MATH401: Lecture 26 (11/18/2025)

Today: * Applications to differential equations
* BFPT for differential equations

4.7 Application to Differential Equations

We show how to use properties we presented (completeness, contraction, Banach's fixed point theorem, etc.) to show existence and uniqueness of solutions to differential equations.

Let's start with a problem to recall concepts related to differential equations.

Problem 1, LSIRA pg 106 Solve the initial value problem

$$y' = 1+y^2, \quad y(0)=0, \qquad y=y(t) \text{ here}$$

and show the solution is defined only on $[0, \pi/2)$.

$$y' = \frac{dy}{dt} = 1+y^2 \implies \int \frac{dy}{1+y^2} = \int dt \quad \text{integrate both sides.}$$

$$\implies \arctan y = t+c$$

$$\implies y(t) = \tan(t+c)$$

$$y(0)=0 \implies \tan(0+c)=0 \implies c=0$$

$y(t)=\tan(t)$ is the solution.

$\tan t = \frac{\sin t}{\cos t}$ is undefined at $t=\frac{\pi}{2}$, so the solution here is defined over $[0, \pi/2)$.

We consider the system "evolving" with time, starting at $t=0$, at which the initial value is specified ($y(0)=0$).

In general, we consider a system of the form

$$(1) \qquad \bar{y}'(t) = \bar{f}(t, \bar{y}(t)), \quad \bar{y}(0) = \bar{y}_0.$$

My **notation**: $\bar{x}, \bar{y}, \bar{\alpha}$ etc. (lower case letters with a bar) are vectors. The book uses bold lower case letters $\mathbf{x}, \mathbf{y}, \boldsymbol{\alpha}$, etc. to denote vectors.

Here, $\bar{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$, $\bar{y}'(t) = \begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix}$, $\bar{y}_0 = \begin{bmatrix} Y_0 \\ \vdots \\ Y_n \end{bmatrix}$ constants

Integrating (1) gives the following integral equation:

$$(2) \text{———} \qquad \bar{y}(t) = \bar{y}_0 + \underbrace{\int_0^t \bar{f}(s, \bar{y}(s))\,ds}_{}$$

A function, $\bar{y}(s)$ is input, $\bar{y}(t)$ is output

(1) and (2) are equivalent, i.e., they have the same solutions.

We can think of the right-hand side of (2) as a function, which takes in $y(s)$ and outputs $y(t)$. To make this concept more clear, we rewrite (2) more generally as

$$u(t) = \bar{y}_0 + \int_0^t \bar{f}(s, \bar{z}(s))\,ds = F(\bar{z}) \text{ or } F(\bar{z}(s)).$$

With $F(\bar{z})(t) = \bar{y}_0 + \int_0^t \bar{f}(s, \bar{z}(s))\,ds$, the equation can be written as

$$\bar{u} = F(\bar{z}).$$

We can now pose this problem as a fixed point problem: find $\bar{y}$ such that $\bar{y} = F(\bar{y})$. We then talk about conditions under which F is a contraction, so that we can get a unique fixed point by Banach's fixed point theorem.

We need one more definition to introduce this condition (when F is a contraction).

Def The function $\bar{f}: \underbrace{[a,b]}_{t,\text{ for time}} \times \mathbb{R}^n \to \mathbb{R}^n$ is **uniformly Lipschitz** with Lipschitz constant K on interval $[a,b]$ if $K \geq 0$ is such that

$$\|\bar{f}(t,\bar{y}) - \bar{f}(t,\bar{z})\| \leq K\|\bar{y}-\bar{z}\| \quad \forall t \in [a,b],\ \forall \bar{y},\bar{z} \in \mathbb{R}^n.$$

We can show that F is a contraction when $\bar{f}$ is uniformly Lipschitz.

Lemma 4.7.1 Let $\bar{y}_0 \in \mathbb{R}^n$ and $\bar{f}: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and uniformly Lipschitz with Lipschitz constant K on $[0,\infty)$. If $a < \frac{1}{K}$, then the map $F: C([0,a],\mathbb{R}^n) \to C(\underbrace{[0,a]}_{t}, \underbrace{\mathbb{R}^n}_{\bar{z}})$ defined as

$$F(\bar{z})(t) = \bar{y}_0 + \int_0^t \bar{f}(t,\bar{z}(s))\,ds$$

is a contraction.

Proof For $\bar{v},\bar{w} \in C([0,a],\mathbb{R}^n)$, $t \in [0,a]$,

$$\|F(\bar{v})(t) - F(\bar{w})(t)\| = \left\| \int_0^t \left[\bar{f}(s,\bar{v}(s)) - \bar{f}(s,\bar{w}(s))\right] ds \right\|$$

$$\leq \int_0^t \|\bar{f}(s,\bar{v}(s)) - \bar{f}(s,\bar{w}(s))\|\,ds$$

$$\leq \int_0^t K\|\bar{v}(s)-\bar{w}(s)\|\,ds \qquad \bar{f}: \text{Unif. Lip.}$$

$$\leq K\int_0^t \rho(\bar{v},\bar{w})\,ds \leq K\int_0^a \rho(\bar{v},\bar{w})\,ds = \underbrace{Ka}_{<1,\text{ as } a<\frac{1}{K}}\rho(\bar{v},\bar{w}).$$

Taking sup over all $t \in [0,a]$, we get

$$\rho(F(\bar{v}),F(\bar{w})) \leq s\,\rho(\bar{v},\bar{w}) \quad \text{for } s = Ka < 1.$$

$\Rightarrow F$ is a contraction. $\square$

We now state the main result about existence of unique solution to an initial value problem.

Theorem 4.7.2 Let $\bar{y}_0 \in \mathbb{R}^n$, $\bar{f} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and uniformly Lipschitz on $[0,\infty)$. Then the $\underbrace{\text{initial value problem}}_{\text{IVP}}$

$$\bar{y}'(t) = \bar{f}(t, \bar{y}(t)), \quad \bar{y}(0) = y_0$$

has a unique solution $\bar{y}$ on $[0,\infty)$.

Proof IDEA: Start with $[0,a]$, apply Lemma 4.7.1. F is a contraction.

So, BFPT $\Rightarrow$ unique solution.

Extend that solution to $[a,2a] \to [2a,3a] \to \cdots \to [0,\infty)$.

We end by giving two examples of IVP — one where we get a unique solution, and another with multiple solutions.

Example 2 IVP: $y' = 0.9y$. $y(0) = 5$. $y = y(t)$, $y' = \frac{dy}{dt}$ here.

Looks very much like a contraction!

$$\frac{dy}{dt} = 0.9y \Rightarrow \int \frac{dy}{y} = \int 0.9\, dt$$ integrate!

$\Rightarrow \ln|y| = 0.9t + C$ $\quad$ $t=0$: $y(0)=5 \Rightarrow$ $5 = e^C$

$\Rightarrow |y| = e^C e^{0.9t}$

$\Rightarrow y = 5e^{0.9t}$

$y > 0$, so can replace $|y|$ with y.

Problem 2, LSIRA Pg 106 Show that all functions

$$y(t) = \begin{cases} 0, & 0 \le t \le a \\ (t-a)^{3/2}, & t > a \end{cases}, \quad \text{for } a \geq 0$$

are solutions to the IVP: $y' = \frac{3}{2} y^{1/3}$, $y(0) = 0$.

We first find y' for all t except at $t=a$ from the definition of $y(t)$.

$$y'(t) = \begin{cases} 0, & 0 \le t < a \\ \frac{3}{2}(t-a)^{1/2}, & t > a \end{cases}$$

We now check the differential equation is satisfied for all $t \geq 0, t \neq a$.

$$t < a: \quad y'(t) = 0 = \frac{3}{2} y^{1/3} = \frac{3}{2}(0)^{1/3} \quad \checkmark$$

$$t > a: \quad y'(t) = \frac{3}{2}(t-a)^{1/2} = \frac{3}{2}\left((t-a)^{3/2}\right)^{1/3} = \frac{3}{2}(y(t))^{1/3} \quad \checkmark$$

We need to be careful at $t=a$: need to find $y'(a)$ using its limit definition.

At $t=a$

$t \searrow a$, $t \nearrow a$ limits should be same

$$y'(a) = \lim_{h \to 0} \frac{y(a \pm h) - y(a)}{h} \qquad y(a) = 0$$

$$\lim_{h \to 0} \frac{y(a-h)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

$$\lim_{h \to 0} \frac{y(a+h)}{h} = \lim_{h \to 0} \frac{(a+h-a)^{3/2}}{h} = \lim_{h \to 0} h^{1/2} = 0.$$

$\Rightarrow y'(a) = 0.$

We check the IVP is satisfied at $t=a$:

$$\Rightarrow y'(a) = 0 = \frac{3}{2}(y(a))^{1/3} = \frac{3}{2}(0)^{1/3} = 0 \quad \checkmark$$