cup A_n^c$.

($\supseteq$) Let $x \in A_1^c \cup \cdots \cup A_n^c$.

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \notin A_j$ for some j

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$. since $x \notin A_j$ for some j, it cannot be in the intersection of all A_i's.

$\Rightarrow x \in (A_1 \cap \cdots \cap A_n)^c$.

Hence $A_1^c \cup \cdots \cup A_n^c \subseteq (A_1 \cap \cdots \cap A_n)^c$. $\square$

Cartesian Products

For A, B: sets, we define

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

cartesian product of A and B

given A_i, $i = 1, \dots, n$ $(A_1, \dots, A_n)$, we define

$$A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \dots, a_n) \mid \underbrace{a_i \in A_i \ \forall i}_{a_1 \in A_1,\, a_2 \in A_2,\, \dots,\, a_n \in A_n}\}.$$

compact notation

$\prod$: product

e.g., $\mathbb{R}^n$: set of n-tuples of real numbers
(or set of real n-vectors)

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

We'll finish the proof in the next lecture...

MATH 401: Lecture 3 (08/26/2025)

Today: * families of sets, properties
* functions, images, pre images

We first do a problem on Cartesian products...

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

'$\subseteq$' Let $(x,y) \in (A \cup B) \times C$.

$\Rightarrow$ $x \in A \cup B$, $y \in C$ (Definition of cartesian product)

$\Rightarrow$ $x \in A$ or $x \in B$, $y \in C$

If $x \in A$ then $(x,y) \in A \times C$, and

if $x \in B$ then $(x,y) \in B \times C$.

$\Rightarrow$ $(x,y) \in A \times C$ or $(x,y) \in B \times C$

$\Rightarrow$ $(x,y) \in (A \times C) \cup (B \times C)$.

'$\supseteq$' Let $(x,y) \in (A \times C) \cup (B \times C)$

$\Rightarrow$ $(x,y) \in A \times C$ or $(x,y) \in B \times C$

$\Rightarrow$ $x \in A$, $y \in C$ or $x \in B$, $y \in C$ $\Rightarrow$ $(x \in A$ or $x \in B)$, $y \in C$.

$\Rightarrow$ $x \in A \cup B$, $y \in C$ $\Rightarrow$ $(x,y) \in (A \cup B) \times C$.

$\square$

LSIRA 1.3 Families of Sets

Recall: $B \cap \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} (B \cap A_i)$. → compact notation for distributive law (from Lecture 2)

We could write, instead, $B \cap \left(\bigcup_{i\in\mathcal{I}} A_i\right) = \bigcup_{i\in\mathcal{I}} (B \cap A_i)$, where $\mathcal{I} = \{1, 2, \dots, n\}$.

We now generalize $\mathcal{I}$ to be infinite in some cases, or indexing more general collections in general.

Def A collection of sets is a **family**.

e.g., $\mathcal{A} = \{[a,b] \mid a, b \in \mathbb{R}\}$ is the family of all closed intervals on $\mathbb{R}$.

Union and Intersection

We extend union, intersection, as well as their distribution to families.

$$\bigcup_{A\in\mathcal{A}} A = \{a \mid a \in A \text{ for some } A \in \mathcal{A}\}.$$

→ collection of elements that belong to at least one set in the family

$$\bigcap_{A\in\mathcal{A}} A = \{a \mid a \in A \text{ for all } A \in \mathcal{A}\}$$

→ collection of elements that belong to every set in the family.

We naturally extend distributive and De Morgan's laws to families..

$$B \cap \left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} (B \cap A), \quad \left(\bigcap_{A\in\mathcal{A}} A\right)^c = \bigcup_{A\in\mathcal{A}} A^c, \text{ etc.}$$

We now work on some problems involving families of sets.

LSIRA 1.3 Prob 1 (Pg 12)

Show that $\bigcup_{n\in\mathbb{N}} [-n,n] = \mathbb{R}$.

('$\subseteq$') $\mathbb{R}$ is the universe here, so $\bigcup_{n\in\mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

Or, observe that $[-n,n] \in \mathbb{R}$ for each $n \in \mathbb{N}$, hence $\bigcup_{n\in\mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

To be more careful, we could consider $x=0$ separately.
Note that $x=0 \in [-n,n]$ $\forall n \in \mathbb{N}$.

('$\supseteq$') Let $x \in \mathbb{R}$.

Let $m = \lceil |x| \rceil$, ceiling of absolute value of x, i.e., the smallest natural number $\geq |x|$.

$\lceil x \rceil = \text{ceil}(x)$
$=$ smallest integer $\geq x$.

Then $x \in [-m,m] = [-\lceil |x| \rceil, \lceil |x| \rceil]$, as

$x \leq |x| \leq \lceil |x| \rceil = m$, and $x \geq -|x| \geq -\lceil |x| \rceil$.

e.g., $x = -3.3 \Rightarrow x \geq -|-3.3| = -3.3$
≥ -4.

$\Rightarrow x \in \bigcup_{n\in\mathbb{N}} [-n,n]$.

□

LSIRA 1.3 Prob 4

Show $\bigcap_{n\in\mathbb{N}} (0, \frac{1}{n}] = \emptyset$ (empty set).

('$\supseteq$') $\emptyset \subseteq A$ for any set A (trivially).

('$\subseteq$') We show $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}] \subseteq \emptyset$.

$\emptyset^c = \mathbb{R}$. Hence we show $x \in \mathbb{R}$ is not in $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

For $x \in \mathbb{R}$, we show $x \notin \bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

If $x \leq 0$, then clearly, $x \notin (0,\frac{1}{n}]$ $\forall n \in \mathbb{N}$.

If $x \geq 1$, then $x \notin (0,\frac{1}{2}]$ for $n=2$, for instance.

Let $0<x<1$. Consider $m = \lceil \frac{1}{x} \rceil + 1$.

Then $x \notin (0, \frac{1}{m}]$ as $x > \frac{1}{m} = \frac{1}{\lceil \frac{1}{x} \rceil + 1}$. $\left(\lceil \frac{1}{x} \rceil + 1 > \frac{1}{x} \right)$

$$\Rightarrow \quad x \notin \bigcap_{n \in \mathbb{N}} (0, \tfrac{1}{n}].$$

Q. Why take $\lceil \frac{1}{x} \rceil + 1$?

Consider $x = \frac{1}{5}$, for instance.
Then $\lceil \frac{1}{x} \rceil = \lceil 5 \rceil = 5$.
Hence $x \in (0, \frac{1}{m}]$ here!

$\square$

LSIRA 1.3 Prob 5 (Pg 12)

Prove that $B \cup \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (B \cup A)$.

($\subseteq$) Let $x \in B \cup \left(\bigcap_{A \in \mathcal{A}} A \right)$

$\Rightarrow x \in B$ or $x \in \bigcap_{A \in \mathcal{A}} A$

$\Rightarrow x \in B$ or $x \in A$ for each $A \in \mathcal{A}$.

$\Rightarrow x \in B \cup A$ for each $A \in \mathcal{A}$.

$\Rightarrow x \in \bigcap_{A \in \mathcal{A}} (B \cup A)$.

('$\supseteq$') Let $x \in \bigcap_{A \in \mathcal{A}} (B \cup A)$

$\Rightarrow x \in B \cup A$ for every $A \in \mathcal{A}$.

$\Rightarrow x \in B$ or $\underbrace{x \in A \text{ for every } A \in \mathcal{A}}$.

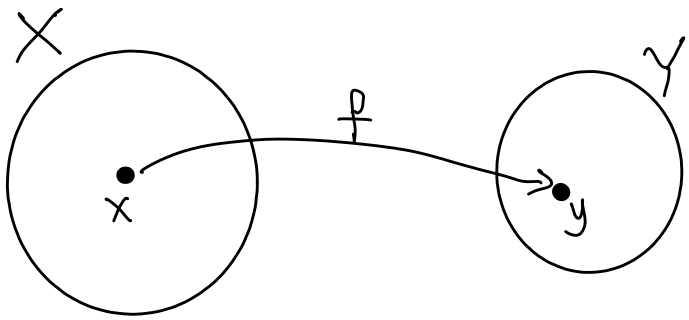
$\Rightarrow x \in B$ or $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in B \cup \left(\bigcap_{A \in \mathcal{A}} A \right)$.

$\square$

LSIRA 1.4 Functions

A function $f: X \to Y$ for sets X, Y is a rule that assigns for each $x \in X$ a **unique** $y \in Y$. We write $f(x) = y$, or

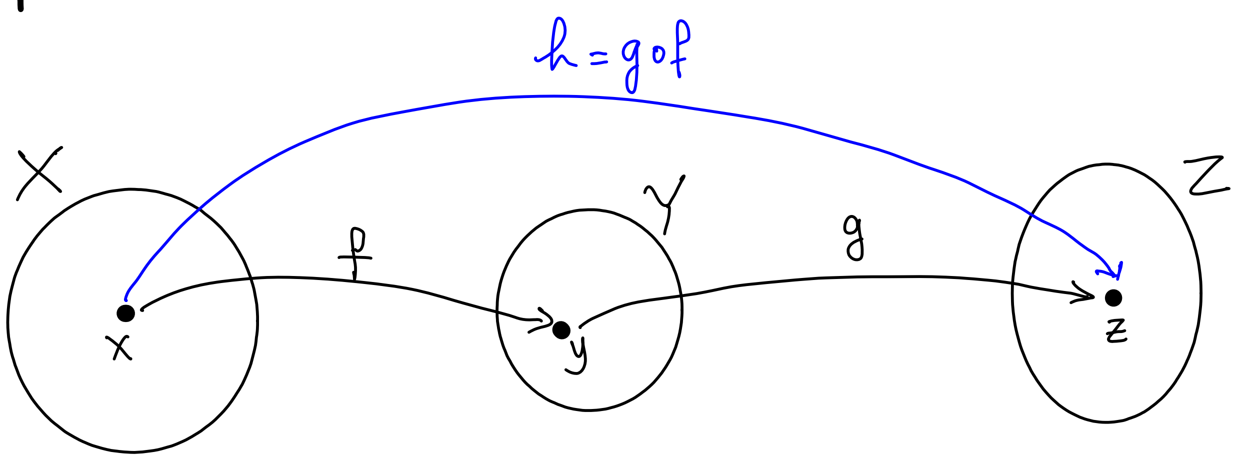
$x \mapsto y$ "maps to".



Rather than the graphs of functions you may have seen previously, we think of such visualizations for functions now.

X is the domain and Y the codomain of f.

Compositions

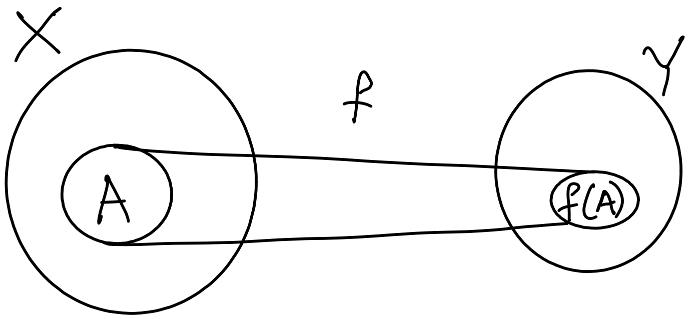


Let $f: X \to Y$ and $g: Y \to Z$ be functions. Then their composition is specified as $h: X \to Z$ defined as $h(x) = g(f(x))$.
The composition is written as $\underline{g \circ f}$, with $g \circ f(x) = g(f(x))$.

"composition of f and g"

$f_1(f_2(\cdots f_n(x)))\cdots)$ → composition of n functions $f_1, f_2, \ldots, f_n$

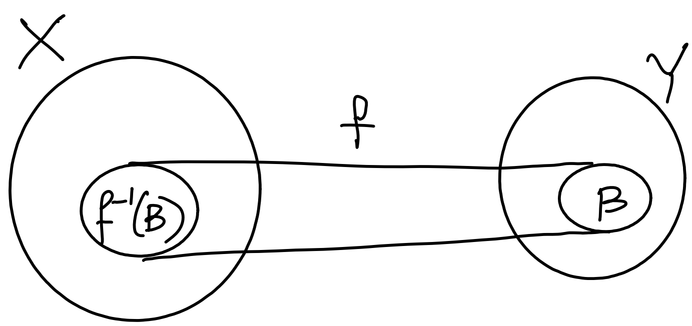
Function: $f: X \to Y$. We now define images and preimages under f.



For $A \subseteq X$, $f(A) \subseteq Y$ is defined as

$$f(A) = \{ f(a) \mid a \in A \},$$

and is called the **image** of A under f.



For $B \subseteq Y$, the set $f^{-1}(B) \subseteq X$ defined as

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}$$

is the **inverse image** or **preimage** of B under f.

In the next lecture, we consider how preimages and images commute with unions and intersections, or not...

MATH 401: Lecture 4 (08/28/2025)

Today: * images/preimages and unions/intersections
* injective/surjective functions
* relations, equivalence relations, partitions

We now consider how images and inverse images commute (or not) with unions and intersections of families of sets.

Prop 1.4.1 Inverse images commute with arbitrary unions and intersections:

$$f^{-1}\Big(\bigcup_{B\in\mathcal{B}} B\Big) = \bigcup_{B\in\mathcal{B}} f^{-1}(B) \quad \text{and}$$

"inverse of union = union of inverses"

$$f^{-1}\Big(\bigcap_{B\in\mathcal{B}} B\Big) = \bigcap_{B\in\mathcal{B}} f^{-1}(B)$$

"inverse of intersection = intersection of inverses"

Proof (of the second statement) $\longrightarrow$ See LSIRA for proof of first statement

($\subseteq$) Let $x \in f^{-1}\big(\bigcap_{B\in\mathcal{B}} B\big) \Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B$

$\Rightarrow f(x) \in B$ for every $B \in \mathcal{B}$.

$\Rightarrow x \in f^{-1}(B)$ for every $B \in \mathcal{B}$.

$\Rightarrow x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$.

($\supseteq$) Let $x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$

$\Rightarrow x \in f^{-1}(B)$ for every $B \in \mathcal{B}$.

$\Rightarrow f(x) \in B$ for every $B \in \mathcal{B}$.

$\Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B \Rightarrow x \in f^{-1}\big(\bigcap_{B\in\mathcal{B}} B\big)$. $\square$

We saw that inverse images commute with unions and intersections. But forward images behave a bit differently.

Prop 1.4.2 $f : X \to Y$ is a function, $\mathcal{A}$ is a family of subsets of X.

Then $f\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} f(A)$, $f\left(\bigcap_{A \in \mathcal{A}} A\right) \subseteq \bigcap_{A \in \mathcal{A}} f(A)$.

Proof

('$\subseteq$') Let $y \in f\left(\bigcup_{A \in \mathcal{A}} A\right)$

"There exists"

$\Rightarrow \exists\, x \in \bigcup_{A \in \mathcal{A}} A$ such that $f(x) = y$.

$\Rightarrow x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x) = y$

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow y \in \bigcup_{A \in \mathcal{A}} f(A)$.

($\supseteq$) Let $y \in \bigcup_{A \in \mathcal{A}} f(A)$.

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow \exists\, x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x) = y$.

$\Rightarrow \exists x \in \bigcup_{A \in \mathcal{A}} A$ such that $f(x) = y$.

$\Rightarrow y \in f\left(\bigcup_{A \in \mathcal{A}} A\right)$

LSIRA gives a slightly different proof for ($\supseteq$):

"for all"

$A \subseteq \bigcup_{A \in \mathcal{A}} A \quad \forall A \in \mathcal{A}$

$\Rightarrow f(A) \subseteq f\left(\bigcup_{A \in \mathcal{A}} A\right) \quad \forall A \in \mathcal{A}$

Since this result holds for every $A \in \mathcal{A}$, we can write

$\bigcup f(A) \subseteq f\left(\bigcup_{A \in \mathcal{A}} A\right)$.

$\square$

We consider intersections now: $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

Proof for ('$\subseteq$')

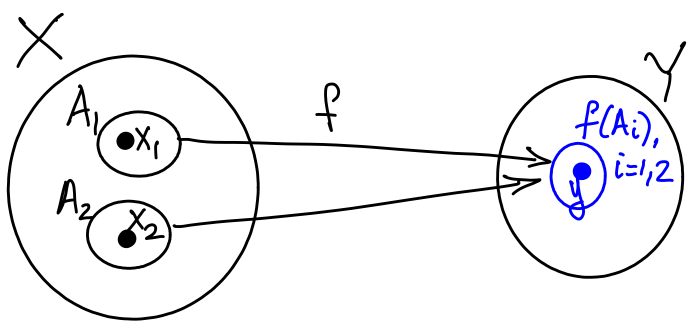
$$\bigcap_{A\in\mathcal{A}} A \subseteq A \quad \forall A\in\mathcal{A}$$

$$\Rightarrow f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq f(A) \ \forall A\in\mathcal{A}.$$

Since this inclusion holds for every $A\in\mathcal{A}$, we get

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A).$$

Counterexample for ('$\supseteq$') for $\cap$



For $x_1 \neq x_2$, $x_1, x_2 \in X$, let

$f(x_i) = y$, $i=1,2$.

Let $A_i = \{x_i\}$, $i=1,2$. $\Rightarrow \bigcap_{i=1,2} A_i = \phi$ (empty set).

But note that $f(A_i) = \{y\}$, $i=1,2$.

$\Rightarrow f\left(\bigcap_{i=1,2} A_i\right) = \phi$. But $\bigcap_{i=1,2} f(A_i) = \{y\} \neq \phi$.

$$\Longrightarrow \bigcap_{i=1,2} f(A_i) \not\subseteq f\left(\bigcap_{i=1,2} A_i\right).$$

But we get this reverse inclusion if we specify that f is injective.

Def Let $f: X \to Y$ be a function.

f is **injective** (1-to-1) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Equivalent definition: For any $y \in Y$, there is **at most** one $x \in X$ s.t. $f(x) = y$.
→ there could be no $x \in X$

f is **surjective** (onto) if for every $y \in Y$, there is **at least** one $x \in X$ such that $f(x) = y$.
→ there could be more than one

f is **bijective** if it is both injective and surjective.

LSIRA 1.4 prob 4 (pg 17)

Let $f: \underset{X}{\mathbb{R}} \to \underset{Y}{\mathbb{R}}$ be a strictly increasing function, i.e.,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for } x_i \in \mathbb{R},\ i = 1, 2.$$

1. Show that f is injective.
2. Does it have to be surjective? → Either give a proof or a counterexample.

1. We show $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
Without loss of generality (WLOG), let $x_1 < x_2$. → The same result holds when $x_2 < x_1$ as well.
Then $f(x_1) < f(x_2)$, as f is strictly increasing.
Hence $f(x_1) \neq f(x_2)$, and so f is injective.

2. No. $f = \arctan(x)$ is strictly increasing.
$f: \mathbb{R} \to \mathbb{R}$, but $\arctan(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}$.
So f need not be surjective.
Another example is $f = e^x$.

Relations (LSIRA 1.5)

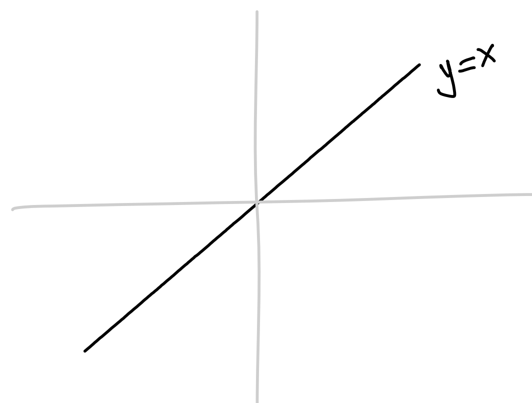
We had seen functions, where a unique $y \in Y$ is assigned for each $x \in X$ by $f: X \to Y$. But entities are related in other ways — numbers are $>$ or $<$ each other, lines are parallel, etc. We define relations formally to describe such dependencies.

Def A relation R on a set X is a subset of $\underbrace{X \times X}$.

We write xRy, $(x,y) \in R$, or $x \sim y$.

Cartesian product of X with itself

e.g., $R = \{ (x,y) \in \mathbb{R}^2 \mid x = y \}$.



Recall, $y = x$ is the 45° line through $(0,0)$. All points are "related" by them belonging to this line.

Here is another relation (on integers):

$$P = \{ (x,y) \in \mathbb{Z}^2 \mid x, y \text{ have same parity} \}.$$

So, all odd integers are related, and so are all even integers.

Some relations have more structure than default – as defined below.

Equivalence Relations

Def A relation $\sim$ on X is an **equivalence relation** if it is

(i) reflexive, i.e., $x \sim x \;\; \forall x \in X$;

(ii) symmetric, i.e., $x \sim y \Rightarrow y \sim x \;\; \forall x, y \in X$; and

(iii) transitive, i.e., $x \sim y,\ y \sim z \Rightarrow x \sim z \;\; \forall x, y, z \in X$.

Note that $<$ is not reflexive, or symmetric; e.g., $5 \not< 5$, and $3 < 5 \not\Rightarrow 5 < 3$.

Def Given an equivalence relation $\sim$ on X, we define the **equivalence class** $[x]$ of $x \in X$ as

$$[x] = \{ y \in X \mid x \sim y \}.$$

→ the set of all "relatives" of x

The collection of equivalence classes forms a partition of X.

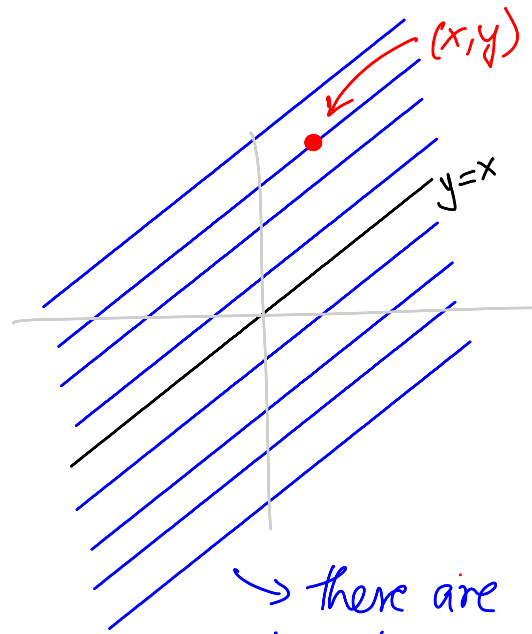
Def A **partition** $\mathcal{P}$ of X is a family of nonempty subsets of X such that $x \in X$ satisfies $x \in P \in \mathcal{P}$ for exactly one P in $\mathcal{P}$ (for every $x \in X$).

The elements P of $\mathcal{P}$ are called **partition classes** of $\mathcal{P}$.

e.g., $\mathcal{P} = \{ \underbrace{\{2k, k \in \mathbb{Z}\}}_{\text{even integers}}, \underbrace{\{2k+1, k \in \mathbb{Z}\}}_{\text{odd integers}} \}$ is a partition of $\mathbb{Z}$.

Here is a direct example of a partition of $\mathbb{R}^2$.

The collection of all lines with slope $= 1$ (45°) is a partition of $\mathbb{R}^2$.



Any point in $\mathbb{R}^2$ belongs to exactly one line with a slope of $m=1$ (i.e., 45° degree slope).

→ there are infinitely many lines with slope $m=1$.

We have not checked that the defining relation is an equivalence relation, but this can be done easily.

→ recall, the point-slope form of the equation of a line: $\frac{y-y_0}{x-x_0} = m$, given slope m and one point (x_0, y_0).

MATH401: Lecture 5 (09/02/2025)

Today: * equivalence relations and partitions
* countability

Recall: * $\sim$ is an equivalence relation on X: $x\sim x$, $x\sim y \Rightarrow y\sim x$, $x\sim y, y\sim z \Rightarrow x\sim z$
* partition of X $\mathcal{P}=\{P\}$

We show that equivalence relations naturally define partitions.

Prop 1.5.3 If $\sim$ is an equivalence relation on X, then the collection of equivalence classes $\mathcal{P}=\{[x] \mid x\in X\}$ is a partition of X.

Proof We show each $x\in X$ belongs to exactly one equivalence class

$x\sim x$ $\quad$ $\sim$ is equivalence relation, so is reflexive ((i))

$\Rightarrow x\in[x] \longrightarrow$ So, each $x\in X$ belongs to at least its own class.

We now show if $x\in[y]$ for $y\in X$, $y\neq x$, then $[x]=[y]$.

We show $[x]\subseteq[y]$ and $[x]\supseteq[y]$.

($\subseteq$) Let $z\in[x]$

$\Rightarrow x\sim z$ $\quad$ Definition of $[x]$

We assumed $x\in[y] \Rightarrow y\sim x$

$\sim$ is an equivalence relation, so $\underbrace{y\sim x, x\sim z \Rightarrow y\sim z}_{\sim \text{ is transitive ((iii))}}$.

$\Rightarrow z\in[y]$.

($\supseteq$) let $z\in[y] \Rightarrow y\sim z$

Also, $x\in[y]\Rightarrow y\sim x$

$\sim$ is equivalence relation $\Rightarrow x\sim y$ $\quad$ ($\sim$ is symmetric (ii))

$\Rightarrow x\sim y, y\sim z \Rightarrow x\sim z$ $\quad$ ($\sim$ is transitive (iii))

$\Rightarrow z\in[x]$. $\square$

LSIRA 1.5 Prob 5 (Pg 20) Let $\sim$ be a relation on $\mathbb{R}^3$ defined as

$$(x,y,z) \sim (x',y',z') \iff 3x-y+2z = 3x'-y'+2z'.$$

Show that $\sim$ is an equivalence relation. Describe its equivalence classes.

We check that $\sim$ is reflexive, symmetric, and transitive.

Reflexive: $(x,y,z) \sim (x,y,z)$, as $3x-y+2z = 3x-y+2z$. ✓

Symmetric: $(x,y,z) \sim (x',y',z') \Rightarrow (x',y',z') \sim (x,y,z)$ holds as

$3x-y+2z = 3x'-y'+2z' \Rightarrow$ $a=b \Rightarrow b=a$ for $a,b \in \mathbb{R}$.

$3x'-y'+2z' = 3x-y+2z$. ✓

Transitive: $(x,y,z) \sim (x',y',z')$ and $(x',y',z') \sim (x'',y'',z'')$

$\Rightarrow (x,y,z) \sim (x'',y'',z'')$ also holds, as

$3x-y+2z = 3x'-y'+2z'$ and $3x'-y'+2z' = 3x''-y''+2z''$

$\Rightarrow 3x-y+2z = 3x''-y''+2z''$. ✓

$$[(x,y,z)] = \{(x',y',z') \in \mathbb{R}^3 \mid 3x-y+2z = 3x'-y'+2z'\}$$

If we set $3x-y+2z = d \in \mathbb{R}$, then

$$[(x,y,z)] = \{\underline{(x',y',z') \in \mathbb{R}^3 \mid 3x'-y'+2z' = d}\}$$

plane with normal vector $(3,-1,2)$ (or $\begin{bmatrix}3\\-1\\2\end{bmatrix}$) through (x,y,z).

We can describe the equivalence classes as follows.
The equivalence class of a point in $\mathbb{R}^3$ is the plane with normal $(3,-1,2)$ passing through that point.

We write $\mathbb{R}^3/\sim$ for the set of all equivalence classes of $\sim$.

Def If $\sim$ is an equivalence relation on X, then $X/\sim$ is the set of all equivalence classes under $\sim$. "X quotient $\sim$"

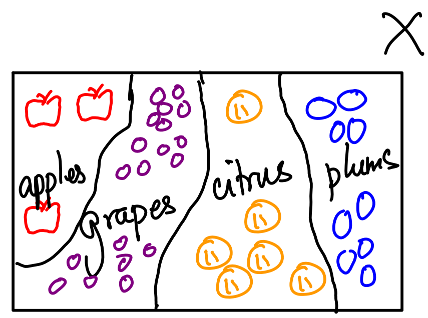
$\mathbb{R}^3/\sim$ here is the set of all planes with normal $(3,-1,2)$.

Note that any point $(x,y,z) \in \mathbb{R}^3$ belongs to exactly one plane with normal $(3,-1,2)$. Also, all such parallel planes together cover all of $\mathbb{R}^3$, i.e., $\mathbb{R}^3/\sim$ is indeed a partition of $\mathbb{R}^3$. Note the similarity to previous example of 45° lines in $\mathbb{R}^2$.

Another example on equivalence classes and Partitions

Let X be the set of all fruits in a grocery store. We can group them into fruit types (classes), e.g., apples, citrus, grapes, tomatoes, plums, etc. Note that apples could include honeycrisp, red delicious, etc. (varities of apples)

$\mathcal{P}$: A partition of X into fruit classes may look like this $\longrightarrow$



$\mathcal{P} = \{ \underbrace{\text{apples}}_{P_1}, \underbrace{\text{grapes}}_{P_2}, \underbrace{\text{citrus}}_{P_3}, \underbrace{\text{plums}}_{P_4}, \dots \}$

Note that any individual fruit belongs to exactly one class. $\mathcal{P}$ is indeed a partition of X.

Equivalence relation $\sim$ on X associated with $\mathcal{P}$

For fruits x, y, $x \sim y$ if x and y are the same fruit type.
$\sim$ is indeed an equivalence relation (can check it's reflexive, symmetric, transitive).

What is the equivalence class $[x]$ of a fruit x?

$[x]$ is the set of all fruits of its type in the store.
e.g., x = Valencia orange, $[x] = \{$set of all citrus fruits$\}$.

What is the quotient space $X/\sim$? $X/\sim$ is the set of all fruit types.

So $X/\sim = \{$apples, citrus, $\dots\}$

Check all problems on equivalence relations from LSIRA.

LSIRA 1.6 Countability

We typically count a set of objects as $1,2,3,\ldots$, i.e., by numbering or indexing the first element, then the second one, etc. We can talk about sets being countable (or not) in general.

Def A set A is **countable** if it is possible to list all elements of A as $a_1, a_2, \ldots, a_n, \ldots$

e.g., $\mathbb{N}$ is countable — just list the elements as $1,2,3,\ldots$.

$\mathbb{N}$ → set of natural numbers

We could use a little more formal definition of a countable set, than the one given above (as listed in LSIRA).

Def A set A is countable if there exists an injective function $f: A \to \mathbb{N}$.

The function f is the "indexing" or "numbering" function that assigns a separate natural number to each element of A.

Note that finite sets are always countable — we can always list the elements in a sequence. Things are more interesting for infinite sets.

Def If f is also surjective, i.e., it is bijective, then A is **countably infinite**, i.e., it is countable and is infinite.

e.g., $\mathbb{Z}$ is countable.

We can list all integers as

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

This is just one way to list all integers. Other ways could be devised as well.

	0	1	-1	2	-2	3	-3	$\ldots$
index	1		3		5		7	$\ldots$
		2		4		6		$\ldots$

Note how the indices are listed. The positive integers are the even entries in the list, and the negative integers ($\&$ 0) are the odd entries in the list

Or, we can define $f: \mathbb{Z} \to \mathbb{N}$ as

$$f(z) = \begin{cases} 2z, & z > 0 \\ 1-2z, & z \le 0 \end{cases}$$

We can specify $f^{-1}(\cdot)$ as follows:

$$f^{-1}(n) = \begin{cases} n/2, & n \text{ even} \\ \frac{-n+1}{2}, & n \text{ odd.} \end{cases}$$

f is bijective, and hence $\mathbb{Z}$ in countably infinite.

Proposition 1.6.1 If A, B are countable, then so is $A \times B$. → Cartesian product

A, B are countable $\Rightarrow \exists$ lists $\{a_n\}, \{b_n\}$ containing all elements of A and B, respectively.

$$\Rightarrow \{\underbrace{(a_1,b_1)}_{1+1}, \underbrace{(a_1,b_2)}_{\substack{1+2\\=3}}, \underbrace{(a_2,b_1)}_{\substack{2+1\\=3}}, \underbrace{(a_1,b_3)}_{\substack{1+3\\=4}}, \underbrace{(a_2,b_2)}_{\substack{2+2\\=4}}, \underbrace{(a_3,b_1)}_{\substack{3+1\\=4}}, \dots\}$$

index

is a list containing all elements of $A \times B$.

Note the index trick: we list pairs of elements (a_i, b_j) with $a_i \in \{a_n\}$ and $b_j \in \{b_n\}$ such that the sum of their indices increase as natural numbers. Thus, $i+j=2$, and then all options for $i+j=3$, followed by all options for $i+j=4$, and so on.

This index trick could be used to show other sets are countable, e.g., the cartesian product of k countable sets is countable. ($A_1 \times A_2 \times \dots \times A_k$, where A_i is countable for $1 \le i \le k$).

LSIRA 1.6 Prob 1 (Pg 22) Show that the subset of a countable set is countable.

Let $B \subset A$, where A is countable.

As A is countable, there is a list $a_1, a_2, \ldots, a_n, \ldots$ such that every $a_i \in A$ is included in the list.

Let $n_1 \in \mathbb{N}$ be the smallest natural number such that $a_{n_1} \in B$. from the list

And let $n_2 \in \mathbb{N}$, $n_2 > n_1$, be the smallest number such that $a_{n_2} \in B$, and let $n_3 > n_2$, $n_3 \in \mathbb{N}$, be the smallest number such that $a_{n_3} \in B$, and so on.

We form a new list with $b_i = a_{n_i}$, $i = 1, 2, 3, \ldots$

$\Rightarrow$ $b_1, b_2, b_3, \ldots$ is a listing of all elements in B, ensuring that B is countable. $\square$

indeed, we will miss no elements of B in this process, and all of them are included in the new list.

Check Prop 1.6.2: (in LSIRA) $\bigcup_{n \in \mathbb{N}} A_n$ is countable when A_n is countable $\forall n$.

We can use a similar indexing trick as in Prop. 1.6.1.

Countability is one way to compare two infinite sets. We know $\mathbb{R} \supseteq \mathbb{Q}$, but both have infinitely many elements. Intuitively, we know $\mathbb{R}$ is bigger as it contains irrational numbers in addition to rationals.

We'll first show that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is, in fact, uncountable. More in the next lecture...

MATH401: Lecture 6 (09/04/2025)

Today: * $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable
* ϵ-δ proofs, convergence

Recall: Proposition 1.6.1 If A, B are countable, then so is $A \times B$.

Proposition 1.6.3 $\mathbb{Q}$ is countable.
set of all rational numbers, $\frac{p}{q}$ for $p \in \mathbb{Z}, q \in \mathbb{N}$

This representation includes all negative rationals. Also, $q \in \mathbb{N}$ avoids $q=0$.

We first observe that $\mathbb{Z} \times \mathbb{N}$ is countable, as we showed that $\mathbb{Z}$ and $\mathbb{N}$ are both countable, and then applying Proposition 1.6.1.

$\Rightarrow \mathbb{Z} \times \mathbb{N}$ can be listed as, for instance,
$\{\{(a_1, b_i)\}_{i=1}^{\infty}, \{(a_2, b_i)\}_{i=1}^{\infty}, \dots, \{(a_k, b_i)\}_{i=1}^{\infty}, \dots\}$ where $\{a_n\}$ and $\{b_n\}$ are listings for $\mathbb{Z}$ and $\mathbb{N}$, respectively.

But $\left\{\left\{\frac{a_1}{b_i}\right\}_{i=1}^{\infty}, \left\{\frac{a_2}{b_i}\right\}_{i=1}^{\infty}, \dots, \left\{\frac{a_k}{b_i}\right\}_{i=1}^{\infty}, \dots\right\}$ is a listing of $\mathbb{Q}$. $\square$

Let's consider any rational number, e.g., $\frac{2}{5}$.
How many times does $\frac{2}{5}$ appear in this listing? Once, exactly as $\frac{2}{5}$.

But infinitely many times as a value, because $\frac{2}{5} = \frac{4}{10} = \frac{20}{50} = \dots$

In fact, every rational number appears infinitely many times in this list. But that is not a problem for countability.

We now show that the set of all reals is uncountable.

Theorem 1.6.4 $\mathbb{R}$ is uncountable.

Consider $[0,1] \subset \mathbb{R}$. We show that $[0,1]$ is uncountable.
To get a contradiction, assume that $[0,1]$ is countable.

As there are infinitely many real #s between 0 and 1.
$[0,1]$ is a countably infinite set (under assumption).

We can list all these real numbers as follows:

Note that each number has infinitely many decimal digits (they could be all zeros after some number of places)

$$\begin{aligned} r_1 &= 0.a_{11}a_{12}a_{13}\cdots \\ r_2 &= 0.a_{21}a_{22}a_{23}\cdots \\ r_3 &= 0.a_{31}a_{32}a_{33}\cdots \\ &\vdots \end{aligned}$$

a_{ij} = j^{th} decimal digit in the i^{th} real number (in the list).

$a_{ij} \in \{0,1,2,\dots,9\}$.

We create a new real number in $[0,1]$ as follows.

$$s = 0.d_1d_2d_3\cdots \quad \text{where}$$

$$d_i = \begin{cases} 1 & \text{if } a_{ii} \neq 1, \text{ and} \\ 2 & \text{if } a_{ii} = 1. \end{cases}$$

e.g.,
$$\begin{aligned} r_1 &= 0.02534\ldots \\ r_2 &= 0.8176\ldots \\ r_3 &= 0.30914\ldots \\ r_4 &= 0.00207\ldots \\ &\vdots \end{aligned}$$

Then $s = 0.1211\ldots$

Note that s has infinitely many decimal digits.

So, s is different from r_i for each i.
This contradicts the assumption that $\{r_i\}$ contains **every** real number in $[0,1]$. Hence $[0,1]$ is uncountable.

Since $\mathbb{R} \supset [0,1]$, and $[0,1]$ is uncountable,
$\mathbb{R}$ is also uncountable. $\square$

This is a standard trick we use to show a set is uncountable. We assume it is countable, and start with a listing. Then we identify an element that is distinct from every element in the listing, contradicting the assumption that the listing includes all such elements.

Corollary. The set of irrational numbers is uncountable.

We showed $\mathbb{Q}$ is countable, and $\mathbb{R}$ is uncountable.
The set of irrationals $= \mathbb{R}/\mathbb{Q}$ is hence uncountable.

2.1. ε-δ Definitions and Proofs

Norms and Distances

Euclidean distance, by default

Def The **distance** between $\bar{x} = (x_1, \dots, x_m)$ $\left(\text{or } \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right)$ and $\bar{y} = (y_1, \dots, y_m)$, two points in $\mathbb{R}^m$ is

$$\|\bar{x} - \bar{y}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_m-y_m)^2}.$$

My notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\theta}$, etc. are vectors → lower case letters with a bar.

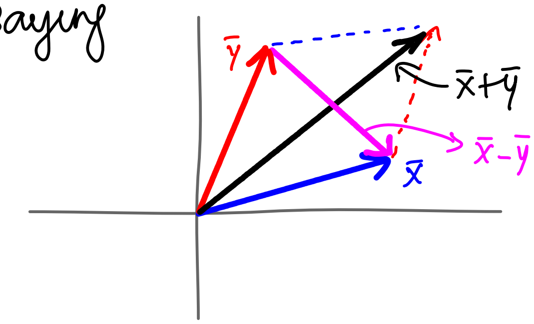
For $m=1$, $\|x-y\| = \sqrt{(x-y)^2} = |x-y|$ → absolute value of $x-y$

think of it as just the distance between two points in $\mathbb{R}$.

Triangle Inequality

$$\forall\, \bar{x}, \bar{y} \in \mathbb{R}^m, \quad \|\bar{x}+\bar{y}\| \le \|\bar{x}\| + \|\bar{y}\|.$$

We could interpret the triangle inequality as saying length of diagonal $\le$ sum of lengths of sides of the parallelogram.

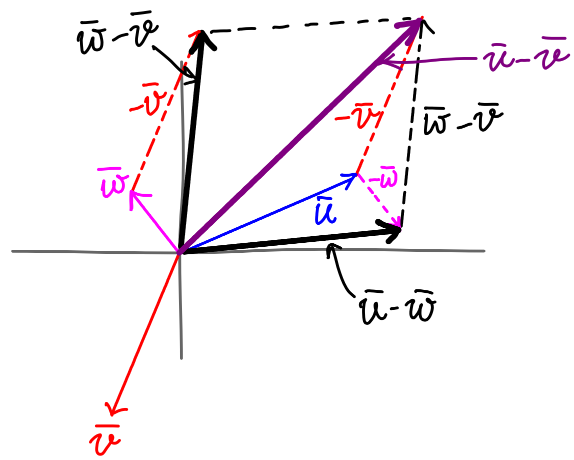


With $\bar{x} = \bar{u} - \bar{w}$, $\bar{y} = \bar{w} - \bar{v}$, we get

$$\|\bar{u}-\bar{v}\| = \|\bar{u} - \bar{w} + \bar{w} - \bar{v}\| \le \|\bar{u}-\bar{w}\| + \|\bar{w}-\bar{v}\|$$

for $\bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^m$

Illustration of the above version in 2D: notice the parallelogram here as well!

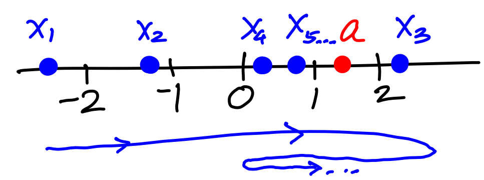


Convergence of Sequences

As a first use of distances, we consider convergence of sequences. How do we say a sequence $\{x_n\}$ converges to a real number a? We should be able to get arbitrarily close to a by going far enough (large n) into the sequence.

Def 2.1.1 A sequence $\{x_n\}$ of real numbers converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$ (no matter how small), there exists an $N \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} x_n = a$.

Here is a pictorial representation of the convergence, with the "path" drawn separately below for clarity.



LSIRA 2.1 Prob 1 (Pg 29)

Show that if $\{x_n\}$ converges to a, then the sequence $\{Mx_n\}$ converges to Ma. Use the definition of convergence to explain how you choose N.

Given $\{x_n\} \to a$ $\left(\lim_{n\to\infty} x_n = a\right)$ $\Rightarrow$ $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - a| < \epsilon \;\; \forall n \geq N$.

We want to show $\{Mx_n\} \to Ma$. We want to show that $\forall \epsilon > 0, \exists N' \in \mathbb{N}$ s.t. $|Mx_n - Ma| < \epsilon \;\; \forall n \geq N$.

Note that when $M=0$, the result holds trivially, as $Mx_n = 0 \;\forall n$, and $Ma = 0$. Hence $|Mx_n - Ma| = 0 < \epsilon$ for any $\epsilon > 0$ for $n \geq 1$.
N'

Also note that both $M > 0$ and $M < 0$ are possible.

Let's assume $M \neq 0$.

First, observe that $|Mx_n - Ma| = |M(x_n - a)| = |M||x_n - a|$.

Note that when $|x_n - a| < \epsilon' = \frac{\epsilon}{|M|}$, $|M||x_n - a| < \epsilon$.

But since $\{x_n\} \to a$, given $\epsilon' = \frac{\epsilon}{|M|} > 0$, $\exists\, N' \in \mathbb{N}$ s.t. $|x_n - a| < \epsilon'$ for all $n \geq N'$. We can choose $N = N'$, and get

$$|x_n - a| < \epsilon' = \frac{\epsilon}{|M|} \quad \forall n \geq N'$$

$$\Rightarrow |M||x_n - a| = \underbrace{|Mx_n - Ma| < \epsilon \quad \forall\, n \geq N'}$$

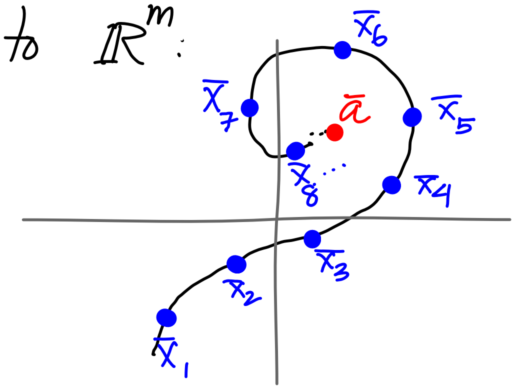
$\Rightarrow \{Mx_n\}$ converges to Ma. $\square$

MATH 401: Lecture 7 (09/09/2025)

Today: * convergence in $\mathbb{R}^m$
* continuity of functions

We extend the notion of convergence in $\mathbb{R}$ to $\mathbb{R}^m$:

The definition naturally extends to $\mathbb{R}^m$ once we think of $|x_n - a|$ as the distance between x_n and a.



Def 2.1.2 A sequence $\{\bar{x}_n\}$ of points in $\mathbb{R}^m$ converges to $\bar{a} \in \mathbb{R}^m$ if $\forall\, \epsilon > 0$, $\exists$ an $N \in \mathbb{N}$ such that $\|\bar{x}_n - \bar{a}\| < \epsilon \;\; \forall\, n \geqslant N$. We write $\lim_{n\to\infty} \bar{x}_n = \bar{a}$.

LSIRA Prob 2.1.3 $\{\bar{x}_n\}$, $\{\bar{y}_n\}$ are two sequences in $\mathbb{R}^m$ where $\{x_n\} \xrightarrow{\text{converges}} \bar{a}$, and $\{\bar{y}_n\} \to \bar{b}$. Then show that $\{\bar{x}_n + \bar{y}_n\}$ converges to $\bar{a} + \bar{b}$.

We want to show: $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that (any given ϵ)

$$\|(\bar{x}_n + \bar{y}_n) - (\bar{a}_n + \bar{b}_n)\| < \epsilon \quad \forall\, n \geqslant N.$$

same ϵ as our target

Hint, hint, hint! $\|\bar{x} + \bar{y} + \bar{z}\| \leq \|\bar{x}\| + \|\bar{y}\| + \|\bar{z}\|$ by applying triangle inequality twice. We often choose $\epsilon/3$ (instead of $\frac{\epsilon}{2}$) with 3 terms!

We are given $\{\bar{x}_n\} \to \bar{a}$, $\{\bar{y}_n\} \to \bar{b}$, so

$\exists\, N_1 \in \mathbb{N}$ s.t. $\|\bar{x}_n - \bar{a}\| < \frac{\epsilon}{2} \;\; \forall\, n \geqslant N_1$ and

$\exists\, N_2 \in \mathbb{N}$ s.t. $\|\bar{y}_n - \bar{b}\| < \frac{\epsilon}{2} \;\; \forall\, n \geqslant N_2$.

$\Rightarrow$ for $N = \max\{N_1, N_2\}$, we get

$$\begin{aligned}
\|(\bar{x}_n + \bar{y}_n) - (\bar{a} + \bar{b})\| &= \|(\bar{x}_n - \bar{a}) + (\bar{y}_n - \bar{b})\| \\
&\leq \|\bar{x}_n - \bar{a}\| + \|\bar{y}_n - \bar{b}\| && \text{by triangle inequality} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon && \text{as } N \geqslant N_1,\ N \geqslant N_2.
\end{aligned}$$

$\Rightarrow \{\bar{x}_n + \bar{y}_n\} \to \bar{a} + \bar{b}$. $\square$

Continuity

$f: \mathbb{R} \to \mathbb{R}$. When is f continuous at $x=a$?

For sequences $\{x_n\} \to a$, we go "far enough out", i.e., $\forall n \geq N \in \mathbb{N}$. Instead of $N \in \mathbb{N}$, here we say $\exists\, \delta > 0$ such that if $|x-a| < \delta$ then $|f(x)-f(a)| < \epsilon$ (for any given $\epsilon > 0$). In other words, $f(x)$ gets close enough to $f(a)$ whenever x is close enough to a!

Def 2.1.4 The function $f: \mathbb{R} \to \mathbb{R}$ is **continuous** at $a \in \mathbb{R}$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $|f(x)-f(a)| < \epsilon$ whenever $|x-a| < \delta$.

Equivalently, if $|x-a| < \delta$ then $|f(x)-f(a)| < \epsilon$.

We naturally extend the definition to $\mathbb{R}^m$ using distances/norms.

LSIRA uses $\mathbf{F}$ (bold upper case F)

Def 2.1.7 The function $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\bar{a} \in \mathbb{R}^n$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $\|\bar{f}(\bar{x})-\bar{f}(\bar{a})\| < \epsilon$ whenever $\|\bar{x}-\bar{a}\| < \delta$.

By restricting our attention to a subset A of $\mathbb{R}^n$, we naturally extend the above definition to subsets of interest.

Def 2.1.8 Let $A \subset \mathbb{R}^n$, and $\bar{a} \in A$.
The function $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\bar{a} \in A$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $\|\bar{f}(\bar{x})-\bar{f}(\bar{a})\| < \epsilon$ whenever $\|\bar{x}-\bar{a}\| < \delta$ and $\bar{x} \in A$.

LSIRA Section 2.1 Prob 4 (extension): If $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,3$ are all continuous at $a \in \mathbb{R}$, the show that so is $f_1 + f_2 - f_3$.
(i.e., show $f_1(x) + f_2(x) - f_3(x)$ is continuous at $x = a$).

Prob 4 considers $f+g$ for two functions f, g.

Let $g(x) = f_1(x) + f_2(x) - f_3(x)$. We want to show that

$\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|g(x) - g(a)| < \epsilon$ whenever $|x - a| < \delta$.

We know: since $f_i(x)$ are continuous at $x = a$,

$\exists \delta_i > 0$ s.t. $|f_i(x) - f_i(a)| < \frac{\epsilon}{3}$ whenever $|x - a| < \delta_i$, $i = 1,2,3$.

Let $\delta = \min_{i=1,2,3} \{\delta_i\}$. Then

We want x to as close to a as required in each case!

e.g., if $\delta_1 = 0.1$, $\delta_2 = 0.05$ and $\delta_3 = 0.08$, then $\delta \leq 0.05$ works!

$$|g(x) - g(a)| = \left| \left(f_1(x) + f_2(x) - f_3(x)\right) - \left(f_1(a) + f_2(a) - f_3(a)\right) \right|$$

$$= \left| \left(f_1(x) - f_1(a)\right) + \left(f_2(x) - f_2(a)\right) + \left(f_3(a) - f_3(x)\right) \right|$$

$$\leq |f_1(x) - f_1(a)| + |f_2(x) - f_2(a)| + \underbrace{|f_3(a) - f_3(x)|}_{= |f_3(x) - f_3(a)|}$$

by triangle inequality (applied twice)

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \text{as } \delta \leq \delta_i \text{ for } i = 1,2,3$$

$$= \epsilon \quad \text{whenever } |x - a| < \delta.$$

$\square$

LSIRA Proposition 2.1.9 Let $g: \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$, and $g(a) \neq 0$. Show that $h(x) = \frac{1}{g(x)}$ is continuous at $x = a$.

Need to show: $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|h(x) - h(a)| < \epsilon$ whenever $|x-a| < \delta$.

We want to show that

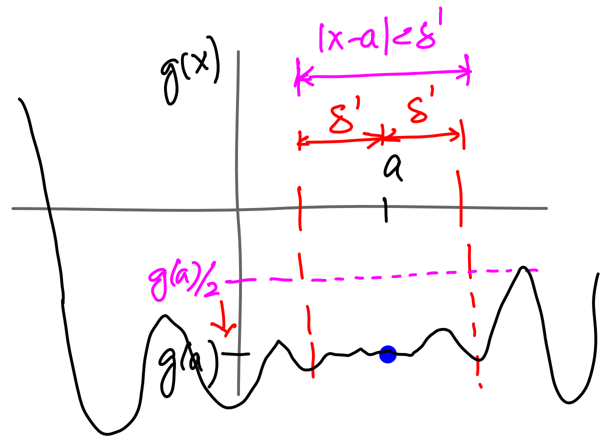
$$|h(x) - h(a)| = \left| \frac{1}{g(x)} - \frac{1}{g(a)} \right| < \epsilon.$$

$$\left| \frac{1}{g(x)} - \frac{1}{g(a)} \right| = \left| \frac{g(a) - g(x)}{g(x)\, g(a)} \right| = \frac{|g(x) - g(a)|}{|g(x)|\, |g(a)|}$$

$\neq 0$

We want to show that $|g(x)|$ is not too small. Else, the fraction could be too large.

There must exist some $\delta' > 0$ such that $|g(x)| > \frac{|g(a)|}{2}$ whenever $|x-a| < \delta'$, as $g(a) \neq 0$.



In the picture here, notice that $g(x)$ lies "below" the $\frac{g(a)}{2}$ level, i.e., far enough away from zero, when $|x-a| < \delta'$.

Also, $g(x)$ is continuous at $x = a \Longrightarrow$

$\exists \delta'' > 0$ s.t. $|g(x) - g(a)| < \epsilon'$ whenever $|x-a| < \delta''$.

Pick $\delta = \min\{\delta', \delta''\}$. Then we get that

$$\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| = \frac{|g(x)-g(a)|}{|g(x)|\,|g(a)|} < \frac{\epsilon'}{\frac{|g(a)|}{2}\,|g(a)|} = \frac{2\epsilon'}{|g(a)|^2}$$

whenever $|x-a|<\delta$.

If we choose $\epsilon' = \frac{|g(a)|^2}{2}\epsilon$, so that $\frac{2\epsilon'}{|g(a)|^2} = \epsilon$,

we get that $\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| < \epsilon$ whenever $|x-a|<\delta$.

Hence $\frac{1}{g(x)}$ is continuous at $x=a$

$\square$

In the next section, we consider the setting where the target or candidate limit (a) is not given to us.
Can we still conclude that $\{\bar{x}_n\}$ converges? When?

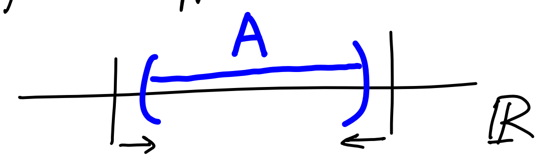
MATH 401: Lecture 8 (09/11/2025)

Today: * completeness
* sup, inf, lim sup, lim inf

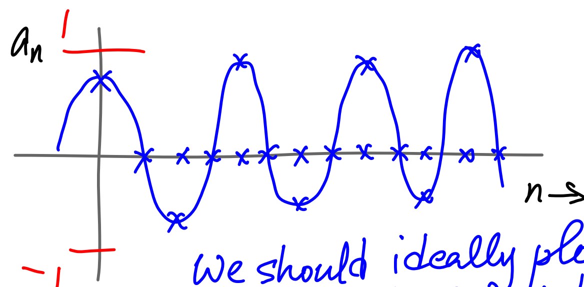
Completeness (LSIRA 2.3)

If we don't know the limit target $\bar{a}$, can we still say $\{\bar{a}_n\}$ converges?
If $\{\bar{a}_n\}$ "behaves nicely" and $\bar{a}_n$'s are in a "nice space", then yes!

Here is an intuition for what we mean by "nice space". Suppose $a_n \in A$ where A is a "finite" interval (open or closed). Then we can be sure that the a_n's cannot become arbitrarily large or arbitrarily small.

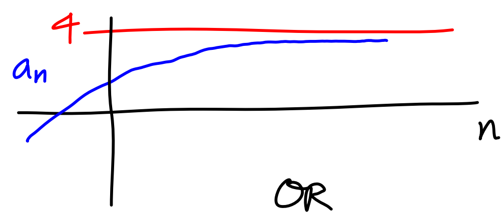


But in this example, the a_n's belong to a bounded interval $[-1,1]$, but they are not "behaving nicely" as the values oscillate between 1 and -1.

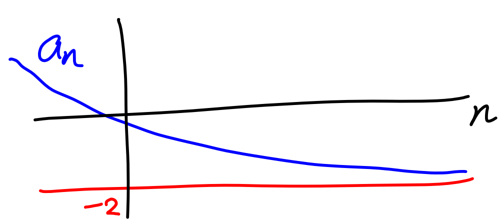


We should ideally plot the discrete pts (x), instead of the continuous curve

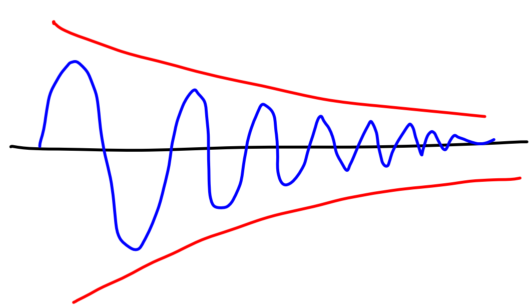
But if the a_n's are increasing and are bounded from above, or decreasing and bounded from below, we can conclude that $\{a_n\}$ converges!



OR



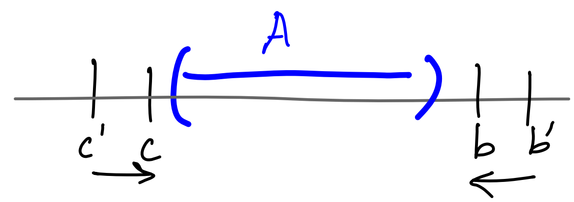
Finally, even if a_n's are oscillating, and hence not increasing/decreasing, it could still be nice if the oscillations become smaller and smaller — as shown here. Intuitively, we want the upper and lower "envelopes" to get closer and closer.



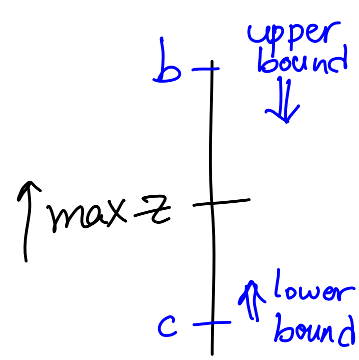
We formalize these intuitive notions of "nice" space and "nice" behavior.

Def A nonempty set $A \subset \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that $a \leq b \ \forall a \in A$, and is **bounded below** if there exists $c \in \mathbb{R}$ such that $a \geq c \ \forall a \in A$. We refer to b as an **upper bound**, and c as a **lower bound**.

If b is an upper bound, then any $b' > b$ is also an upper bound. Similarly, and $c' < c$ is also a lower bound.



We usually want to find a smallest upper bound, and a largest lower bound. This idea is ubiquitous in optimization, where finding the correct maximum value for a function $z = f(\bar{x})$ may be hard, but it may be easier to obtain lower/upper bounds. In order to get as best a handle on the actual max z value, we try to find the smallest upper bound, and the biggest lower bound that work.



In the same way, we want to "estimate" A as accurately as possible by finding the smallest upper bound and the largest lower bound for the set.

The Completeness Principle

Every nonempty subset A of $\mathbb{R}$ that is bounded above has a least upper bound. This bound is called the **supremum of A**, written $\sup A$.

Similarly every nonempty subset A of $\mathbb{R}$ that is bounded below has a greatest lower bound, called the **infimum of A**, written $\inf A$.

LSIRA 2.2 Problem 1 Argue that $\sup [0,1) = 1$ and $\sup [0,1] = 1$.

Let $A = [0,1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$. So $x \in A$ can be arbitrarily close to 1, i.e., $x = 1-\epsilon$, $\epsilon > 0$, arbitrarily small. Hence any $1-\epsilon$ cannot be an upper bound for A, since $\forall \epsilon > 0$, $\exists\ 1-\epsilon' \in A$ s.t. $1-\epsilon' > 1-\epsilon$.

$\Rightarrow$ $b \geq 1$ satisfies $x \leq b\ \forall x \in A$, and hence $\sup A = 1$.

The same argument holds for $[0,1]$ too. Note that the sup is in A in the latter case, but $\sup A \notin A$ for $A = [0,1)$.

So, what is the big deal about the completeness principle?

First, it does not hold over $\mathbb{Q}$ (rationals), as, e.g.,

$$A = \{x \in \mathbb{R} \mid x^2 < 3\} \text{ has } \sup A = \sqrt{3}. \text{ But}$$

$$B = \{x \in \mathbb{Q} \mid x^2 < 3\} \text{ has } \underline{\text{no}} \text{ supremum in } \mathbb{Q}!$$

$\rightarrow$ $\sqrt{3}$ is irrational, and we can get arbitrarily close to $\sqrt{3}$ using rational numbers!

We say that $\mathbb{Q}$ does not satisfy completeness principle.

Monotone Sequences, lim sup, lim inf

We now describe sequences that behave "nicely" like the bounded sets introduced earlier. We then consider how to handle sequences that are not as "nice".

"nondecreasing" if you want $a_{n+1} > a_n$ to mean "increasing"

Def A sequence $\{a_n\}$ in $\mathbb{R}$ is **increasing** if $a_{n+1} \geq a_n \; \forall n$.
A sequence $\{a_n\}$ in $\mathbb{R}$ is **decreasing** if $a_{n+1} \leq a_n \; \forall n$.
$\{a_n\}$ is **monotone** if it is either increasing or decreasing.
$\{a_n\}$ is **bounded** if $\exists \, M \in \mathbb{R}$ s.t. $|a_n| \leq M \; \forall n$.

LSIRA Theorem 2.2.2 Every monotone bounded sequence in $\mathbb{R}$ converges to a number in $\mathbb{R}$.
we do not specify which number!

Proof (for increasing sequences). We proceed in two steps.

1. $\{a_n\}$ is bounded $\Rightarrow$ $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ is bounded. set
$\Rightarrow \exists \, a \in \mathbb{R}$ such that $\sup A = a$. $\rightarrow$ using completeness of $\mathbb{R}$

2. a is the least upper bound. $\rightarrow$ We show $\{a_n\} \rightarrow a$
$\Rightarrow a - \epsilon$ is not an upper bound for any $\epsilon > 0$.
$\{a_n\}$ is increasing $\Rightarrow$ $a - \epsilon < a_n \leq a \;\; \forall n \geq N$ for some N.

$\Rightarrow |a - a_n| < \epsilon \;\; \forall \, n \geq N$, i.e., $\{a_n\}$ converges.
$a_n - a > -\epsilon$ and $a - a_n < \epsilon$ $\square$

But what if $\{a_n\}$ is not monotone and/or not bounded? Can we still say something about $\{a_n\}$ as $n \to \infty$?

Given a general sequence $\{a_n\}$, we define two related sequences that are monotone themselves.

Def Given $\{a_k\}$, $a_k \in \mathbb{R}$, we define two new sequences $\{M_n\}$ and $\{m_n\}$ as follows.

$$M_n = \sup \{a_k \mid k \geq n\} \quad \text{and}$$

$$m_n = \inf \{a_k \mid k \geq n\}.$$

$M_n = \infty$, $m_n = -\infty$ are allowed here.

M_n "captures" how large $\{a_k\}$ can be "after" n, and m_n captures how small $\{a_k\}$ can be "after" n.

Note that $\{M_n\}$ and $\{m_n\}$ are monotone!

$\{M_n\}$ is decreasing, as suprema are taken over smaller subsets. and $\{m_n\}$ is increasing, as infima are taken over smaller subsets.

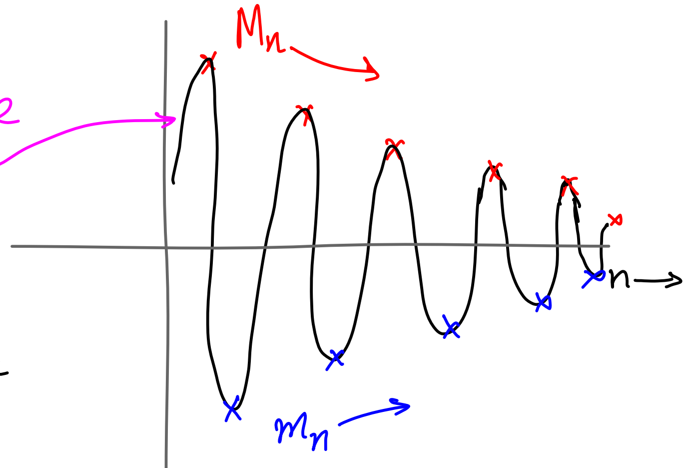
e.g., consider $A = \{1, 2, \ldots, 10\}$. The largest number in A cannot be bigger than the largest number in $A' = \{1, 2, \ldots, 7\}$, or in any $A' \subset A$, in general.

$$\Rightarrow \lim_{n \to \infty} M_n \text{ and } \lim_{n \to \infty} m_n \text{ exist!}$$

Def The **limit superior** or lim sup of the original sequence $\{a_n\}$ is $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} M_n$.

The **limit inferior** of $\{a_n\}$ is $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n$.

We ideally want to draw a sequence of points "......" in place of the continuous curve here



It appears while $\{x_n\}$ may be "oscillating" the upper bounds M_n and lower bounds m_n appear to be converging. Hence, $\{a_n\}$ also appears to converge! But we could have $\{a_n\}$ oscillate forever, even when M_n and m_n are finite $\forall n \geq N$.

LSIRA 2.2 Problem 4

Let $a_n = (-1)^n$. What is $\limsup_{n\to\infty} a_n$? $\liminf_{n\to\infty} a_n = ?$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} M_n = 1.$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n = -1.$$

Note than $a_n = 1 \ \forall\ n = 2k$, and $a_n = -1 \ \forall\ n = 2l+1$.

Hence $a_n \leq 1 \ \forall\ n$, and $a_n \geq -1 \ \forall\ n$.

In fact, $\{M_n\}$ and $\{m_n\}$ behave identical to $\{a_n\}$ here!

In the above problem, even though lim sup and lim inf are both finite, they are not equal, and we cannot say anything about $\{a_n\}$ converging to a limit. But when the lim sup and lim inf are equal, we get the picture drawn earlier, with $\{a_n\}$ converging to that value!

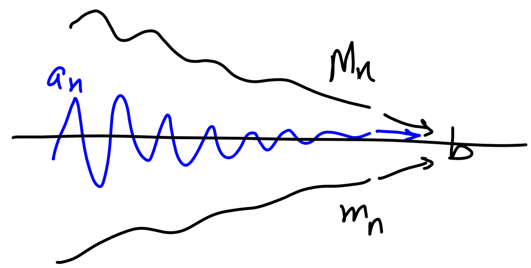
LSIRA Proposition 2.2.3 Let $\{a_n\}$ be a sequence of real numbers.

Then $\lim\limits_{n\to\infty} a_n = b$ if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b.$$

b can $\pm\infty$ here!

($\Leftarrow$) Assume $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$



$$\Rightarrow \lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = b$$

Also, $m_n \le a_n \le M_n \quad \forall n$

$$\Rightarrow \lim_{n\to\infty} a_n = b.$$

(by "squeeze law" or "squeeze theorem"; LSIRA 2.2 Problem 2 — assigned in HW4!)

We'll finish the proof in the next lecture..

MATH 401: Lecture 9 (09/16/2025)

Today: * Cauchy sequences
* Intermediate value theorem (IVT)

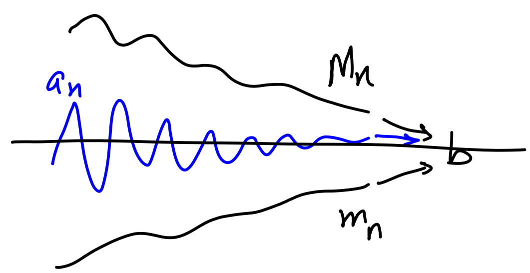
We first present the proof of Proposition 2.2.3...

LSIRA Proposition 2.2.3 Let $\{a_n\}$ be a sequence of real numbers. Then $\lim\limits_{n\to\infty} a_n = b$ iff $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$.

b can be $\pm\infty$ here!

($\Leftarrow$) Assume $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$

$$\Rightarrow \lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = b$$



Also, $m_n \leq a_n \leq M_n \;\; \forall n$

$$\Rightarrow \lim_{n\to\infty} a_n = b.$$

(by "squeeze law" or "squeeze theorem"; LSIRA 2.2 Problem 2 — assigned in HW4!)

($\Rightarrow$) Assume $\lim\limits_{n\to\infty} a_n = b$, and $b \in \mathbb{R}$.

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - b| < \epsilon \;\; \forall n \geq N$.

($\Leftrightarrow$ $a_n - b < \epsilon$ and $b - a_n < \epsilon$)

$|x| < 5 \Rightarrow -x < 5$ and $x < 5$

$$\Rightarrow b - \epsilon < a_n < b + \epsilon \quad \forall n \geq N$$

$$\Rightarrow b-\epsilon < m_n < b+\epsilon \quad \text{and}$$
$$b-\epsilon < M_n < b+\epsilon \quad \forall n \geq N$$

We will repeatedly use this trick of splitting $|x-y| < \epsilon$ into $x-y < \epsilon$ and $y-x<\epsilon$

Since the result holds for any $\epsilon > 0$,

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b.$$

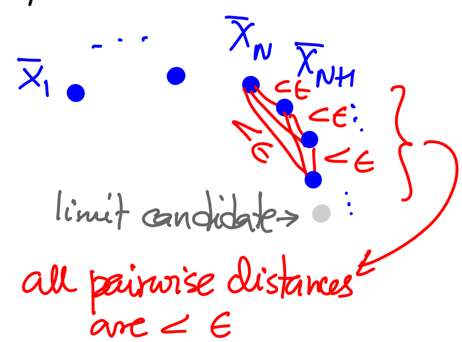
$\square$

Cauchy Sequences

We extend the idea of completeness in $\mathbb{R}$ to $\mathbb{R}^m$. But there is no natural way to order points in $\mathbb{R}^m$ (as in $\mathbb{R}$). Instead, we say the points get closer and closer to each other.

Def 2.2.4 A sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$ is called a **Cauchy sequence** if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\|\bar{x}_n - \bar{x}_k\| < \epsilon \quad \forall n, k \geq N$.

n, k are two indices, and represent any two points that are both far out enough into the sequence $(n, k \geq N)$



Completeness Result in $\mathbb{R}^m$

Theorem 2.2.5 The sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$ converges **iff** it is cauchy.

This is an iff result. We prove both directions, but one of them is easier than the other. We show the easy direction in $\mathbb{R}^m$, but the reverse direction in $\mathbb{R}$ (and can be extended to $\mathbb{R}^m$).

Proposition 2.2.6 All convergent sequences in $\mathbb{R}^m$ are cauchy.

Proof Let $\{\bar{a}_n\}$ converge to $\bar{a}$ in $\mathbb{R}^m$.

We want to show $\|\bar{a}_n - \bar{a}_k\| < \epsilon \quad \forall n, k \geq N$ for some $N \in \mathbb{N}$.

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|\bar{a}_n - \bar{a}\| < \frac{\epsilon}{2}, \forall n \geq N$.

Ideally, we use ϵ' here, and then choose $\epsilon' = \frac{\epsilon}{2}$.

$\Rightarrow$ If $n, k \geq N$, then

$$\|\bar{a}_n - \bar{a}_k\| = \|\bar{a}_n - \bar{a} + \bar{a} - \bar{a}_k\| \leq$$

triangle inequality

$$\|\bar{a}_n - \bar{a}\| + \|\bar{a} - \bar{a}_k\| < \underbrace{\frac{\epsilon}{2} + \frac{\epsilon}{2}}_{} = \epsilon \quad \forall n \geq N.$$

now we see why we chose $\frac{\epsilon}{2}$!

$\Rightarrow \{\bar{a}_n\}$ is cauchy. $\square$

We present proof for the reverse direction in $\mathbb{R}$. We can repeat this argument for each dimension to prove the result in $\mathbb{R}^m$. We need a lemma first.

Lemma 2.2.7 Every Cauchy sequence $\{a_n\}$ in $\mathbb{R}$ is bounded.

Want to show: $|a_n| \le M$ for some $M \in \mathbb{R}$. note, $M \geq 0$

$\{a_n\}$ is cauchy $\Rightarrow |a_n - a_k| < \epsilon \ \forall n,k \geq N \in \mathbb{N}$ for any $\epsilon > 0$.

$\Rightarrow |a_n - a_N| < 1$ (for $\epsilon = 1$) the definition applies for any ϵ, so we choose $\epsilon = 1$. After all, we just need to find a valid bound

$\Rightarrow a_n - a_N < 1$ and $a_N - a_n < 1$

$\Rightarrow a_n < a_N + 1$ and $a_n > a_N - 1 \ \forall n \geq N$.

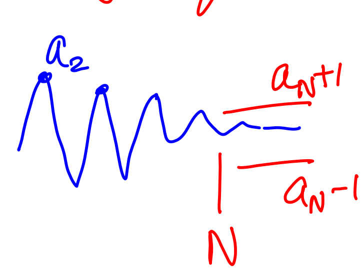
$\Rightarrow M = \max\{a_1, a_2, \dots, a_{N-1}, a_N + 1\}$ is an upper bound, and

$m = \min\{a_1, a_2, \dots, a_{N-1}, a_N - 1\}$ is a lower bound. □

Could also get $|a_n| - |a_N| \le |a_n - a_N| < 1$

$\Rightarrow |a_n| \le |a_N| + 1$.

We could have a larger number among $a_1, a_2, \dots, a_{N-1}$, which are not considered earlier since the Cauchy definition stipulates $n,k \geq N$.



Proposition 2.2.8 All Cauchy sequences in $\mathbb{R}$ converge.

Proof $\{a_n\}$ is cauchy $\Rightarrow$ $\{a_n\}$ is bounded (by Lemma 2.2.7).
$\Rightarrow M = \limsup_{n\to\infty} a_n$ and $m = \liminf_{n\to\infty} a_n$ are both finite.

We can use Proposition 2.2.3 now, if we can show $M = m$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - a_k| < \epsilon \;\; \forall n, k \geq N$.

In particular, $|a_n - a_N| < \epsilon \;\; \forall n \geq N$. (taking $k = N$)

$\Rightarrow \quad a_n - a_N < \epsilon \quad$ and $\quad a_N - a_n < \epsilon \;\; \forall n \geq N$

$\Rightarrow \quad a_n < a_N + \epsilon \quad$ and $\quad a_n > a_N - \epsilon$

i.e., $a_N - \epsilon < a_n < a_N + \epsilon \;\; \forall n \geq N$ holds for any $\epsilon > 0$.

$\Rightarrow \quad M_n = \sup\{a_k \mid k \geq n\} < a_N + \epsilon$ ADD

$-\left(m_n = \inf\{a_k \mid k \geq n\} > a_N - \epsilon\right) \Rightarrow \quad -m_n < -a_N + \epsilon$

$\Rightarrow \quad M_n - m_n < 2\epsilon \;\; \forall n \geq N$ and for any $\epsilon > 0$, arbitrary.

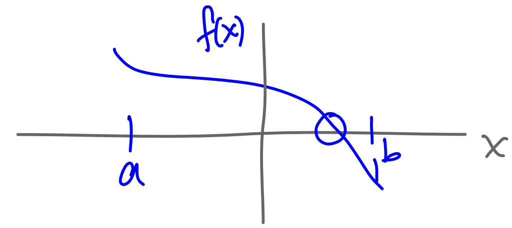
$\Rightarrow \quad M = m$ (as $n \to \infty$).

$\square$

We now present four fundamental theorems, the proofs of which use many of the results we have presented. These theorems are quite fundamental in analysis, and also finds use in many applied domains as well.

Intermediate Value Theorem

This is a rather straightforward result to understand — if a function goes from above the x-axis to below it, and is continuous, then it must cross the x-axis.



Theorem 2.3.1 (Intermediate Value Theorem) Assume $f : [a,b] \to \mathbb{R}$ is continuous, and $f(a)$ and $f(b)$ have opposite signs. Then there exists $c \in (a,b)$ such that $f(c) = 0$.

We will use a characterization of continuity using sequences in the proof (from LSIRA 2.1, actually!).

Proposition 2.1.5 $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ iff $\lim\limits_{n\to\infty} f(x_n) = f(a)$ for all sequences $\{x_n\}$ that converge to a.

Proof

($\Rightarrow$) Assume f is continuous at $x = a$.

Consider $\{x_n\} \to a$, i.e., $\lim\limits_{n\to\infty} x_n = a$.

Need to show: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|f(x_n) - f(a)| < \epsilon \ \forall n \geq N$.

$\Rightarrow \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

same

$\exists N' \in \mathbb{N}$ s.t. $|x_n - a| < \delta$ whenever $n \geq N'$.

$\rightarrow$ plays the "role of ϵ," i.e., the convergence definition must hold for any $\epsilon > 0$, and here we choose $\epsilon = \delta$.

$\Rightarrow$ If $n \geq N'$, then $|f(x_n) - f(a)| < \epsilon$, as $|x_n - a| < \delta$.

$\Rightarrow \{f(x_n)\} \to f(a)$. Reverse direction in the next lecture...

MATH401: Lecture 10 (09/18/2025)

Today: * Intermediate Value Theorem (IVT)
* Bolzano-Weierstrass (BW) theorem
* Extreme Value theorem (EVT)

Recall: **Proposition 2.1.5** $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x=a$ iff $\lim_{n\to\infty} f(x_n) = f(a)$ for all sequences $\{x_n\}$ that converge to a.

($\Leftarrow$) Contrapositive argument

We assume f not continuous at $x=a$, and show there must exist a sequence $\{x_n\}$ that converges to a, but $\{f(x_n)\}$ does not converge to $f(a)$.

If f is not continuous at $x=a$, we take the converse of what is implied by it being continuous.

$\Rightarrow \exists\, \epsilon > 0$ s.t. no matter how small you choose $\delta > 0$,

$\exists\, x$ s.t. $|x-a| < \delta$ but $|f(x) - f(a)| \geq \epsilon$.

x here depends on δ

Also, note that δ can be chosen arbitrarily small.

Pick $\delta = \frac{1}{n}$. $\Rightarrow \exists\, x_n$ s.t. $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \epsilon$. as f is not continuous at $x=a$.

$\Rightarrow \{x_n\} \to a$ (as $n \to \infty$), but $\{f(x_n)\} \not\to f(a)$. $\square$

This notion of continuity defined in terms of sequences can be quite useful in many contexts, especially when we try to generalize results to higher dimensions.

We now get back to the proof of intermediate value theorem.

Recall:

Theorem 2.3.1 (Intermediate Value Theorem) Assume $f:[a,b]\to\mathbb{R}$ is continuous, and $f(a)$ and $f(b)$ have opposite signs. Then there exists $c\in(a,b)$ such that $f(c)=0$.

Proof Consider $f(a)<0<f(b)$. The other case of $f(a)>0>f(b)$ can be argued similarly.

Let $A=\{x\in[a,b]\mid f(x)<0\}$ and $c=\sup A$. A is bounded (subset of $[a,b]$), hence $\sup A$ exists.

We show $f(c)=0$.

f is continuous and $f(b)>0$. $\Rightarrow c<b$.

$\Rightarrow$ The sequence $x_n=c+\frac{1}{n}\in[a,b]\ \forall n\geq N$, for sufficiently large N. $c<b$,

$\Rightarrow \{x_n\}\to c$ as $n\to\infty$.

Also, $f(x_n)>0\ \forall$ such n. as $x_n\notin A$, since $x_n>c$.

By Proposition 2.1.5, as f is continuous, $\lim\limits_{n\to\infty}f(x_n)=f(c)$, and since $f(x_n)>0\ \forall n$, $f(c)\geq 0$. We can be sure that $f(c)\neq 0$.

On the other hand, by definition of c, consider $z_n=c-\frac{1}{n}$ for sufficiently large n. when n is large enough, $c-\frac{1}{n}\geq a$.

$\Rightarrow z_n\leq c\ \forall n$ large enough, and $\{z_n\}\to c$ (as $n\to\infty$).

Also, $z_n\in A\subset[a,b]$ for n large enough. $\Rightarrow f(z_n)<0$.

Again, by Proposition 2.1.5, $f(c)=\lim\limits_{n\to\infty}f(z_n)$ and since $f(z_n)<0\ \forall n$, we get $\underline{f(c)\leq 0}$. Hence $f(c)\geq 0$ and $f(c)\leq 0$, i.e., $f(c)=0$. □

Again, we can be sure that $f(c)\neq 0$.

The Intermediate Value Theorem does not hold in $\mathbb{Q}$!

Consider $f(x) = x^2 - 3 \Rightarrow f(0) = -3$ and $f(2) = 1$.

But $\nexists\, x \in [0,2] \cap \mathbb{Q}$ s.t. $f(x) = 0$, as $\sqrt{3} \notin \mathbb{Q}$.

The **Bolzano-Weierstrass** (BW) Theorem

We saw that every Cauchy sequence converges. But what if a sequence is not Cauchy, and hence does not converge? Can we still say something nice about its structure? It turns out yes, when the sequence is bounded! We need the notion of a subsequence first.

Def (Subsequence) Given a sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$, we choose an infinite subset of its terms to form another sequence $\{\bar{y}_k\}$. (Of course, it is interesting only when we do not choose all terms of $\{\bar{x}_n\}$).

$$\bar{x}_1, \bar{x}_2, \ldots, \bigcirc, \bar{x}_n, \bigcirc \ldots \bigcirc \ldots \bigcirc$$
$$\bar{y}_1 \quad \bar{y}_2 \quad \bar{y}_3 \quad \bar{y}_4 \quad \cdots$$

$\{\bar{y}_k\} \to$ subsequence of $\{\bar{x}_n\}$.

If $n_1 < n_2 < \cdots < n_k < \cdots$ are indices of terms picked to form a new sequence, then

$$\{\bar{y}_k\} = \{\bar{x}_{n_k}\} \text{ is a } \textbf{subsequence} \text{ of } \{\bar{x}_n\}.$$

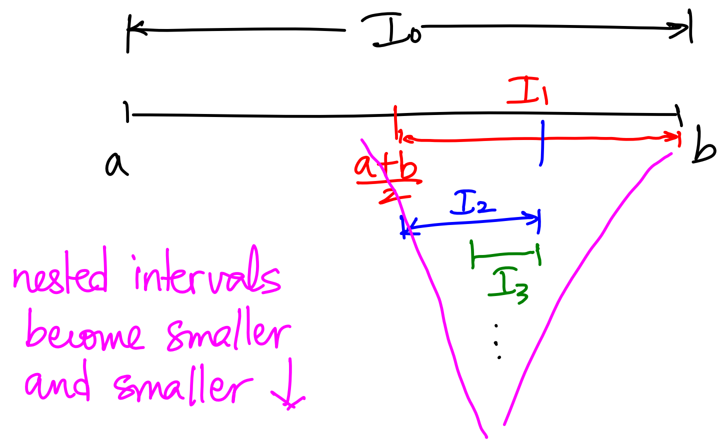
We state and prove BW theorem in $\mathbb{R}$.

Proposition 2.3.2 Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

$\{x_n\}$ is bounded $\Rightarrow \exists\, a \le b$ s.t. $x_n \in [a,b] = I_0\ \forall n$.

We identify a Cauchy subseq.

We argue we can pick smaller and smaller subintervals of I_0, each of which has infinitely many terms of $\{x_n\}$.



Let $I_1 = \left[\frac{a+b}{2}, b\right]$ be such that it has infinitely many terms of $\{x_n\}$.

It could happen that $I_1' = \left[a, \frac{a+b}{2}\right]$ is the one with infinitely many terms, or both I_1 and I_1' have infinitely many terms of $\{x_n\}$. But since $\{x_n\}$ has infinitely many terms, at least one of the two half intervals is guaranteed to have infinitely many terms. We always choose a half interval with infinitely many terms, and continue the process.

In general, I_k is a half interval of I_{k-1} that has infinitely many terms of $\{x_n\}$. Note that I_k is a subinterval of I_{k-1} for each k ($k \geq 1$), and we get a sequence of nested subintervals that are shrinking in size by a factor of $\left(\frac{1}{2}\right)$ in each step.

Since $|I_0| = |[a,b]| = b-a$ is finite, $|I_k| \longrightarrow 0$ as $k \longrightarrow \infty$.

We can now specify how to construct the convergent subsequence. Essentially, we pick one term of $\{x_n\}$ from each subinterval I_k as follows.

Let y_1 be the first element of $\{x_n\}$ in I_1. And let y_2 be the first element of $\{x_n\}$ **after** y_1 that is in I_2.

$\vdots$

In general, let y_k be the first element of $\{x_n\}$ **after** y_{k-1} that is in I_k, for $k \geq 1$.

Note that the y_k's are included in nested, shorter and shorter subintervals, and hence are getting closer and closer to each other.

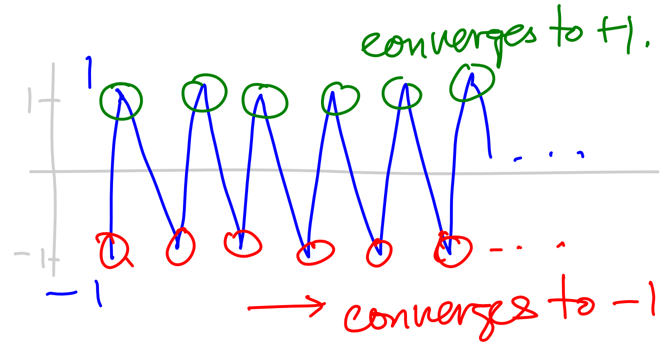
we could make this argument more formal

$\Rightarrow \{y_k\}$ is Cauchy!

$\Rightarrow \{y_k\}$ converges by Proposition 2.2.8 $\square$

Consider a somewhat trivial example. Let $x_n = (-1)^n$, $n \in \mathbb{N}$.

One can immediately see that $a_n \leq 1 \ \forall n$ and $a_n \geq -1 \ \forall n$ hold, i.e., we can choose $[a,b] = [-1,1]$ in the proof above.



Then $y_k = (-1)^k$ for $k = 2n$, $n \in \mathbb{N}$ defines a subsequence $\{y_k\} \to 1$, and $z_k = (-1)^k$ for $k = 2n-1$, $n \in \mathbb{N}$ defines a subsequence $\{z_k\} \to -1$.

The BW theorem naturally extends to $\mathbb{R}^m$ — we essentially repeat the above argument one dimension at a time! See LSIRA for details.

We now present two theorems that use the results on sequences to specify properties of "good" (continuous or differentiable) functions defined on the sequences.

The **Extreme Value Theorem** (EVT) in $\mathbb{R}$

Theorem 2.3.4 Let $f:[a,b]\to\mathbb{R}$ be a continuous function on the closed bounded interval $[a,b]$. Then $\exists$ points $c, d \in [a,b]$ such that $f(d) \le f(x) \le f(c)$ $\forall\, x \in [a,b]$.

In words, f has maximum and minimum points in $[a,b]$.

Proof (for maximum) → A similar argument can be made for minimum

Let $M = \sup\{f(x) \mid x \in [a,b]\}$. → We're not sure yet whether M is finite

Choose sequence $\{x_n\}$ in $[a,b]$ such that $f(x)\to M$.

As f is continuous, such a sequence exists. → irrespective of whether M is finite or not

$[a,b]$ is bounded $\Rightarrow$ By BW Theorem, $\{x_n\}$ has a convergent subsequence $\{y_k\}$.

$[a,b]$ is closed $\Rightarrow$ $c = \lim_{k\to\infty} y_k \in [a,b]$.

$\Rightarrow f(y_k)\to M$ by construction. → We chose $\{x_n\}$ so that $f(x_n)\to M$ in the first place.

f is continuous $\Rightarrow$ by Proposition 2.1.5, $f(y_k)\to f(c)$.

$\Rightarrow f(c) = M$, i.e., M is the maximum, and $c\in[a,b]$ is the corresponding maximum point for f. $\square$

MATH401: Lecture 11 (09/23/2025)

Midterm exam: Oct 7
Take-home exam; sections 1.1–1.6, 2.1, 2.2

Today: * Mean value theorem
* Metric spaces

Mean Value Theorem (MVT) on $\mathbb{R}$

For the final theorem (4^{th} one, after IVT, BW, EVT), we assume the function is much "nicer", i.e., it's differentiable, to be able to present a stronger result on its structure. We recall the definition of derivative first.

Recall: **Derivative** of function f at $x=a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

f is **differentiable** at $x=a$ if the above limit exists.

The mean value theorem says for a differentiable and continuous function f on $[a,b]$, there exists a point inside the interval where the instantaneous slope of the function is equal to the "mean" slope of f over the interval. We need two results to be used as building blocks first.

Lemma 2.3.5 Let $f:[a,b]\to\mathbb{R}$ have a maximum or minimum at an inner point $c\in(a,b)$ where the function is differentiable. Then $f'(c)=0$.

Proof We show $f'(c)>0$ or $f'(c)<0$ is not possible.

Assume $f'(c)>0$. A similar argument works for $f'(c)<0$.

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\quad \text{by definition.}$$

as $f'(c)>0$

$$\Rightarrow \frac{f(x)-f(c)}{x-c}>0 \quad \text{for all } x \text{ sufficiently close to } c.$$

$$\Rightarrow \quad x>c \Rightarrow f(x)>f(c), \text{ and}$$
$$x<c \Rightarrow f(x)<f(c)$$

If $x=c$ is a maximum, then $f(x)\le f(c)$ for $\forall x$.

$\Rightarrow$ $x=c$ is neither a maximum nor minimum.

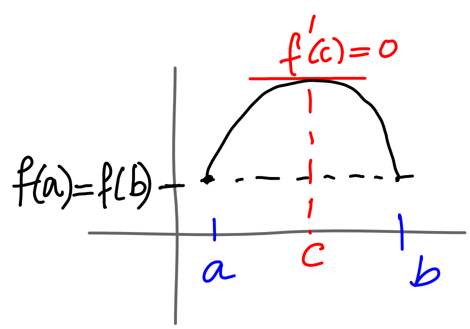
The result follows by the contrapositive argument now. $\square$

Lemma 2.3.6 (Rolle's Theorem) Let $f:[a,b]\to\mathbb{R}$ be continuous at all $x\in[a,b]$ and is differentiable at all inner points $x\in(a,b)$. If $f(a)=f(b)$, then there exists a point $c\in(a,b)$ such that $f'(c)=0$.

Proof EVT (Theorem 2.3.4) $\Rightarrow$

f has a maximum and minimum in $[a,b]$. Since $f(a)=f(b)$, at least one of these optima must be at an inner point c.

So Lemma 2.3.5 $\Rightarrow f'(c)=0$. $\square$



Trivial case: $f(x)=f(a)\ \forall x\in[a,b]$

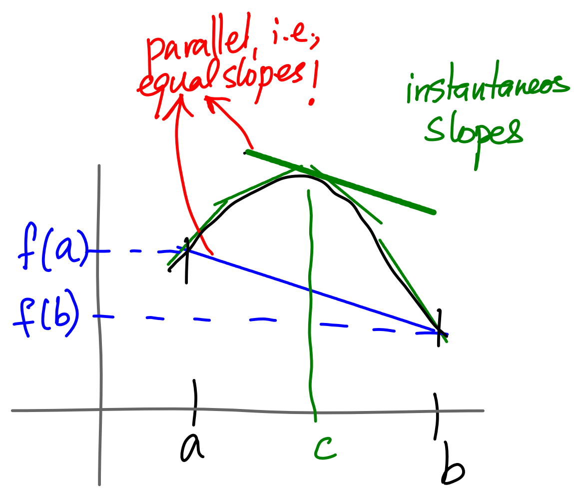
↳ straight line!

Theorem 2.3.7 (The Mean Value Theorem (MVT))

Let $f:[a,b]\to\mathbb{R}$ be continuous in all $[a,b]$ and differentiable at all inner points $x\in(a,b)$. Then there exists $c\in(a,b)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

The mean (or average) slope of $f(x)$ over $[a,b]$ is $f(b)-f(a)/(b-a)$. This theorem says there is a point $c\in(a,b)$ where the instantaneous slope, i.e., slope of tangent, is equal to the mean slope!



Proof

Let $g(x)=f(x)-\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$. → How did we come up with this function?!! See next page...

$g(a)=f(a)$, and

$g(b)=f(b)-\left(\frac{f(b)-f(a)}{\cancel{b-a}}\right)(\cancel{b-a})=f(a)$. → $b>a$ by assumption, and hence $b-a\neq 0$.

We can show that $g(x)$ is indeed continuous in $[a,b]$ and differentiable at all $x\in(a,b)$. → $g(x)=f(x)+m(x-a)$ for constant m; $f(x)$ is continuous and differentiable, and so is $(x-a)$; their sum is so as well.

So, Rolle's theorem (Lemma 2.3.6) $\Rightarrow\ \exists\, c\in(a,b)$ s.t. $g'(c)=0$.

$$\Rightarrow\ g'(x)=f'(x)-\left(\frac{f(b)-f(a)}{b-a}\right)=0 \text{ at } x=c.$$

$$\Rightarrow\ f'(c)=\frac{f(b)-f(a)}{b-a}. \qquad \square$$

Now, how did we come up with the $g(x)$ function?!

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \overset{\text{want to show!}}{=} \frac{f(b)-f(a)}{b-a}$$

$$f'(x)\Big|_{x=c} =$$

$$\Longrightarrow \underbrace{f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)}\Bigg|_{x=c} = 0$$

Looks like $g'(x)=0$ for some function $g(x)$.

We want to find $g(x)$ such that $g(a)=g(b)=0$, and then we could use Rolle's theorem!

So we take antiderivative of $f'(x) - \dfrac{f(b)-f(a)}{b-a}$ to get

$$f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)x + C \longrightarrow \text{constant}$$

With $g(x) = f(x) - \left(\dfrac{f(b)-f(a)}{b-a}\right)x + C$, we choose C such that $g(a)=g(b)$! Note that

$$\underbrace{f(b) - \left(\frac{f(b)-f(a)}{b-a}\right)b + \left(\frac{f(b)-f(a)}{b-a}\right)a}_{g(b)} = f(a) = \underbrace{f(a) - \left(\frac{f(b)-f(a)}{b-a}\right)a + \left(\frac{f(b)-f(a)}{b-a}\right)a}_{g(a)}.$$

Chapter 3 Metric Spaces

We have showed several results on sequences and functions in $\mathbb{R}$ and $\mathbb{R}^m$. But many of these results could be shown for far more general spaces which have many of the nice properties of $\mathbb{R}$ (or $\mathbb{R}^m$). We define metric spaces with this goal in mind.

3.1 Definitions

nonempty

Def A **metric space** (X, d) consists of a set $X \neq \phi$, and a function $d : X \times X \longrightarrow [0, \infty)$ such that

(i) (positivity) $d(x,y) \geq 0 \;\; \forall \; x, y \in X$, and
$d(x,y) = 0$ iff $x = y$;

(ii) (symmetry) $d(x,y) = d(y,x) \;\; \forall \; x, y \in X$; and

(iii) (triangle inequality) $d(x,y) \leq d(x,z) + d(z,y) \;\; \forall \; x, y, z \in X$.

hold. A function d satisfying (i)–(iii) on X is a **metric** on X.

We sometimes write just X, when the metric d is evident. At the same time, note that a space X could have multiple metrics defined on it. The first example we consider studies a metric on $\mathbb{R}^2$ that is different from the usual Euclidean metric.

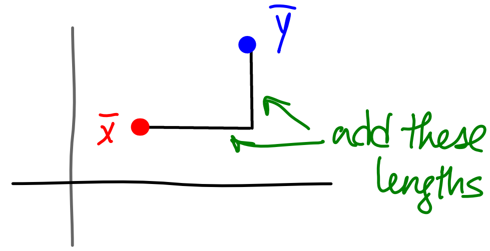
Examples

LSIRA eg. 3 **Manhattan or taxicab metric (in $\mathbb{R}^2$).** X

For $\bar{x}, \bar{y} \in \mathbb{R}^2$, let $\qquad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\bar{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$d(\bar{x}, \bar{y}) = |y_1 - x_1| + |y_2 - x_2|.$$

Check that this is a metric space.



The intuition is that if a taxi is to go from point X to point y in downtown Manhattan with perpendicular streets, it will have to go East/West (horizontal) and then North/South (vertical). We add these two straight line distances to get the taxicab distance between x and Y.

(i) $d(\bar{x}, \bar{y}) \geq 0$ holds, as $|x_1 - y_1| \geq 0$ and $|x_2 - y_2| \geq 0$.
The only way we get $d(\bar{x}, \bar{y}) = 0$ is when both absolute differences are zero, i.e., when $x_1 = y_1$ and $x_2 = y_2$, i.e., when $\bar{x} = \bar{y}$.

(ii) $d(\bar{x}, \bar{y}) = d(\bar{y}, \bar{x})$ follows from absolute differences being symmetric, i.e., $|x_i - y_i| = |y_i - x_i|$ for $i = 1, 2$.

(iii) Triangle inequality:

$$\begin{aligned} d(\bar{x}, \bar{y}) &= |y_1 - x_1| + |y_2 - x_2| \\ &= |y_1 - z_1 + z_1 - x_1| + |y_2 - z_2 + z_2 - x_2| \\ &\leq |y_1 - z_1| + |z_1 - x_1| + |y_2 - z_2| + |z_2 - x_2| \\ &= d(x, z) + d(z, y). \end{aligned}$$

standard triangle inequality in $\mathbb{R}$

Hence $d(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{z}) + d(\bar{z}, \bar{y}) \quad \forall \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^2$. $\square$

MATH 401: Lecture 12 (09/25/2025)

Today: * Examples of metric spaces
* isometry

Recall (X,d): Metric space (i) positivity: $d(x,y) \geq 0 \ \forall x,y \in X$, and $d(x,y)=0 \iff x=y$.

(ii) symmetry: $d(x,y)=d(y,x) \ \forall x,y \in X$

(iii) triangle inequality: $d(x,y) \leq d(x,z)+d(z,y) \ \forall x,y,z \in X$

One more requirement: $d(x,y) < \infty \ \forall x,y \in X$. (finiteness)

The finiteness requirement is usually satisfied. But you should use your judgement to decide in which cases this property needs to be proved.

LSIRA 3.1 Example 4 (Problem 1)

Let X be the space of messages, where each message is a vector (k is fixed)

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \text{ with } x_i \in \underbrace{S}_{\text{alphabet}} = \{\underbrace{s_1, \dots, s_N}_{\text{symbols in the alphabet}}\}.$$

Let $d(\bar{x},\bar{y}) = \#$ indices i where $x_i \neq y_i$. Show (X,d) is a metric space.

Note that shorter messages could be padded up with empty cells, or a "dummy" symbol (equivalent to 0 when using numbers). And larger messages can be chopped up into pieces of length k each (with padding up if needed for the last piece).

An example: Let $S=\{R,G,B\}$ for colors red, green, blue. Consider vectorizing each image by assigning to each pixel its predominant color, and stacking these color symbols into a vector. For instance, a 12×12 pixel image is represented by a 144-vector of color values from $\{R,G,B\}$.

(i) $d(\bar{x},\bar{y}) \geq 0$, and $d(\bar{x},\bar{y}) = 0 \iff \bar{x} = \bar{y}$ as messages.

↳ $d(\bar{x},\bar{y})$ is the # places (or indices) where the messages differ, and hence is ≥ 0.

$d(\bar{x},\bar{y}) = 0 \iff \bar{x}$ and $\bar{y}$ are identical in all entries, i.e., they do not differ at all. Hence $\bar{x} = \bar{y}$.

(ii) symmetry ✓ $d(\bar{x},\bar{y}) = $ # indices where $x_i \neq y_i$

$= $ # indices where $y_i \neq x_i$

$= d(\bar{y},\bar{x})$

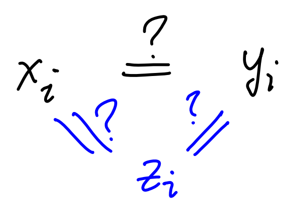
(iii) triangle inequality.

$d(\bar{x},\bar{y})$ counts # indices i where $x_i \neq y_i$

$x_i \neq y_i \implies$ cannot have $x_i = z_i$ and $z_i = y_i$.

Combining with z_i, here are the possibilities:

1. $x_i \neq z_i$, $z_i = y_i$
2. $x_i = z_i$, $z_i \neq y_i$
3. $x_i \neq z_i$, $z_i \neq y_i$



$\implies d(x,y) \leq d(x,z) + d(z,y)$, as there are three possible cases for each index contributing 1 to the right-hand side sum corresponding to the one case possibly contributing 1 to the left-hand side distance.

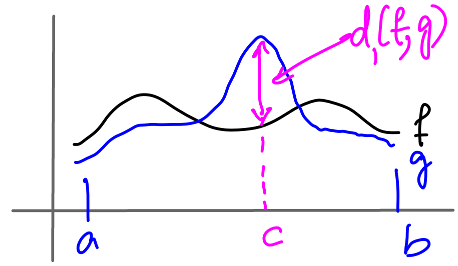
□

LSIRA Pg 47, Problem 2 **Distance between Functions.**

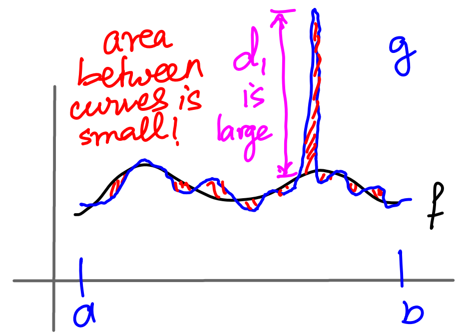
Let X = set of all continuous functions from $[a,b] \to \mathbb{R}$, and let

$$d_1(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\}.$$

Show that d_1 is a metric over X.



Measuring distance between functions is a widely studied problem in analysis as well as applications. We illustrate use of d_1 here (top right). But d_1 may not be the best choice in all cases — in the second instance, f and g are quite close to each other except at one point $x=c$, where g shoots up a lot. $d_1(f,g)$ will be quite large here, even though f and g are near equal. Measuring distance using the area between f and g may be better here.



Proof : We first show $d_1(f,g)$ is finite for any $f, g \in X$.

finitness is not obvious in this case!

f, g are continuous over $[a,b]$

$\Rightarrow f-g$ is continuous over $[a,b]$.

By the Extreme Value Theorem (Theorem 2.3.4), $h = f-g$ has a maximum and minimum over $[a,b]$.

$\Rightarrow \sup\{|h(x)| : x \in [a,b]\}$ is finite.

$\Rightarrow d_1(f,g)$ is finite.

(i) (positivity) $d_1(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\} \geq 0$, as the supremum of a set of ≥ 0 values is ≥ 0.

Need to also show $d_1(f,g) = 0 \iff f = g$ over $[a,b]$

$(\Rightarrow)$ $f(x) = g(x)\ \forall x \in [a,b]$

$\Rightarrow |f(x)-g(x)| = 0\ \forall x \in [a,b]$

$\Rightarrow \sup\{|f(x)-g(x)| : x \in [a,b]\} = 0 \Rightarrow d_1(f,g) = 0.$

$(\Leftarrow)$ $d_1(f,g) = 0 \Rightarrow \sup\{|f(x)-g(x)| : x \in [a,b]\} = 0$

The supremum of a set of ≥ 0 is zero $\Rightarrow$ each element $= 0$!

$\Rightarrow f(x) = g(x)\ \forall x \in [a,b]$.

(ii) (symmetry) $|f(x)-g(x)| = |g(x)-f(x)|$

$\Rightarrow d(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\} =$

$\sup\{|g(x)-h(x)| : x \in [a,b]\} = d_1(g,f)$.

(iii) <u>Triangle inequality</u>

$|f(x)-g(x)| = |f(x)-h(x)+h(x)-g(x)|$

$\leq |f(x)-h(x)| + |h(x)-g(x)|$ by standard triangle inequality over $\mathbb{R}$.

$d_1(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\}$

$\leq \sup\{|f(x)-h(x)| + |h(x)-g(x)| : x \in [a,b]\}$

$\leq \sup\{|f(x)-h(x)| : x \in [a,b]\} +$ as $\sup\{a+b\} \leq \sup\{a\} + \sup\{b\}$

$\sup\{|h(x)-g(x)| : x \in [a,b]\}$

$= d_1(f,h) + d_1(h,g) \quad \forall f,g,h \in X$.

$\square$

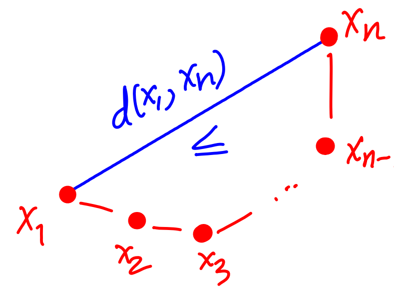
LSIRA Pg48, Problem 7

Let (X, d) be a metric space, and $x_i \in X$, $i = 1, \dots, n$.

Show $\quad d(x_1, x_n) \leq \sum_{i=1}^{n-1} d(x_i, x_{i+1})$.

$d(x_1,x_2) + d(x_2,x_3) + \dots + d(x_{n-1},x_n)$

Can use induction



$n=2$ (base case)

$d(x_1,x_2) \leq d(x_1,x_2)$ holds, as both sides are the same.

$n=3$ case could be considered as the base case as well:

$d(x_1,x_3) \leq d(x_1,x_2) + d(x_2,x_3)$ follows from triangle inequality for (X,d).

Assume result is true for $n=k$, i.e.,

$$d(x_1,x_k) \leq \sum_{i=1}^{k-1} d(x_i, x_{i+1}) \qquad (*)$$

For $n=k+1$,

$$\begin{aligned} d(x_1, x_{k+1}) &\leq d(x_1, x_k) + d(x_k, x_{k+1}) && \text{by triangle inequality in } (X,d) \\ &\leq \sum_{i=1}^{k-1} d(x_i,x_{i+1}) + d(x_k,x_{k+1}) && \text{by } (*) \\ &= \sum_{i=1}^{k} d(x_i,x_{i+1}) \quad \checkmark \end{aligned}$$

Hence the result holds for all n.

$\square$

We now talk about comparing two metric spaces, and functions between metric spaces. When are two metric spaces "the same"? As metric spaces are about pairwise distances between points, we want these distances to be preserved.

Def 3.1.2 Let (X, d_X) and (Y, d_Y) are metric spaces.

An **isometry** between the spaces is a bijection $i : X \to Y$ such that $d_X(x,y) = d_Y(i(x), i(y))$ $\forall x, y \in X$.

The two spaces are isometric if an isometry exists between them.

Since i is a bijection, its inverse exists, and i^{-1} is an isometry from (Y, d_Y) to (X, d_X). Hence we can just say isometry between the spaces.

LSIRA Pg48, Problem 11 For $a \in \mathbb{R}$, let $f(x) = x+a$. Show f is an isometry from $\mathbb{R}$ to $\mathbb{R}$.

$$d(x,y) = |x-y| \quad \forall x, y \in \mathbb{R}.$$

To show f is an isometry, we need to show

1. f is a bijection; and
2. $d(f(x), f(y)) = d(x,y)$ $\forall x, y \in \mathbb{R}$

1. $f(x) = x+a$ is a bijection as $\overbrace{x_1 \neq x_2 \Rightarrow f(x_1) = x_1 + a \neq x_2 + a = f(x_2)}^{\text{injection}}$; and $\underbrace{\forall y \in \mathbb{R}, \exists x = f^{-1}(y) = y - a}_{\text{surjection}}$.

2. $d(f(x), f(y)) = |f(x) - f(y)| = |x+a-(y+a)| = |x-y|$
$= d(x,y)$ $\forall x, y \in \mathbb{R}$.

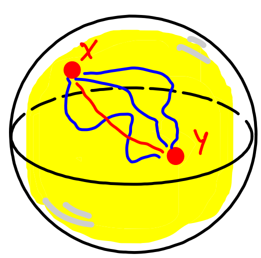
□

The requirement of i being a bijection is too strict in some settings. There may be spaces that look otherwise quite similar, even if they are not isometric.

If $i : X \to Y$ in Definition 3.1.2 is only an injection (and not a bijection), we call i an <u>embedding</u>.

You may have heard about embeddings in other geometric settings. For instance, think of a sphere (surface of a ball) in 3D space. We can work with the sphere as a metric space — the distance between any two points is the length of the shortest curve connecting the points that lies entirely on the surface (of the sphere). This is called the shortest geodesic distance. We can prove it is a metric.

S^2 (2-sphere) surface of a ball



in $\mathbb{R}^3$

There are many curves between x and y that lie on the surface of the sphere. Length of a shortest geodesic curve defines the distance between x and y.

The embedding here is literally the positioning of the sphere in $\mathbb{R}^3$.

MATH401: Lecture 13 (09/30/2025)

Today: * Convergence and continuity in metric spaces.

Convergence and Continuity (LSIRA 3.2)

We can naturally extend the concepts of convergence, functions, and their continuity from $\mathbb{R}$ or $\mathbb{R}^m$ to metric spaces. The only difference is that the distances bounded by ϵ and δ are now measured using the metrics in the metric spaces.

Def 3.2.1 Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to $a \in X$ if $\forall \epsilon > 0$ (no matter how small), $\exists N \in \mathbb{N}$ such that $d(x_n, a) < \epsilon \;\forall\, n \geq N$. We write $\lim_{n\to\infty} x_n = a$, $\{x_n\} \to a$, or $x_n \to a$.

Notice the correspondence to the definition(s) of convergence we have seen previously in $\mathbb{R}$ or $\mathbb{R}^m$. There, $d(x_n, a)$ was replaced by $|x_n - a|$ (in $\mathbb{R}$) or $\|\bar{x}_n - \bar{a}\|$ in $\mathbb{R}^m$.

Def A sequence $\{x_n\}$ in the metric space (X, d) converges to $a \in X$ iff $\lim_{n\to\infty} d(x_n, a) = 0$. (given as Lemma 3.2.2)

We can provide a proof using the standard definition of limit. See LSIRA.

We now talk about functions from one metric space to another, and when they are continuous. We essentially extend the definitions from $\mathbb{R}$ (or $\mathbb{R}^m$) to metric spaces.

Def 3.2.4 Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is continuous at $a \in X$ if

$\forall \epsilon > 0 \; \exists \delta > 0$ such that $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x, a) < \delta$.

When talking about $f: \mathbb{R} \to \mathbb{R}$ being continuous, we had both these distances measured as simply $|f(x) - f(a)|$ and $|x - a|$. We are just generalizing those distances to using the corresponding metrics in the spaces here.

LSIRA gives an equivalent definition of continuity in terms of convergence of $\{f(x_n)\}$ to $f(a)$ when $\{x_n\} \to a \in X$. See Proposition 3.2.5.

A Direct Application

Proposition 3.2.6 Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces. Let $f: X \to Y$ and $g: Y \to Z$ be functions, and $h: X \to Z$ be defined as $h(x) = g(f(x))$. If f is continuous at $a \in X$ and g is continuous at $b = f(a) \in Y$, then h is continuous at $a \in X$.

LSIRA presents a proof using Proposition 3.2.5. Here, we give a direct ϵ-δ proof

Problem 2 (pg 51) Prove Proposition 3.2.6 using direct definition of continuity.

Want to show: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $d_Z(h(x), h(a)) < \epsilon$ whenever $d_X(x, a) < \delta$.

Given f, g are continuous at a and $b = f(a)$, respectively.

$\Rightarrow \forall \epsilon_Y > 0, \exists \delta_X > 0$ s.t. $d_Y(f(x), \underbrace{f(a)}_{=b}) < \epsilon_Y$ whenever $d_X(x, a) < \delta_X$. —(1)

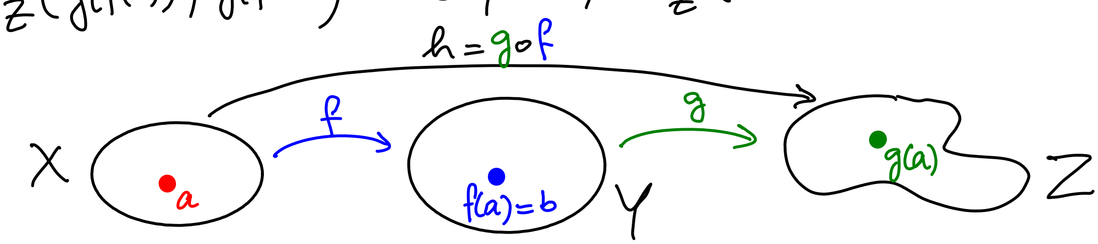
$\forall \epsilon_Z > 0, \exists \delta_Y > 0$ s.t. $d_Z(g(y), g(b)) < \epsilon_Z$ whenever $d_Y(y, b) < \delta_Y$. —(2)

Let $\epsilon > 0$. (2) $\Longrightarrow$ with $\epsilon_Z = \epsilon$, $\exists \delta_Y > 0$ s.t. $d_Z(g(y), g(b)) < \epsilon$.

(1) $\Longrightarrow$ with $\epsilon_Y = \delta_Y$, $\exists \delta_X$ s.t. $d_Y(f(x), \overset{=b}{f(a)}) < \delta_Y$ whenever $d_X(x, a) < \delta_X$.

$\Rightarrow d_X(x, a) < \delta_X \Rightarrow d_Y(\underset{y}{f(x)}, \underset{b}{f(a)}) < \delta_Y$.

$\Rightarrow d_Z(g(f(x)), g(f(a))) < \epsilon$, i.e., $d_Z(h(x), h(a)) < \epsilon$ as desired. $\square$



A geometric definition of continuity

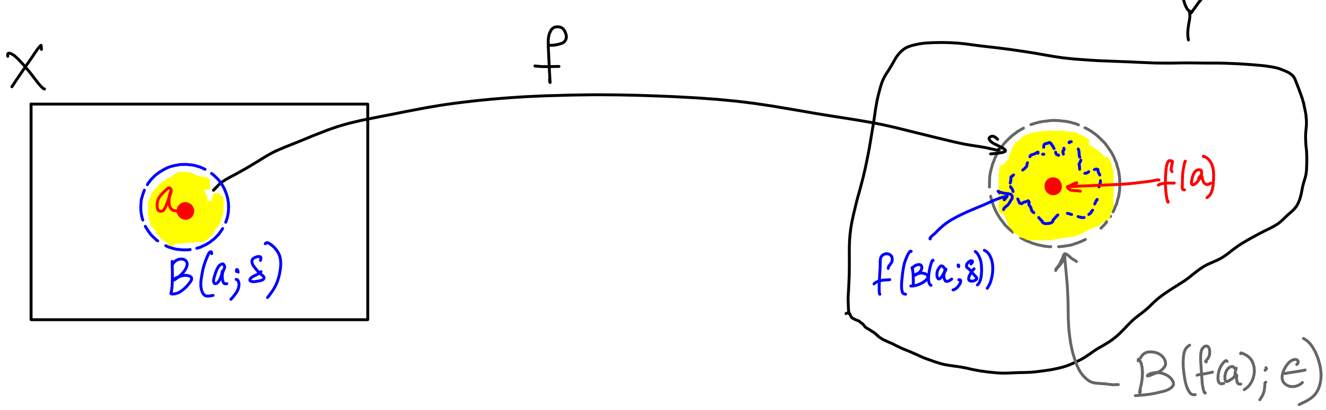
In general, continuous functions map open sets to open sets. We make this notion more precise here.

Def (open ball) Let (X,d) be a metric space and $r>0$, then

$$\overline{B}(a;r) = \{x \in X \mid d(x,a) < r\} \quad \leq$$

is the **open ball** (closed) of radius r centered at $a \in X$.

Some books use $\overline{B}$ to denote the closed ball



Def $f: X \to Y$ is continuous at $a \in X$ if for every open ball $B_Y(f(a);\epsilon)$, $\epsilon>0$, there is an open ball $B_X(a;\delta)$, $\delta>0$, such that $f(B(a;\delta)) \subseteq B(f(a);\epsilon)$.

We will use this definition of continuity later on.

Def The function $f: X \to Y$ is **continuous** if it is so at every $x \in X$.

instead of at just one $a \in X$.

<u>LSIRA Problem 1, pg 51</u> Let (X,d) be the <u>discrete metric space</u>, defined as follows (Example 6, 3.1, pg 46): Let $X \neq \emptyset$, and let

$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y. \end{cases}$$

We can show $d(\cdot)$ is indeed a metric.

Show that the sequence $\{x_n\} \to a$ <u>iff</u> $\exists N \in \mathbb{N}$ such that $x_n = a \ \forall n \geq N$.

$(\Rightarrow)$ $\exists N \in \mathbb{N}$ s.t. $x_n = a \ \forall n \geq N$.

$$\Rightarrow \underbrace{d(x_n,a) = d(a,a) = 0}_{<\epsilon \text{ for any } \epsilon>0.} \ \forall n \geq N \Rightarrow \{x_n\} \to a.$$

$(\Leftarrow)$ $\{x_n\} \to a \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $d(x_n,a) < \epsilon$ whenever $n \geq N$.

Choose $\epsilon = \frac{1}{2}$, and let N_ϵ be its corresponding N. any number <1 will do here!

$$\Rightarrow d(x_n,a) < \frac{1}{2} \ \forall n \geq N_\epsilon.$$

But d is the discrete metric, so $d(x_n,a) < \frac{1}{2} \Rightarrow d(x_n,a) = 0$!

But $d(x_n,a) = 0 \Rightarrow x_n = a \ \forall n \geq N_\epsilon$.

$\square$

Problem 5 pg 52 Let (X, d) be a metric space. Choose $a \in X$. Show $f: X \to \mathbb{R}$ where $f(x) = d(x,a)$ is a continuous function.

Need to show $f(x)$ is continuous at all points in X.

Let $b \in X$; need to show $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(b)| < \epsilon$ whenever $d(x,b) < \delta$. Since b is any point in X, f is continuous.

But $|f(x) - f(b)| = |d(x,a) - d(b,a)| \leq d(x,b)$ We know we will have $d(x,b) < \delta$

by inverse triangle inequality (LSIRA Proposition 3.1.4).

By triangle inequality $d(x,a) \leq d(x,b) + d(b,a)$

$$\Rightarrow d(x,b) \geq d(x,a) - d(b,a) \qquad (1)$$

Also, $d(a,b) \leq d(a,x) + d(x,b)$

$$\Rightarrow d(x,b) \geq d(a,b) - d(a,x)$$
$$= d(b,a) - d(x,a) \qquad (2)$$

by symmetry

(1) and (2) $\Rightarrow d(x,b) \geq |d(x,a) - d(b,a)|$.

Hence by choosing $\delta = \epsilon$, we have

$|f(x) - f(b)| < \epsilon$ whenever $d(x,b) < \delta$.

$\Rightarrow f(x)$ is continuous at $b \in X$.

But b is an arbitrary point in X.

$\Rightarrow f(x)$ is continuous. $\square$

MATH401: Lecture 14 (10/02/2025)

Today: * open and closed sets
* review for midterm exam

But first Inverse triangle inequality (LSIRA Proposition 3.1.4)

$$|d(x,a)-d(b,a)| \leq d(x,b) \equiv d(x,b) \geq |d(x,a)-d(b,a)|, \text{ i.e.,}$$

show $d(x,b) \geq d(x,a)-d(b,a)$ and $d(x,b) \geq d(b,a)-d(x,a)$

Proof

By triangle inequality,

$$d(x,a) \leq d(x,b) + d(b,a)$$

$$\Rightarrow \quad d(x,b) \geq d(x,a) - d(b,a) \quad \text{——— (1)}$$

Also, $d(b,a) \leq d(b,x) + d(x,a)$

$$\Rightarrow \underbrace{d(b,x)}_{= d(x,b) \text{ by symmetry}} \geq d(b,a) - d(x,a) \quad \text{——— (2)}$$

$$(1) \,\&\, (2) \Rightarrow \quad d(x,b) \geq |d(x,a)-d(b,a)|.$$

$\square$

3.3 Open and Closed Sets (in metric spaces)

Recall Ball (open by default): For $a \in (X,d)$, $r>0$

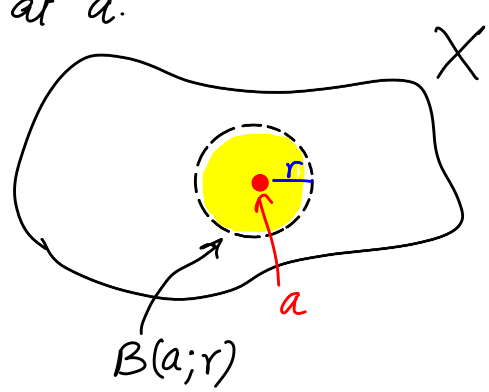
$$B(a;r) = \{x \in X : d(x,a) < r\}$$ is the

open ball of radius r centered at a. Also,

$$\bar{B}(a;r) = \{x \in X : d(x,a) \leq r\}$$ is the

closed ball of radius r centered at a.

We draw open balls with dashed border curves, and closed balls with solid boundary/border curves.



Points and Sets

Def Given $x \in X$ and $A \subseteq X$, there are three possibilities.

(i) $\exists\ B(x;r) \subset A$ for $r>0$; the r-ball at x is contained fully in A

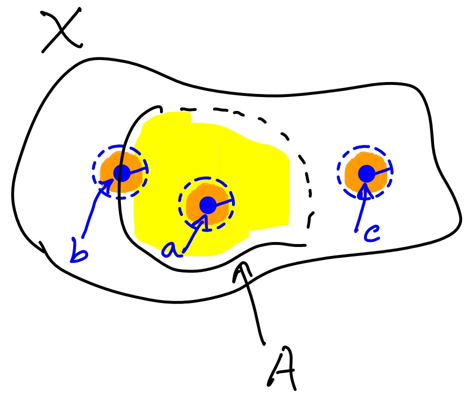
x is an **interior point** of A. e.g., a.

(ii) $\exists\ B(x;r) \subset A^c\ (= X \setminus A)$

x is an **exterior point** of A, e.g., c.

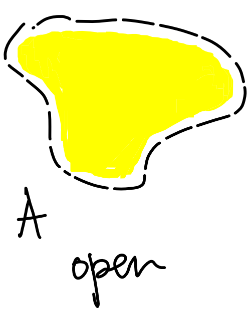
(iii) All balls $B(x;r)$, $r>0$, intersect both A and A^c.

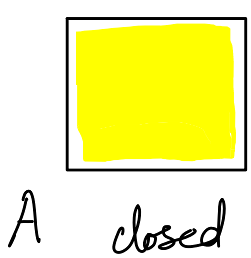
x is a **boundary point** of A, e.g, b.

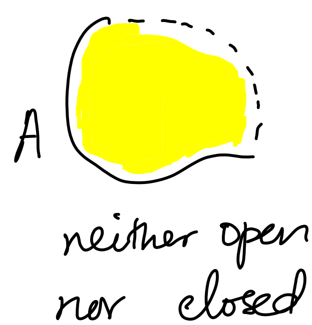


The set of all boundary points of A is denoted ∂A, called "boundary of A".

Def A subset A of a metric space (X,d) is **open** if it does not contain any of its boundary points, and it is **closed** if it contains all its boundary points.







ϕ, X are both open and closed, as they do not have any boundary points.

Proposition 3.3.3 A set $\underbrace{A \subset (X,d)}_{\text{set } A \text{ in a metric space}}$ is open iff it consist of only interior points, i.e., $\forall a \in A,\ \exists r > 0$ s.t. $B(a;r) \subset A$.

Proposition 3.3.4 A set $A \subset (X,d)$ is open iff $\underline{A^c}$ is closed.

↳ complement; taken w.r.t. X

Proof $(\Rightarrow)$ $(\Leftarrow)$ A is open

$\Rightarrow$ $\Leftarrow$ $A \not\ni$ boundary points of A

$\Rightarrow$ $\Leftarrow$ All $\underline{\text{boundary points of } A}$ are in A^c.

$\Rightarrow$ $\Leftarrow$ A^c is closed.

↳ note that boundary points of A are also boundary points of A^c, as every ball centered at these points intersect both A and A^c.

Can present the statements in reverse order for proof in the other direction $(\Leftarrow)$

Given any set A, we can study an associated open set and an associated closed set.

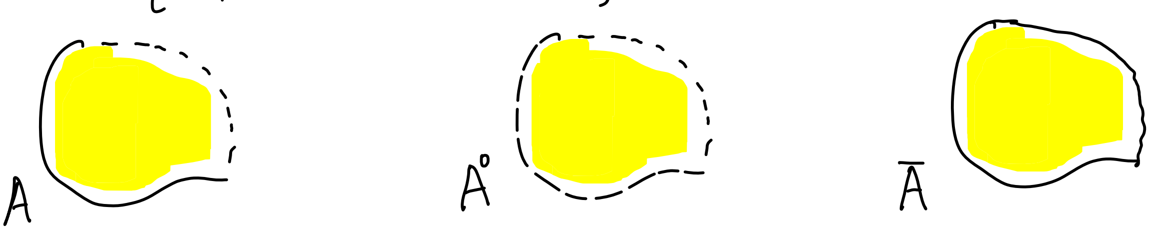
Def The **interior** of $A \subset (X,d)$ is

$$A^\circ = \{x \mid x \text{ is an interior point of } A\},$$

and the **closure** of A is

$$\bar{A} = \{x \mid x \in A \text{ or } x \text{ is a boundary point of } A\}, \text{ or}$$

$$\bar{A} = \{x \mid x \in A \text{ or } x \in \partial A\}.$$



A $\qquad$ A° $\qquad$ $\bar{A}$

Proposition For any set $A \subseteq (X,d)$, we have $A^\circ \subseteq A \subseteq \bar{A}$.

Think about how you can prove this result.

Proposition 3.3.5 (Problem 4a, Pg 58) A° is open, $\bar{A}$ is closed.

A° is open: A° is the set of interior points of A.

$\Rightarrow \forall x \in A^\circ$, $\exists B(x;r) \subset A$, $r>0$.

$\Rightarrow B(x,r) \cap A^c = \phi$.

$\Rightarrow$ x cannot be a boundary point of A.

Note that $\partial(A^\circ) = \partial A$, as the open balls that intersect A must also intersect A°, by definition.

$\Rightarrow A^\circ$ cannot contain any of its boundary points $\Rightarrow A^\circ$ is open.

Also follows directly from Proposition 3.3.3.

To prove $\bar{A}$ is closed, we prove $\bar{A}^c$ is open. By definition,

$$\bar{A}^c = \{x \in X \mid x \notin A \text{ and } x \notin \partial A\}.$$

follows from the definition of $\bar{A} = \{x \mid x \in A \text{ or } x \in \partial A\}$.

Let $x \in \bar{A}^c$. $\Rightarrow \exists r>0$ s.t. $B(x;r) \cap A = \phi$. But we want $B(x;r) \subset \bar{A}^c$.

Suppose $y \in B(x;r)$ be s.t. $y \in \partial A$. $\Rightarrow \exists \epsilon>0$ s.t. $B(y;\epsilon) \cap A \neq \phi$.

definition of boundary point

But $B(y;\epsilon) \subset B(x;r) \Rightarrow B(x;r) \cap A \neq 0$, a contradiction.

$\Rightarrow \forall y \in B(x;r)$, $y \notin A$, $y \notin \partial A \Rightarrow B(x;r) \subset \bar{A}^c$.

$\Rightarrow x$ is an interior point of $\bar{A}^c$.

$\Rightarrow \bar{A}^c$ is open (by Proposition 3.3.3). $\Rightarrow \bar{A}$ is closed.

$\square$

Quick Review for Midterm

Recall: * injective & surjective functions...

↳ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

* relations and equivalence relations.

↳ reflexive, symmetric, transitive

* countability

↳ may not be necessary to work with a decimal representation to construct a proof for uncountability in all cases.

Check problem from HW3!

* Convergence

$\{x_n\} \to a$: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|x_n - a| < \epsilon \ \forall n \geq N$.

* Continuity $f(x)$ is continuous at $x = a$:

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Recall: fg is continuous when f and g are so.

Want to show: $|f(x)g(x) - f(a)g(a)| < \epsilon$

* Choose ϵ_f, ϵ_g, etc., independent of x & $f(x), g(x)$.

Consider $(|g(a)| + \cancel{\epsilon_g})\epsilon_f + |f(a)|\epsilon_g$

If one uses ϵ_g here as well, things could be trickier!

e.g., when $g(a) = 0$, $f(a) \neq 0$, we get

$$\underline{\epsilon_g(\epsilon_f + |f(a)|)} \longrightarrow \epsilon$$

↳ harder to choose ϵ_g, ϵ_f to get ϵ!

MATH401: Lecture 16 (10/09/2025)

Today: * open/closed sets
* continuity using open sets
* completeness in metric spaces

Recall: open and closed sets, interior, boundary, closure of A...

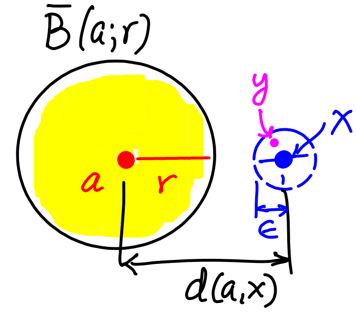
Lemma 3.3.6 $B(a;r)$ is open, and $\bar{B}(a;r)$ is closed.

see LSIRA

We show $\bar{B}(a;r)^c$ is open.

Let $x \notin \bar{B}(a;r) \Rightarrow d(a,x) > r$ —— (1)

Let $\epsilon = \dfrac{d(a,x)-r}{2}$. —— (2)



Consider $y \in B(x;\epsilon) \Rightarrow d(x,y) < \epsilon$.

$$d(a,x) \leq d(a,y) + d(x,y) \quad \text{(triangle inequality)}$$

$$\begin{aligned}
\Rightarrow d(a,y) &\geq d(a,x) - d(x,y) \\
&> d(a,x) - \epsilon \\
&= d(a,x) - \left(\frac{d(a,x)-r}{2}\right) \quad \text{by (2)} \\
&= \frac{d(a,x)+r}{2} \\
&> \frac{r+r}{2} = r \quad \text{by (1)}.
\end{aligned}$$

$\Rightarrow y \notin \bar{B}(a;r)$; this result holds for any $y \in B(x;\epsilon)$.

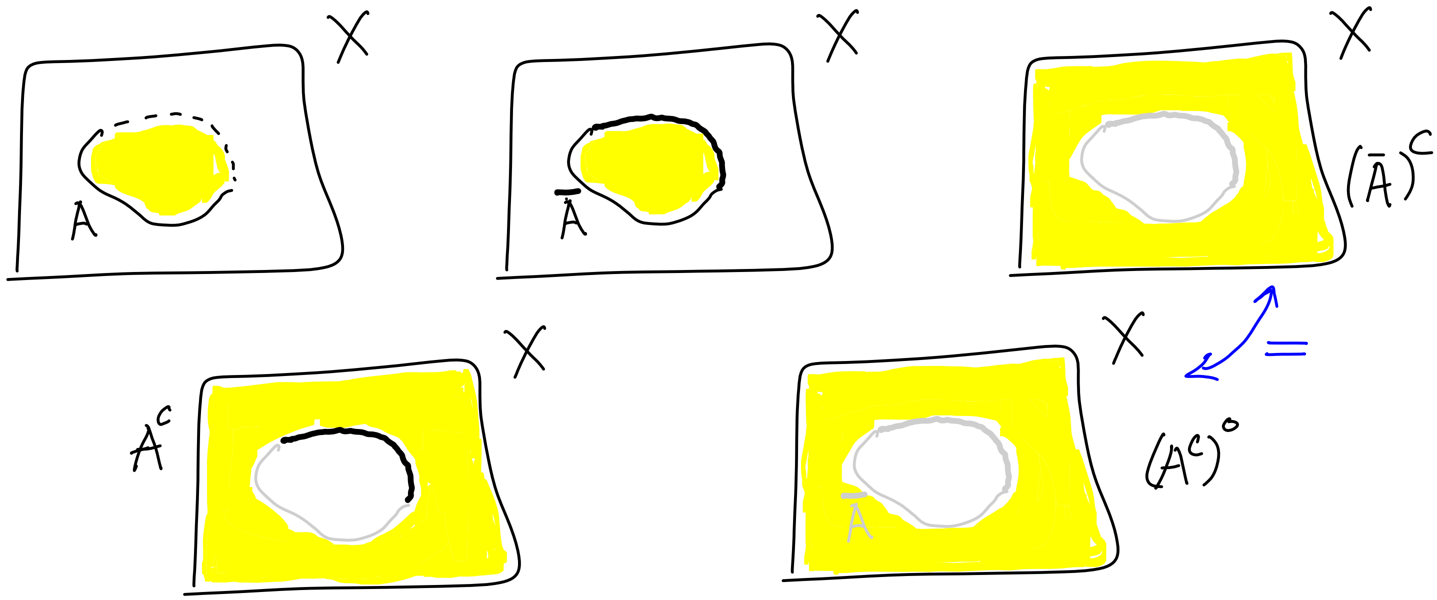
$\Rightarrow B(x;\epsilon) \subseteq \bar{B}(a;r)^c \Rightarrow \bar{B}(a;r)^c$ is open.

$\Rightarrow \bar{B}(a;r)$ is closed. $\square$

We do one more problem before talking about continuity as defined using open sets in metric spaces.

Proposition $(\bar{A})^c = (A^c)^\circ$, where A is a subset of a metric space X.

Here are some illustrations.



$(\subseteq)$ Let $x \in (\bar{A})^c = X \setminus \bar{A}$

$\Rightarrow x \notin A,\ x \notin \partial A$ $\longrightarrow$ $x \in A^c$

$\Rightarrow \exists r>0$ s.t. $B(x;r) \cap A = \phi$

$\Rightarrow B(x;r) \subset A^c \Rightarrow x \in (A^c)^\circ$.

$(\supseteq)$ Let $x \in (A^c)^\circ$ $\longrightarrow$ $x \in A^c$

$\Rightarrow \exists r>0$ s.t. $B(x;r) \subset A^c$

$\Rightarrow B(x;r) \cap A = \phi$.

$\Rightarrow x \notin \partial A$, and $x \notin A$

$\Rightarrow x \in (\bar{A})^c$.

$\square$

Proposition 3.3.7 Let $F \subset (X,d)$. The following are equivalent:

(i) F is closed.

(ii) $\forall \{x_n\}$ convergent in F with $a = \lim_{n\to\infty} x_n$, we have $a \in F$.

Proof in LSIRA. Intuitively, a closed set contains all its limit points.

Continuity

We generalize the notion and definitions of continuity in $\mathbb{R}^m$ to metric spaces.

metric spaces

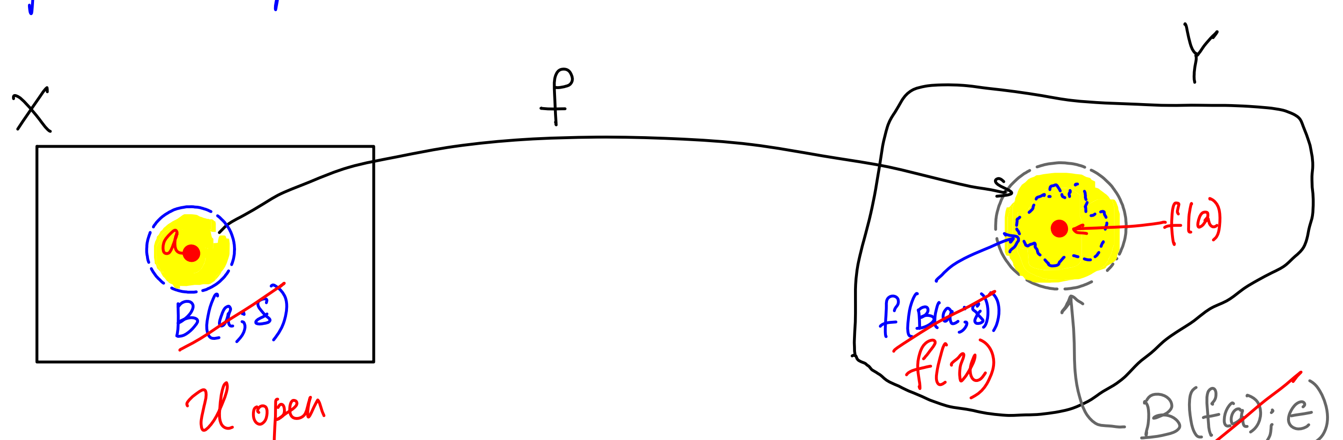
Proposition 3.3.9 Let $f: X \to Y$ be a function, and $x_0 \in X$.

The following statements are equivalent.

(i) f is continuous at x_0.

(ii) $\forall$ open sets $V \ni f(x_0)$ in Y, $\exists$ open set $\mathcal{U} \ni x_0$ in X s.t. $f(\mathcal{U}) \subseteq V$.

Recall the picture from Lecture 13 — we can consider open sets in place of open balls, and the concepts carry through.



We use x_0 here instead of a, but that is a trivial change...

Proof

(i) $\Rightarrow$ (ii)

f is continuous at $x_0 \Longrightarrow$

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$.

Let V be an open set in Y with $f(x_0) \in V$.

$\Rightarrow \exists \ \epsilon > 0$ s.t. $B_Y(f(x_0); \epsilon) \subset V$.

Consider $B_X(x_0; \delta)$; by definition of continuity,

$$f(B_X(x_0; \delta)) \subseteq B_Y(f(x_0); \epsilon) \subset V.$$

$\Rightarrow \mathcal{U} = B_X(x_0; \delta)$ works for (ii).

(ii) $\Rightarrow$ (i)

Consider $V = B_Y(f(x_0); \epsilon)$ → open set containing $f(x_0)$

The result holds for any open set $V \ni f(x_0)$ in Y, so we take $V = B_Y(f(x_0); \epsilon)$

$\exists \ \mathcal{U}$ open, $\mathcal{U} \ni x_0$, s.t. $f(\mathcal{U}) \subseteq V$.

$\mathcal{U}$ open $\Rightarrow \exists \ \delta > 0$ s.t. $B_X(x_0; \delta) \subset \mathcal{U}$.

Take x s.t. $d_X(x, x_0) < \delta \Rightarrow x \in B_X(x_0; \delta) \subseteq \mathcal{U}$

and hence $f(x) \in V = B_Y(f(x_0); \epsilon)$

$\Rightarrow d_Y(f(x), f(x_0)) < \epsilon$.

$\Rightarrow f$ is continuous at x_0, i.e., (i) holds.

$\square$

Continuous functions also map closed sets to closed sets, and this fact is formalized in Proposition 3.3.11.

metric spaces

Proposition 3.3.~~9~~11 Let $f: X \to Y$ be a function, and $x_0 \in X$.

(i) f is continuous at x_0.

(ii) $\forall$ ~~open~~ closed sets $V \ni f(x_0)$ in Y, $\exists$ ~~open~~ closed set $\mathcal{U} \ni x_0$ in X s.t. $f(\mathcal{U}) \subseteq V$.

See LSIRA for proof.

In words, we can replace "open sets" in Prop 3.3.9 with "closed sets" to get Prop 3.3.11.

The book LSIRA specifies definitions of continuity in terms of neighborhoods of x_0 in X and $f(x_0)$ in Y. A neighborhood of x_0 is just an open set containing x_0. But many books define neighborhoods to be either open or closed, but contains an open set that contains x_0.

To avoid any confusion, we refer to open sets containing x_0 (or $f(x_0)$) directly, rather than talk about neighborhoods.

Completeness (LSIRA 3.4)

Recall $\mathbb{R}$ is complete (Section 2.2)
limsup, liminf, Cauchy, ...

We generalize the notion of completeness to metric spaces. It is easier to try and generalize the notion of Cauchy sequences to metric spaces first.

metric space

Def 3.4.1 A sequence $\{x_n\}$ in (X,d) is a Cauchy sequence if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

Proposition 3.4.2 Every convergent sequence in (X,d) is Cauchy.

Let $\{x_n\} \to a$ in (X,d) $\Rightarrow \exists N \in \mathbb{N}$ s.t.
$d(x_n, a) < \frac{\epsilon}{2}$ for any $\epsilon > 0$.

We directly start with $\frac{\epsilon}{2}$ here, instead of ϵ

$$\Rightarrow d(x_n, x_m) \leq d(x_n, a) + d(x_m, a) \quad \text{by } \Delta\text{le ineq.}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever } n, m \geq N.$$

$\Rightarrow \{x_n\}$ is Cauchy. $\square$

MATH401: Lecture 17 (10/14/2025)

Today: * complete metric spaces
* Banach's Fixed Point Theorem (BFPT)

Recall: Every convergent sequence in (X,d) is Cauchy.

But the converse does not always hold.

Example 1

We saw in LSIRA Section 2.2 that $\mathbb{Q}$ is not complete.
With $X=\mathbb{Q}$, and $d(x,y)=|x-y|$, (X,d) is a metric space. can show all properties of metric spaces hold.

Consider $\{x_n\}=\{1.0, 1.4, 1.41, 1.412, \dots,\} \longrightarrow \sqrt{2} \notin \mathbb{Q}$.
closer and closer approximations of $\sqrt{2}$

Each $x_n \in \mathbb{Q}$, and $\{x_n\}$ is Cauchy. (Why?)

Any pair of elements x_n and x_m are identical up to the $(d-1)^{st}$ decimal digit whenever $n,m \geqslant d$; so $|x_n - x_m| < \frac{1}{10^{(d-1)}}$.

Example 2 $\{\frac{1}{n}\}$, $n \geqslant 2$ is Cauchy in $X=(0,1)$ with $d(x,y)=|x-y|$.

$$|x_n - x_k| = \left|\frac{1}{n}-\frac{1}{k}\right| < \frac{1}{N} \quad \text{whenever } n,k \geqslant N. \text{ So, } N=\left\lceil \frac{1}{\epsilon}\right\rceil$$

will do (for proof that $\{x_n\}$ is Cauchy).

But $\{\frac{1}{n}\} \longrightarrow 0$ as $n \to \infty$, and $0 \notin X=(0,1)$.

So we define a metric space as complete when it includes all limit points.

Def 3.4.3 A metric space (X,d) is called **complete** if all Cauchy sequences in X converge in X.

We are throwing in all limit points to "complete" the space, starting with $X=\mathbb{Q}$, we get $\mathbb{R}$. (Example 1).

Example 2: (continued..) $X = [0,1]$ is complete. Note that $\{x_n\} = \{1 - \frac{1}{n}\} \to 1$ as $n \to \infty$.

In fact, we can formalize this observation — if $A \subset X$ is closed, then it will be complete on its own!

Proposition 3.4.4 Assume (X,d) is a complete metric space. If $A \subset X$, then (A, d_A) is complete iff A is closed.

$d_A \to$ restriction of d to A.

$(\Leftarrow)$ A closed.

Consider a Cauchy sequence $\{a_n\}$ in A.

$\{a_n\}$ is a sequence in X as well, as $A \subset X$.

X is complete $\Rightarrow \{a_n\} \to a \in X$.

A is closed $\Rightarrow a \in A$ (by Prop 3.3.7).

$\Rightarrow (A, d_A)$ is complete.

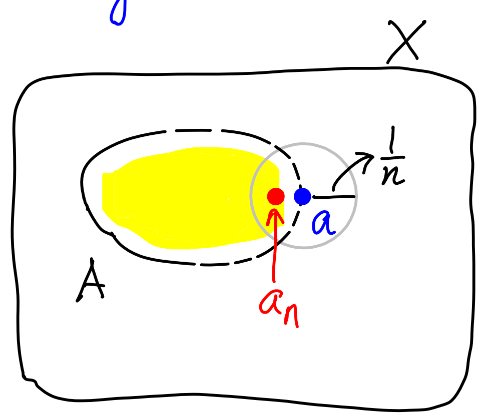
$(\Rightarrow)$ let A be not closed. $\to$ contrapositive argument

$\Rightarrow \exists\, a \in \partial A,\ a \notin A$ $\to$ We identify a Cauchy sequence in A that converges to a.

Pick each $a_n \in B(a; \frac{1}{n})$ such that $a_n \in A$.

$\Rightarrow \{a_n\}$ is Cauchy. (Why?) Also,

$\{a_n\} \to a$ in X, as X is complete, but $\{a_n\}$ does not converge in A (as $a \notin A$).



$\Rightarrow \exists$ a Cauchy sequence in A that does not converge in A.

$\Rightarrow (A, d_A)$ is not complete.

$\square$

Banach's Fixed Point Theorem (BFPT)

We now present a central result in many areas of mathematics – a fixed point theorem. The theorem will depend crucially on completeness of metric spaces. We first define a fixed point.

Def Let $f: X \to X$ be a function, where (X,d) is a metric space. A point $a \in X$ is a **fixed point** for f if $f(a) = a$.

Motivation In many areas of pure and applied mathematics, we often want to solve $g(\bar{x}) = \bar{0}$. → system of equations

If we can write $g(\bar{x}) = f(\bar{x}) - \bar{x} = \bar{0}$, and study $f(\bar{x}) = \bar{x}$, we are solving a fixed point problem!

For example, $\underbrace{x^5 + 4x^3 - 2}_{g(x)} = 0 \implies x = \underbrace{\left(\frac{2 - x^5}{4}\right)^{1/3}}_{f(x)}$.

We can try to find a sequence $\{\bar{x}_n\}$ where $\bar{x}_{n+1} = f(\bar{x}_n)$ instead of solving $g(x) = 0$ directly. And even if we do not know for sure that $g(x) = 0$ has a (unique) solution, we can take $\bar{x}_n$ as our approximate solution when $n \geq N$ for some large N.

Note that $f(\bar{x})$ may not be unique above – e.g., we could write $x = \underbrace{(2 - 4x^3)^{1/5}}_{f(x)}$ and use a different $f(x)$ to still get $f(x) = x$.

We need one more property of f so as to be able to guarantee the existence of a fixed point.

Def $f: X \to X$ is a **contraction** if $\exists\ 0 < s < 1$ such that $d(f(x), f(y)) \le s\, d(x,y)\ \forall\ x, y \in X$. We say that there is a constant s with $0 < s < 1$

s is the **contraction factor** for f.

Note (i) All contractions are continuous. (Why?)
Can use open ϵ-δ ball definition; choose $\delta = \frac{\epsilon}{s}$.

(ii) $d\left(f^{\circ n}(x), f^{\circ n}(y)\right) \le s^n d(x,y)$ where

$$f^{\circ n}(x) = \underbrace{f(f(\cdots f(x)))}_{n \text{ times}} \to n\text{-fold composition of } f$$

We now state and prove Banach's fixed point theorem.

Theorem 3.4.5 (Banach's Fixed Point Theorem)
Let (X,d) be a complete metric space, and $f: X \to X$ be a contraction. Then f has a unique fixed point $a \in X$, and the sequence $\{x_n\}$ converges to a, where $x_0 \in X$ and $x_n = f^{\circ n}(x_0)$, $\forall n \in \mathbb{N}$.

$x_1 = f(x_0)$, $x_2 = f(f(x_0))\ldots$

Proof

We show uniqueness first.
Assume there exist two fixed points $a, b \in X$, $a \neq b$. Then

$$d(a,b) = d(f(a), f(b)) \le s\, d(a,b), \qquad s < 1$$

as a, b are fixed points; as f is a contraction

$$\Rightarrow d(a,b) = 0 \Rightarrow a = b.$$

We prove $\{x_n\}$ is Cauchy. Then $\{x_n\} \to a$, as (X,d) is complete. some $a \in X$

Also, $x_{n+1} = f(x_n) \Rightarrow$ as $n \to \infty$, we get
$a = f(a) \Rightarrow a$ is a fixed point.

So we're done if we prove $\{x_n\}$ is Cauchy.

$$d(x_n, x_k) \leq \sum_{i=0}^{k-1} d(x_{n+i}, x_{n+i+1})$$ by $\triangle$le ineq. (see Lecture 12 for a similar result, showed using induction)

$$= \sum_{i=0}^{k-1} d\left(f^{\circ(n+i)}(x_0), f^{\circ(n+i)}(x_1)\right)$$

$$\leq \sum_{i=0}^{k-1} s^{n+i} d(x_0, x_1)$$ Recall, $0<s<1$.

$$= \frac{s^n(1-s^k)}{1-s} d(x_0, x_1)$$ sum of geometric series

$$\leq \frac{s^n}{1-s} \underbrace{d(x_0, x_1)}_{\text{finite}}$$ $s<1$

We can choose $N \in \mathbb{N}$ large enough such that this expression is $< \epsilon$ for any $\epsilon > 0$ whenever $n, k \geq N$ (as $0<s<1$).

$\Rightarrow$ $\{x_n\}$ is Cauchy! $\square$

$$\frac{s^n}{1-s} d(x_0, x_1) < \epsilon$$

$$\Rightarrow s^n < \frac{(1-s)\epsilon}{d(x_0, x_1)}$$

$$-n \log s \;\overset{?}{>}\; -\log\left(\frac{(1-s)\epsilon}{d(x_0, x_1)}\right)$$

$$n \log\left(\frac{1}{s}\right) > \log\left(\frac{d(x_0, x_1)}{(1-s)\epsilon}\right)$$

$$\Rightarrow N \geq \left\lceil \frac{\log\left(d(x_0,x_1)/(1-s)\epsilon\right)}{\log(1/s)} \right\rceil + 1$$ will do.

MATH401: Lecture 18 (10/16/2025)

Today: * compactness
* relation to closed, bounded, complete sets

Recall BFPT: (X,d) complete, $f: X \to X$ is a contraction:
$\{x_n\} \to a$, where $x_0 \in X$, $x_n = f^{\circ n}(x_0)$ $\forall n \in \mathbb{N}$.

LSIRA Prob 5 (pg 63)

Let $f: [0,1] \to [0,1]$ be differentiable and $\exists$ $0 < s < 1$ s.t. $|f'(x)| < s$ $\forall x \in (0,1)$. Show $\exists$ exactly one $a \in [0,1]$ s.t. $f(a) = a$.

It is evident we want to use Banach's fixed point theorem. In fact, the result is a direct statement of BFPT. So we just need to show f is a contraction, so that BFPT applies.

Let $x, y \in [0,1]$ s.t. $x < y$. Since f is differentiable, we can apply the mean value theorem (MVT). We get that

$$\exists\, c \in (x,y) \text{ s.t. } f'(c) = \frac{f(y) - f(x)}{y - x}.$$

$$\Rightarrow |f(y) - f(x)| = |f'(c)|\,|y-x| \qquad \text{(note that } y > x \text{, so } y - x \neq 0)$$
$$< s\,|y - x|$$

$$\Rightarrow d(f(y), f(x)) < s\, d(y, x) \quad \text{for } s < 1.$$

$$\Rightarrow f \text{ is a contraction.}$$

Hence by BFPT, f has a unique fixed point in $[0,1]$.

We do one more problem on completeness before starting the next topic.

Problem 1, LSIRA Pg 62

1. Show that the discrete metric space is complete. $d(x,y)=\begin{cases}1, & \text{if } x\neq y\\ 0, & \text{if } x=y\end{cases}$

Need to show all Cauchy sequences converge.

Let $\{x_n\}$ be Cauchy. Choose $\epsilon=\frac{1}{2}$. Could use any value <1 here!

$\Rightarrow \exists N\in\mathbb{N}$ s.t. $d(x_n,x_m)<\epsilon=\frac{1}{2}$ whenever $m,n\geqslant N$. choosing the right ϵ directly

But d is the discrete metric $\Rightarrow d(x_n,x_m)=0\Rightarrow x_n=x_m$.

So, with $x_N=a$, we get $x_n=a$ $\forall n\geqslant N$.

$\Rightarrow \{x_n\}$ converges to a. $\square$

3.5 Compact Sets

We now talk about another property of sets that make them "nice", and describe its relation to the concepts we have already seen—closedness, boundedness, completeness, etc.

Recall Theorem 2.2.2: Every monotone bounded sequence in $\mathbb{R}$ converges. We then introduced the concept of Cauchy sequences, and subsequently looked at closed and complete spaces. Recall

Proposition 3.4.4 $A\subseteq X$ (metric space), (A,d_A) is complete iff A is closed.

Now, we introduce the concept of compact spaces. We use the idea of subsequences — which we introduced as part of Bolzano-Weierstrass theorem!

Def Let $\{x_n\}$ be a sequence in (X,d) (metric space). For $n_i \in \mathbb{N}$ s.t. $n_1 < n_2 < \cdots < n_i < n_{i+1} < \cdots$ the sequence $\{y_k\} = \{x_{n_k}\}$ is a **subsequence** of $\{x_n\}$.

$$x_1, \underset{1}{x_2}, x_3, \underset{2}{\cdot}, \cdots, \underset{k}{x_{n_k}}, \cdots$$

$\{y_k\}$: subsequence

Proposition 3.5.1 If $\{x_n\} \to a$, then so do all its subsequences.

Proof (LSIRA Problem 2, Pg 67) Let $\{y_k\} = \{x_{n_k}\}$ be a subsequence.

Note that $k \leq n_k$.

So, $k \geq N \Rightarrow n_k \geq k \geq N$.

$$\Rightarrow d(y_k, a) = d(x_{n_k}, a) < \epsilon \quad \forall k \geq N.$$

$$\Rightarrow \{y_k\} \to a. \qquad \square$$

We use the notion of a convergent subsequences to define compact sets.

Def A set $K \subseteq (X,d)$ is a **compact set** if for every sequence in K, there is a subsequence that converges to a point in K.

(X,d) is compact if X is compact in the above sense.

The main point to note is that the limit points are all in K.

Recall: BW theorem: (Proposition 2.3.2) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (also the same result in $\mathbb{R}^m$)

We first explore the relation between compactness and closedness and boundedness. Recall that finite intervals, e.g., $[a,b] \in \mathbb{R}$, are bounded. We define boundedness for metric spaces.

Def $A \subseteq (X,d)$ is **bounded** if there exists $M \in \mathbb{R}$ such that $d(a,b) \leq M \quad \forall a,b \in A$.

only $0 \leq M < \infty$ makes sense

Recall: A closed set A contains all pts in ∂A.

Compactness implies closedness and boundedness.

Proposition 3.5.4 Every compact set K of (X,d) is closed and bounded.

Contrapositive proofs

1. Assume K is not closed.

$\Rightarrow \exists a \in \partial K$ s.t. $a \notin K$.

Let $\{x_n\}$ be s.t. $d(x_n, a) < \frac{1}{n}$ and $x_n \in K$.

So $\{x_n\} \to a \notin K$. $\Rightarrow$ All $\{y_k\} = \{x_{n_k}\} \to a \notin K$.

So no subsequence converges to a point in K, i.e., K is not compact.

2. Assume K is not bounded.

$\Rightarrow \nexists M$ s.t. $d(a,b) \le M \ \forall a, b \in K$.

b is an arbitrary pt.

Let $b \in K$. $\forall n \in \mathbb{N}$, $\exists x_n \in K$ s.t. $d(x_n, b) > n$.

$\Rightarrow$ For any $\{y_k\} = \{x_{n_k}\}$ $\lim_{k\to\infty} d(y_k, b) = \infty$.

$\Rightarrow \{y_k\}$ cannot converge to a point in K, since

$d(y_k, b) \le d(y_k, a) + d(a,b)$ for any point $a \in K$.

by triangle inequality.

fixed, as a and b are fixed (do not depend on n, k)

$\Rightarrow d(y_k, a) \to \infty$ since $d(y_k, b) \to \infty$ as $k \to \infty$.

$\Rightarrow \{x_n\}$ has no convergent subsequence $\Rightarrow K$ is not compact. $\square$

What about the converse? Holds in $\mathbb{R}^m$!

Corollary 3.5.5 A subset of $\mathbb{R}^m$ is compact iff it is closed and bounded.

Prove only the converse here. → We already saw compact $\Rightarrow$ closed & bounded.

Let A be a closed and bounded set in $\mathbb{R}^m$.

Let $\{\bar{x}_n\}$ be a sequence in A. $\{x_n\}$ is bounded, as A is so.

By BW (Theorem 2.3.3), $\{\bar{x}_n\}$ has a convergent subsequence.

A is closed, so the limit point is in $A \Rightarrow A$ is compact. $\square$

But the converse result does not hold in general for metric spaces.

As an example, consider $(\mathbb{N}, d)$ where d is the discrete metric.

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

$\mathbb{N}$ is complete, closed, bounded.

bounded → as $d \leq 1$ for discrete metric

closed → all candidate limit points are natural numbers

complete → all Cauchy sequences converge

But $\{n\}$ does not converge, and nor does any of its subsequences.

MATH 401: Lecture 19 (10/21/2025)

Today: * compact v/s complete
* compactness under functions
* total boundedness

Recall Compact $\Rightarrow$ closed and bounded ($\equiv$ in $\mathbb{R}^m$)

What is the relation between compactness and completeness?
e.g., $\mathbb{R}$ is complete, but is not compact!
But all compact sets are complete, as we show below.
In this sense, compactness is the strongest "niceness" property we've seen so far.

Lemma 3.5.6 Let $\{x_n\}$ be a Cauchy sequence in (X,d). If $\exists$ a subsequence $\{x_{n_k}\} \to a$, then $\{x_n\} \to a$ also. not necessarily complete

Need to show: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $d(x_n, a) < \epsilon \;\; \forall n \geq N$.

Given 1. $\{x_n\}$ is Cauchy.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \frac{\epsilon}{2} \;\; \forall n, m \geq N$.

2. $\{x_{n_k}\} \to a \Rightarrow \exists K$ s.t. $n_K \geq N$ and

$d(x_{n_K}, a) < \frac{\epsilon}{2}$.

we are directly choosing desired ϵ values here!

$\Rightarrow \forall n, n_K \geq N$

$d(x_n, a) \leq d(x_n, x_{n_K}) + d(x_{n_K}, a)$ (by triangle inequality)

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. $\quad n_K \geq K \geq N$ $\quad \square$

Proposition 3.5.7 Every compact metric space X is complete.

Proof Let $\{x_n\}$ be a Cauchy sequence. Since X is compact, $\{x_n\}$ has a convergent subsequence that converges to $a \in X$ (say). By Lemma 3.5.6, we get that $\{x_n\} \to a$ also. Thus all Cauchy sequences converge, and hence X is complete. $\square$

We next study how compact sets are preserved or not by continuous functions and their inverse images. We get the forward result directly:

Proposition 3.5.9 Let $f: X \to Y$ be continuous.
metric spaces

If $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.

Proof Let $\{y_n\}$ be a sequence in $f(K)$. We want to show it has a convergent subsequence.

We have $y_n \in f(K) \Rightarrow \exists\, x_n \in K$ s.t. $f(x_n) = y_n$.

Consider the sequence $\{x_n\}$ in K. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some $x \in K$. Then $\{y_{n_k}\} = \{f(x_{n_k})\}$ is a subsequence of $\{y_n\}$ that converges to $y = f(x) \in f(K)$ by Proposition 3.2.5 (LSIRA Pg 50). $\square$

This proposition says that for a continuous function $f: X \to Y$ (where X, Y are metric spaces), for all sequences $\{x_n\}$ in X converging to $a \in X$, the sequence $\{f(x_n)\}$ in Y converges to $f(a) \in Y$.

Proposition 3.5.9 says that compact sets get mapped to compact sets by a continuous function. We use this setting to extend the Extreme Value Theorem to arbitrary metric spaces.

Theorem 3.5.10 (Extreme Value Theorem) Let K be a nonempty compact subset of metric space (X,d) and $f: K \to \mathbb{R}$ be continuous. Then f has maximum and minimum points in K, i.e., $\exists\, c, d \in K$ s.t.

$$f(d) \le f(x) \le f(c) \quad \forall\, x \in K.$$

Proof K is compact, f is continuous. So Proposition 3.5.9 gives that $f(K)$ is compact, so it is $\underbrace{\text{closed and bounded}}_{f(K) \subseteq \mathbb{R}}$.

$\Rightarrow \sup f(K), \inf f(K) \in f(K)$ and $\exists\, c, d \in K$ s.t. $f(d) = \inf f(K)$ and $f(c) = \sup f(K)$, i.e., d is a minimum and c is a maximum. $\square$

Compactness may not be preserved under inverse images, as the next problem shows.

LSIRA Problem 8, Pg 68 $f: X \to Y$ is continuous, and let $K \subseteq Y$ is compact. Show that $f^{-1}(K)$ is closed. Find an example where $f^{-1}(K)$ is not compact.

Proof

$K \subseteq Y$ compact $\Rightarrow K$ is closed.

K is closed $\Rightarrow$ $f^{-1}(K)$ is closed.

follows from Proposition 3.3.11, which says continuous functions map closed sets to closed sets.

For the counterexample, consider the following function.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$. could be any constant

$\Rightarrow K = \{0\}$ is closed and bounded, and hence compact (in $\mathbb{R}$).

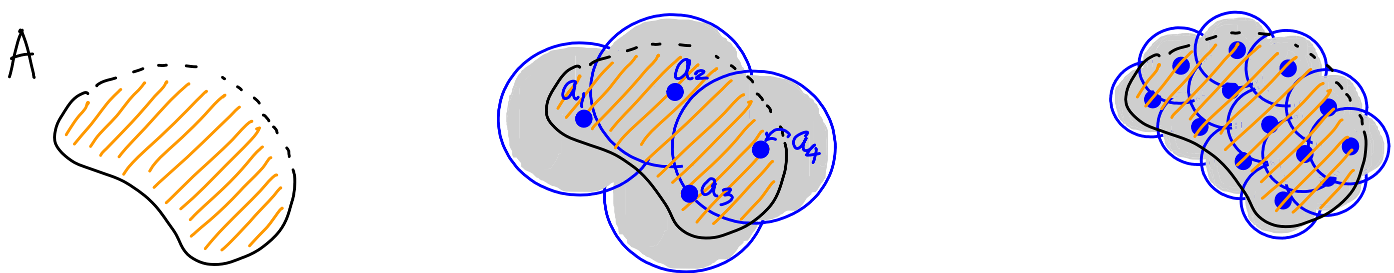
But $f^{-1}(K) = f^{-1}(\{0\}) = \mathbb{R}$ is not bounded, and hence not compact.

Def If f is s.t. $f^{-1}(K)$ is compact whenever K is compact, f is called a **proper** function.

We now introduce a different notion of compactness using open ball covers, which generalizes easily to more general spaces.

Def A set $A \subset (X,d)$ is **totally bounded** if $\forall \epsilon > 0$, there exist finite balls $B(a_i, \epsilon)$, $i = 1, \dots, n$ with $a_i \in A\ \forall i$ that cover A, i.e.,

$$\bigcup_{i=1}^{n} B(a_i, \epsilon) \supseteq A.$$



As ϵ gets smaller, we need to pick more centers $a_i \in A$ to cover A, but it will still be a finite # a_i's.

What is the relationship between compactness and total boundedness? It turns out we get implication in one direction easily.

Proposition 3.5.12 If $A \subseteq (X,d)$ is compact, then it is totally bounded.

Proof We provide a contrapositive proof. We assume A is not totally bounded, and show A is not compact.

We do so by constructing a sequence such that none of its subsequences converge.

let A be not totally bounded. Then there exists some $\epsilon > 0$ such that no finite collection of ϵ-balls centered in A cover A.

To show A is not compact, we construct a sequence $\{x_n\}$ in A that cannot have a convergent subsequence. We pick

$x_1 \in A$ arbitrarily.

As there is no finite collection covering A.

$B(x_1, \epsilon) \not\supseteq A \Rightarrow$ Can pick $x_2 \in A \setminus B(x_1, \epsilon)$.

$B(x_1, \epsilon) \cup B(x_2, \epsilon) \not\supseteq A \Rightarrow$ Can pick $x_3 \in A \setminus \bigcup_{i=1,2} B(x_i, \epsilon)$.

In general, pick $x_n \subseteq A \setminus \bigcup_{i=1}^{n-1} B(x_i, \epsilon)$.

$\Rightarrow \underline{d(x_n, x_m) \geq \epsilon} \; \forall \, n, m \in \mathbb{N}, \; n \neq m.$

Each point x_n is chosen outside of all previous $(n-1)$ ϵ-balls, and hence is $\geq \epsilon$ away from $a_1, \dots, a_{n-1}$.

So $\{x_n\}$ is not Cauchy, and hence not convergent.

Proposition 3.4.2: Every convergent sequence is Cauchy.

But we need to ensure that none of its subsequences converge as well. And we do get that for the same reason!

$\{x_n\}$ cannot have a convergent subsequence.

We can pick any subsequence of $\{x_n\}$ here, say $\{y_k\} = \{x_{n_k}\}$.

$$\Rightarrow d(y_k, y_l) = d(x_{n_k}, x_{n_l}) \geq \epsilon \;\; \forall \, k, l \in \mathbb{N}$$

$\Rightarrow \{y_k\}$ is not Cauchy. $\Rightarrow \{y_k\} = \{x_{n_k}\}$ is not convergent. $\square$

MATH 401: Lecture 20 (10/23/2025)

Today: * compact v/s totally bounded
* open cover property (OCP)

Recall Prop 3.5.12: compact $\Rightarrow$ totally bounded.

What about the converse? Does total boundedness with some extra structure imply compactness?

Recall Corollary 3.5.5: Closed and bounded $\Longleftrightarrow$ compact in $\mathbb{R}$, but equivalence does not hold for all metric spaces.

need completeness also, with closed & totally bounded.

Theorem 3.5.13 A subset $A \subseteq (X,d)$ of a complete metric space (X,d) is compact iff A is closed and totally bounded.

See LSIRA for proof.

What is the relation between total boundedness and boundedness?

LSIRA Problem 9, Pg 68 Show that a totally bounded subset of (X,d) is always bounded. Find a bounded set in some (X,d) that is not totally bounded.

Let $A \subseteq (X,d)$ be totally bounded.

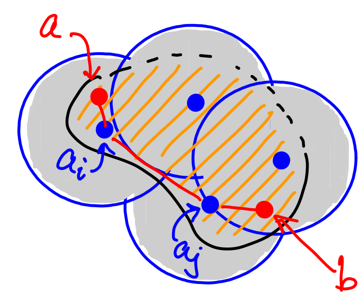
$\Rightarrow$ For $\epsilon = 1$, say, $\exists$ points $a_1, \dots, a_n \in A$ s.t. any finite ϵ will do.

$$\bigcup_{i=1}^{n} B(a_i, 1) \supseteq A.$$

<u>Want to show</u>

$\exists M$ s.t. $d(a,b) \le M \;\forall\, a,b \in A$.

WLOG, let $a \in B(a_i, 1)$ and $b \in B(a_j, 1)$, $i \neq j$.



$$d(a,b) \le d(a,a_i) + d(a_i,a_j) + d(a_j,b) \quad (\triangle\text{le ineq})$$

$$< 1 + \underbrace{\max_{1\le k<l\le n} d(a_k,a_l)}_{\text{finite}} + 1 = M \text{ works!}$$

$d(a_i,a_j) \le \max\limits_{\substack{1\le k,l\le n \\ k\neq l}} \{d(a_k,a_l)\}$ is finite, as it is the largest of $\binom{n}{2}$ pairwise distances (of centers).

Take any infinite set A in (X,d) where d is the discrete metric.

$d(a,b) \le 1 \;\forall\, a,b \in A \Rightarrow A$ is bounded.

A cannot be totally bounded since for $0<\epsilon<1$, $B(a_i,\epsilon) = \{a_i\}$, so we need infinitely many a_i to have ϵ-balls that cover A.

the only values for d are 0 and 1, and $d(a_i,x)=1$ whenever $x \neq a_i$.

LSIRA 3.6 Compactness using Finite Covers

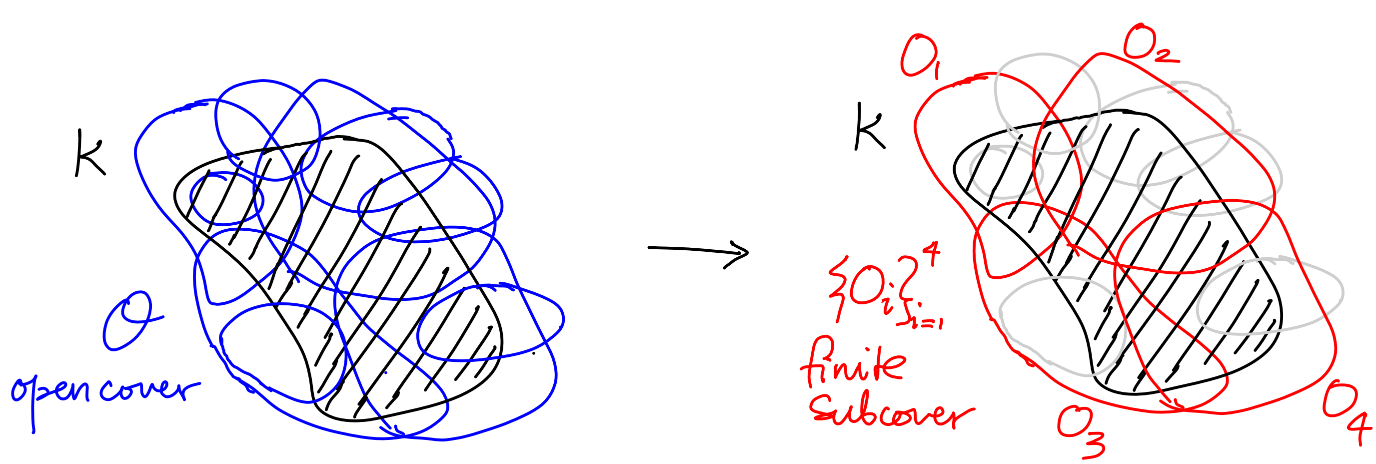
Recall how we extended the notion of continuous functions mapping open balls to (subsets of) open balls to that of mapping open sets to open sets. We can define the concept of total boundedness (with finite # of centers a_i) more generally for open sets.

Def (open cover) An **open cover** of $K \subseteq (X,d)$ is a collection $\mathcal{O}$ (finite or infinite) of open sets, i.e., $\mathcal{O} = \{O_i\}_{i \in I}$ s.t. $K \subseteq \bigcup_{O_i \in \mathcal{O}} O_i$.

→ index set I is infinite in the nontrivial cases

Def (open cover property) Let $K \subseteq (X,d)$. If for every open cover of K $\mathcal{O} = \{O_i\}$, $\exists$ a finite # elements $O_1, O_2, \ldots, O_n$, $O_i \in \mathcal{O}$ s.t. $K \subseteq \bigcup_{i=1}^{n} O_i$, then K has the **open cover property**.

In words, every open cover has a finite subcover.



Note: Each '$\bigcirc$' is supposed to be an open set, even though it's draw with a solid line as opposed to a dashed one...

We get one direction of the equivalence between OCP and compactness readily.

Proposition 3.6.2 If $K \subseteq (X,d)$ has the O.C.P., then it is compact.

Proof Show K not compact $\Longrightarrow$ K does not have OCP. → contrapositive argument

K not compact $\Rightarrow \exists \{x_n\}$ without a convergent subsequence in K.

$\Rightarrow \exists\, x \in K$ and $B(x, \overset{>0}{r_x})$ that contains only finitely many terms of $\{x_n\}$. → as no subseq. converges to x

Note that $\mathcal{O} = \{B(x,r_x)\}_{x\in K}$ is an open cover of K.

But $\mathcal{O}$ cannot have a finite sub cover, as any $\{B(x_i,r_{x_i})\}_{i=1}^{n}$ for $n<\infty$ (finite subcollection of the balls) can have only finitely many terms of $\{x_n\}$. So K does not have the OCP. → each ball has only finitely many terms. □

What about the converse result? We need a lemma first.

Lemma 3.6.3 Let $\mathcal{O}$ be an open cover of $A \subseteq (X,d)$. Let $f: A \to \mathbb{R}$ be defined as

$$f(x) = \sup\{r \in \mathbb{R} \mid r<1 \text{ and } B(x,r) \subseteq O \text{ for some } O \in \mathcal{O}\}.$$

Then f is continuous and is strictly positive.

→ upper bound on the radius of an open ball at x that sits entirely inside a single cover element O of $\mathcal{O}$.

Proof (Strictly positive) $\mathcal{O}$ is an open cover of A. → follows from definition!

$\Rightarrow \exists\, O \in \mathcal{O}$ s.t. $x \in O$ for any $x \in A$.

O is open (by definition) $\Rightarrow \exists\, r > 0$ s.t. $B(x;r) \subseteq O$.

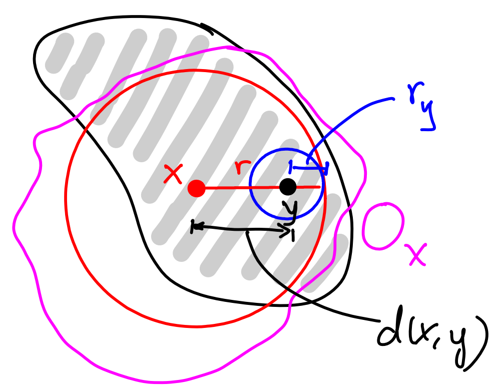
Can also take $r < 1$ here. → helps to keep $f(\cdot)$ bounded.

Why continuity? We want to use EVT to argue that $f(\cdot)$ has a minimum!

(Continuity) We show $|f(x) - f(y)| \leq d(x,y)$ by choosing $\delta = \epsilon$ in the definition of continuity.

If $f(x), f(y) \leq d(x,y)$, the result follows directly.

So assume, WLOG, $f(x) > d(x,y)$, and $f(y) < d(x,y)$.



$\Rightarrow \exists\, r > d(x,y)$ s.t $B(x,r) \subseteq O_x$.

$\Rightarrow$ with $r_y = r - d(x,y)$, we have

$B(y, r_y) \subseteq O_x$ as $B(y, r_y) \subseteq B(x, r_x)$.

$\Rightarrow f(y) \geq r_y = r - d(x,y)$

Since this inequality holds for all such r, it holds for its supremum as well, and hence we get

$$f(y) \geq f(x) - d(x,y).$$

We assumed $f(x)$ is larger $\Rightarrow$

$f(x) - f(y) \leq d(x,y) \Rightarrow |f(x) - f(y)| \leq d(x,y)$ as desired.

We can consider $f(y) > d(x,y) \geq f(x)$, or $f(x), f(y) \geq d(x,y)$ in a similar fashion.

$\square$

We are now ready to present the main theorem, which specifies the equivalence of compactness and O.C.P. This theorem is called the Heine-Borel theorem, but some other books/authors refer to the corresponding result in $\mathbb{R}$ (or $\mathbb{R}^m$) as the Heine-Borel theorem. See Problem 1 in the next lecture...

Theorem 3.6.4 $K \subseteq (X,d)$ is compact iff it has the O.C.P.

We will present the proof in the next lecture...

MATH 401: Lecture 21 (10/28/2025)

Today: * compact $\iff$ OCP
* modes of continuity

Recall OCP: $\mathcal{O} = \{O_i\}_{i \in I}$, $K \subseteq \bigcup_{i \in I} O_i$, $\exists \{O_i\}_{i=1}^{n}$ s.t. $K \subseteq \bigcup_{i=1}^{n} O_i$.

Prop 3.6.2: OCP $\Rightarrow$ compact.

Theorem 3.6.4 compact iff OCP.

Proof Need to prove: If K is compact, and $\mathcal{O}$ is an open cover, then $\mathcal{O}$ has a finite subcover.

We already showed the reverse implication in Prop. 3.6.2.

By the Extreme Value Theorem (EVT, Theorem 3.5.10), $f(x)$ defined in Lemma 3.6.3 taken over K has a minimum value r over K. We can conclude that $r > 0$, as $f(x) > 0 \ \forall x \in K$.

$$\Rightarrow B(x, \tfrac{r}{2}) \subset O_x \in \mathcal{O} \quad \forall x \in K.$$

$\exists\, O_x \in \mathcal{O}$ s.t. $B(x, r/2) \subset O_x \ \forall x \in K$.

Now, K is compact $\Rightarrow K$ is totally bounded.

By Proposition 3.5.12.

$\Rightarrow \exists$ a finite collection of balls $\{B(x_i, \tfrac{r}{2})\}_{i=1}^{n}$ that cover K.

And each such $B(x_i, r/2) \subset O_i \in \mathcal{O}$.

$\Rightarrow \{O_i\}_{i=1}^{n}$ is a finite subcover of $\mathcal{O}$. $\square$

sometimes referred to as the

Problem 1, LSIRA pg 71 (Heine-Borel theorem) Let $\mathcal{I}$ be a collection of open intervals in $\mathbb{R}$ s.t. $[0,1] \subseteq \bigcup_{I \in \mathcal{I}} I$. Show that there is a finite collection $\{I_i\}_{i=1}^{n}$ of intervals from $\mathcal{I}$ s.t. $[0,1] \subseteq \bigcup_{i=1}^{n} I_i$.

Proof $[0,1]$ is a closed and bounded set in $\mathbb{R}$.

$\Rightarrow$ it is compact. $\Rightarrow$ it has the open cover property.

$\Rightarrow$ We can find a finite subset $\{I_i\}_{i=1}^{n}$, $I_i \in \mathcal{I}$ s.t. $[0,1] \subseteq \bigcup_{i=1}^{n} I_i$.

One more problem on open cover property and compactness.

Problem 4, LSIRA pg 71 Let $K_1, \ldots, K_n$ be compact subsets of (X,d). Use the O.C.P to show that $\bigcup_{i=1}^n K_i$ is compact.

Need to show: Any open cover $\mathcal{O} = \{O_i\}$ of $\bigcup_{i=1}^n K_i$ has a finite subcover.

Each K_j, $j=1,\ldots,n$, is compact. For any K_j, as $K_j \subset \bigcup_{i=1}^n K_i$, $\mathcal{O}$ is an open cover for K_j as well. as an open cover of K_j

K_j is compact $\Rightarrow \exists$ finite subcover of $\mathcal{O}$, say, $K_j \subseteq \{O_r^j\}_{r=1}^{n_j}$ with $O_r^j \in \mathcal{O}$, and $n_j < \infty$, for $j=1,\ldots,n$.

Now consider $\mathcal{F} = \bigcup_{j=1}^n \{O_r^j\}_{r=1}^{n_j}$, or $\mathcal{F} = \bigcup_{j=1}^n \bigcup_{r=1}^{n_j} O_r^j$.

$\mathcal{F}$ is a finite collection of open sets, and $K_j \subseteq \mathcal{F}\ \forall j$. # open sets $\leq \sum_{j=1}^n n_j$.

$\Rightarrow \bigcup_{j=1}^n K_j \subseteq \mathcal{F}$.

$\Rightarrow \bigcup_{j=1}^n K_j$ has the O.C.P, and hence it is compact. $\square$

The result may not hold if we have an infinite collection $\{K_j\}_{j \in J}$, $|J|$ not finite, of compact sets.

What about (in)finite intersections of compact sets?

4. Spaces of Continuous Functions

Chapter 4 in LSIRA

We generalize notions of continuity and convergence now to spaces of functions — where elements of the space are functions.

LSIRA 4.1 Modes of Continuity

→ we first generalize notions of continuity of functions

Recall Definition of continuity: $f: X \to Y$ is continuous at $a \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x,a) < \delta$.

$\delta \to \delta(\epsilon, a)$

Q. Can we use the same δ for all $a \in X$?

This is the first generalization we consider...

Def 4.1.1 A function $f: X \to Y$ (metric spaces) is ***uniformly continuous*** if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in X$ with $d_X(x,y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

$\delta \to \delta(\epsilon)$ → independent of $x, y \in X$ (x is used in place of a)

Example 1 Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = 3x+2$. Then f is uniformly continuous

$\forall \epsilon > 0$, we can choose $\delta = \frac{\epsilon}{3}$, as $\forall x, y \in \mathbb{R}$ with $|x-y| < \frac{\epsilon}{3}$ we get $|f(x) - f(y)| = |3x - 3y| = 3|x-y| < 3 \cdot \frac{\epsilon}{3} = \epsilon$.

Note that the choice of $\delta \left(= \frac{\epsilon}{3}\right)$, while independent of $x, y \in X$, depends on the function. If $f(x)$ were $5x+2$, we would have chosen $\delta = \frac{\epsilon}{5}$.

A function that is continuous at all points but not uniformly continuous is called ***pointwise continuous***.

Example 2 $X = (0,2) \subset \mathbb{R}$, $f(x) : X \to \mathbb{R}$ is $f(x) = x^2$.

Show $f(x)$ is uniformly continuous over X.

$\forall \epsilon > 0$, let $\delta = \frac{\epsilon}{4}$.

$\forall x, y \in (0,2)$ and $|x-y| < \delta$, we get

as $x, y \in (0,2)$

$$|f(x)-f(y)| = |x^2-y^2| = |(x+y)(x-y)| < (2+2)|x-y| < 4 \cdot \frac{\epsilon}{4} = \epsilon.$$

Note that we get the same result for $X = [0,2]$ here.

<u>Problem 1, LSIRA pg 80</u> Show $f(x) = x^2$ is not uniformly continuous over $\mathbb{R}$.

<u>Proof by contradiction</u> Assume f is uniformly continuous. Thus, for any $\epsilon > 0$, $\exists \delta > 0$ s.t. $|f(x)-f(y)| < \epsilon$ whenever $|x-y| < \delta$. Then find an x, y and a δ (for a given ϵ) that violates this condition (to get a contradiction).

Goal: Start with x, y s.t. $|x-y| < \delta$ and "solve" $|f(x)-f(y)| \geq \epsilon$, and use $|f(x)-f(y)| = |x^2-y^2| = |(x+y)(x-y)|$. → hint given in LSIRA

Let $y = x + \frac{\delta}{2} \Rightarrow |x-y| = |-\delta/2| < \delta$.

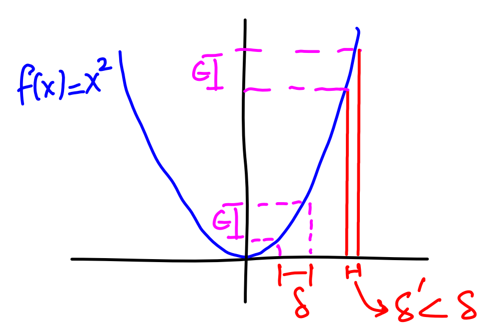
$$\Rightarrow |f(x)-f(y)| = |x^2-y^2| = |(x+y)(x-y)| = \left|2x + \frac{\delta}{2}\right| \left|\frac{\delta}{2}\right|$$

$$= \left| x\delta + \frac{\delta^2}{4} \right| \geq \epsilon$$ want!

Can choose $x = \frac{\epsilon}{\delta} \Rightarrow \left|x\delta + \frac{\delta^2}{4}\right| = \left|\frac{\epsilon}{\delta}\delta + \frac{\delta^2}{4}\right| = \epsilon + \frac{\delta^2}{4} > \epsilon$. ($\frac{\delta^2}{4} > 0$)

Contradiction!

From the graph of $f(x) = x^2$, we can see that we have to choose smaller and smaller δ's to get the same ϵ bound as we go higher.



How are compactness and uniform continuity related?

Proposition 4.1.2 Let X, Y be metric spaces. If X is compact, then all continuous functions $f: X \to Y$ are uniformly continuous.

See LSIRA for proof.

MATH 401: Lecture 22 (10/30/2025)

Today: * pointwise and uniform convergence

Recall: uniform continuity uses same δ for continuity at all $x \in X$.

We now generalize the definition of continuity to use the same δ for a collection of functions (and hence for all points in the domain for each function in the collection).

Def 4.1.3 Let (X, d_X), (Y, d_Y) be metric spaces. $\mathcal{F}$, a collection of functions $f: X \to Y$, is **equicontinuous** if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall f \in \mathcal{F}$, $\forall x, y \in X$ with $d_X(x,y) < \delta$ we have $d_Y(f(x), f(y)) < \epsilon$.

same δ for all $f \in \mathcal{F}$

Problem 3, LSIRA pg 80 $f: X \to Y$ is Lipschitz continuous with Lipschitz constant K if $d_Y(f(x), f(y)) \le K\, d_X(x,y)$ $\forall x, y \in X$. Assume $\mathcal{F}$ is a collection of functions $f: X \to Y$ that are Lipschitz continuous with the same Lipshitz constant K. Show that $\mathcal{F}$ is equicontinuous.

Want to show: $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x,y) < \delta$ $\forall f \in \mathcal{F}$ $\forall x, y \in X$.

Given: $\forall f \in \mathcal{F}$, f is Lipschitz continuous with Lipschitz constant K

$$\Rightarrow d_Y(f(x), f(y)) \le K\, d_X(x,y).$$

Only $K > 0$ makes sense here ($K = 0$ gives result trivially).

Choose $\delta = \dfrac{\epsilon}{K}$, and we get $d_Y(f(x), f(y)) < \epsilon$. $\square$

LSIRA 4.2 Modes of Convergence for Functions

Similar to how we generalized modes of continuity, we now generalize notions of convergence.

Def 4.2.1 Let $(X, d_X), (Y, d_Y)$ be metric spaces, and $\{f_n\}$ be a sequence of functions $f_n : X \to Y$. We say that $\{f_n\}$ **converges pointwise** to $f : X \to Y$ if $f_n(x)$ converges to $f(x)$ $\forall x \in X$. This means $\forall \epsilon > 0, x \in X, \exists N_x \in \mathbb{N}$ s.t. $d_Y(f_n(x), f(x)) < \epsilon$ whenever $n \geq N_x$.

N_x depends on x

We get uniform convergence if the same N can be used $\forall x \in X$.

x

Def 4.2.2 Let $(X, d_X), (Y, d_Y)$ be metric spaces, and $\{f_n\}$ be a sequence of functions $f_n : X \to Y$. We say that $\{f_n\}$ **converges uniformly** to $f : X \to Y$ if $\forall \epsilon > 0, x \in X, \exists N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon$ whenever $n \geq N$.

$\hookrightarrow$ N is independent of $x \in X$.

Let's do a problem on pointwise v/s uniform convergence.

Problem 1, LSIRA pg 85 Let $f_n(x) = \frac{x}{n}$. Show $\{f_n\}$ converges pointwise, but not uniformly to 0.

Pointwise Convergence

Need to show: $\forall x, \epsilon > 0 \ \exists N_x \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \epsilon$ when $n \geq N_x$.

But what is $f(x)$?

For a given x, $\lim_{n \to \infty} \frac{x}{n} = 0 \Rightarrow f(x) = 0$.

$\Rightarrow$ We want $\left|f_n(x)\right| = \left|\frac{x}{n}\right| = \frac{|x|}{n} < \epsilon. \Rightarrow N_x > \frac{|x|}{\epsilon}$ works.

Choose $N_x = \left\lceil \frac{|x|}{\epsilon} \right\rceil + 1$, for instance.

Not converging uniformly: Show that $\forall N \in \mathbb{N}$, and given $\epsilon > 0$, $\exists x$ s.t. $|f_N(x)| > \epsilon$. thus violating the definition of uniform convergence

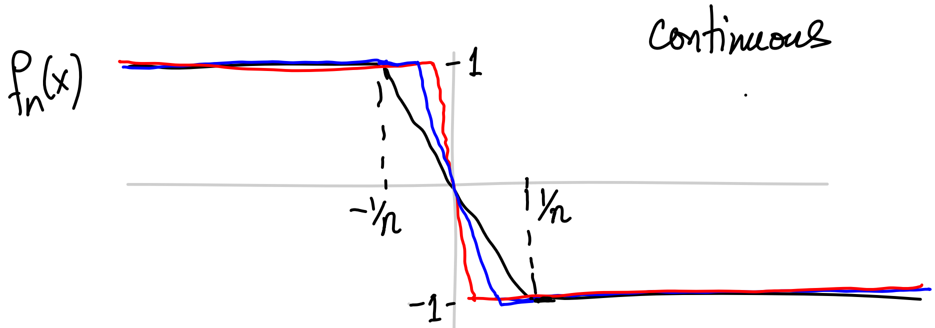
Pick $\epsilon = 1$. For any $N \in \mathbb{N}$, pick $x > N$. Then

$$f_N(x) = \frac{x}{N} > 1 = \epsilon.$$

$\square$

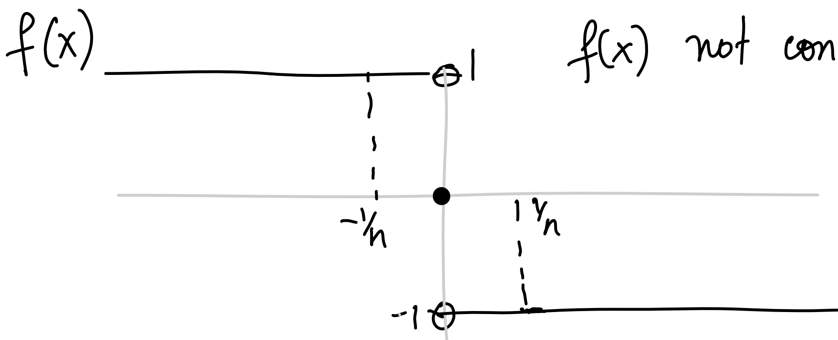
We can have a sequence $\{f_n\}$ of continuous functions converging pointwise to function f that is not continuous!

Consider the sequence of functions shown below for $n \in \mathbb{N}$. Note that each $f_n(x)$ is continuous.



$$f_n(x) = \begin{cases} 1, & x \leq -1/n \\ -nx, & -1/n < x < \frac{1}{n} \\ -1, & x \geq \frac{1}{n} \end{cases}$$

$\{f_n\}$ converges pointwise to the following function $f(x)$, which has a discontinuity at $x=0$!



$f(x)$ not continuous at $x=0$!

$$f(x) = \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ -1, & x > 0 \end{cases}$$

But if we insist on uniform convergence, then continuity is preserved.

Proposition 4.2.4 Let (X, d_X), (Y, d_Y) be metric spaces, and $\{f_n\}$ be a sequence of continuous functions $f_n : X \to Y$ converges uniformly to $f : X \to Y$. Then f is continuous.

Proof: See LSIRA. Uses triangle inequality & $\frac{\epsilon}{3}$-technique.

Problem 5, LSIRA pg 85 Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and the sequence $\{f_n\}$ converges uniformly to $f : \mathbb{R} \to \mathbb{R}$ on all intervals $[-k, k]$, $k \in \mathbb{N}$. Show that $f(x)$ is continuous.

Need to show f is continuous at all $x \in \mathbb{R}$.

Note that $\forall x \in \mathbb{R}$, $\exists\, k \in \mathbb{N}$ s.t. $x \in [-k, k]$.

By Proposition 4.2.4, $f(x)$ is continuous on $[-k, k]$, and hence at x.
But x is arbitrary, and hence $f(x)$ is continuous. $\square$

We now present a way to show uniform convergence by presenting an iff characterization of the same

Proposition 4.2.3 Let (X, d_X), (Y, d_Y) be metric spaces, and $\{f_n\}$ be a sequence of functions $f_n : X \to Y$, with $f : X \to Y$ being another function. The following statements are equivalent.

(i) $\{f_n\}$ converges uniformly to f.

(ii) $\sup \{ d_Y(f_n(x), f(x)) \mid x \in X \} \longrightarrow 0$ as $n \to \infty$.

In words, unif. convergence $\iff$ "max distance" between f_n and $f \to 0$.
This result gives us a way to show uniform convergence.

<u>Problem 3, LSIRA pg 85</u> Let $f_n : [0,\infty) \to \mathbb{R}$ be defined as $f_n(x) = e^{-x}\left(\frac{x}{n}\right)^{ne}$.

a) Show $\{f_n\}$ converges pointwise.

b) Find maximum of $f_n(x)$. Does $\{f_n\}$ converge uniformly?

a) Pointwise convergence to $f(x)$. what is $f(x)$?

$x \in [0,\infty)$, $\epsilon > 0$ are given.

As $n \to \infty$, $\frac{x}{n} \to 0$ and $\left(\frac{x}{n}\right)^n \to 0$ if $\frac{x}{n} < 1$.

$\Rightarrow f(x) = 0$.

<u>Need to show</u> $\forall \overset{\geq 0}{x}, \epsilon > 0$, $\exists N_x \in \mathbb{N}$ s.t. $|f_n(x)| < \epsilon$ $\forall n \geq N_x$.

<u>Note</u>: When $n > x$, $\frac{x}{n} < 1 \Rightarrow \left(\frac{x}{n}\right)^{ne} < \left(\frac{x}{n}\right)$.

$\Rightarrow$ Choose n s.t. $\frac{x}{n} < \min(1, e^x \epsilon)$ to get

$$e^{-x}\left(\frac{x}{n}\right)^{ne} < e^{-x}\left(\frac{x}{n}\right) < e^{-x} e^x \epsilon = \epsilon.$$

$$\Rightarrow N_x = \left\lceil \frac{x}{\min(1, e^x \epsilon)} \right\rceil + 1 \text{ will do.}$$

(b) To find $\max\{f_n(x)\}$, solve $f_n'(x) = 0$.

$$f_n(x) = e^{-x}\left(\frac{x}{n}\right)^{ne}$$ product rule!

$$\Rightarrow f_n'(x) = e^{-x} \cancel{n}e\left(\frac{x}{n}\right)^{(ne-1)}\left(\frac{1}{\cancel{n}}\right) - e^{-x}\left(\frac{x}{n}\right)^{ne}$$

$$= e^{-x}\left(\frac{x}{n}\right)^{ne-1}\left(e - \frac{x}{n}\right) = 0 \Rightarrow x = 0,\ x = ne.$$

$$f_n(x)\big|_{x=0}=0 \quad , \quad f_n(x)\big|_{x=ne}= e^{-ne}\left(\frac{\cancel{n}e}{\cancel{n}}\right)^{ne} = 1.$$

Check $f''(x)\big|_{x=ne}<0$ to ensure it is a local maximum.

$$f''(x) = \underbrace{e^{-x}\left(\frac{x}{n}\right)^{ne-1}\left(-\frac{1}{n}\right)}_{<0 \text{ at } x=ne} + \overbrace{\cancel{\left(e-\frac{x}{n}\right)}}^{=0 \text{ at } x=ne}\left(e^{-x}(ne-1)\left(\frac{x}{n}\right)^{ne-2}\cdot\frac{1}{n} - e^{-x}\left(\frac{x}{n}\right)^{ne-1}\right)$$

$$\Rightarrow \quad \max f_n(x) = 1.$$

$$\Rightarrow \quad \sup\{|f_n(x)-\overbrace{f(x)}^{=0}|\} = 1 \not\to 0 \quad \text{as } n\to\infty$$

Hence by Proposition 4.2.3, $\{f_n\}$ does not converge uniformly.

MATH 401: Lecture 23 (11/04/2025)

Today: $\int$ or $\frac{d}{dx}$ of $\{f_n(x)\}$

4.3 Integrating and Differentiating Sequences

We consider more general questions about properties of $\{f_n(x)\}$ that are "preserved" under convergence. In particular, given $\{f_n(x)\} \xrightarrow[\text{pointwise}]{\text{uniform or}} f(x)$, we consider the following questions:

1. Does $\left\{\int_a^b f_n(x)\,dx\right\} \xrightarrow[\text{pointwise}]{\text{uniform or}} \int_a^b f(x)\,dx$?

2. Does $\{f_n'(x)\} \xrightarrow[\text{pointwise}]{\text{uniform or}} f'(x)$?

Problem 3, LSIRA pg 91 Let $f_n: [0,1] \to \mathbb{R}$ be $f_n(x) = nx(1-x^2)^n$.

Show that $f_n(x) \to 0 \;\forall x \in [0,1]$ as $n \to \infty$, but $\int_0^1 f_n(x)\,dx \to \frac{1}{2}$.

This is a problem directly from your Calculus I & II classes!

For $x \in [0,1]$, we ask $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n = ?$

We consider the end points first:

$x=0 \Rightarrow f_n(0) = n\cdot 0 \cdot (1-0^2)^n = 0.$

$x=1 \Rightarrow f_n(1) = n\cdot 1\cdot (1-1^2)^n = 0.$

For $x \in (0,1)$, $(1-x^2) \in (0,1) \Longrightarrow (1-x^2)^n \to 0$ as $n \to \infty$.

One could guess that the rate of increase on n is smaller than the rate of decrease of $(1-x^2)^n$ as $n\to\infty$, and hence the product $\to 0$. But we can compute the limit exactly!

We use L'Hôpital's rule (L'H):

$$\lim_{n\to\infty} nx(1-x^2)^n = \lim_{n\to\infty}\frac{nx}{(1-x^2)^{-n}} \overset{L'H}{=} \lim_{n\to\infty}\frac{x}{\ln(1-x^2)\,(1-x^2)^{-n}(-1)}$$

$$= \lim_{n\to\infty}\frac{-x\overbrace{(1-x^2)^n}^{0}}{\ln(1-x^2)} = 0.$$

Let's look at the integral now...

$$\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx$$

$$= \left(-\frac{n}{2}\right)\int_0^1 (1-x^2)^n(-2x\,dx)$$

$$= \left(-\frac{n}{2}\right)\int_1^0 u^n\,du = -\frac{n}{2}\left(\frac{u^{n+1}}{n+1}\right)\Bigg|_1^0$$

$$= \frac{n}{2(n+1)}$$

take $u = 1-x^2$

$\Rightarrow \dfrac{du}{dx} = -2x$

$du = -2x\,dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=0$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}.$$

This problem illustrates the situation where $f_n(x)$ converges pointwise to $f(x)$ (=0 here), but $\int f_n(x)\,dx$ does not converge to $\int f(x)\,dx$.

We get "preservation of integrals" when we have uniform convergence!

Proposition 4.3.1 Let $\{f_n(x)\}$ be a sequence of continuous functions that converge uniformly to $f(x)$ on $[a,b]$. Then the functions

$$F_n(x) = \int_a^x f_n(t)\,dt \quad \text{converge uniformly to}$$

$$F(x) = \int_a^x f(t)\,dt \quad \text{on } [a,b]. \longrightarrow \forall x \in [a,b].$$

Given: $\{f_n(x)\}$ converges uniformly to $f(x)$ $\Rightarrow \exists N \in \mathbb{N}$ s.t.

$$|f_n(t) - f(t)| < \frac{\epsilon}{b-a} \quad \forall t \in [a,b].$$

choice of ϵ will become clear below!

$$\Rightarrow |F_n(x) - F(x)| = \left| \int_a^x (f_n(t) - f(t))\,dt \right|$$

$$\leq \int_a^x |f_n(t) - f(t)|\,dt$$

$$< \int_a^x \frac{\epsilon}{b-a}\,dt \leq \int_a^b \frac{\epsilon}{b-a}\,dt = \frac{\epsilon(b-a)}{(b-a)} = \epsilon.$$

$\Rightarrow \{F_n(x)\}$ converges uniformly to $F(x)$ on $[a,b]$. $\square$

It is often helpful to state the above result with variable lower limits for the integrals (rather than a).

Corollary 4.3.2 Let $\{f_n(x)\}$ be a sequence of continuous functions that converge uniformly to $f(x)$ on $[a,b]$. Then the functions

$$F_n(x) = \int_{x_0}^{x} f_n(t)\,dt \quad \text{converge uniformly to}$$

$$F(x) = \int_{x_0}^{x} f(t)\,dt \quad \text{for any } x_0 \in [a,b], \text{ and } \forall x \in [a,b].$$

We now state these results in terms of series. Just as we did with sequences, we would like "nice" properties of individual functions in the series carry over to the limit...

Reformulation in terms of Series

Def A series of functions $\sum_{n=0}^{\infty} v_n(x)$ **converges pointwise** to $f(x)$ on a set I if the sequence $\{S_N(x)\}$ of partial sums $S_N(x) = \sum_{n=0}^{N} v_n(x)$ converge to $f(x)$ $\forall x \in I$. Similarly, the series $\sum_{n=0}^{\infty} v_n(x)$ **converges uniformly** to $f(x)$ on I if $\{S_N(x)\}$ converges uniformly to $f(x)$ on I.

Corollary 4.3.3 Let $\{v_n(x)\}$ be a sequence of continuous functions such that the series $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on $[a,b]$.

Then $\forall\, x_0 \in [a,b]$, the series

$$\sum_{n=0}^{\infty} \int_{x_0}^{x} v_n(t)\,dt \quad \text{converges uniformly, and}$$

$$\int_{x_0}^{x} \sum_{n=0}^{\infty} v_n(t)\,dt = \sum_{n=0}^{\infty} \int_{x_0}^{x} v_n(t)\,dt.$$

Key message: We can integrate the series term-by-term! We cannot always interchange the order of $\int$ and $\sum$, but can do so here due to uniform convergence of the input series.

In order to use this result, we need a way to test if a given series converges uniformly. How can we check if $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly?

Recall Proposition 4.2.3, which could be used to test uniform convergence of a sequence.

Proposition 4.3.4 (Weierstrass' M-test) Let $\{v_n(x)\}$ be a sequence of functions $v_n : A \to \mathbb{R}$, and let there exist a convergent series $\sum_{n=0}^{\infty} M_n$ such that $M_n \geq 0$ and $|v_n(x)| \leq M_n \;\forall n \in \mathbb{N}, \forall x \in A$.

Then $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on A.

Show existence of a series of term-wise (upper) bounds!

Problem 1, LSIRA pg 91 Show that $\sum_{n=0}^{\infty} \frac{\cos(nx)}{n^2+1}$ converges uniformly on $\mathbb{R}$.

$$|v_n(x)| = \left| \frac{\cos(nx)}{n^2+1} \right| \leq \frac{1}{n^2+1} \leq \frac{1}{n^2} = M_n \quad (n \geq 1).$$

Note that the term for $n=0$ is $\frac{\cos(0)}{1} = 1$, which can be handled separately.

Now, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Why?

Hence the input series converges uniformly by Weierstrass' M-test.

$$\underbrace{S_n}_{\text{partial sums}} = \sum_{k=1}^{n} \frac{1}{k^2}.$$

To show a series converges, show the corresponding sequence of partial sums converges.

Note: $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.

$$\Rightarrow S_n \leq 1 + \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2.$$

$$\Rightarrow \lim_{n \to \infty} S_n = 2.$$

Hence by Corollary 4.3.3, we get that

this observation is not part of the solution to Problem 1 above!

$$\int_0^x \sum_{n=0}^{\infty} \frac{\cos(nt)}{n^2+1}\, dt = \sum_{n=0}^{\infty} \underbrace{\int_0^x \frac{\cos(nt)}{n^2+1}\, dt}_{\left.\frac{\sin(nt)}{n(n^2+1)}\right|_0^x} = \sum_{n=0}^{\infty} \frac{\sin(nx)}{n(n^2+1)}.$$

We now consider the same question on derivatives. If a sequence/series of functions converges (uniformly or pointwise) to a function, do their derivatives also converge to the derivative of the limit function?

We immediately get the answer is no in general!

Example

It can be shown that the sequence $\left\{\frac{\sin nx}{n}\right\}$ converges uniformly to 0, but the sequence of derivatives $\{\cos nx\}$ does not converge.

oscillates between -1 and 1