MATH 524 : Lecture 1 (08/19/2025)

This is Algebraic Topology.
I'm Bala Krishnamoorthy (Call me Bala).

Today:
* syllabus, logistics
* neighborhoods, continuous functions
* topology using neighborhoods
* homeomorphism

I will be teaching computational topology (Math 529) next semester. The two classes – Math 524 and Math 529 will be kept independent. In particular, we will spend nearly no focus on computational aspects in Math 524.

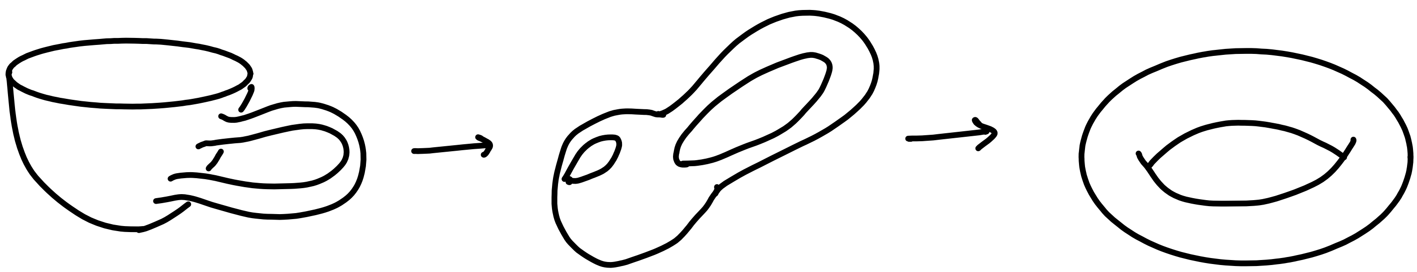
Check the course web page at

https://bala-krishnamoorthy.github.io/Math524.html

All documents, important updates, homework assignments etc. will be posted there. Check the class page frequently.

More about the video assignment to come soon. But you're encouraged to start looking for topics that you might want to make the video on as we proceed in the course.

Topology studies how "space is connected". You might have heard the (true!) joke that the topologist cannot distinguish between a coffee cup and a donut! Indeed, they both are connected the same way.



In algebraic topology, we cast problems on how space is connected as equivalent problems on algebraic objects — groups, rings, etc., and maps between them (homomorphisms).

As a subfield of mathematics, algebraic topology started in late 19^{th} and early 20^{th} century. Poincaré introduced the fundamental group first. Later Betti introduced homology groups, which are much easier to compute (both by hand as well as algorithmically) than the former.

We will spend a lot of time talking about homology groups, and the dual concept of cohomology. We will not be spending much attention on the fundamental group. There are several (equivarnt) ways to define homology groups Perhaps the "nicest" way to do so is using simplicial complexes. We will spend a fair bit on time studying simplicial homology.

We will introduce/refresh background concepts as needed. First, we will talk about continuous functions and topological spaces, defined in terms of neighborhoods.

Continuous functions

We first give the classical ε-δ definition in Euclean spaces.

Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$. f is continuous at $\bar{x} \in \mathbb{R}^n$ if there exists $\delta > 0$ for every $\varepsilon > 0$ such that $\|f(\bar{y}) - f(\bar{x})\| < \varepsilon$ whenever $\|\bar{y} - \bar{x}\| < \delta$ for $\bar{y} \in \mathbb{R}^n$. f is continuous (in all of $\mathbb{R}^n$) if it is so at every $\bar{x} \in \mathbb{R}^n$.

my notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\mu}$, etc., are all vectors – lower case letters with a bar.

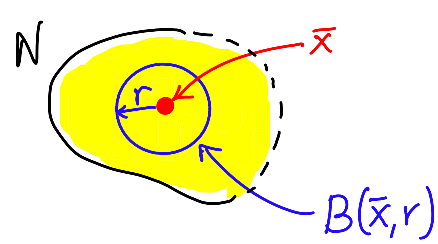
We give an equivalent definition based on neighborhoods.

Def A subset N of $\mathbb{R}^n$ is a **neighborhood** of $\bar{x} \in \mathbb{R}^n$ if for some $r > 0$, the closed ball $B(\bar{x}, r)$ centered at $\bar{x}$ is contained entirely within N.

Notice that neighborhood N can be open or closed.

$$B(\bar{x}, r) = \{\bar{y} \in \mathbb{R}^n \mid \|\bar{x} - \bar{y}\| \leq r\}$$

closed Ball of radius r centered at $\bar{x}$



Def $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous if given any $\bar{x} \in \mathbb{R}^n$ and a neighborhood N of $f(\bar{x})$ in $\mathbb{R}^m$, $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in $\mathbb{R}^n$.

Now we define what a topological space (or topology) is. We give the definition in terms of neighborhoods first. In most textbooks, you will see the definition given in terms of open sets. Later, we will see that both definitions are equivalent.

Topological space (or topology)

more notation: Upper case letters, e.g., A, B, X, Y, etc, denote sets or matrices.

Def I We are given a set X and a nonempty collection of subsets of X for each $\bar{x} \in X$ called the neighborhoods of $\bar{x}$. This is a topological space if it satisfies the following axioms.

(a) $\bar{x}$ lies in each of its neighborhood.

(b) Intersection of two neighborhoods of $\bar{x}$ is itself a neighborhood of $\bar{x}$.

(c) If N is a neighborhood of $\bar{x}$, and $\mathcal{U} \subseteq X$ contains N, then $\mathcal{U}$ is a neighborhood of $\bar{x}$.

(d) If N is a neighborhood of $\bar{x}$, $\mathring{N}$, the **interior** of N is also a neighborhood of $\bar{x}$.

The interior of N is $\mathring{N} = \{\bar{y} \in N \mid N \text{ is a neighborhood of } \bar{y}\}$.

Intuitively, every point of N not on its boundary is in its interior.

We can extend the definition of continuous functions to functions defined between topological spaces.

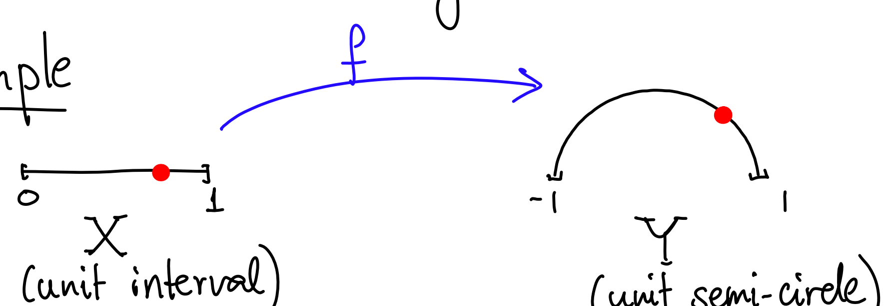
Def Let X, Y be topological spaces. $f: X \to Y$ is continuous if $\forall\, \bar{x} \in X$ and for every neighborhood N of $f(\bar{x})$ in Y, the set $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in X.

We are interested in studying when two topological spaces are similar. There are a few different notions of topological similarity, and the strongest notion is that of homeomorphism. For two spaces to be homeomorphic, we need a function between them that is "nicer" than just a continuous function.

Def A function $f: X \to Y$ is a **homeomorphism** if it is one-to-one, onto, continuous, and has a continuous inverse.

When such a function exists between two spaces X and Y, we say they are **homeomorphic**, or are topologically equivalent. We denote this fact by $X \approx Y$.

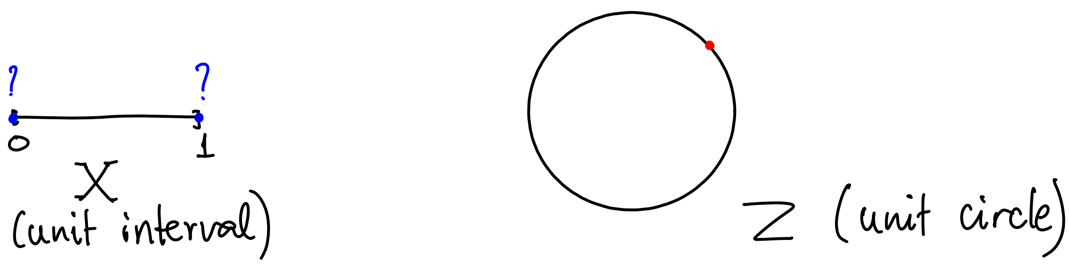
Example



$X \approx Y$. Can you define the function f?

Think of X & Y as subsets of $\mathbb{R}^2$, and write down the form of f, as well as f^{-1}. You can show f satisfies all requirements for being a homeomorphism.

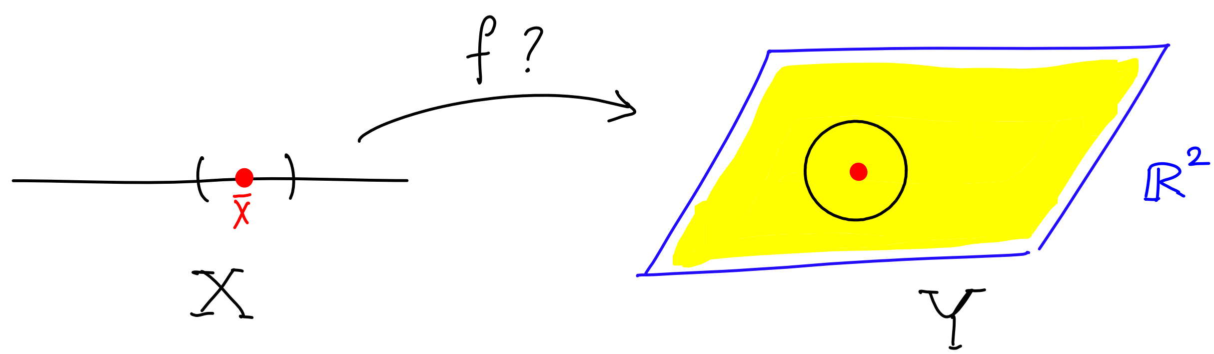
Showing two spaces are **not** homeomorphic could be harder — we need to show that no such function exists between X and Y.



Here, $X \not\cong Z$. Where do things breakdown?

Intuitively, one can notice the two end points of X behave distinctly from any point in Y.

Here is another example. Perhaps the simplest example of a topological space is $\mathbb{R}^d$ under the usual definition of neighborhoods, which specifies that any set $N \subseteq \mathbb{R}^d$ big enough to contain $B(\bar{x}, r)$ for some $r>0$ is a neighborhood of $\bar{x} \in \mathbb{R}^d$. But notice that $\mathbb{R}^1 \not\cong \mathbb{R}^2$, for instance. It is not straightforward to prove this fact rigorously. But, how would one "argue" for it?



One method is to appeal to how the two spaces are connected. Recall that topologically similar spaces are "connected" the same way. Here, if we remove one point from both $X = \mathbb{R}^1$ and $Y = \mathbb{R}^2$, we can see that it affects the connectivity differently. Removing one point leaves X disconnected (into two pieces). But removing a point from Y still leaves it connected – its just like poking a hole in the "sheet" that is $\mathbb{R}^2$, which remains connected.

More formally, we could try to define a homeomorphism from X to Y. But we can observe that neighborhoods in X are 1-dimensional, while those in Y are 2D. Hence we cannot define a bijection between them.

We will talk about open sets in the next lecture, and define a topology using open sets. That definition is equivalent to the one introduced earlier today, i.e., Def I.

MATH 524 : Lecture 2 (08/21/2025)

Today: * open sets, topology using open sets
* simplices, properties of simplices

We now consider topology defined in terms of open sets. This is the default approach taken in most textbooks. We first define open sets using the concept of neighborhoods.

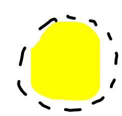
Def $O \subseteq X$ is **open** if it is a neighborhood of each of its points. By (c) of **Def I**, union of any collection of open sets is also open. Also, by (b) of **Def I**, the intersection of any finite number of open sets is open.

We mention unions and finite intersections of open sets as they are both required to be open in a topology. See below.

Notice, $\mathring{N}$ (interior of neighborhood N) is always open.

Alternatively, we can start by defining open sets directly.

Def A set $A \subseteq \mathbb{R}^n$ is **open** if each $\bar{x} \in A$ can be surrounded by a ball of positive radius that lies entirely inside the set.



We can also define open sets more generally, starting with collections of subsets of some set X.

We could define neighborhoods in terms of open sets.

Def A subset $N \subseteq X$ is a neighborhood of $\bar{x}$ if there exists an open set O s.t. $\bar{x} \in O \subseteq N$.

We now formally state the definition of topology in terms of open sets. This definition sees more use than the one using neighborhoods.

Def II A **topology** on a set X is a collection of open sets of X such that any union and finite intersection of open sets is open, and ϕ (empty set) and X are open.
The set X along with the topology is called a **topological space**.

We can define continuous functions also in terms of open sets.

Def $f: X \to Y$ is **continuous** iff (if and only if) the inverse image of each open set of Y is open in X.

We now start the discussion of homology, which is a less strict version of topological similarity than homeomorphism. We study in detail simplicial homology, where the spaces are made of "gluing" "nice" objects called simplices together, and are hence are very "regular".

As we will see, it is also much easier to algebraize questions about homology (than those about homeomorphism).

There is a "continuous" version of homology defined on spaces not composed to regular pieces (simplices), termed singular homology. It turns out singular homology is equivalent to simplicial homology.

We start by defining simplices, which are the building blocks.

Simplices

We define simplices in the usual geometric setting first, and then define them abstractly. We need some concepts from geometry first.

Def The set $\{\bar{a}_0, \dots, \bar{a}_n\}$ of points in $\mathbb{R}^d$ is **geometrically independent** (**GI**) if for any scalars $t_i \in \mathbb{R}$, the equations
$$\sum_{i=0}^{n} t_i = 0, \quad \sum_{i=0}^{n} t_i \bar{a}_i = \bar{0} \quad \text{imply that } t_0 = t_1 = \dots = t_n = 0.$$

Here are some observations about GI sets.

* $\{\bar{a}_i\}$ is GI $\forall i$. (singleton sets)

* $\{\bar{a}_0, \dots, \bar{a}_n\}$ is GI $\Longleftrightarrow$ ← if and only if
  $\{\bar{a}_1 - \bar{a}_0, \bar{a}_2 - \bar{a}_0, \dots, \bar{a}_n - \bar{a}_0\}$ is linearly independent (LI).

$\bar{a}_0$ is chosen as the "origin", so to speak. But any $\bar{a}_i$ could play the role of $\bar{a}_0$ here.

IDEA: $\sum_{i=1}^{n} t_i (\bar{a}_i - \bar{a}_0) = \bar{0} \Longrightarrow t_i = 0\ \forall i$ (LI)

$$\sum_{i=1}^{n} t_i \bar{a}_i + \underbrace{\left(-\sum_{i=1}^{n} t_i\right)}_{t_0} \bar{a}_0 = \bar{0}$$

$\sum_{i=0}^{n} t_i \bar{a}_i = 0$ & $\sum_{i=0}^{n} t_i = 0 \Longrightarrow t_i = 0\ \forall i$

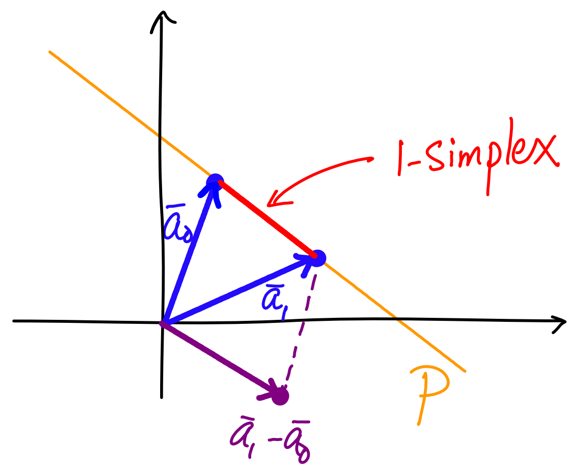
* 2 distinct points in $\mathbb{R}^d$ are GI,
  3 non-collinear points are GI,
  4 non-coplanar points are GI, and so on.

Notice the relationship/correspondence to LI vectors. For instance, $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ is GI, but of course the set is not LI.

Def Given GI set $\{\bar{a}_0, \ldots, \bar{a}_n\}$, the **n-plane** P spanned by these points consists of all $\bar{x}$ such that
$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i \quad \text{for scalars } t_i \text{ with } \sum_{i=0}^{n} t_i = 1.$$
The scalars t_i are uniquely determined by $\bar{x}$.

Notice that t_i could be ≥ 0 or ≤ 0 here.

P can also be described as the set of $\bar{x}$ such that
$$\bar{x} = \bar{a}_0 + \sum_{i=1}^{n} t_i (\bar{a}_i - \bar{a}_0).$$



Hence P is the plane through $\bar{a}_0$ parallel to the vectors $\bar{a}_i - \bar{a}_0$.

Going back to the previous example with $\{[1\ 1], [2\ 2]\}$, the plane P is the line generated by one of the two vectors.

Q. What is the set described by $\bar{x} = \sum_{i=0}^{n} t_i a_i$, $\sum t_i = 0$?

e.g., consider $n=1$: $\bar{x} = t_0\bar{a}_0 + t_1\bar{a}_1$ with $t_0 + t_1 = 0 \Rightarrow t_0 = -t_1$.

$\Rightarrow \bar{x} = t_0(\bar{a}_0 - \bar{a}_1)$, i.e., it's the line generated by $\bar{a}_0 - \bar{a}_1$.

We now define a simplex as the set "spanned" by a set of GI points.

Def Let $\{\bar{a}_0, \dots, \bar{a}_n\}$ be a GI set in $\mathbb{R}^d$. The **n-simplex** σ spanned by $\bar{a}_0, \dots, \bar{a}_n$ is the set of points $\bar{x} \in \mathbb{R}^d$ s.t. $\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i$ with $\sum_{i=0}^{n} t_i = 1$, $t_i \geq 0 \; \forall i$.

The t_i are uniquely determined by $\bar{x}$, and are called the **barycentric coordinates** of $\bar{x}$ (in σ) w.r.t. $\bar{a}_0, \dots, \bar{a}_n$.

we will later extend definition of t_i to $\bar{x} \notin \sigma$. the

0-simplex : a point

1-simplex : line segment

2-simplex

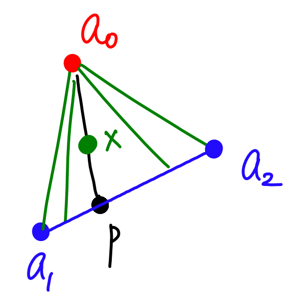
$\bar{x} = \bar{a}_0$ is trivial to consider.

Assume $\bar{x} \neq \bar{a}_0$, i.e., $t_0 \neq 1$. Now consider

$$\bar{x} = \sum_{i=0}^{2} t_i a_i = t_0 \bar{a}_0 + (1-t_0) \underbrace{\left[\frac{t_1}{1-t_0} \bar{a}_1 + \frac{t_2}{1-t_0} \bar{a}_2 \right]}_{\bar{p}}$$

Since $\sum_{i=0}^{2} t_i = 1$, $1 - t_0 = t_1 + t_2$. Hence $\left(\frac{t_1}{1-t_0}\right)\bar{a}_1 + \left(\frac{t_2}{1-t_0}\right)\bar{a}_2$ is a point $\bar{p}$ on the line segment $\overline{\bar{a}_1 \bar{a}_2}$, and $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{p}$ is a point on the line segment $\overline{\bar{a}_0 \bar{p}}$.

Hence the 2-simplex is the union of such line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\overline{a_1 a_2}$, i.e., the triangle $a_0 a_1 a_2$ ($\triangle a_0 a_1 a_2$).



This result extends to higher order simplices. For instance, a tetrahedron is the union of all line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\triangle a_1 a_2 a_3$.

Properties of Simplices

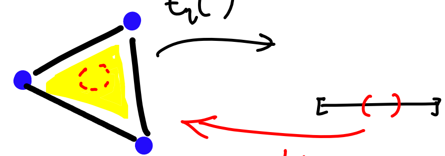
(1) $t_i(\bar{x})$ are continuous functions of $\bar{x}$.

IDEA: $t_i : R^d \to \mathbb{R}$

convex hull $\{\bar{x} \mid \bar{x} = \sum_{i=0}^{n} t_i a_i,\ t_i \geq 0,\ \sum_i t_i = 1\}$

domain $\to$ $\mathrm{Dom}(t_i) = \mathrm{conv}(\{\bar{a}_0, \ldots, \bar{a}_n\})$

$\mathrm{Range}(t_i) = [0, 1]$



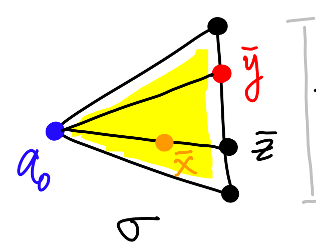
Prove that $t_i^{-1}(\text{open set in } [0,1])$ is open in σ.

(2) σ is the union of all line segments joining $\bar{a}_0$ to points of the simplex spanned by $\{\bar{a}_1, \ldots, \bar{a}_n\}$.
Two such line segments intersect only at $\bar{a}_0$.

$\hookrightarrow$ proof?

Assume two such line segments from $\bar{a}_0$ to $\bar{y}, \bar{z} \in \tau$, the simplex spanned by $\{\bar{a}_1, \ldots, \bar{a}_n\}$, meet at $\bar{x} \neq \bar{a}_0$.

$\bar{y} \neq \vec{z}$



Then $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{y} = s_0 \bar{a}_0 + (1-s_0)\bar{z}$, for $t_0, s_0 \in [0,1]$,
where $t_0 \neq s_0$ by assumption (else $\bar{y} = \bar{z}$!).

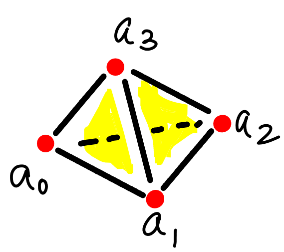
$\Rightarrow \bar{a}_0 = u\bar{y} + v\bar{z}$, where $u, v \in \mathbb{R}$ with $u+v=1$.

$\Rightarrow \bar{a}_0 \in P(\{\bar{y}, \bar{z}\}) \in P(\tau)$ $\to$ $(n-1)$-plane spanned by $\{\bar{a}_1, \ldots, \bar{a}_n\}$.

which contradicts the GI of $\{\bar{a}_0, \ldots, \bar{a}_n\}$.

Def The points $\bar{a}_0, \dots, \bar{a}_n$ which span σ are called its **vertices**. The **dimension** of σ is n $(\dim(\sigma) = n)$.
A simplex spanned by a nonempty subset of $\{\bar{a}_0, \dots, \bar{a}_n\}$ is a **face** of σ. The face spanned by $\{\bar{a}_0, \dots, \hat{\bar{a}}_i, \dots \bar{a}_n\}$ where $\hat{\bar{a}}_i$ means $\bar{a}_i$ is not included, is the **face opposite** $\bar{a}_i$.
Faces of σ distinct from σ itself are its **proper faces**, their union is its **boundary**, Bd σ or $\partial\sigma$.

$\partial(\bar{a}_0) = \phi \longrightarrow$ there are no proper faces of a vertex.



tetrahedron $a_0a_1a_2a_3 = \sigma$ $\rightarrow$ a 3-simplex

proper faces : $\Delta a_0a_1a_2, \Delta a_0a_2a_3, \dots$ (4)

edges $\rightarrow$ $\overline{a_0a_1}, \overline{a_0a_2}, \dots$ (6)

vertices $\rightarrow$ $\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3$ (4)

$\partial\sigma = \bigcup(\text{proper faces})$ (triangles, edges, vertices)
$\longrightarrow$ the "hollow" tetrahedron

Def The **interior** of σ, $\text{Int}(\sigma)$ or $\mathring{\sigma}$, is $\text{Int}(\sigma) = \sigma - \text{Bd}\,\sigma$.
$\text{Int}(\sigma)$ is called an open simplex.

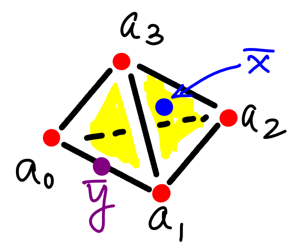
$\text{Int}(\bar{a}_0) = \bar{a}_0 \longrightarrow$ as $\partial a_0 = \phi$.

Bd σ consists of all $\bar{x} \in \sigma$ with at least one $t_i(\bar{x}) = 0$.
Int σ consists of all $\bar{x} \in \sigma$ with $t_i(\bar{x}) > 0 \;\forall i$.

could be σ itself!

Given $\bar{x} \in \sigma$, there is exactly one face τ s.t. $\bar{x} \in \operatorname{Int} \tau$.
τ is that face of σ spanned by those $\bar{a}_i$ for which $t_i(\bar{x}) > 0$.

$\bar{x}$ is interior to $\triangle a_1 a_2 a_3$
$\bar{y}$ is interior to $\overleftrightarrow{a_0 a_1}$



(3) σ is a compact, convex set in $\mathbb{R}^d$, and is the intersection of all convex sets in $\mathbb{R}^d$ containing $\bar{a}_0, \dots, \bar{a}_n$.

(4) There exists one and only one GI set of points $\{\bar{a}_0, \dots, \bar{a}_n\}$ spanning σ.

(5) $\operatorname{Int} \sigma$ is convex, and is open in P, and $Cl(\operatorname{Int} \sigma) = \sigma$. $\operatorname{Int} \sigma$ is the union of all "open line segments" joining $\bar{a}_0$ with points in $\operatorname{Int} \tau$, where τ is the face opposite $\bar{a}_0$.

closure

MATH 524: Lecture 3 (08/26/2025)

Today: * simplicial complexes
* underlying space

One more property of simplices first...

Def Unit ball: $\mathbb{B}^n = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| \leq 1\}$

Unit sphere: $\mathbb{S}^{n-1} = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| = 1\}$

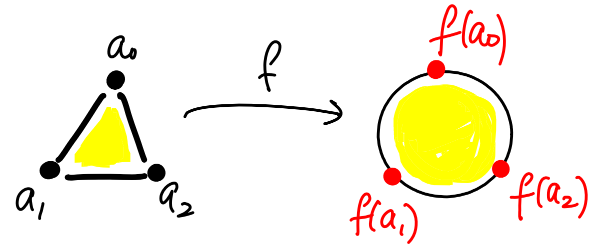
upper/lower hemisphere: $\mathbb{E}_+^{n-1} / \mathbb{E}_-^{n-1} = \{\bar{x} \in \mathbb{S}^{n-1} \mid x_n \geq 0 / x_n \leq 0\}$.

We will use these definitions later on

e.g., $\mathbb{B}^0 = \{\bar{0}\}$, $\mathbb{B}^1 = [-1, 1]$, $\mathbb{S}^0 = \{-1, 1\}$. two points

(6) There is a homeomorphism of σ (n-simplex) with $\mathbb{B}^n$ that carries $\partial\sigma$ to $\mathbb{S}^{n-1}$.

(proof in Munkres [M] EAT)



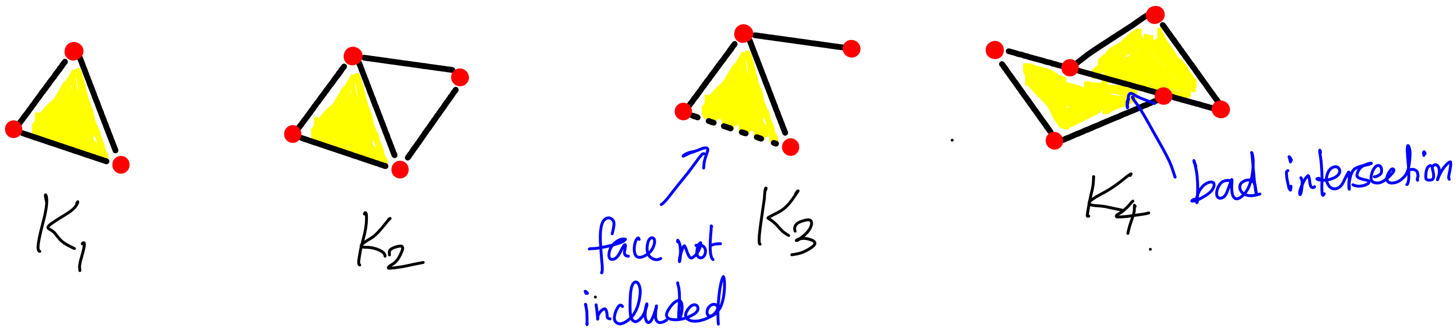
See [M] (Munkres - Elements of Algebraic Topology) for the proof.

In summary, simplices are "nice" elementary objects that can be used as building blocks to build larger spaces or objects. We will now introduce these larger objects, which are quite general, but are still "nice" since we "glue" simplices together nicely to build them.

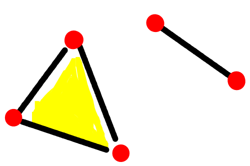
Simplicial Complexes

Def A simplicial complex K in $\mathbb{R}^d$ is a collection of simplices in $\mathbb{R}^d$ such that

(1) every face of a simplex in K is in K, and
(2) the intersection of any two simplices of K, when non-empty is a face of each of them.



K_1, K_2 are simplicial complexes, while K_3, K_4 are not.



K_5

K_5 is a simplicial complex – in particular, a simplicial complex need not be a single connected component.

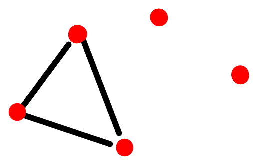
Here is another equivalent definition:

Lemma 2.1 [M] A collection of simplices K is a simplicial complex iff

(1) every face of a simplex in K is in K; and
(2) every pair of distinct simplices in K have disjoint interiors.

A simplex σ and all its proper faces together is a simplicial complex.

Def If L is a subcollection of K that contains all faces of its elements, then it is a simplicial complex on its own, called a **subcomplex** of K.

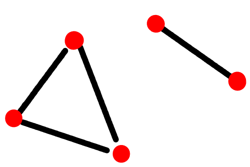


L_5.

A subcomplex of K_5

Def The subcomplex of K that is the collection of all simplices in K of dimension at most p is the **p-skeleton** of K, denoted $K^{(p)}$.

$K^{(0)}$ are the vertices of K.

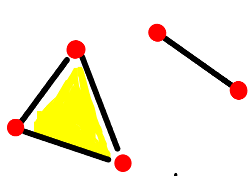


$K_5^{(1)}$ (the 1-skeleton of K_5).

Def The **dimension** of a simplicial complex K is the largest dimension of any simplex in K.

$$\dim(K) = \max_{\sigma \in K} \{\dim(\sigma)\}.$$

e.g.,



K_5

$\dim(K_5) = 2$, also referred to as a 2-complex.

A p-dimensional simplicial complex is referred to, in short, as a p-complex.

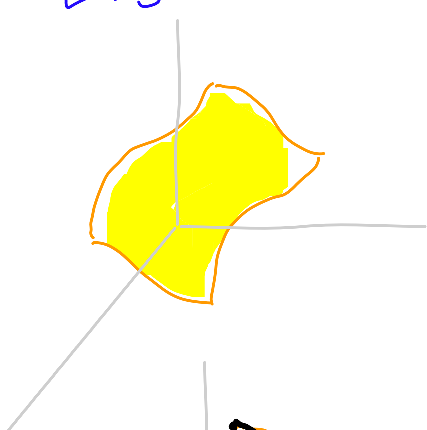
Q. What is $\dim(K^{(p)})$? $\longrightarrow$ p-skeleton of K

One can immediately conclude $\dim(K^{(p)}) \leq p$. But notice that $\dim(K^{(p)})$ need not always be $= p$. For instance, $\dim(K_5^{(3)}) = 2$, since $K_5^{(3)} = K_5$ itself. But if we avoid this somewhat trivial case, $\dim(K^{(p)}) = p$, typically.

Or, more generally, $\dim(K^{(p)}) = \min(p, \dim(K))$.

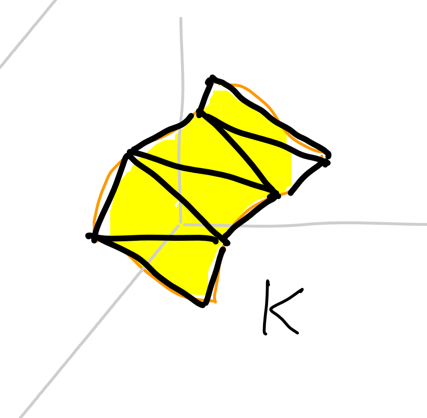
Recall that we want to use simplicial complexes as a "nice" structured way to model spaces. We now outline the somewhat subtle distinction between the simplicial complex and the (sub)space that it models.

Let's start with an illustration.



Consider a subspace of, say, $\mathbb{R}^3$ modeled by a sheet of paper. We could capture this space by a simplicial complex K consisting of six triangles.

Complementarily, if we start with K, we could talk about the subspace of $\mathbb{R}^3$ that it captures. We can specify the usual topology on this subspace (as inherited from $\mathbb{R}^3$).



Def Let $|K|$ be the subset of $\mathbb{R}^d$ which is the union of all simplices in K. Give each simplex its natural topology as a subspace of $\mathbb{R}^d$. Then we can topologize $|K|$ by declaring a subset A of $|K|$ is closed in $|K|$ if $A \cap \sigma$ is closed in K $\forall \sigma \in K$. $|K|$ is called the **underlying space** of K, or the **polytope** of K.

also referred to as "polyhedron"

Some people use the word polytope only when K is finite, i.e., it has a finite number of simplices, while using the word polyhedron more generally, i.e., even for the case where K is not finite.

In convex geometry, $P = \{\bar{x} \in \mathbb{R}^d \mid A\bar{x} \leq \bar{b}\}$ is a polyhedron, and a closed polyhedron is referred to as a polytope.

The two topologies — one as a subspace of $\mathbb{R}^d$, and the other defined using the simplices as above — need not be identical in all cases. But if K is finite, they usually concide. In fact, typical examples where they differ come from infinite simplicial complexes K.

$|K|$ topologized in two different ways: here is an example where the two topologies are different.

<u>Example</u> $K = \{\bigcup [m, m+1] \;\forall m \in \mathbb{Z}\setminus\{0\}\} \cup$ → notice that $[-1,0]$ is included

$\{[\frac{1}{n+1}, \frac{1}{n}] \;\forall n \in \mathbb{Z}_{>0}\}$ and all faces.

K is an infinite 1-complex. → infinitely many simplices

$|K| = \mathbb{R}$ as a set, but not as a topological space. Indeed,
$A = \{\frac{1}{n}, n \in \mathbb{Z}_{>0}\}$ is closed in $|K|$, but not in $\mathbb{R}$.
→ A does not include 0.

But if K is finite, the topologies are the same.

<u>Properties of $|K|$</u> → Munkres – Elements of Algebraic Topology

<u>Lemma 2.2 [M]</u> If $L \subseteq K$ is a subcomplex, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.
→ $|\sigma|$ to be precise, but notice σ and $|\sigma|$ are identical!

<u>Lemma 2.3 [M]</u> A map $f : |K| \to X$ is continuous iff $f|_\sigma$ is continuous for each $\sigma \in K$.
→ f restricted to σ

Recall the barycentric coordinates of $\bar{x} \in \sigma$ ($t_{\bar{a}_i}(\bar{x})$ for vertices $\bar{a}_i$).
We can naturally extend the barycentric coordinates to $\bar{x} \notin \sigma$.

Def If $\bar{x} \in |K|$, then $\bar{x}$ is interior to precisely one simplex in K, whose vertices are, say, $\bar{a}_0, \dots, \bar{a}_n$. Then

$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i, \quad \text{where} \quad t_i > 0 \ \forall i, \ \sum_{i=0}^{n} t_i = 1.$$

If $\bar{v}$ is an arbitrary vertex of K, then the barycentric coordinate of $\bar{x}$ w.r.t $\bar{v}$, $t_{\bar{v}}(\bar{x})$, is defined as $t_{\bar{v}}(\bar{x}) = 0$ if $\bar{v} \notin \{\bar{a}_0, \dots, \bar{a}_n\}$, and $t_{\bar{v}}(\bar{x}) = t_i$ if $\bar{v} = \bar{a}_i$.

Notice that $t_{\bar{v}}(\bar{x})$ is continuous on $|K|$, as $t_{\bar{a}_i}(\bar{x})$ are continuous, as we noted in the last lecture, and then by Lemma 2.3.

Lemma 2.4 [M] $|K|$ is Hausdorff.

A space X is Hausdorff if every pair of distinct points $\bar{x}, \bar{y} \in X$ can be surrounded by open sets $\mathcal{U}, \mathcal{V} \subseteq X$ s.t. $\bar{x} \in \mathcal{U}$, $\bar{y} \in \mathcal{V}$, $\mathcal{U} \cap \mathcal{V} = \phi$.

Proof For $\bar{x}_i \neq \bar{x}_j$ in $|K|$, by definition, there exists at least one $\bar{v}$ (vertex) s.t. $t_{\bar{v}}(\bar{x}_i) \neq t_{\bar{v}}(\bar{x}_j)$. Choose r in between $t_{\bar{v}}(\bar{x}_i)$ and $t_{\bar{v}}(\bar{x}_j)$ and define $\mathcal{U} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) < r\}$ and $\mathcal{V} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) > r\}$ as the required open sets.

We now study some important subspaces of $|K|$.

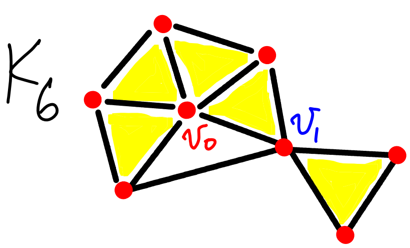
MATH 524: Lecture 4 (08/28/2025)

Today: * star, closed star, link
* simplicial maps
* abstract simplicial complexes

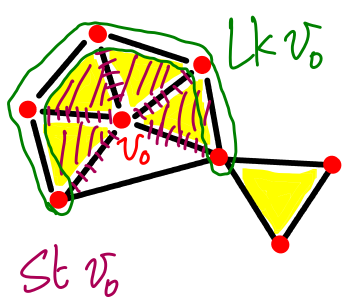
We now study some important subspaces of $|K|$.

Three Subspaces of $|K|$

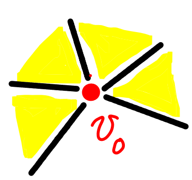
Def If $\bar{v}$ is a vertex of K, then the **star** of $\bar{v}$ in K, denoted $\mathrm{St}\,\bar{v}$ (or $\mathrm{St}(\bar{v},K)$) is the union of the interiors of all simplices in K that contain $\bar{v}$ as a vertex. The closure of $\mathrm{St}\,\bar{v}$, denoted $\overline{\mathrm{St}}\,\bar{v}$ or $\mathrm{Cl}\,\mathrm{St}\,\bar{v}$, is the **closed star** of $\bar{v}$. It is the union of all simplices of K which have $\bar{v}$ as a vertex. $\mathrm{Cl}\,\mathrm{St}\,\bar{v}$ is a polytope of a subcomplex of K. $\mathrm{Cl}\,\mathrm{St}\,\bar{v} - \mathrm{St}\,\bar{v}$ is called the **link** of $\bar{v}$, denoted $\mathrm{Lk}\,\bar{v}$.



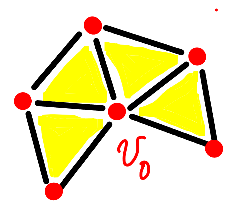
We illustrate these subcomplexes on K_6 for vertices v_0 and v_1. Note that the unshaded triangle below v_0 is not part of K_6.



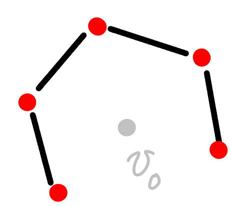
add to get $\mathrm{Cl}\,\mathrm{St}\,v_0$



$\mathrm{St}\,v_0$



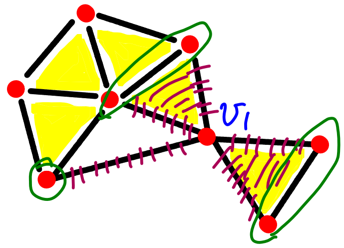
$\mathrm{Cl}\,\mathrm{St}\,v_0$



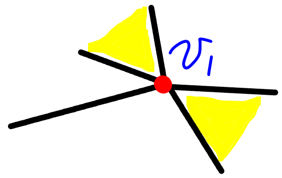
$\mathrm{Lk}\,v_0$

Note that $\mathrm{Lk}\,v_0 = \mathrm{Cl}\,\mathrm{St}\,v_0 - \mathrm{St}\,v_0$.

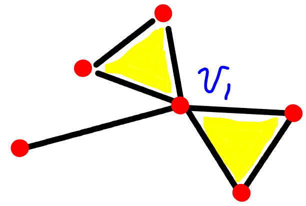
Also note that $v_0 \in \mathrm{St}\,v_0$ (indeed, $\mathrm{Int}\,v_0 = v_0$, and v_0 is a simplex that contains v_0 as a vertex, trivially).



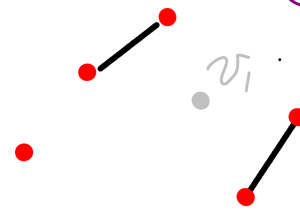
$St\, v_1$



$st\, v_1$



$Cl\,St\, v_1$



$Lk\, v_1$

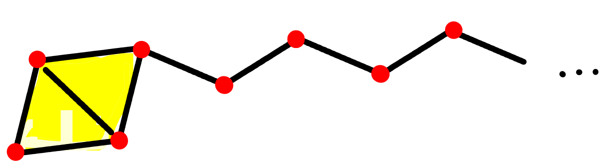
Properties of star, closed star, link

* $St\,\bar{v}$ is open in $|K|$. → We could use $t_{\bar{v}}(\cdot)$ to prove.
* The complement of $St\,\bar{v}$ is the union of all simplices that do not contain $\bar{v}$ as a vertex, and hence it is the polytope of a subcomplex of K.
* $Lk\,\bar{v}$ is the polytope of a subcomplex of K.
* $Lk\,\bar{v} = Cl\,St\,\bar{v} \cap (\text{complement of } St\,\bar{v})$.
* $St\,\bar{v}$ and $Cl\,St\,\bar{v}$ are both path-connected.
  X is path-connected if $\forall \bar{u}, \bar{v} \in X$, $\bar{u} \neq \bar{v}$, $\exists$ a path connecting $\bar{u}$ and $\bar{v}$ in X.
* $Lk\,\bar{v}$ need not be connected.

Def A simplicial complex K is **locally finite** if each vertex of K belongs to only finitely many simplices of K. Equivalently, K is locally finite iff each closed star is the polytope of a finite subcomplex of K.

Note: A locally finite simplicial complex could be infinite, e.g, K_7.

K_7



(the edges continue forever)

Simplicial Maps

We study maps between simplicial complexes as a first step toward developing the tools to compare spaces modeled by the simplicial complexes.

Def Let K, L be simplicial complexes. A function $f: |K| \to |L|$ is a (linear) **simplicial map** if it takes simplices of K linearly onto onto simplices of L. In other words, if $\sigma \in K$, then $f(\sigma) \in L$.

linearly: If $\sigma = \text{conv}\{\bar{v}_0, \ldots, \bar{v}_n\}$ and $\bar{x} = \sum_{i=0}^{n} t_i \bar{v}_i$, $t_i \geq 0$, $\sum_{i=0}^{n} t_i = 1$, then $f(\bar{x}) = \sum_{i=0}^{n} t_i f(\bar{v}_i)$.

Note that $\{f(\bar{v}_0), \ldots, f(\bar{v}_n)\}$ span a simplex τ of L, which could be of a lower dimension than σ.

Munkres takes a slightly different approach in defining simplicial maps.
[M]: Starts with $f: K^{(0)} \to L^{(0)}$, then insist that when $\{\bar{v}_0, \ldots, \bar{v}_n\}$ span $\sigma \in K$, $\{f(\bar{v}_0), \ldots f(\bar{v}_n)\}$ span $\tau \in L$.

f is a continuous map of σ onto τ, and hence as a map of σ onto $|L|$. Then by Lemma 2.3, it is a continuous map from $|K|$ to $|L|$.

If $g: |K| \to |L|$ and $h: |L| \to |M|$ are simplicial maps, then $f = h \circ g$ is a simplicial map from $|K|$ to $|M|$.

If we further insist that $f: K^{(0)} \to L^{(0)}$ is a **bijective** correspondence such that vertices $\bar{v}_0, \ldots, \bar{v}_n$ of K span a simplex of K iff $f(\bar{v}_0), \ldots, f(\bar{v}_n)$ span a simplex of L, then the induced simplicial map $g: |K| \to |L|$ is a homeomorphism. We call this map an **isomorphism** of K with L (or a simplicial homeomorphism).

Abstract Simplicial Complexes (ASCs)

Def An abstract simplicial complex (ASC) is a collection $\mathcal{S}$ of finite nonempty sets such that if $A \in \mathcal{S}$, then so is every nonempty subset of A.

Note: $\mathcal{S}$ itself could be infinite, but each $A \in \mathcal{S}$ is finite.

Example: $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ is an ASC.

We specify several more definitions related to ASCs.

Def A (any element of $\mathcal{S}$) is a **simplex** of $\mathcal{S}$. Its **dimension** is given as $\dim(A) = |A| - 1$.

↳ # elements in A, or size of A

The **dimension of the ASC** is defined as follows.

$\dim(\mathcal{S}) =$ largest dimension of any simplex in $\mathcal{S}$, or ∞ if no such largest dimension exists.

The **vertex set** V of $\mathcal{S}$ (or $V(\mathcal{S})$) is the union of all singleton elements (simplices) of $\mathcal{S}$. We do not distinguish between the individual vertices and the singleton sets they represent.

$$v_0 \text{ (vertex)} \equiv \{v_0\} \quad \text{0-simplex of } \mathcal{S}.$$

A subcollection of $\mathcal{S}$ that is a simplicial complex by itself is a **subcomplex** of $\mathcal{S}$.

We can now talk about when two ASCs are "similar".

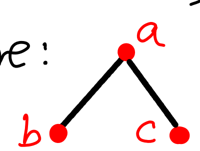
Def Two ASCs $\mathcal{S}$ and $\mathcal{T}$ are **isomorphic** if there exists a bijective correspondence f mapping $V(\mathcal{S})$ to $V(\mathcal{T})$ such that $\{a_0, \dots, a_n\} \in \mathcal{S}$ iff $\{f(a_0), \dots, f(a_n)\} \in \mathcal{T}$.

e.g., With $\mathcal{T} = \{\{d\}, \{e\}, \{f\}, \{d,e\}, \{d,f\}\}$, $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.

It turns out the previous notion of simplicial complexes (in $\mathbb{R}^d$) and ASC are directly related.

Def Let K be a (geometric) simplicial complex. Let V be its vertex set. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, \dots, a_n\}$ of V such that $\bar{a}_0, \dots, \bar{a}_n$ span a simplex of K. Then $\mathcal{K}$ is an ASC called the **vertex scheme** of K. Symmetrically, we call K a **geometric realization** of $\mathcal{K}$.

e.g., (continued) $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ has a geometric realization S as shown here:



This complex could be sitting in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Theorem 3.1 [M] (a) Every ASC $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K.

A version of this result is given as the **geometric realization theorem** which states that every abstract d-complex has a geometric realization in $\mathbb{R}^{2d+1}$.

IDEA: If $\dim(\mathcal{S}) = d$ then let $f: V(\mathcal{S}) \to \mathbb{R}^{2d+1}$ be an injective function whose image is a set of GI points in $\mathbb{R}^{2d+1}$. Specify that for each abstract simplex $\{a_0, \dots, a_n\} \in \mathcal{S}$, $\{f(a_0), \dots, f(a_n)\} \in K$. Then $\mathcal{S}$ is isomorphic to the vertex scheme of K.

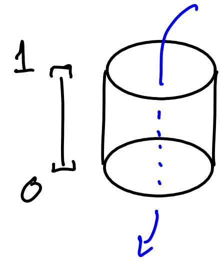
MATH 524: Lecture 5 (09/02/2025)

Today: * Examples of ASCs
* Review of algebra

Examples of ASCs

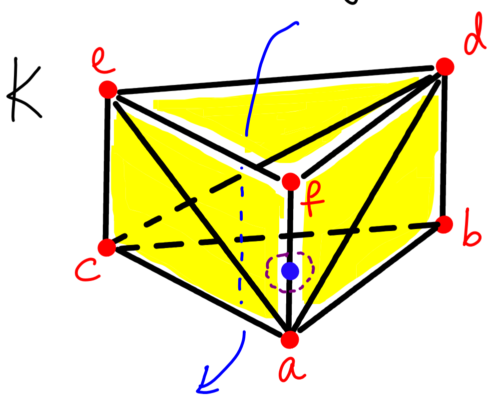
1. Cylinder

$$\underbrace{S^1}_{\text{circle}} \times \underbrace{I}_{[0,1]}$$



We want to describe a simplicial complex K such that $|K|$ is homeomorphic to the cylinder.

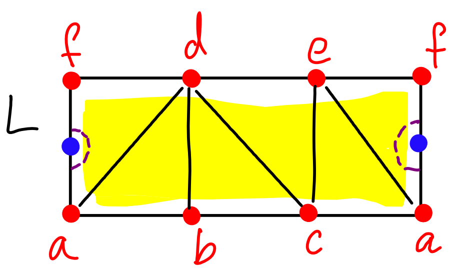
We first describe a geometric simplicial complex K, which could be sitting in $\mathbb{R}^3$, for instance.



K comprises of the six triangles adf, abd, bcd, cde, ace, and aef.

Indeed, $|K| \approx$ cylinder.

But we now specify an abstract simplicial complex whose underlying space is homeomorphic to the cylinder. We start with a rectangle L, and then assign labels to specific vertices in L. Thus, L along with the labels is the ASC.



Notice that both the left and right border edges of L are labeled af going from bottom to top.

We can describe the required map between K and L as follows.

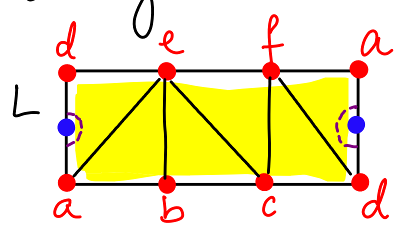
Let $f: K^{(0)} \to L^{(0)}$ is the vertex map that assigns vertices in K the labels in L. We can extend f to a simplicial map $g: |K| \to |L|$. This map g is a "pasting map", or a quotient map.

indeed, we are starting with the rectangular strip (of paper, say) L, and pasting its end edges together (af).

Notice how we can visualize a neighborhood of a point on edge af in K and correspondingly on L.

2. Möbius strip

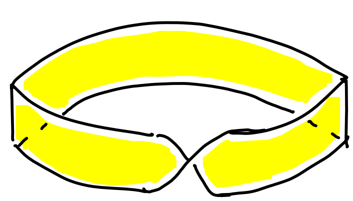
We now start with the rectangular space L and a specific vertex labeling as shown here.



The ASC $\mathcal{S}$ here has 6 triangles ade, abe, bce, cef, cdf, adf, as well as their faces.

We're again gluing the end edges, but now with a "twist."

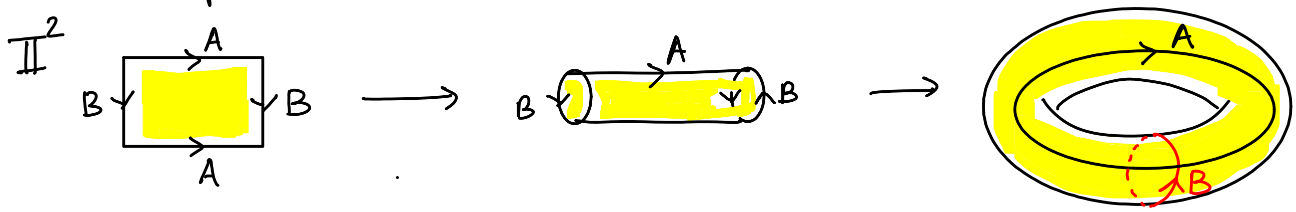
Let K be a geometric realization of $\mathcal{S}$. We can consider a simplicial map $g: |L| \to |K|$, which maps vertices in L to vertices in K. Again, g is a quotient (or "pasting") map that maps the left edge of |L| to the right edge, but with a "twist."

$L \xrightarrow{g}$ 

Notice that we do want a homeomorphism from $|L|$ to K, and just a vertex map is not enough. But of course, the vertex map is naturally (linearly) extended to the desired map from $|L|$ to $|K|$.

\mathbb{T}^2 in LaTeX!

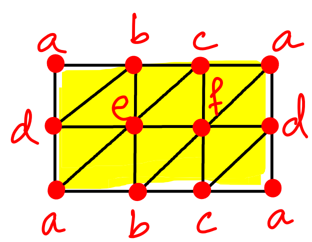
3. Torus ($\mathbb{T}^2$) The quotient space obtained by making identifications on the sides of a rectangle as follows.



Notice that this is an example of a quotient map defined on a general space, and not on an ASC.
This is the surface of a "donut", and not the solid donut itself.

Now, let us find an ASC K such that $|K| \approx \mathbb{T}^2$.
Let's start with a rectangular space as before, and assign labels that could work. Here is a first try.

K_1:

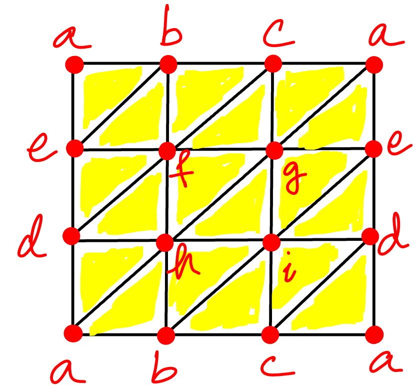


Is $|K_1| \approx \mathbb{T}^2$? No!
We are doing too much gluing!

Notice that $\overline{ad}$ is part of 4 triangles ade, adb, adc, adf, for instance.
The gluings specified above glue only two edges together at a time.
With this gluing, edge ad is part of four triangles, i.e., we get a "fan" of four "flaps" meeting at ad. But notice that there are no such 4-way junctions in the torus.

We need to "spread out" more!

K_2:



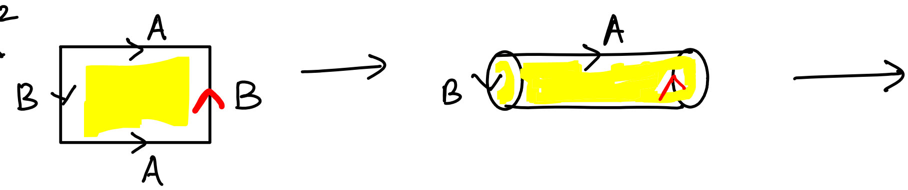
$|K_2| \approx \mathbb{T}^2$

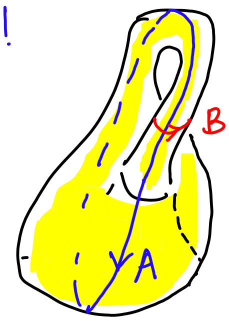
We can show that $|K_2| \approx \mathbb{T}^2$. See [M] for details, but on a complex similar to K_2.

Every edge is face of exactly two triangles.

4. Klein bottle ($\mathbb{K}^2$) → \mathbb(K)^2 in LaTeX!

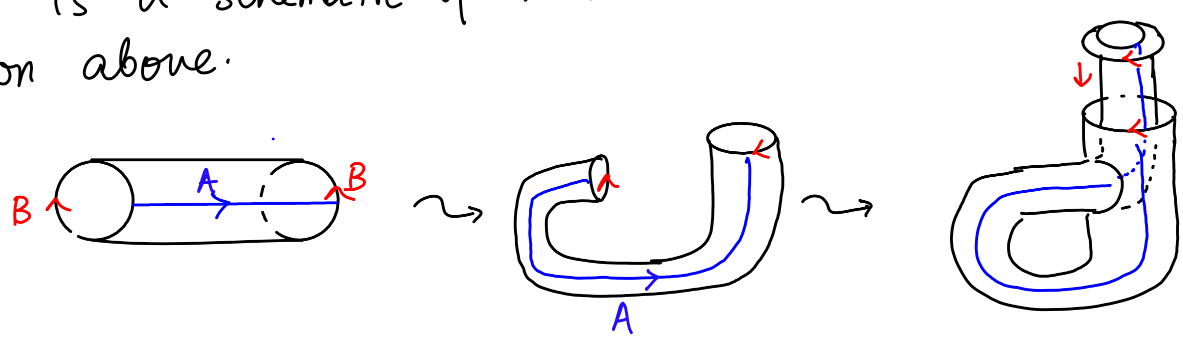
$\mathbb{K}^2$





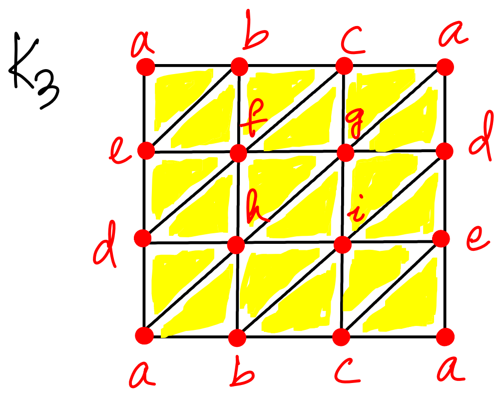
Here, we identify the opposite pairs of edges – one pair with a twist as in the Möbius strip (B here) and the other without (A here; similar to torus or cylinder). The Klein bottle does not have an embedding in $\mathbb{R}^3$, but has in $\mathbb{R}^4$. We must go to the higher dimension to avoid self-intersections.

We do get an **immersion** in $\mathbb{R}^3$, which allows self intersection. Here is a schematic of how one arrives at the immersion shown above.



This instance illustrates the difficulty faced when working with geometric embeddings. We could instead work with the abstract space along with the quotient map!

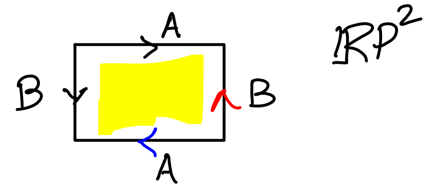
We now construct an ASC for $\mathbb{K}^2$.



$|K_3| \approx \mathbb{K}^2 \rightsquigarrow$ one can check to make sure we are not gluing more than two edges anywhere.

Of course, $|K_3| \not\approx |K_2|$, and in general, $\mathbb{K}^2 \not\approx \mathbb{T}^2$.

Notice that we could start with the rectangular space (for L) and identify pairs of edges in several ways. For instance, when we glue both pairs of opposite edges with twists, we get the real projective plane ($\mathbb{R}P^2$).



The related question now is how to identify homeomorphic simplicial complexes K for any such quotient space. In particular, when do we get "nice" labelings (or gluings)?

See Lemma 3.2 in [M] for a condition given in terms of closed stars of vertices in K. This result is left as a **candidate for video tutorial**.

Review of Abelian Groups

We now review several properties and results from groups and homomorphisms between groups. The idea is to cast questions about similarity of topological spaces as corresponding questions on homomorphisms between groups defined on simplicial complexes that are homeomorphic to the spaces in question.

A good book – Fraleigh (First course in Abstract Algebra).

closure is assumed, i.e., $a+b \in G$ $\forall a,b \in G$.

Group: set G with an operation $+$ "addition", such that

(1) there exists an **identity**, $0 \in G$, s.t.

$$a+0 = 0+a = a \quad \forall a \in G;$$

(2) $\forall a \in G$, there is an **inverse**, i.e., $-a \in G$ s.t.

$$a+(-a) = (-a)+a = 0; \text{ and}$$

(3) $a+(b+c) = (a+b)+c \quad \forall a,b,c \in G$; i.e., $+$ is **associative**.

(4) Further, if $a+b = b+a$ $\forall a,b \in G$, then G is an **abelian group**.

In general, we will work with abelian groups in this class.

Notation: $\quad ng = \underbrace{g+g+\cdots+g}_{n \text{ times}}$ for $g \in G$.

<u>Homomorphisms</u> $f: G \to H$, G, H are groups is a homomorphism if $f(g_1 +_G g_2) = f(g_1) +_H f(g_2)$.

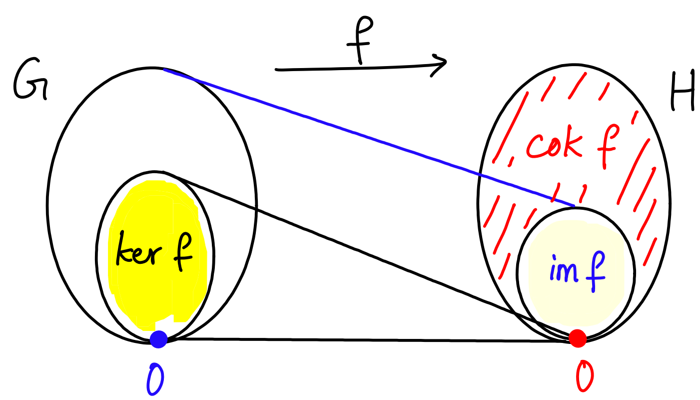
Intuitively, homomorphisms "preserve the structure" of groups.

We study subgroups specified by homomorphism f:

kernel of f: $f^{-1}(0)$, is a subgroup of G, denoted $\mathbf{ker} f$.

image of f: $f(G)$, is a subgroup of H, denoted **im f**.

cokernel of f: quotient group of H given as $H/f(G)$, denoted **cok f**.



f is a monomorphism (injection) iff $\ker f = 0$.

f is an epimorphism (surjection) iff $\operatorname{cok} f = 0$, and in this case, f defines an isomorphism $G/\ker f \simeq H$.

An abelian group G is **free** if it has a basis $\{g_\alpha\}$ of elements in G such that $\forall g \in G$, $g = \sum n_\alpha g_\alpha$ is a unique finite sum, for $n_\alpha \in \mathbb{Z}$.

This uniqueness (for the free abelian group) implies that each basis element g_α generates an infinite cyclic group $H = \{ng_\alpha \mid n \in \mathbb{Z}\}$.

notation used in [M]

Note: $\mathbb{Z}/n$ (or $\mathbb{Z}_n$) has elements $\{0, 1, \ldots, n-1\}$ with addition mod n.

More generally, if each $g \in G$ can be written as $\sum n_\alpha g_\alpha$, but not necessarily uniquely, then $\{g_\alpha\}$ **generates** G. If $\{g_\alpha\}$ is finite, we say that G is **finitely generated**.

We will work mostly with finitely generated abelian groups

Def If G is free, and has a basis of n elements, then every basis of G has n elements. The number of elements in a basis of G is its **rank**, denoted $\mathrm{rk}(G)$.
or $\mathrm{rank}(G)$

The **order** of G is the # elements in G, denoted $|G|$.

A crucial property: If $\{g_\alpha\}$ is a basis of G, any function f from $\{g_\alpha\}$ to abelian group H extends uniquely to a homomorphism from G to H.

Somewhat similar in flavor to a vertex map extending to the corresponding simplicial map

MATH 524: Lecture 6 (09/04/2025)

Today: * two results on abelian groups
* orientation of simplices

More results on groups...

Let G be an abelian group. $g \in G$ has **finite order** if $ng = 0$ for some $n \in \mathbb{Z}_{>0}$. The set of all elements of finite order in G is a subgroup T of G, called the **torsion subgroup**. If T vanishes, we say G is **torsion-free**.

Notice that $0 \in G$ is a trivial case in this context, as $n0 = 0$ for any $n \in \mathbb{Z}$.

We now consider how to "combine" (abelian) groups to form bigger (abelian) groups. The intuition is similar to combining multiple individual dimensions to form a higher dimensional space.

[M] defines internal direct sums, direct products, and external direct sums. We discuss them all for the sake of completeness.

Internal direct sums

Let G be an abelian group, and let $\{G_\alpha\}_{\alpha \in J}$ be a collection of subgroups of G indexed bijectively by the index set J. If each $g \in G$ can be written uniquely as finite sum $g = \sum_\alpha g_\alpha$, where $g_\alpha \in G_\alpha$ for each $\alpha \in J$, then G is the **internal direct sum** of the groups G_α, and is written $G = \bigoplus_{\alpha \in J} G_\alpha$.

If $J = \{1, 2, \ldots, n\}$ for finite n, say, we also write

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \quad \text{or} \quad G = \bigoplus_{\alpha=1}^{n} G_\alpha$$

There is a similar distinction here to a basis vs generating set of a group.

If each $g \in G$ can be written as a finite sum $g = \sum_\alpha g_\alpha$, but not necessarily uniquely, then G is simply the **sum** of groups $\{G_\alpha\}$.

We write $G = \sum_\alpha G_\alpha$, or $G = G_1 + \cdots + G_n$ (if finite).

internal sum, to be precise

Here, we say $\{G_\alpha\}$ **generates** G.

Notice that if G is free abelian with basis $\{g_\alpha\}$, then G is the direct sum of subgroups $\{G_\alpha\}$, where G_α is the infinite cyclic group generated by g_α.

The converse is also true here, i.e., if G is the direct sum of $\{G_\alpha\}$ where G_α is the infinite cyclic group generated by g_α, then G is free abelian with basis $\{g_\alpha\}$.

Direct Products and External direct Sums

Def Let $\{G_\alpha\}_{\alpha\in J}$ be an indexed family of abelian groups. The **direct product** $\prod_{\alpha\in J} G_\alpha$ is the group whose set is the cartesian product of sets G_α, and the operation is compoment-wise addition.

J can be infinite here; you could assume it is finite, though, to get the intuition. There is technical work required to extend the results and definitions to the infinite case – but it's not critical for us.

The **external direct sum** G is the subgroup of the direct product $\prod_{\alpha\in J} G_\alpha$ consisting of all tuples

$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{\alpha_i} \\ \vdots \end{pmatrix}$ such that $g_{\alpha_i} = 0_{\alpha_i}$ for all but finitely many values of α_i.

Examples

1. $G_1 = \mathbb{Z}\times\mathbb{Z}$ G_1 has rank 2; basis is $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$. $rk(G_1)=2$.

   operation is compomentwise addition.

2. $G_2 = \mathbb{Z}/_2 \times \mathbb{Z}/_3$ (or $\mathbb{Z}_2\times\mathbb{Z}_3$)

   compoment wise addition mod 2 and mod 3.

   $1+_2 1 = 0$ $1+_3 1 = 2$, $1+_3 2 = 0$

   G_2 is a cyclic group, $|G_2| = 6$, order

   $G_2 = \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$.

   $rk(G_2) = 1$, as $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a basis.

   $1\times\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, $2\times\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$, $3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

   $4\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, $5\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, $6\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

Theorem The group $\prod_{i=1} \mathbb{Z}/t_i$ for $t_i \in \mathbb{Z}_{>0}$ is cyclic and is isomorphic to $\mathbb{Z}_{t_1 t_2 \cdots t_n}$ iff $\gcd(t_i, t_j) = 1 \ \forall i, j$.

t_i and t_j are relatively prime

Back to example 2: $\mathbb{Z}/2 \times \mathbb{Z}/3 \simeq \mathbb{Z}_6$.

If $n = (p_1)^{n_1}(p_2)^{n_2} \cdots (p_r)^{n_r}$ for primes $p_1, \ldots, p_r$, then

$$\mathbb{Z}_n \simeq \mathbb{Z}/(p_1)^{n_1} \times \mathbb{Z}/(p_2)^{n_2} \times \cdots \times \mathbb{Z}/(p_r)^{n_r}.$$

Structure of finitely generated abelian groups

Two main results that we will use in characterizing the structure of homology groups on simplicial complexes.

Theorem 4.2 [M] Let F be a free abelian group. If R is a subgroup of F, then R is a free abelian group. If $\operatorname{rank}(F) = n$, then $\operatorname{rank}(R) = r \leq n$. Furthermore, there is a basis $e_1, \ldots, e_n$ of F and numbers $t_1, \ldots, t_k$ ($t_i \in \mathbb{Z}_{>0}$) such that

(1) $t_1 e_1, \ldots, t_k e_k, e_{k+1}, \ldots, e_r$ is a basis for R, and

(2) $t_1 | t_2 | \cdots | t_k$, i.e., t_i divides t_{i+1} $\forall i \geq 1$. ($i \leq k-1$).

The t_i's are uniquely determined by F and R.

Intuitively, the subgroup inherits the structure of the original group...

Theorem 4.3 [M] (Fundamental theorem of finitely generated abelian groups).
Let G be a finitely generated abelian group, and let T be its torsion subgroup. The following results hold.

(a) There is a free abelian subgroup H of G such that $G = H \oplus T$. The rank of H $rk(H) = \beta$, a finite number.

(b) There exist finite cyclic groups $T_1, \dots, T_k$ with $|T_i| = t_i > 1$, and $t_1 | t_2 | \cdots | t_k$ such that

$$T = T_1 \oplus \cdots \oplus T_k.$$

(c) The numbers β and $t_1, \dots, t_k$ are uniquely determined by G.

β is the **betti number** of G, and $t_1, \dots, t_k$ are the **torsion coefficients** of G.

"torsion" meaning "twistedness" or "cyclic nature"; as opposed to the free part.

A quick example on torsion ...

Example What is the torsion subgroup of the multiplicative group $\mathbb{R}^*$ of all nonzero real numbers?

$G = \mathbb{R}/\{0\}$, operation is $*$ (multiplication), identity is 1, $g^{-1} = \frac{1}{g}$ $\forall g \in G$.

The answer is $\{1, -1\}$.

Here is the main consequence of the previous theorem:

Any finitely generated abelian group G can be written as a direct sum of cyclic groups, i.e.,

↪ is isomorphic to

$$G \cong \underbrace{(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})}_{\beta} \oplus \mathbb{Z}/t_1 \oplus \mathbb{Z}/t_2 \oplus \cdots \oplus \mathbb{Z}/t_k$$

where $\beta \geq 1$, $t_i > 1$, and $t_i \mid t_{i+1}$ $\forall i$. This is a canonical form, called the **invariant factor decomposition** of G.

We can also get the **primary decomposition**, which is another canonical form:

$$G \cong \underbrace{(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})}_{\beta} \oplus \mathbb{Z}/(p_1)^{n_1} \oplus \cdots \oplus \mathbb{Z}/(p_r)^{n_r}$$ for primes $p_1, \ldots, p_r$.

Examples

1. What are the beti number and torsion coefficients of
$$G = \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z} \ ?$$
$$G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4,$$ so $\beta = 2$.

Update! Since $\gcd(3,4) = 1$ (3 and 4 are coprime), we get that $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{12}$. Hence the torsion coefficient is $t_1 = 12$ here.

2. Find the primary and invariant factor decompositions of $\mathbb{Z}/4 \times \mathbb{Z}/12 \times \mathbb{Z}/18$.

$$\begin{array}{ccc} \mathbb{Z}/4 & \mathbb{Z}/12 & \mathbb{Z}/18 \\ \downarrow & \downarrow & \downarrow \\ \cancel{(2)^2} & \mathbb{Z}/3 \times \mathbb{Z}/4 & \mathbb{Z}/2 \times \mathbb{Z}/9 \end{array}$$

We do not get $\mathbb{Z}/4 \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, as 2 and 2 are not coprime.

Primary decomposition: $\mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/9$.

invariant factor decomposition:

$$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/36$$

$$\begin{array}{ccc} 2 & 4 & 4 \\ & \times\, 3 & \times\, 9 \\ \hline 2 & 12 & 36 \end{array}$$

Notice that $2 | 12 | 36$.

A standard "trick" is to write the factors for each prime in a line in a right justified fashion. Then multiply the numbers in each column to get the torsion coefficients.

Homology Groups

We now study groups and homomorphisms defined on simplicial complexes. Questions about topological similarity are posed as equivalent questions on corresponding groups' structure.

We need a few foundational concepts.

Orientation of a simplex

Let σ be a simplex (geometric or abstract). We define two orderings of its vertex set to be equivalent if they differ by an even permutation, i.e., you can go from one ordering to the other using an even number of pairwise swaps.

If $\dim(\sigma) > 0$, the orderings fall into two equivalence classes. Each class is an **orientation** of σ.

If $\dim(\sigma) = 0$, it has only one orientation.

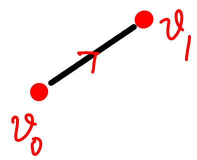
An **oriented simplex** is a simplex σ together with an orientation of σ.

Notation: Let $v_0, \ldots, v_p$ be independent. Then $\sigma = v_0 v_1 \cdots v_p$ is the simplex spanned by $v_0, \ldots, v_p$, and $[v_0, \ldots, v_p]$ denotes the oriented simplex σ with the orientation $(v_0, \ldots, v_p)$.

$\rightarrow$ GI if $\bar{v}_0, \ldots, v_p \in \mathbb{R}^d$ and distinct if $v_0, \ldots, v_p$ are (just) labels in the abstract setting.

When it is clear from the context, we will use σ to denote both the simplex as well as its orientation (or the oriented simplex).

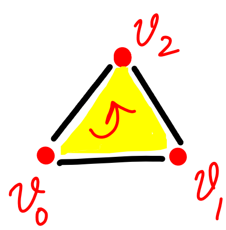
1-simplex $[v_0, v_1]$, $[v_1, v_0]$ → opposite orientation



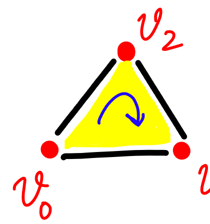
equivalent to orienting the edge from v_0 to v_1.
$[v_1 v_0]$ → draw the arrow the other way.

2-simplex Notice that $[v_0, v_1, v_2]$ is the same as $[v_1, v_2, v_0]$.

$(v_0 v_1 v_2) \to (v_1 v_0 v_2) \to (v_1 v_2 v_0)$ two pairwise swaps

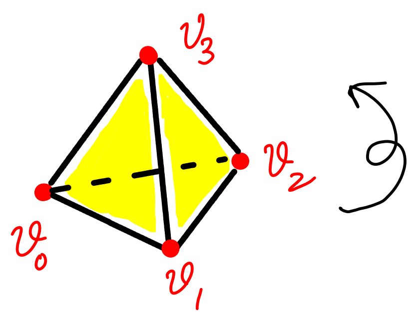


$[v_0 v_1 v_2]$ → can be the counterclockwise orientation



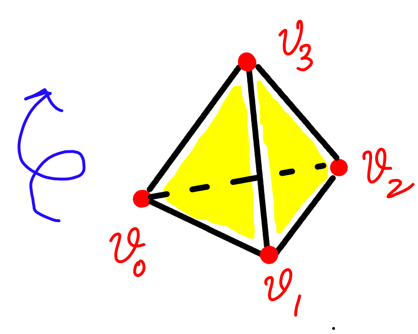
$[v_0 v_2 v_1]$ → is the clockwise orientation

3-simplex $[v_0 v_1 v_2 v_3]$



We could imagine orienting the tetrahedron as per the right-hand thumb rule – $v_0 \to v_1 \to v_2$ as the fingers of your right hand curl around, and $v_2 \to v_3$ points up along your thumb.

Notice that $[v_0 v_2 v_1 v_3]$, the opposite orientation, then corresponds to the left-hand thumb rule.



MATH 524: Lecture 7 (09/09/2025)

Today: * chains
* boundary homomorphism, fundamental lemma of homology

Recall orientation, $\sigma = [v_0 \ldots v_p] \to$ oriented simplex, opposite orientation...

Starting with oriented simplices, we define collections of them as functions, and then consider adding these functions — to define groups.

Def Let K be a simplicial complex. A **p-chain** on K is a function c from the set of oriented p-simplices of K to $\mathbb{Z}$ such that

(1) $c(\sigma) = -c(\sigma')$ if σ, σ' are opposite orientations of the same simplex; and

(2) $c(\sigma) = 0$ for all but finitely many p-simplices σ.

Thus, even on infinite simplicial complexes, each p-chain has nonzero values on only finitely many p-simplices.

We can add two p-chains by adding their values. The resulting group is the group of oriented p-chains of K, $C_p(K)$. If $p<0$ or $p>\dim(K)$, $C_p(K)$ is trivial.

One can indeed check that $C_p(K)$ is a group - identity (0), inverse ($c(\sigma')$), and associativity all hold. In fact $C_p(K)$ are abelian groups, as adding the functions is commutative.

Are there really only two orientations of higher dimensional simplices?

YES! Consider a 3-simplex $\sigma = [v_0 v_1 v_2 v_3]$ and $-\sigma = [v_1 v_0 v_2 v_3]$, its reverse orientation. What about $[v_2 v_0 v_3 v_1]$, for instance?

$$[v_2 v_0 v_3 v_1] \xrightarrow{1,2} [v_0 v_2 v_1 v_3] \xrightarrow{3} [v_0 v_1 v_2 v_3]$$ 3 swaps, i.e., odd.

Hence $[v_2 v_0 v_3 v_1]$ should be the opposite orientation to $[v_0 v_1 v_2 v_3]$. But then it should be the same orientation as $[v_1 v_0 v_2 v_3]$.

check: $[v_2 v_0 v_3 v_1] \xrightarrow{1} [v_1 v_0 v_3 v_2] \xrightarrow{2} [v_1 v_0 v_2 v_3]$ 2 swaps, i.e., even!

For oriented simplex σ, the **elementary chain** c corresponding to σ is the function defined as follows:

$$c(\sigma) = 1,$$
$$c(\sigma') = -1, \text{ where } \sigma' \text{ is the opposite orientation of } \sigma,$$
$$c(\tau) = 0, \quad \forall \tau \neq \sigma.$$

The correspondence to unit vectors in a Euclidean space is indeed direct here. The elementary chains have value +1 for exactly one p-simplex. Later on, we will see that these elementary chains correspond to unit vectors representing each p-simplex — at least in the case when K is finite.

Notation: σ denotes the simplex, oriented simplex, or the elementary chain corresponding to the simplex. Then we can write $\sigma' = -\sigma$ (where σ' is the simplex with orientation opposite to that of σ).

Lemma 5.1 [M] $C_p(K)$ is free abelian, and a basis for $C_p(K)$ can be obtained by orienting each p-simplex, and using the corresponding elementary chains.

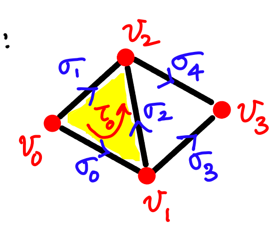
Notice that $C_0(K)$ has a "natural" basis, since each 0-simplex has only one orientation. But we do need to choose an orientation for each p-simplex to get a basis when $p > 0$. And there exist many bases when $p > 0$.

Corollary [M] Any function f from oriented p-simplices of K to abelian group G extends naturally to a homomorphism from $C_p(K)$ to G provided $f(-\sigma) = -f(\sigma)$ for all oriented p-simplices σ in K.

reverse orientation of σ.

Let's consider a small example.

K:



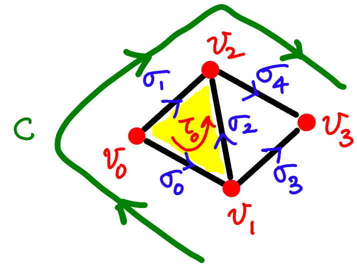
Let K be a shown here, with 4 vertices (v_0, v_1, v_2, v_3), 5 edges $(\sigma_0 - \sigma_4)$, and 1 triangle (τ_0). We orient the edges lexicographically, i.e., $[v_i v_j]$ with $i < j$. The triangle τ_0 is oriented as $[v_0 v_1 v_2]$, or CCW as shown here.

A 1-chain c can be specified as follows:

$c(\sigma_0) = -1$

$c(\sigma_1) = 1 \quad c(\sigma_2) = 0,$

$c(\sigma_4) = 1 \quad c(\sigma_3) = 0$

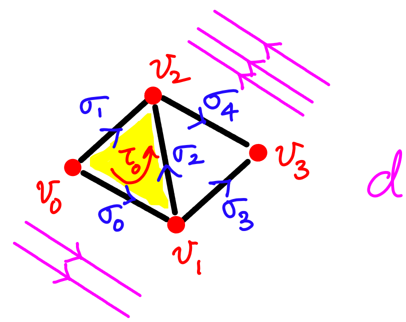


Notice that a value of -1 on σ_0 could be interpreted as "traversing" σ_0 once in its reverse orientation, i.e., going from v_1 to v_0 once. Following the same logic, we see that c here represents the piecewise linear "curve" going $v_1 \to v_0 \to v_2 \to v_3$ (once).

Here is another 1-chain d:

$d(\sigma_0) = 2, \quad d(\sigma_4) = -3$

$d(\sigma_j) = 0, \; j = 1, 2, 3.$



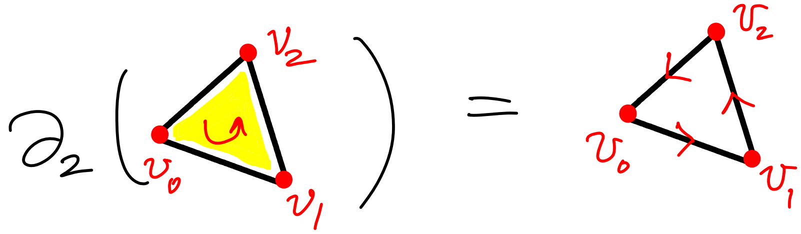
In particular, notice that the chains need not represent single connected pieces all the time.

A 2-chain can be $g(\tau_0) = 2$, which represents two copies of the single triangle in K.

Now that we have defined the chain groups $C_p(K)$ for each p, we now talk about how to connect/relate the $C_p(K)$ for various p. In particular, how are $C_p(K)$ and $C_{p-1}(K)$ related?

We define a homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ called the **boundary operator** (or boundary homomorphism). → Called the "p-boundary"

Intuitively, the boundary of a triangle is made of its three edges. But now we take the orientation also into account.



Def We define the homomorphism

$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$$

called the **boundary operator** as follows. If $\sigma = [v_0, \dots, v_p]$, $p > 0$, then

$$\partial_p \sigma = \partial_p [v_0, \dots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_p] \qquad (1)$$

where $\hat{v}_i$ means vertex v_i is deleted from $[v_0, \dots, v_p]$.

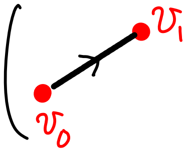
As $C_p(K)$ is trivial for $p < 0$, ∂_p is the trivial homomorphism for $p \leq 0$.

Since ∂_p is a homomorphism, we naturally extend the definition of boundary from p-simplices to p-chains. If $c = \sum n_i \sigma_i$ is a p-chain, then $\partial_p c = \partial_p\left(\sum n_i \sigma_i\right) = \sum n_i (\partial_p \sigma_i)$.

Examples

1-simplex

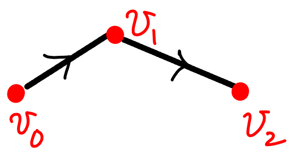
$\partial_1 [v_0 v_1] = v_1 - v_0$

$\partial_1 \Big($ 

$\Big) = v_1 - v_0$

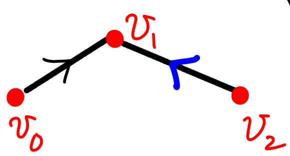
Notice that $\partial_1 [v_1 v_0] = v_0 - v_1$;

head − tail, if you think of the oriented edge as an "arrow".

$\partial_1 \Big($ 

$\Big) = v_1 - v_0 + v_2 - v_1 = v_2 - v_0$

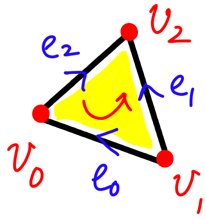
Notice that the computations are sensitive to the choice of orientations.

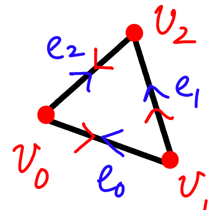
$\partial_1 \Big($ 

$\Big) = 2v_1 - v_0 - v_2$. $\quad (v_1 - v_0 + v_1 - v_2)$

2-simplex

$\partial_2 [v_0 v_1 v_2] = (-1)^0 [v_1 v_2] + (-1)^1 [v_0 v_2] + (-1)^2 [v_0 v_1] = [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$.

$\partial_2 \Big($ 

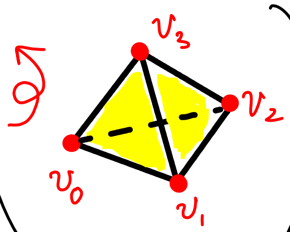
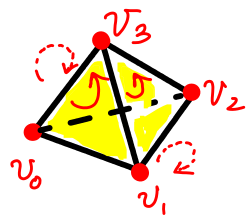
$\Big) =$ 

The 1-boundary is $-e_0 + e_1 - e_2$

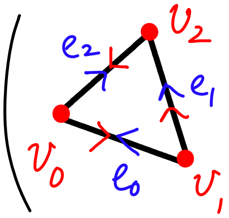
Notice that the orientation induced from the 2-simplex onto its faces (1-simplices) by the boundary operation could be distinct from the individual orientations of the 1-simplices themselves.

3-simplex

$$\partial_3 [v_0 v_1 v_2 v_3] = [v_1 v_2 v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 v_1 v_2]$$

$\partial_3 \Big($  $\Big) =$ 

We observe that $\partial_1 (\partial_2 [v_0 v_1 v_2]) = 0.$ (both algebraically and geometrically)

$\partial_1 \Big($  $\Big) = \partial_1 (-e_0 + e_1 - e_2) = -(v_0 - v_1) + (v_2 - v_1) - (v_2 - v_0) = 0.$

A similar observation can be made for the tetrahedron:

$$\partial_2 (\partial_3 [v_0 v_1 v_2 v_3]) = \partial_2 \big([\underline{v_1 v_2} v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 \underline{v_1 v_2}] \big) = 0$$

$+[v_1 v_2]$ $\qquad\qquad$ $-[v_1 v_2]$

every edge cancels in pairs.

Indeed, this result holds in general — $\partial_p \partial_{p+1} \sigma = 0$. And we can prove it using the definition of ∂_p.

Before that, let's make sure ∂_p is well-defined. In particular, we need to check that $\partial_p(-\sigma) = -\partial_p(\sigma)$.

We check what happens in Sum (1) when we swap v_j & v_{j+1}.

Consider $\sum_{i=0}^{p}(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_p]$ and $\sum_{i=0}^{p}(-1)^i(-[v_0,\dots,\hat{v}_i,\dots v_p])$. If $i\neq j,j+1$, the corresponding terms do differ by a sign. When $i=j$, compare terms in

$$\partial_p[v_0,\dots,v_{j-1},v_j,v_{j+1},v_{j+2},\dots,v_p] \text{ ——— (1a)}$$

and

swapped

$$\partial_p[v_0,\dots,v_{j-1},v_{j+1},v_j,v_{j+2},\dots,v_p]. \text{ ——— (1b)}$$

before we leave out one vertex at a time...

We have $(-1)^j[v_0,\dots v_{j-1},\hat{v}_j,v_{j+1},v_{j+2},\dots,v_p]$ in (1a), and

$(-1)^{j+1}[v_0,\dots,v_{j-1},v_{j+1},\hat{v}_j,v_{j+2},\dots,v_p]$ in (1b)

These two terms do differ by a sign: $(-1)^j$ and $(-1)^{j+1}$. Argument for $i=j+1$ is similar.

We now prove the general result on taking the boundary of a boundary. Indeed, we will use this result to define homology groups as subgroups of $C_p(K)$. Hence this result is called the fundamental lemma of homology.

Lemma 5.3 [M] $\quad \partial_{p-1}\circ\partial_p=0. \longrightarrow$ Fundamental lemma of homology

Proof

$$\partial_{p-1}\partial_p[v_0,\dots,v_p]$$

$$=\sum_{i=0}^{p}(-1)^i\partial_{p-1}[v_0,\dots,\hat{v}_i,\dots v_p]$$

$$=\sum_{j<i}(-1)^i(-1)^j[\dots\hat{v}_j,\dots,\hat{v}_i,\dots]+\sum_{j>i}(-1)^i(-1)^{j-1}[\dots\hat{v}_i,\dots\hat{v}_j\dots]$$

$=0$, as the terms cancel in pairs! $\square$

MATH 524: Lecture 8 (09/11/2025)

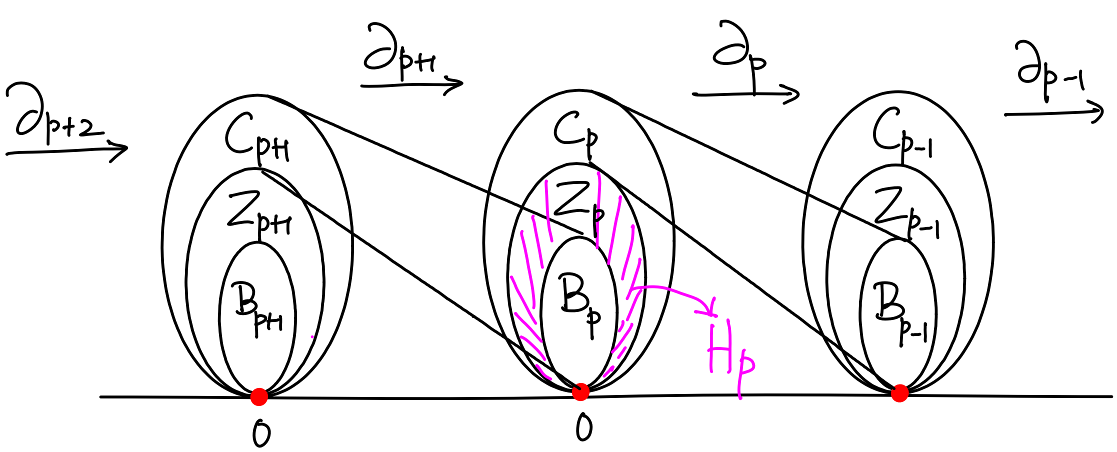
Today: * cycles, boundaries, homology group
* Examples

Recall: $\partial_p \circ \partial_{p+1} = 0$

Def The kernel of $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$ is the group of **p-cycles**, denoted $Z_p(K)$. The image of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is the group of **p-boundaries**, denoted $B_p(K)$.
Since $\partial_p \partial_{p+1} = 0$ by the above lemma, each boundary of a $(p+1)$-chain is automatically a p-cycle. Hence, $B_p(K) \subset Z_p(K) \subset C_p(K)$.

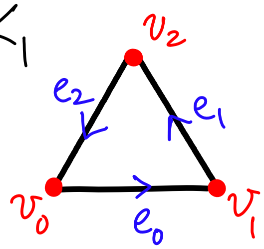
We now define $H_p(K) = Z_p(K) / B_p(K)$,
and call it the **p-th homology group** of K.

The various groups and ∂ homomorphisms have the following structure:



Examples

1. K_1



$C_1(K_1)$ is free abelian, generated by, e.g., $\{e_0, e_1, e_2\}$.

We use vector notation for chains

Any 1-chain in K can be given as $\bar{c} = n_0 e_0 + n_1 e_1 + n_2 e_2$ for $n_i \in \mathbb{Z}$.

ideally, $\bar{c} = n_0\bar{e}_0 + n_1\bar{e}_1 + n_2\bar{e}_2$

When is $\bar{c}$ a cycle?

$$\partial_1 \bar{c} = n_0(v_1 - v_0) + n_1(v_2 - v_1) + n_2(v_0 - v_2)$$
$$= (n_2 - n_0)v_0 + (n_0 - n_1)v_1 + (n_1 - n_2)v_2.$$

So, $\partial_1 \bar{c} = 0$ iff $n_0 = n_1 = n_2$. Thus $\bar{c}$ is a 1-cycle iff $n_0 = n_1 = n_2$.

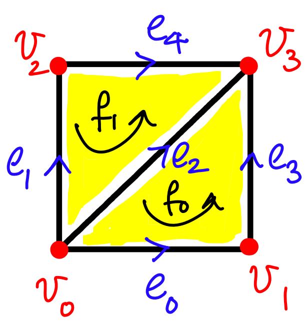
We can see that $Z_1(K_1)$ is infinite cycle, generated by $\bar{e}_0 + \bar{e}_1 + \bar{e}_2$. In other words, we can pick an integer, and that number tells us how many times we go around the cycle. If $n_0 = n_1 = n_2 = -3$, for instance, we go around in the opposite direction (i.e., clockwise) 3 times.

There are no 2-simplices, so $B_1(K_1)$ is trivial. In other words, there are no 1-boundaries. Hence $H_1(K_1) = Z_1(K_1) \simeq \mathbb{Z}$.

Also, $\beta_1(K) = rk(H_1(K_1)) = 1$.

Example 2

K_2



This is the same example 2 in [M], but with different choices of orientations.

$|K_2|$ is a square.

A general 1-chain in K_2 is $\bar{c} = \sum_{i=0}^{4} n_i \bar{e}_i$. Then $\partial_1 \bar{c} = \sum_i n_i \partial_1(\bar{e}_i)$.

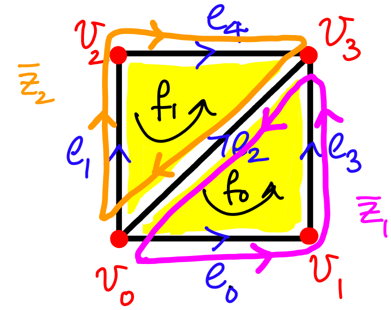
When is $\bar{c}$ as 1-cycle?

We need $n_0 = n_3$ (at v_1), and $n_1 = n_4$ (at v_2).

Similarly, $n_2 = -(n_0 + n_1)$ at v_0, and $n_2 = -(n_3 + n_4)$ at v_3.

Hence we can choose n_0, n_1 arbitrarily, and the other n_i's are fixed. So $Z_1(K_2)$ is free abelian with rank 2. One basis is $\{ \underbrace{\bar{e}_0 + \bar{e}_3 - \bar{e}_2}_{n_0 = 1,\, n_1 = 0},\ \underbrace{\bar{e}_1 + \bar{e}_4 - \bar{e}_2}_{n_0 = 0,\, n_1 = 1} \}$.

Let's call these cycles as $\{\bar{z}_1, \bar{z}_2\}$.

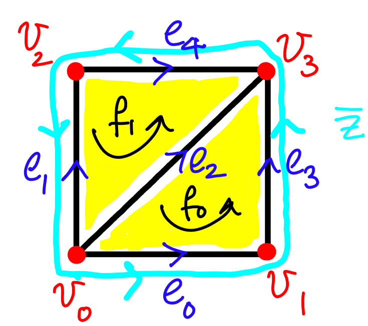


Indeed, any other cycle in K_1 can be written as a sum of $\bar{z}_1$ and $\bar{z}_2$. For instance, let $\bar{z}$ be the 1-cycle $e_0 + e_3 - e_4 - e_1$

Indeed, $\bar{z}$ can be written as $\bar{z}_1 - \bar{z}_2$

$$= (\bar{e}_0 + \bar{e}_3 - \bar{e}_2) - (\bar{e}_1 + \bar{e}_4 - \bar{e}_2).$$

note that the e_2 portions from $\bar{z}_1$ and $\bar{z}_2$ cancel.



Now let's characterize $B_1(K_2)$. Notice that both 1-cycles $\bar{z}_1$ and $\bar{z}_2$ are also 1-boundaries. Indeed, we have

$$\partial_2 \bar{f}_0 = \bar{e}_0 + \bar{e}_3 - \bar{e}_2 = \bar{z}_1 \quad \text{and} \quad \partial_2 \bar{f}_1 = \bar{e}_2 - \bar{e}_4 - \bar{e}_1 = -\bar{z}_2.$$

So $B_1(K_2) = Z_1(K_2)$. Hence $H_1(K_2) = Z_1(K_2)/B_1(K_2) = 0$.

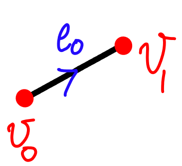
(i.e., the first homology group is trivial; there are no 1-cycles that are not 1-boundaries).

Likewise, $H_2(K_2) = 0$. The general 2-chain is $\bar{d} = m_0 \bar{f}_0 + m_1 \bar{f}_1$. And $\partial_2 \bar{d} = 0$ iff $m_0 = m_1 = 0$. There are no 2-cycles. And $H_p(K_2) = 0$ for $p \geqslant 3$ trivially.

We now present some definitions we will use subsequently.

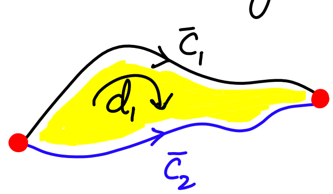
Def A chain $\bar{c}$ is **carried by a subcomplex** L if $\bar{c}(\sigma) = 0 \ \forall \sigma \notin L$. Two p-chains $\bar{c}_1, \bar{c}_2$ are **homologous** if $\bar{c}_1 - \bar{c}_2 = \partial_{p+1} \bar{d}$ for some $(p+1)$-chain $\bar{d}$. In particular, if $\bar{c} = \partial_{p+1} \bar{d}$, then $\bar{c}$ is homologous to zero, or we say that $\bar{c}$ **bounds**, i.e., $\bar{c}$ is a **boundary**.

→ we write $\bar{c}_1 \sim \bar{c}_2$.

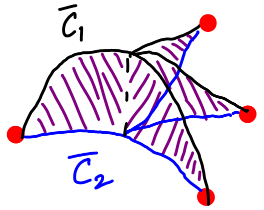


Here, the 2 0-chains v_0 and v_1 are homologous, since $v_1 - v_0 = \partial_1 e_0$.

Consider two 1-chains $\bar{c}_1, \bar{c}_2$ representing two 1D curves starting and ending at the same pair of vertices as shown.

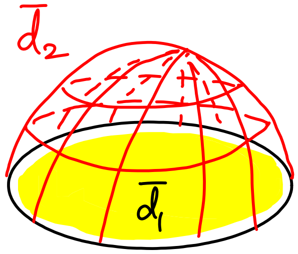


Then $\bar{c}_1$ and $\bar{c}_2$ are homologous here, as $\bar{c}_1 - \bar{c}_2 = \partial_2 \bar{d}_1$, where $\bar{d}_1$ is the 2-chain representing the 2D patch in between $\bar{c}_1$ & $\bar{c}_2$.



Notice that $\bar{c}_1$ and $\bar{c}_2$ need not be just simple open curves. Here, $\bar{c}_1$ and $\bar{c}_2$ both represent Y-shaped 1D curves. Again, $\bar{c}_1 - \bar{c}_2$ is the boundary of the 2D patch in between the two Y-shaped curves.

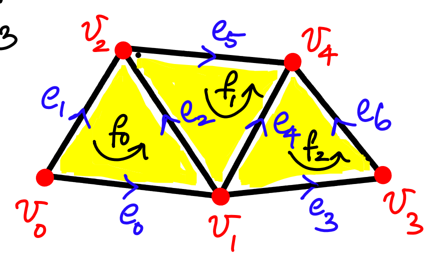
Consider two 2-chains $\bar{d}_1$ and $\bar{d}_2$, one representing a disc, and another representing the upper hemispherical surface that has the same boundary as the disc.



$\bar{d}_1$ and $\bar{d}_2$ are homologous, as $\bar{d}_1 - \bar{d}_2$ represents the boundary of the 3D solid hemisphere bounded by the two surfaces.

<u>Example 3</u>

K_3

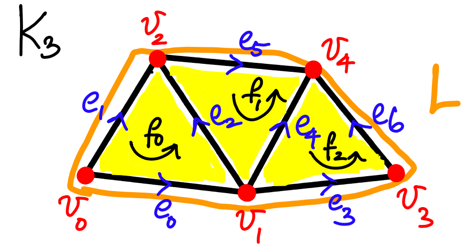


$$\bar{c} = \sum_{i=0}^{6} n_i e_i$$

Let $\bar{c}$ be a general 1-chain on K_3, and let its value on e_2 be c_2, i.e., $\bar{c}(e_2) = c_2$. Then the 1-chain $\bar{c}' = \bar{c} + \partial_2(c_2 \bar{f}_1)$ has $\bar{c}'(e_2) = 0$. Thus, we have "pushed $\bar{c}$ off e_2". Now, let $\bar{c}'(e_4) = c_4$. Then $= \bar{c}' + \partial_2(c_4 \bar{f}_2)$ has $\bar{c}''(e_4) = 0$. Notice that $\bar{c}''(\bar{e}_2) = 0$ still, as $\bar{e}_2 \notin \partial_2 \bar{f}_2$. So we have pushed $\bar{c}'$ off of e_4.

By combining the two steps, we have pushed $\bar{c}$ off of e_2 and e_4.

So, a 1-chain $\bar{c}$ on K_3 is homologous to $\bar{c}''$ carried by L_3, which is the subcomplex of K_3 made of $\{e_0, e_1, e_3, e_5, e_6\}$. Hence, $\bar{c}$ is a 1-cycle iff $\bar{c}''$ is.



But $\bar{c}''$ is a 1-cycle iff it is a multiple of $e_0+e_3+e_6-e_5-e_1$.

Hence $Z_1(K_3)$ has rank 1 (a basis is $\{e_0+e_3+e_6-e_5-e_1\}$).

But notice that this 1-cycle is also a 1-boundary. Precisely, it is $\partial_2(f_0+f_1+f_2)$. So $B_1(K_3)=Z_1(K_3)$.

Hence $H_1(K_3) = Z_1(K_3)/B_1(K_3) = 0$.

We also get that $H_p(K_3)=0$ for $p\geq 2$.

We used the two triangles to push off the general chain to the subcomplex which consists of the boundary of K_3. It's simpler to come up with the criterion for when a chain carried by this subcomplex is a cycle.

Notice that $H_p(K_2)=H_p(K_3)$ for $p\geq 1$, and that $|K_2|\approx|K_3|$.

This follows from the fundamental result that the homology groups depend only on the underlying space, and not on the particular simplicial complex chosen.

We could study homology in the "continuous" setting, i.e., without considering simplicial complexes. This homology, termed singular homology, can be shown to be equivalent to simplicial homology.

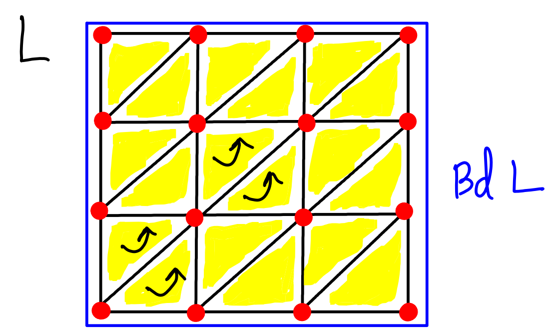
We can apply the techniques illustrated so far to compute the homology groups of more complicated simplicial complexes...

Homology Groups of Surfaces

If K is finite, $C_p(K)$ has finite rank, so does $Z_p(K)$ and $B_p(K)$. Also, $H_p(K)$ is finitely generated, and we can apply the fundamental theorem of finitely generated abelian groups to find the structure of $H_p(K)$. In particular, we want to compute the betti number and torsion coefficients of finite simplicial complexes representing surfaces, e.g., torus, Klein bottle, Möbius strip etc.

We first formalize the idea of "pushing chains off to the boundary of the simplicial complex."

Lemma 6.1 [M]



Let L be the simplicial complex such that $|L|$ is a rectangle. Let $Bd\,L$ be the subcomplex of L representing the edges making up the boundary of the rectangle. Orient each triangle counterclockwise (CCW), and the edges arbitrarily. Then the following results hold.

(1) Every 1-cycle of L is homologous to a 1-cycle carried by $Bd\,L$.

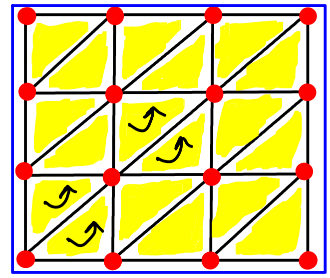
(2) If $\bar{d}$ is a 2-chain of L, and if $\partial \bar{d}$ is carried by $Bd\,L$, then $\bar{d}$ is a multiple of the chain $\sum \sigma_i$, where σ_i are all the triangles.

MATH 524: Lecture 9 (09/16/2025)

Today: * Homology groups of torus, Klein bottle

Lemma 6.1 [M]

Let L be the simplicial complex such that $|L|$ is a rectangle. Let $\mathrm{Bd}\, L$ be the subcomplex of L representing the edges making up the boundary of the rectangle.



Orient each triangle counter clockwise (CCW), and the edges arbitrarily. Then the following results hold.

(1) Every 1-cycle of L is homologous to a 1-cycle carried by $\mathrm{Bd}\, L$.

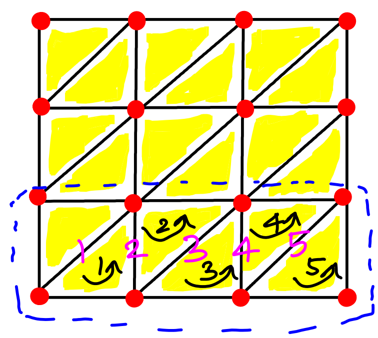
(2) If $\bar{d}$ is a 2-chain of L, and if $\partial\bar{d}$ is carried by $\mathrm{Bd}\, L$, then $\bar{d}$ is a multiple of the chain $\sum \sigma_i$, where σ_i are all the triangles.

Proof of Lemma 6.1 [M]

(2) follows because if σ_i and σ_j in $\bar{d}$ have an edge in the middle in common, then $\partial\bar{d}$ has coefficient 0 on that edge. Hence, $\bar{d}$ has the same value on σ_i & σ_j. We can extend this argument to all σ_i's, as $\partial\bar{d}$ is carried by $\mathrm{Bd}\, L$, giving that $\bar{d}$ has the same coefficient on all of them.

(1) We use the idea of pushing the chain off of edges in the middle (as we did in Example 3).

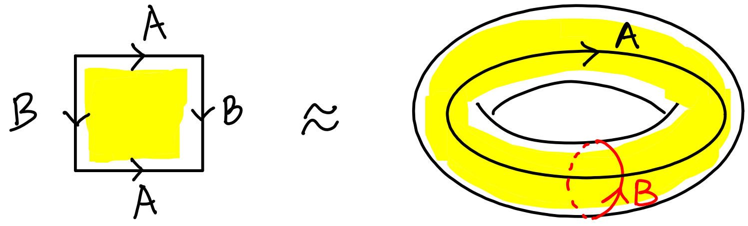
We can use the triangles in the order shown here—from $1a$ to $5a$—to push the input chain off of edges marked 1 to 5 (in that order). After these steps, the chain will be pushed on to the edges shown with the dashed outline.



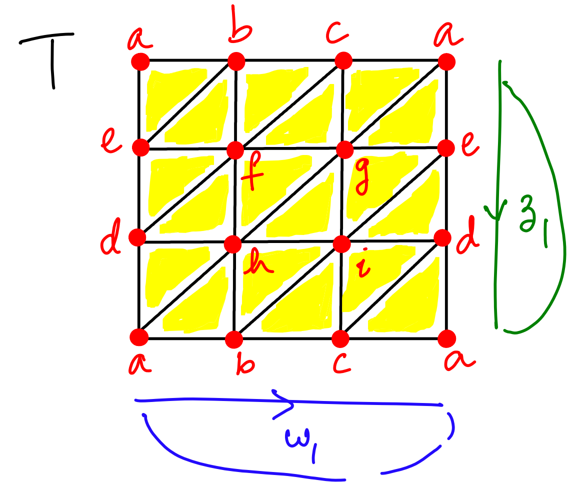
We then repeat the process on the second "horizontal strip" of triangles, and then the top strip. Ultimately, the chain is carried by $\mathrm{Bd}\, L$. □

Theorem 6.2 [M] $(\mathbb{T}^2)$

(homology of torus)



Let T be the simplicial complex representing L, the rectangle, along with the vertex labels. $|T|$ is the torus.



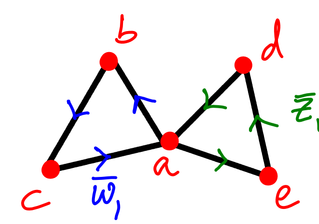
Then $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $H_2(T) \simeq \mathbb{Z}$.

Orient each 2-simplex CCW; let $\bar{\gamma}$ denote their sum (2-chain). Let $\bar{w}_1 = [a,b] + [b,c] + [c,a]$, and $\bar{z}_1 = [a,e] + [e,d] + [d,a]$.

Then $\bar{\gamma}$ generates $H_2(T)$ and $\{\bar{w}_1, \bar{z}_1\}$ is a basis for $H_1(T)$.

Proof

Let $g : |L| \to |T|$ be the pasting map (labeling), and let $A = g(|\text{Bd} L|)$. Then A is homeomorphic to the union of two circles with one point in common (also called a wedge of two circles).



We apply the same "pushing off" arguments as in the proof of the last Lemma. We get the following results.

(1) Every 1-cycle of T is homologous to a 1-cycle carried by A.

(2) If $\bar{d}$ is a 2-chain of T and $\partial\bar{d}$ is carried by A, then $\bar{d}$ is a multiple of $\bar{r}$.

Notice the correspondence of statements (1) and (2) above to those in Lemma 6.1 [M].

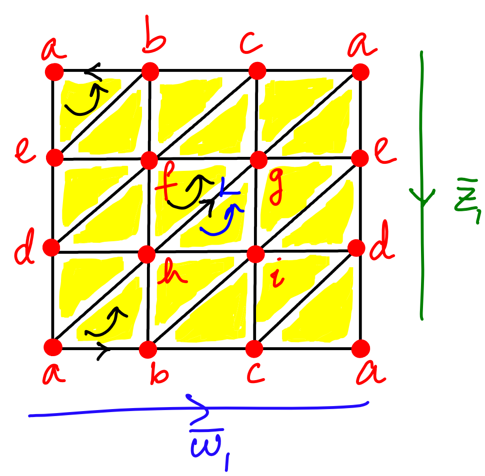
But we get two more results here.

(3) If $\bar{c}$ is a 1-cycle carried by A, then $\bar{c} = m\bar{w}_1 + n\bar{z}_1$ for $m, n \in \mathbb{Z}$; and

(4) $\partial_2 \bar{r} = 0$.

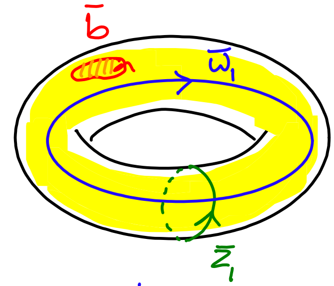
To see these results, check the orientations induced on each edge by the two triangles it is a face of.

e.g., $[a,b]$ has $+1$ from $\partial_2[abh]$ and -1 from $\partial_2[aeb]$.

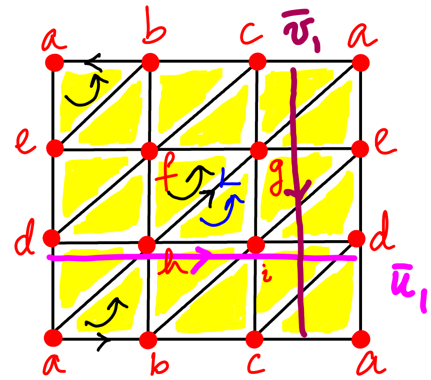


A 1-cycle in A bounds only if it is trivial, as every 1-cycle is $\bar{c} = m\bar{w}_1 + n\bar{z}_1$, and if $\bar{c} = \partial_2\bar{d}$ then by (2) $\bar{d} = p\bar{r}$, $p \in \mathbb{Z}$. Hence by (4), since $\partial_2\bar{d} = 0$, we must have $m=n=p=0$ here.

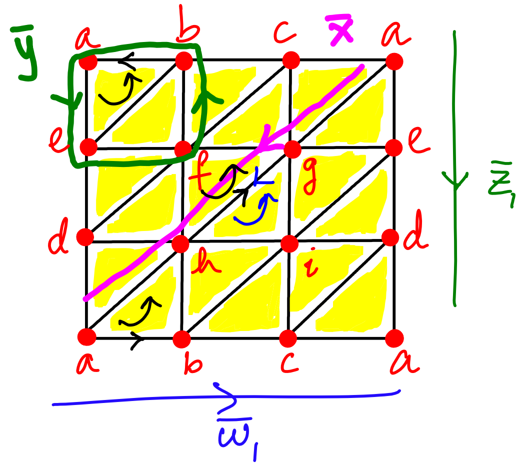
Hence we can conclude that $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $\{\bar{w}_1, \bar{z}_1\}$ is a basis.



Imagine elastic bands wrapping around the torus along $\bar{w}_1$ and $\bar{z}_1$ as shown here. These bands cannot be shrunk to a point, while another cycle/band represented by $\bar{b}$ (as shown) can be shrunk to a point. Indeed $\bar{b}$ bounds the patch of surface it encloses.

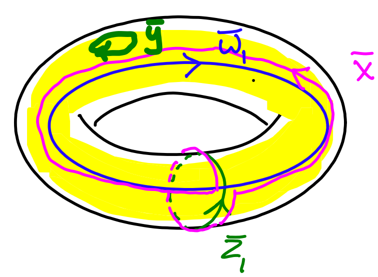


Also, there are many other choices for a basis for $H_1(T)$. For instance, we could use $\{\bar{u}_1, \bar{v}_1\}$, where $\bar{u}_1 = [d,h]+[h,i]+[i,d]$ and $\bar{v}_1 = [c,g]+[g,i]+[i,c]$.



What about the cycle $\bar{x} = [a,g]+[g,h]+[h,a]$?

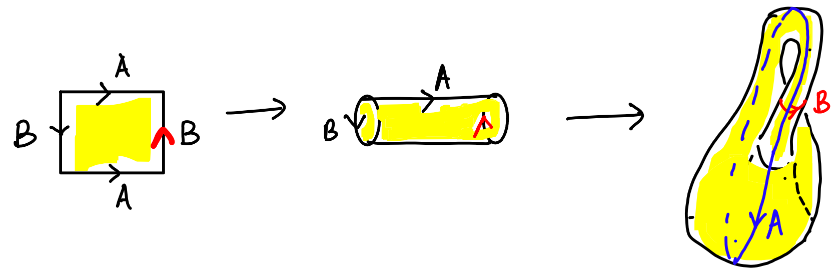
We can see that $\bar{w}_1 - \bar{z}_1 + \bar{x}$ is the boundary of the 2-chain consisting of all triangles below $\bar{x}$ in the diagram. In other words, $\bar{x}$ is homologous to $\bar{w}_1 + (-1)\bar{z}_1$, or equivalently, we can write $\bar{x}$ as a combination of $\bar{w}_1$ and $\bar{z}_1$.



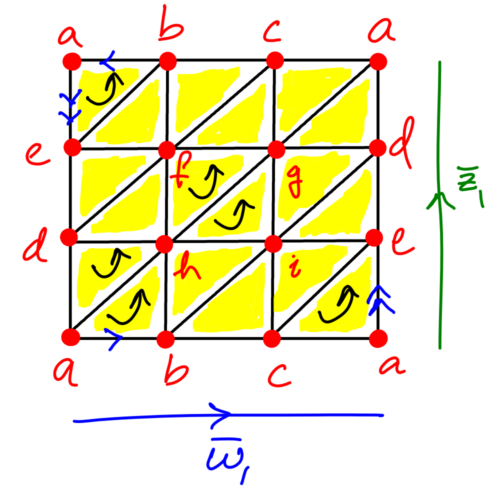
And, $\bar{y}$ is a boundary — it is the boundary of the 2-chain consisting of triangles aeb and bef. So $\bar{y} \in B_1(T)$, and hence $\notin H_1(T)$.

$H_2(T)$: By (1) and (2), if $\bar{d}$ is a 2-cycle of T then $\bar{d} = p\bar{r}$, $p \in \mathbb{Z}$. But (4) says $\partial_2 \bar{r} = 0$, and hence every such 2-chain is a 2-cycle. There are no tetrahedra in T, and hence no 2-boundaries. So, $H_2(T) = Z_2(T) \simeq \mathbb{Z}$, with $\bar{r}$ being a generator. $\square$

Theorem 6.3 [M] (Klein bottle)



K



Let K be the complex shown, and L the rectangle with the labels. $|K| \approx$ Klein bottle. Then

$$H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}/_2 \quad \text{and} \quad H_2(K) = 0.$$

Let
$$\bar{w}_1 = [a,b] + [b,c] + [c,a] \quad \text{and}$$
$$\bar{z}_1 = [a,e] + [e,d] + [d,a]$$

Then the torsion subgroup of $H_1(K)$ is represented by $\bar{z}_1$, and the free part is generated by $\bar{w}_1$.

Proof

We follow the same technique as in the case of the torus. Indeed, we can push an input chain off of all the edges in the middle, as before. But notice that the edges in A (boundary of the square) do not all behave identically. For instance, $[a,b]$ does get opposite orientations from $\partial_2[abh]$ and $\partial_2[aeb]$. But $[a,e]$ gets $+1$ from both $\partial_2[aec]$ and $\partial_2[aeb]$. As such, $\partial_2 \bar{r} \neq 0$ here!

Similar to the previous case, we get the following results.

(1) Every 1-cycle of K is homologous to a 1-cycle carried by A.

(2) If $\bar{d}$ is a 2-chain of K and $\partial\bar{d}$ is carried by A, then $\bar{d}$ is a multiple of $\bar{r}$.

We also get (3):

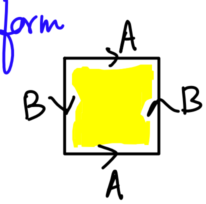
(3) If $\bar{c}$ is a 1-cycle carried by A, then $\bar{c} = m\bar{w}_1 + n\bar{z}_1$ for $m, n \in \mathbb{Z}$.

But instead of (4), we get

(4') $\partial_2 \bar{r} = 2\bar{z}_1$.

Like last time, we get that $\bar{c}$, a 1-cycle of K, is homologous to $m\bar{w}_1 + n\bar{z}_1$. If $\bar{c} = \partial_2 \bar{d}$, then $\bar{c} = \partial_2 \bar{d} = 2p\bar{z}_1$, $p \in \mathbb{Z}$. Hence, $\bar{c}$ is a boundary iff $m = 0$, n is even. Hence we get $H_1(K) \simeq \underset{\text{free}}{\mathbb{Z}} \oplus \underset{\text{torsion.}}{\mathbb{Z}/_2}$, with $\bar{w}_1$ generating the free part and $\bar{z}_1$ generating the torsion part.

Intuitively, one can see from the "pasting" picture itself that the boundary of the square space is $2B$. In the case of the torus, both A and B do not form boundaries, but here, $2B$ is the boundary.



$H_2(K)$: Since $\partial_2 \bar{r} = 2\bar{z}_1 \neq 0$, $Z_2(K) = 0$, and hence $H_2(K) = 0$.

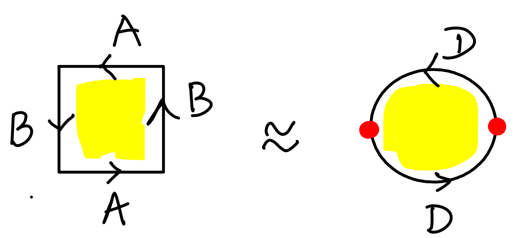
Intuitively, the Klein bottle does not enclose a 3D space like the torus. Hence, its 2nd homology group is trivial.

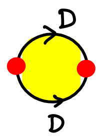
MATH 524: Lecture 10 (09/18/2025)

Today:
* Homology of $\mathbb{R}P^2$
* connected sum
* 0-dimensional homology

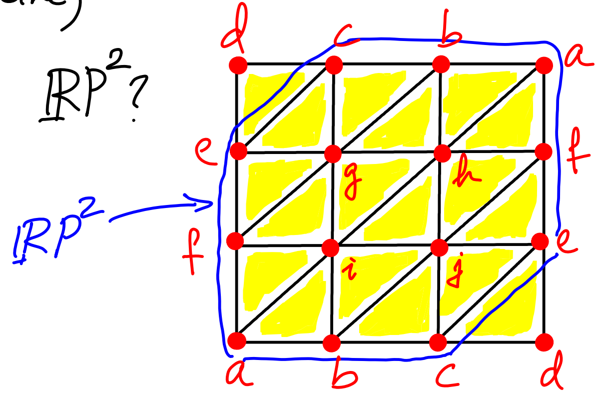
Theorem 6.4 [M]

$\mathbb{R}P^2$ (projective plane)



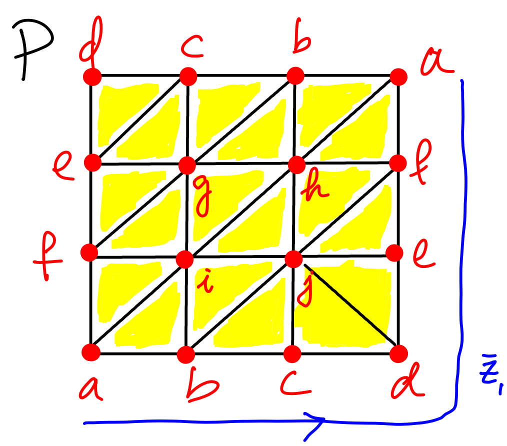
(Note:  $\approx S^2$!)

$\mathbb{R}P^2$?



Is there a problem with the simplicial complex? Notice that edge $[c,e]$ is shared by three triangles $[cde]$, $[ceg]$, $[cej]$. If you work it out, this simplicial complex is in fact almost correct, i.e., its underlying space is $\mathbb{R}P^2$ with a "flap" which is triangle $[cde]$.

But we can fix the simplicial complex by flipping one copy of $\overline{ce}$ for an off-diagonal edge, e.g., $\overline{dj}$. You can check that every edge in the simplicial complex P is shared by exactly two triangles.



Let P be the simplicial complex, and L is the underlying space (rectangle). $|P| \approx \mathbb{R}P^2$. We get $H_1(P) \simeq \mathbb{Z}/2$, and $H_2(P) = 0$.

Proof

Let $g: |L| \to |P|$ be the pasting map, and let $A = g(|\mathrm{Bd}\,L|)$.

Here, A is a circle. Let $z_1 = [a,b]+[b,c]+[c,d]+[d,e]+[e,f]+[f,a]$.

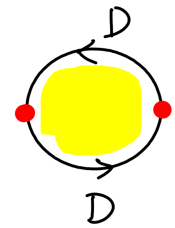
We get corresponding results to (1)-(4) as for torus & Klein bottle. (1) and (2) hold as before. Again, we push chains off of the edges in the middle, and can give criteria describing the homology in terms of structure of cycles carried by A, the boundary of $|L|$. In place of (3), (4), we get the following results.

(3') Every 1-cycle carried by A is a multiple of $\bar{z}_1$.

(4'') $\partial_2 \bar{\gamma} = 2\bar{z}_1$.

Hence $H_1(P) \simeq \mathbb{Z}/2$, $H_2(P) = 0$. as $\partial_2 \bar{\gamma} \neq 0$

We could come to the same conclusion directly from this diagram - notice that ∂D is the boundary of the 2D space modeled by the disc.

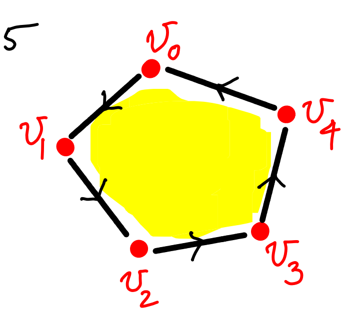


Hence, $H_1(P)$ has no free part, but only the torsion part.

So, $\beta_1(P) = 0$, and $\bar{z}_1$ generates the torsion part.

The projective plane is a special case of the k-fold dunce hat for $k=2$. The general space is obtained by taking a k-sided polygon (k-gon) with vertices $v_0, \ldots, v_{k-1}$ and edges $v_i v_{i+1}$ for $i = 0, \ldots, k-2$ along with $v_{k-1} v_0$. We then identify consecutive pairs of edges $[(v_i v_{i+1})$ and $(v_{i+1} v_{i+2})]$.

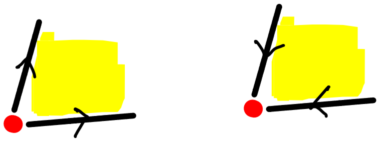
$k=5$



D_K

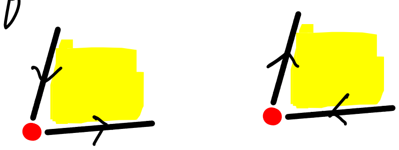
The arrows here indicate how you identify the edges, and not their orientations.

When "gluing" edges connected to a single node, with the arrows indicating the order of identification/gluing, if the arrows both come in or both go out, then we glue them without a twist.



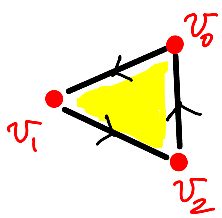
no twist

But if one arrow comes in and the other goes out, we glue with a twist, as we do in the case of Möbius strip.



with twist

D_3
$(k=3)$

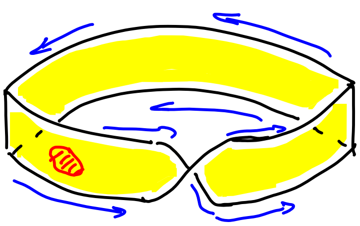


$H_1(D_k) \simeq \mathbb{Z}/k$.

Essentially, the boundary of the 2D space modeled by the k-gon is the cycle going k-times around.

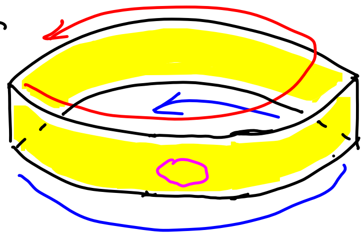
Here are two more examples — Möbius strip and cylinder.

M



$H_1(M) \simeq \mathbb{Z}$.

C



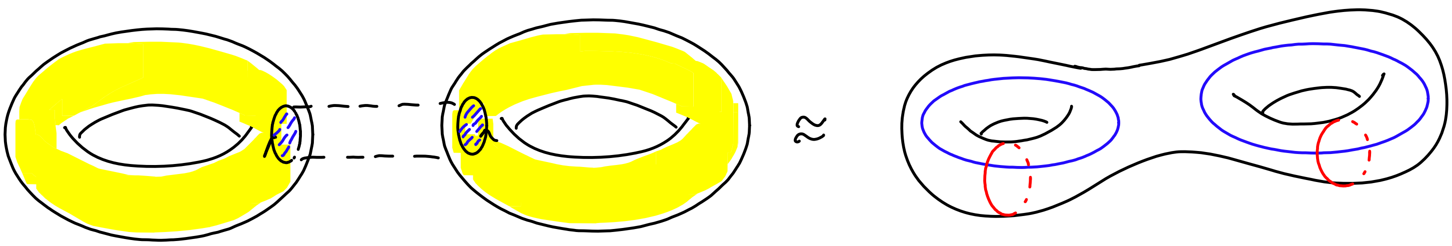
$H_1(C) \simeq \mathbb{Z}$

Indeed, both the cylinder and Möbius strip have the same homology groups (and not just H_1).

We now talk about how to join surfaces to get more general surfaces, and how to figure out their homology groups

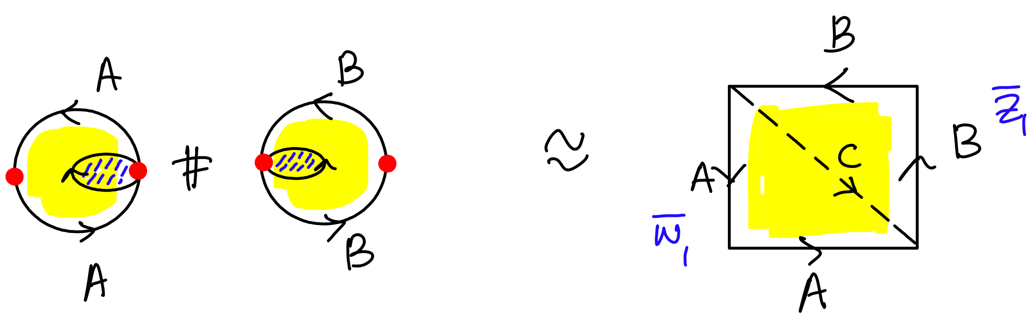
Def The **connected sum** of two surfaces S_1 and S_2 is the space obtained by deleting an open disc from each, and pasting the remaining pieces along the edge of the removed disc. We denote this connected sum as $S_1 \# S_2$.

e.g., $\mathbb{T}^2 \# \mathbb{T}^2$ double torus



$H_1(\mathbb{T}^2 \# \mathbb{T}^2) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

<u>$\mathbb{R}P^2 \# \mathbb{R}P^2$</u>



Theorem 6.5 [M] $H_1(\mathbb{R}P^2 \# \mathbb{R}P^2) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$ and

$$H_2(\mathbb{R}P^2 \# \mathbb{R}P^2) = 0.$$

We can figure out the structure of these homology groups directly from the diagram above, rather than use a simplicial complex.

Let $\overline{w}_1$ be the 1-cycle represented by "A" (left and below), and $\overline{z}_1$ be the 1-cycle represented by "B" (right and above). We get (1) and (2) as before, and (3') & (4') as follows.

(3') Every 1-cycle carried by A is of the form $m\overline{w}_1 + n\overline{z}_1$, $m, n \in \mathbb{Z}$.

(4') $\partial_2 \overline{r} = 2\overline{w}_1 + 2\overline{z}_1$.

So $H_2(\mathbb{R}P^2 \# \mathbb{R}P^2) = 0$. What about $H_1(\mathbb{R}P^2 \# \mathbb{R}P^2)$?

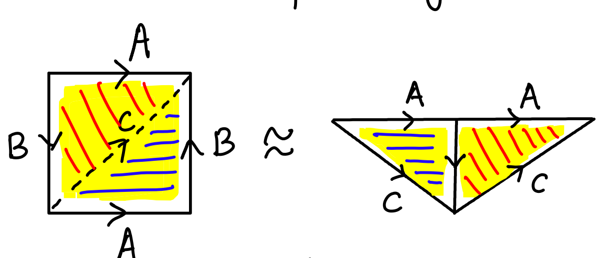
$\{\overline{w}_1, \overline{z}_1\}$ is a basis, but the torsion part is not "sarated" in the basis. We can use $\{\overline{w}_1, \underbrace{\overline{w}_1+\overline{z}_1}_{\overline{z}_1'}\}$ as another basis instead, as $\overline{z}_1 = -(\overline{w}_1) + (\overline{w}_1 + \overline{z}_1)$.

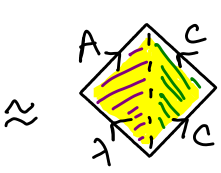
With $\{\overline{w}_1, \overline{z}_1'\}$ as the basis, we can directly see that $2\overline{z}_1'$ is a boundary, so $H_1(\mathbb{R}P^2 \# \mathbb{R}P^2) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

We could've used $\{3\overline{w}_1 + 4\overline{z}_1, \overline{w}_1 + \overline{z}_1\}$ also! Or $\{\overline{z}_1, \overline{w}_1 + \overline{z}_1\}$.

Notice that $H_1(\mathbb{R}P^2 \# \mathbb{R}P^2) \simeq H_1(\mathbb{K}^2)$. In fact, $\mathbb{R}P^2 \# \mathbb{R}P^2 \approx \mathbb{K}^2$!

Here's a proof by picture.

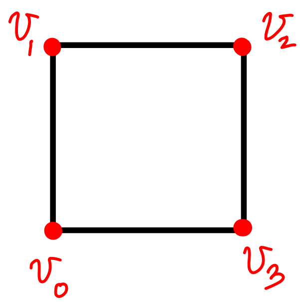




To glue the "B" edges with a twist, we first take the mirror image of the right triangle across the horizontal axis and then slide it over to the left side to glue. The resulting space is indeed the connected sum of two $\mathbb{R}P^2$'s, as shown below.

Zero-dimensional Homology

We start with an example.



Consider a general 0-chain of the form $\bar{c} = \sum_{i=0}^{3} n_i v_i$.

We can use the similar idea to pushing edges off; here, we push some vertices off. For instance, we can push v_3 off of the 0-chain using edge $\overline{v_2 v_3}$. Continuing this process, we see that $\bar{c} = \sum_{i=0}^{3} n_i v_i \sim n v_0$ for some $n \in \mathbb{Z}$.

With $\bar{c}' = n_3(v_2 - v_3) = n_3(\partial[v_2, v_3])$,
$\bar{c} + \bar{c}' = n_0 v_0 + n_1 v_1 + (n_2 + n_3) v_2$.

Theorem 7.1 [M] The group $H_0(K)$ of simplicial complex K is free abelian. If $\{v_\alpha\}$ is a collection of vertices such that there is one vertex from each connected component of $|K|$, then $\{v_\alpha\}$ is a basis for $H_0(K)$.

MATH 524: Lecture 11 (09/23/2025)

Today:
* 0-dimensional homology
* reduced homology
* relative homology

Recall

Theorem 7.1 [M] The group $H_0(K)$ of simplicial complex K is free abelian. If $\{v_\alpha\}$ is a collection of vertices such that there is one vertex from each connected component of $|K|$, then $\{v_\alpha\}$ is a basis for $H_0(K)$.

Proof (ideas)

Step 1

0-skeleton, i.e., vertices

homologous

(i) For $v, w \in K^{(0)}$, we define $v \sim w$ if there is a sequence $a_0, \dots, a_n$, with $a_i \in K^{(0)}$ such that $a_0 = v, a_n = w$, and $(a_i, a_{i+1}) \in K^{(1)} \, \forall i$.

the orientation does not matter; we just need (a_i, a_{i+1}) as an edge.

We also define $C_v = \bigcup \{ \text{St}\, w \mid w \sim v \}$.

(ii) Show C_v is path-connected $\forall v \in K^{(0)}$.

(iii) If $C_v \neq C_{v'}$ (i.e., are distinct), then they are disjoint.

It follows that C_v are the connected components of $|K|$.

$\square$

Step 2 Let $\{v_\alpha\}$ be a collection of vertices of K with one vertex from each component. Each 0-chain in a single component is homologous to v_α. Hence every 0-chain on K is homologous to a linear combination of elementary 0-chains v_α.

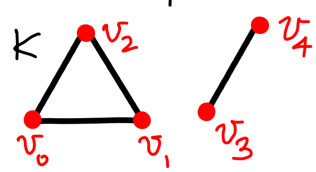
Let $\bar{c} = \sum' n_\alpha v_\alpha$ be a general 0-chain. Suppose $\bar{c} = \partial \bar{d}$ for 1-chain. We can write $\bar{d} = \sum' \bar{d}_\alpha$ where $\bar{d}_\alpha$ has terms of $\bar{d}$ carried by C_α. Consider one such component: $\partial \bar{d}_\alpha \sim n_\alpha v_\alpha$. It follows that $n_\alpha = 0$ here.

Why?

Let $\varepsilon : C_0(K) \to \mathbb{Z}$ be the homomorphism defined by $\varepsilon(v) = 1 \ \forall v \in K^{(0)}$. Then $\varepsilon(\partial[v,w]) = \varepsilon(w - v) = 1 - 1 = 0$. Hence $0 = \varepsilon(\partial \bar{d}_\alpha) = \varepsilon(n_\alpha v_\alpha) = n_\alpha$. $\square$

Note: $\beta_0 = \operatorname{rk}(H_0(K))$ counts the number of connected components

Another example: $\beta_0(K) = 2$ here $\longrightarrow$
$\{v_0, v_3\}$ is a basis for $H_0(K)$



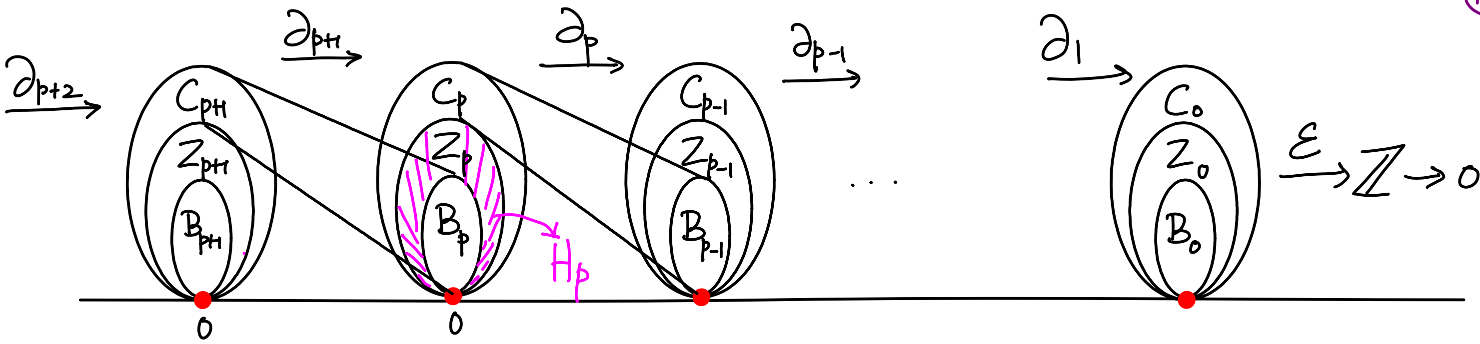
To follow the intuition for $p \geq 1$ of $\beta_p = 1$ when 1 $(p+1)$-dimensional "patch" is missing (thus creating a p-dim hole), we want $\beta_0 = 1$ when 1 edge, for instance, is missing, i.e., when there are two components (not 1). To this end, we define reduced homology groups.

Reduced Homology Groups

Let $\varepsilon : C_0(K) \to \mathbb{Z}$ be a surjective homomorphism defined by $\varepsilon(v) = 1 \ \forall v \in K^{(0)}$. For a 0-chain $\bar{c}$, $\varepsilon(\bar{c})$ is the sum of the values of $\bar{c}$ on vertices of K. ε is the **augmenting map** for $C_0(K)$. Also, $\varepsilon(\partial_1 \bar{d}) = 0$ for all 1-chains $\bar{d}$. So we define the the **reduced homology group** of K in dimension 0 as

$$\tilde{H}_0(K) = \ker \varepsilon / \operatorname{im} \partial_1 .$$

Also, if $p > 0$, $\tilde{H}_p(K) = H_p(K)$.



Theorem 7.2 [M] $\tilde{H}_0(K)$ is free abelian, and $\tilde{H}_0(K) \oplus \mathbb{Z} \simeq H_0(K)$.

So, $\tilde{H}_0(K)$ vanishes if K is connected. Else $\{v_\alpha - v_{\alpha_0}\}$ for $\alpha \neq \alpha_0$ form a basis for $\tilde{H}_0(K)$. Here v_{α_0} is any one of the v_α's, which are from each connected component.

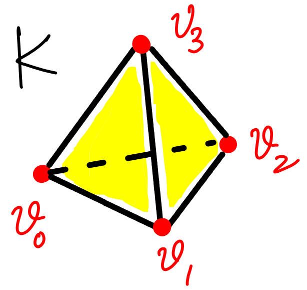
Proof If $\bar{c} \in \ker \varepsilon$, then $\varepsilon(\bar{c}) = \varepsilon(\bar{c}') = 0$, where $\bar{c}' \sim \bar{c}$ and $\bar{c}' = \sum n_\alpha v_\alpha$. But $\varepsilon(\bar{c}') = \sum n_\alpha \varepsilon(v_\alpha) = \sum n_\alpha$.

If $|K|$ has only one component, $\bar{c}' = 0$. If $|K|$ has more than one component, then $\bar{c}'$ is a linear combination of $\{v_\alpha - v_{\alpha_0}\}$.

We refer to $rk(\tilde{H}_0(K)) = \tilde{\beta}_0$ as the reduced 0-th betti number of K

We get $\tilde{\beta}_0 = \beta_0 - 1$ and $\tilde{\beta}_p = \beta_p \ \forall p \geq 1$.

Homology of a p-simplex



solid tetrahedron

K: σ (3-simplex) and all its faces.

$\tilde{H}_i(K) = 0 \ \forall i$.

$\tilde{H}_3(K) = 0$, $\tilde{H}_2(K) = 0$, $\tilde{H}_1(K) = 0$, but $H_0(K) \simeq \mathbb{Z}$, and hence $\tilde{H}_0(K) = 0$.

Let Σ^{p-1} be the simplicial complex whose polytope is $\mathrm{Bd}\,\sigma$.
Then, $\tilde{H}_i(\Sigma^{p-1}) = 0$ for $i \neq p-1$, and $\tilde{H}_{p-1}(\Sigma^{p-1}) \simeq \mathbb{Z}$.

Here (for $p=3$), Σ^2 consists of the four triangles that are faces of σ, and their own faces. There are no tetrahedra in Σ^2, so $\bar{r} = n \sum_{i=0}^{3} \tau_i$, where τ_i are the triangles, is a 2-cycle which is not a 2-boundary for each $n \in \mathbb{Z}$, $n \neq 0$. Hence $\{\bar{r}\}$ is a basis for $\tilde{H}_2(\Sigma^2)$.

Relative Homology

We often want to talk about homology groups restricted to some parts of the given simplicial complex K. In particular, we want to "avoid" a subcomplex from consideration. Given a subcomplex K_0, a chain carried by K_0 is trivially extended to a chain in all of K by assigning zero as the coefficient for all simplices in K but not in K_0. Intuitively, we want to "zero out" all chains in K_0, and talk about homology groups in K modulo K_0.

Def If $K_0 \subseteq K$ is a subcomplex, the quotient group $C_p(K)/C_p(K_0)$ is the group of **relative p-chains of K modulo K_0**, denoted $C_p(K, K_0)$.

Notice that $C_p(K_0)$ can be naturally considered as a subgroup of $C_p(K)$, by assigning coefficients of zero to simplices not in K_0.

$C_p(K,K_0)$ is free abelian, and has as a basis all cosets of the form

$$\{\sigma_i\} = \sigma_i + C_p(K_0)$$

where σ_i is a p-simplex of K not in K_0.

Intuitively, adding any chain from $C_p(K_0)$ to σ_i is like adding "zero" as far as $C_p(K,K_0)$ is concerned.

the "absolute" boundary operator ∂

$\partial : C_p(K_0) \longrightarrow C_{p-1}(K_0)$ is just the restriction of ∂ on $C_p(K)$ to K_0. This homomorphism induces a homomorphism

$$\partial : C_p(K,K_0) \longrightarrow C_{p-1}(K,K_0).$$

we will use ∂ to denote both the absolute and the relative boundary operators

This is the **relative boundary operator**.

Like the absolute boundary operator, the relative boundary operator also satisfies $\partial \circ \partial = 0$. We let

$$Z_p(K,K_0) = \ker \partial_p : C_p(K,K_0) \to C_{p-1}(K,K_0),$$

$$B_p(K,K_0) = \operatorname{im} \partial_{p+1} : C_{p+1}(K,K_0) \longrightarrow C_p(K,K_0), \text{ and}$$

$$H_p(K,K_0) = Z_p(K,K_0)/B_p(K,K_0).$$

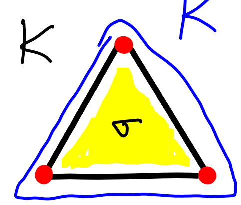
These groups are called the **relative p-cycle, relative p-boundary**, and the **relative homology group** of dimension p of K modulo K_0.

A relative p-chain $\bar{c}$ is a relative p-cycle iff $\partial_p \bar{c}$ is carried by K_0. Furthermore, it's a relative p-boundary iff there exists a $(p+1)$-chain $\bar{d}$ of K such that $\bar{c} - \partial_{p+1}\bar{d}$ is carried by K_0.

Recall that in the absolute case, we wanted $\partial_p \bar{c}$ and $\bar{c} - \partial_{p+1}\bar{d}$ to be empty, respectively.

<u>Example 1</u> Let K be the p-simplex σ and all its faces. Let K_0 be $K^{(p-1)}$, i.e., the simplicial complex made of all proper faces of σ. Then

$H_i(K, K_0) = 0 \;\forall\, i \neq p$, and



$H_p(K, K_0) \simeq \mathbb{Z}$, and $\{\sigma\}$ is a basis.

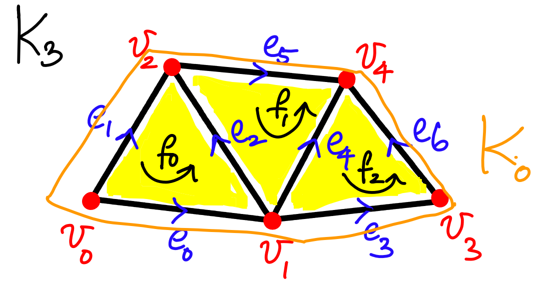
$H_2(K, K_0) \simeq \mathbb{Z}$

as $\partial_p \sigma$ is carried by K_0, and hence it's a relative p-cycle. There are no $(p+1)$-simplices to bound, so it's not a relative p-boundary.

Notice that $\bar{c} = n\sigma$ for $n \in \mathbb{Z}$ is not an absolute cycle, as $\partial_2 \bar{c} \neq 0$.

Example 2 (Example 3 in Lecture 9)

Let K_0 be the subcomplex consisting of e_0, e_1, e_3, e_5, e_6 and all the vertices. Then we get



$H_2(K_3, K_0) \simeq \mathbb{Z}$, and $\bar{r} = \sum_{i=0}^{2} f_i$ is a generator.

We use the same techniques as before. The triangles are oriented CCW. Then $\bar{r}$, the 2-chain which is the sum of the triangles taken with multipliers of 1 each, has $\partial \bar{r}$ carried by K_0. Hence it is a relative 2-cycle.

There are no tetrahedra in K_1 and hence there are no 2 boundaries (absolute or relative). Hence $\bar{r}$ generates $H_2(K_1 K_0)$.

We now consider $H_1(K, K_0)$. Using the same "pushing off edges in the middle" argument as before, we get that any 1-chain in K is homologous to a 1-chain carried by K_0, and hence is a relative 1-cycle that is trivial. In more detail, every 1-chain in K not in K_0 is a relative 1-cycle, and is also a relative 1-boundary since we can find a 2-chain generated by f_1 and f_2 whose difference with this 1-chain is carried by K_0.

Thus, $H_1(K_3, K_0) = 0$, as any 1-chain in K is homologous to a 1-chain carried by K_0.

Here, $H_0(K_3, K_0) = 0$ as well, as $v_i \in K_0 \ \forall i$.

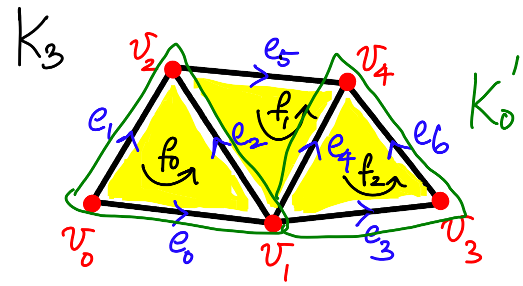
Notice the similarity between Examples 1 (for $p=2$) and 2 – the homology groups are the same. Also, notice that $|K_3|$ and $|K|$ are homeomorphic (both are discs), and $|K_0|$'s are also homeomorphic (to a circle in each case). These examples seem to indicate that relative homology groups are determined by the underlying space, and not by the choice of the simplicial complexes – indeed, this is true in general, but the proof is technical.

MATH 524: Lecture 12 (09/25/2025)

Today: * More examples of relative homology
* Excision theorem

Example 2 (continued..)

Now, consider K_0' as the subcomplex made of $\{e_0, e_1, e_2, e_3, e_4, e_6\}$, and all vertices.



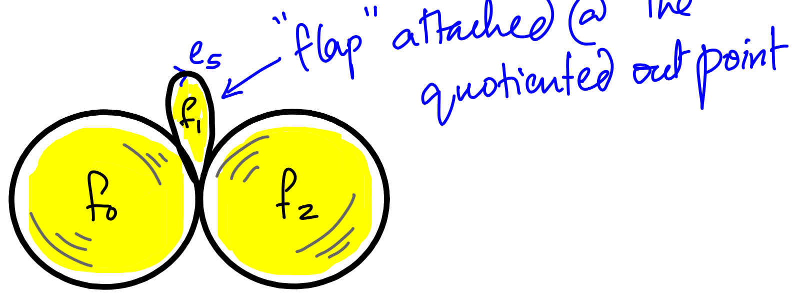
$H_2(K_3, K_0') \simeq \mathbb{Z} \oplus \mathbb{Z}$ here!

If $\bar{r}' = n_0\bar{f}_0 + n_2\bar{f}_2$, then $\partial\bar{r}'$ is carried by K_0, and hence is a relative 2-cycle. And $n_0, n_2 \in \mathbb{Z}$ could be chosen arbitrarily. Indeed, $\{\bar{f}_0, \bar{f}_1\}$ is a basis.

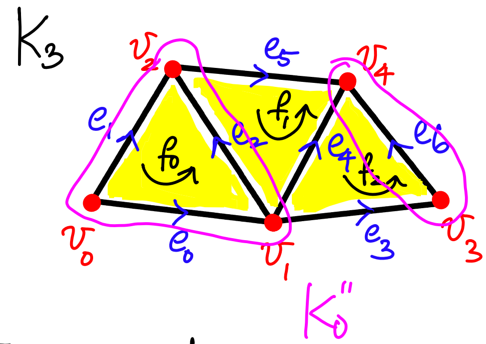
But $H_1(K_3, K_0') = 0$ still. All relative 1-chains are generated by $\{\bar{e}_5\}$, which happens to be a relative 1-cycle as $\partial_1\bar{e}_5$ is carried by K_0'. But $\bar{e}_5$ is also a relative 1-boundary as $\bar{e}_5 + \partial_2\bar{f}_1$ is carried by K_0'.

Similarly, $H_0(K_3, K_0') = 0$, as all $v_i \in K_0'$.

Intuitively, one could think of K_3/K_0' as comprised of two spheres touching each other at a point, along with a "flap" (disc) attached to the same point of contact between the spheres.



Now consider K_0'' as shown:

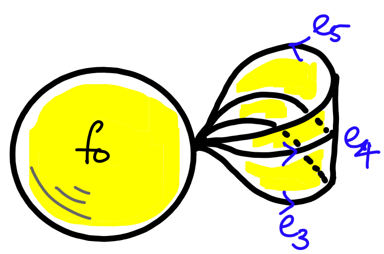


K_0'' : $\{e_0, e_1, e_2, e_6\}$ and all vertices.

We get that $H_2(K_3, K_0'') \simeq \mathbb{Z}$, and $\{\bar{f}_0\}$ is a basis. Notice that $n_1\bar{f}_1 + n_2\bar{f}_2$ is not a relative 2-cycle for any $n_1, n_2 \in \mathbb{Z}$, except $n_1 = n_2 = 0$.

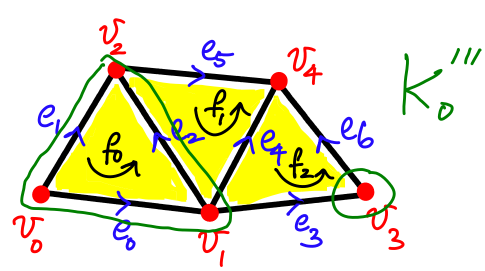
$H_1(K_3, K_0'') \simeq \mathbb{Z}$. We can push off any relative 1-chain in K_3/K_0'' of $\bar{e}_3$ and $\bar{e}_4$, for instance, leaving $\bar{e}_5$ as a generator of $H_1(K_3, K_0'')$.

Intuitively, one could imagine "shrinking" all of $|K_0''|$ to a point, and consider homology of K modulo that point. In this sense, one could think of



Also, notice that different choices of K_0 lead to different $H_p(K, K_0)$ groups.

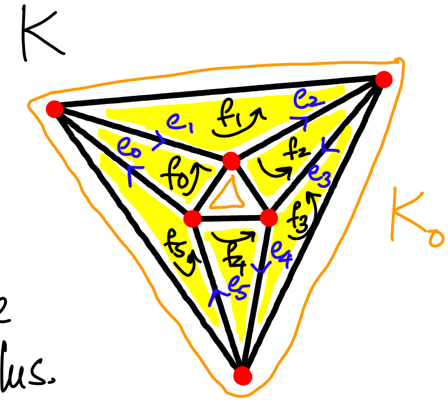
Now consider K_0''' as shown.



What is $H_0(K_3, K_0''') = ?$

Think! Think!

Example 3 (Annulus)



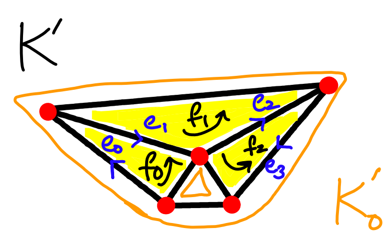
Let K consist of the six triangles $f_0, \ldots, f_5$ as shown here, with the triangle in the middle missing. Hence $|K|$ is homeomorphic to the 2D annulus.

Let K_0 consist of all the boundary edges and their vertices, i.e., both the inner and outer circular boundaries.

Then $H_2(K, K_0) \simeq \mathbb{Z}$. Notice that $\bar{r} = \sum_{i=0}^{5} f_i$ has $\partial_2 \bar{r}$ carried by K_0. Indeed, $\bar{r}$ generates $H_2(K, K_0)$.

What about $H_1(K, K_0)$? Notice that we can push any relative 1-chain off of $\bar{e}_1$ using $\bar{f}_1$, and then $\bar{e}_2$ using $\bar{f}_2$, and so on, all the way around. But we will be left with $\bar{e}_0$ in this case. Thus, $\{\bar{e}_0\}$ is a relative 1-cycle which is not a relative 1-boundary. Thus, $H_1(K, K_0) \simeq \mathbb{Z}$.

But now consider a modified complex as shown here.

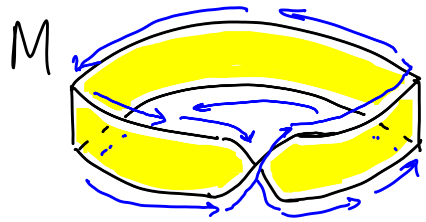


Notice that e_0 is carried by K_0'.

Indeed, $H_1(K', K_0') = 0$ here.

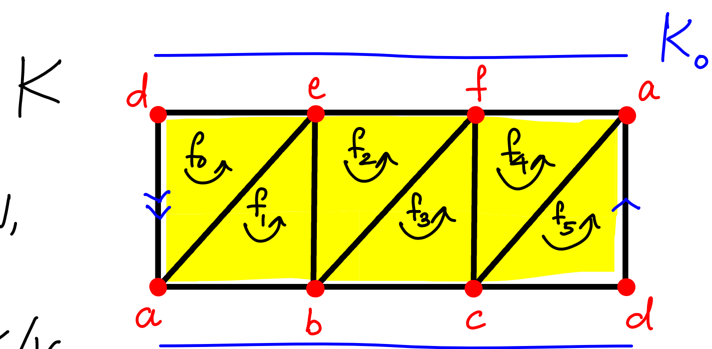
Example 4 Torsion in relative homology groups of Möbius strip:

Recall:



$H_1(M) \simeq \mathbb{Z}$ no torsion! despite the twist.

Let K triangulate the Möbius strip, as shown. With all triangles oriented ccw, notice that for the 2-chain $\bar{r} = \sum_{i=0}^{5} f_i$, $\underline{\partial_2 \bar{r} = 2(\overline{da})}$ in K/K_0.



$\longrightarrow \partial_2 \bar{r}$ is $2\overline{da}$ + edges in K_0

Let K_0 be the "edge" of the Möbius strip, as shown. Then $H_1(K, K_0) \simeq \mathbb{Z}_2$, as $2(\overline{da})$ is a relative boundary, but $(\overline{da})$ is not. Of course, $\overline{da}$ is a relative 1-cycle here.

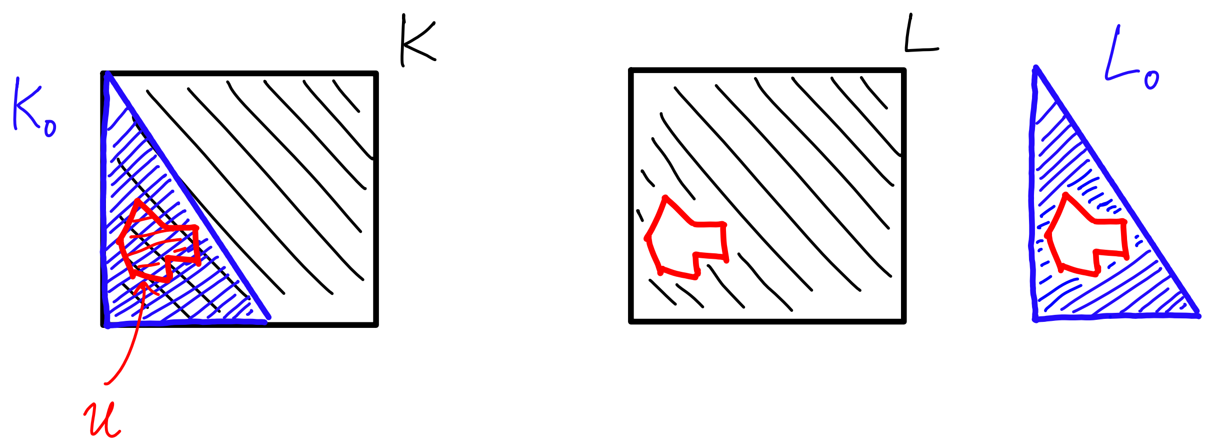
Note that every edge "going across" is a relative 1-cycle here, e.g., $\overline{ae}$, $\overline{bf}$, $\overline{ca}$, etc.

Intuitively, we can "shrink" all of K_0 to a point, and consider $|K|/|K_0|$, after this reduction. This point of view affords some powerful applications/tools. In particular, we could make changes to the interior of K_0, without affecting $H_p(K, K_0)$. We make this notion precise in the following theorem.

Theorem 9.1 [M] (Excision theorem) Let K_0 be a subcomplex of K. Let $\mathcal{U} \subset |K_0|$ (strict subset) be an open set such that $|K| - \mathcal{U}$ is the polytope of a subcomplex L of K and let L_0 be the subcomplex whose polytope is $|K_0| - \mathcal{U}$. Then inclusion induces an isomorphism

$$H_p(L, L_0) \simeq H_p(K, K_0).$$

Here is a schematic illustration. The spaces here are supposed to represent simplicial complexes.



In many cases, L/L_0 is much nicer, or easier to compute with, than K/K_0. In particular, if $\mathcal{U}$ is chosen to be large (but still contained in K_0), L and L_0 might be much simpler than K and K_0. We will encounter applications of the excision theorem later on...

Proof idea: Consider the composite map ϕ

$$C_p(L) \to C_p(K) \longrightarrow C_p(K)/C_p(K_0)$$

defined as inclusion followed by projection. "project out" parts in K_0

a p-chain in L is extended to a p-chain in K by setting the weights on p-simplices in K/L to zero.

ϕ is surjective, as $C_p(K)/C_p(K_0)$ has as basis all cosets $\{\sigma_i\}$ for p-simplices σ_i in K not in K_0, and all such $\sigma_i \in L$. Also, $\ker \phi$ is $C_p(L_0)$.

So, ϕ induces an isomorphism $C_p(L)/C_p(L_0) \simeq C_p(K)/C_p(K_0)$ $\forall p$.

And ∂ is preserved under this isomorphism.

The p-simplex σ is mapped to empty (i.e., to zero) if it is in L_0 by the projection part of ϕ.

Hence, $H_p(L, L_0) \simeq H_p(K, K_0)$. $\square$

We now turn to simplicial maps, and how the groups we have studied – chains, cycles, boundaries, and homology groups – behave under them. We introduce several useful algebraic tools in this process.

Homomorphisms induced by Simplicial Maps

§12 in [M]

Recall Simplicial map: Given simplicial complexes K and L, $f: K \to L$ is a simplicial map if f is a continuous map of $|K|$ to $|L|$ that maps each simplex of K linearly onto a simplex of L.

We could start with the corresponding vertex map, and extend the same linearly to the simplicial map.

Note that a simplex in K could be mapped to a lower dimensional simplex in L by f. We define a homomorphism from f by "staying in the same dimension."

$(\dim(f(\sigma)) \le \dim \sigma)$.

Def Let $f: K \to L$ be a simplicial map. If $(v_0, \dots, v_p)$ is a simplex of K, then $f(v_0), \dots, f(v_p)$ span a simplex of L. We define a homomorphism $f_{\#}: C_p(K) \to C_p(L)$ by defining it on oriented p-simplices as follows.

$$f_{\#}([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_i) \text{ are distinct,} \\ 0, & \text{otherwise.} \end{cases}$$

This map is indeed well-defined, i.e., $f_{\#}(-\sigma) = -f_{\#}(\sigma)$.

If we swap v_i and v_j in $[v_0, \dots, v_p]$, the sign of the right-hand side expression is changed.

The family of homomorphisms $\{f_{\#}\}$, one in each dimension, is called the **chain map induced by the simplicial map f**.

MATH 524: Lecture 13 (09/30/2025)

Today: simplicial maps and induced homomorphisms

Recall For $f: K \to L$, $f_\#([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_i) \neq f(v_j), \\ 0 & \text{o.w.} \end{cases}$

Notation: Ideally, we should write $(f_\#)_p : C_p(K) \to C_p(L)$, for each dimension p. But we avoid writing p, when the dimension is evident. We will work with $f_\#$ and ∂ in both K and L. We use ∂ to denote the boundary in K and in L. We could be particular, and write ∂_K and ∂_L, when necessary.

Notice that $f_\#$ is a homomorphism for C_p groups. What about the homology groups H_p? It turns out that $f_\#$ induces a homomorphism f_* from $H_p(K)$ to $H_p(L)$.

Lemma 12.1 [M] The homomorphism $f_\#$ commutes with ∂; therefore $f_\#$ induces a homomorphism $f_* : H_p(K) \to H_p(L)$.

composing the maps along either path gives the same result

$$\begin{tikzcd} C_p(K) \arrow[r, "(f_\#)_p"] \arrow[d, "\partial_K"'] & C_p(L) \arrow[d, "\partial_L"] \\ C_{p-1}(K) \arrow[r, "(f_\#)_{p-1}"] & C_{p-1}(L) \end{tikzcd}$$

To be exact, we say $\partial_L \circ (f_\#)_p = (f_\#)_{p-1} \circ \partial_K$, or just briefly, $\partial f_\# = f_\# \partial$.

Proof We first show that

$$\partial f_{\#}([v_0,\dots,v_p]) = f_{\#}(\overbrace{\partial [v_0,\dots,v_p]}^{\sum_{i=0}^{p}(-1)^i [v_0 \dots \hat{v}_i \dots v_p]}) \quad \text{———} \quad (*).$$

Let τ be the simplex spanned by $f(v_0),\dots,f(v_p)$. We consider three cases, based on the dimension of τ.

Case 1. $\dim(\tau) = p$. Here, $f(v_0),\dots,f(v_p)$ are distinct, and hence the result follows directly since $f_{\#}$ and ∂ are homomorphisms.

Case 2. $\dim(\tau) \le p-2$. → three or more vertices are mapped to one vertex, or two or more pairs are identified.

LHS of $(*)$ vanishes, as $f(v_i)$ are not all distinct.
RHS of $(*)$ is also 0, as $\forall i$, two or more terms out of $f(v_0),\dots,f(v_{i-1}), f(v_{i+1}),\dots, f(v_p)$ are the same.

Case 3 $\dim(\tau) = p-1$. → exactly one pair of v_i's are mapped to the same vertex in L

WLOG, assume vertices are ordered such that

$f(v_0) = f(v_1) \neq f(v_2) \neq \dots \neq f(v_p)$. → $f(v_0)=f(v_1)$, while the remaining $f(v_i)$ are all distinct, and also distinct from $f(v_0)$

Again, LHS of $(*)$ vanishes by definion.
RHS of $(*)$ has only two nonzero terms:
$[f(v_1), f(v_2),\dots,f(v_p)]$ and $-[f(v_0), f(v_2),\dots, f(v_p)]$.

As $f(v_0)=f(v_1)$, these two terms cancel.

Further, $f_\#$ carries cycles to cycles and boundaries to boundaries.

Let $\bar{z} \in Z_p(K)$. Then $\partial(\bar{z}) = 0$. By Lemma, $\partial f_\#(\bar{z}) = f_\# \partial(\bar{z}) = 0$. So $f_\#(\bar{z})$ is a cycle, i.e., $f_\#(\bar{z}) \in Z_p(L)$.

Similarly, if $\bar{b} \in B_p(K)$, then $\bar{b} = \partial_{p+1}\bar{d}$ for $\bar{d} \in C_{p+1}(K)$.

But $\partial f_\#(\bar{d}) = f_\# \partial \bar{d} = f_\#(\bar{b})$, and hence $f_\#(\bar{b}) \in B_p(L)$.

Thus, $f_\#$ induces a homomorphism of the homology groups, $f_* : H_p(K) \to H_p(L)$. $\square$

We can naturally combine the homomorphisms induced by multiple simplicial maps, as the following theorem describes.

Theorem 12.2 [M] (a) Let $i : K \to K$ be the identity simplicial map. Then $i_* : H_p(K) \to H_p(K)$ is the identity homomorphism.

(b) Let $f : K \to L$ and $g : L \to M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$, i.e., the following diagram commutes.

$$\begin{tikzcd} H_p(K) \arrow[rr, "(g\circ f)_*"] \arrow[dr, "f_*"'] & & H_p(M) \\ & H_p(L) \arrow[ur, "g_*"'] & \end{tikzcd}$$

This theorem presents the **functorial** property of the induced homomorphism. Think of H_p as an operator that assigns to each simplicial complex an abelian group, and $*$ as another operator that assigns to each simplicial map of one complex to another, a homomorphism between the corresponding abelian groups.

We say that $(H_p, *)$ is a "functor" from the "category" of simplicial complexes and simplicial maps to the "category" of abelian groups and homomorphisms.

Intuitively, a "category" consists of a collection of sets (or "objects") along with maps between them. A functor assigns pairs of such structure, i.e., (object, map) pairs, in one category to those in the other category such that it "preserves the structure" of the category.

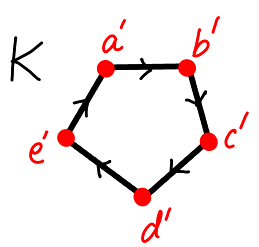
The ideas of commuting diagrams in particular, and functoriality in general, are used widely in algebraic topology. We will see more of these concepts in the upcoming lectures.

We study further the homomorphisms induced by simplicial maps. In particular, we talk about when distinct simplicial maps induce equal homomorphisms on homology groups.

Lemma 12.3 [M] $f_\#$ preserves the augmentation map ϵ; therefore it induces a homomorphism f_* of reduced homology groups.

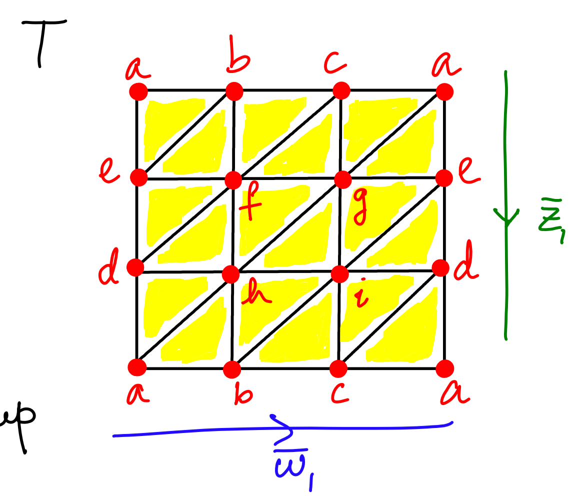
Proof. Let $f: K \to L$ be a simplicial map. Then $\epsilon f_\#(v) = 1$, and $\epsilon(v) = 1$ $\forall v \in K^{(0)}$. Hence $\epsilon \circ f_\# = \epsilon$. Thus $f_\#$ carries the kernel of $\epsilon_K : C_0(K) \to \mathbb{Z}$ into the kernel of $\epsilon_L : C_0(L) \to \mathbb{Z}$, and so it induces a homomorphism $f_* : \tilde{H}_0(K) \to \tilde{H}_0(L)$.

Example
Consider K that is a loop, and T, which represents $\mathbb{T}^2$.



$\bar{z}$: $a'b'c'd'e'$

$H_1(K) \simeq \mathbb{Z}$, $\{\bar{z}\}$ generates this group



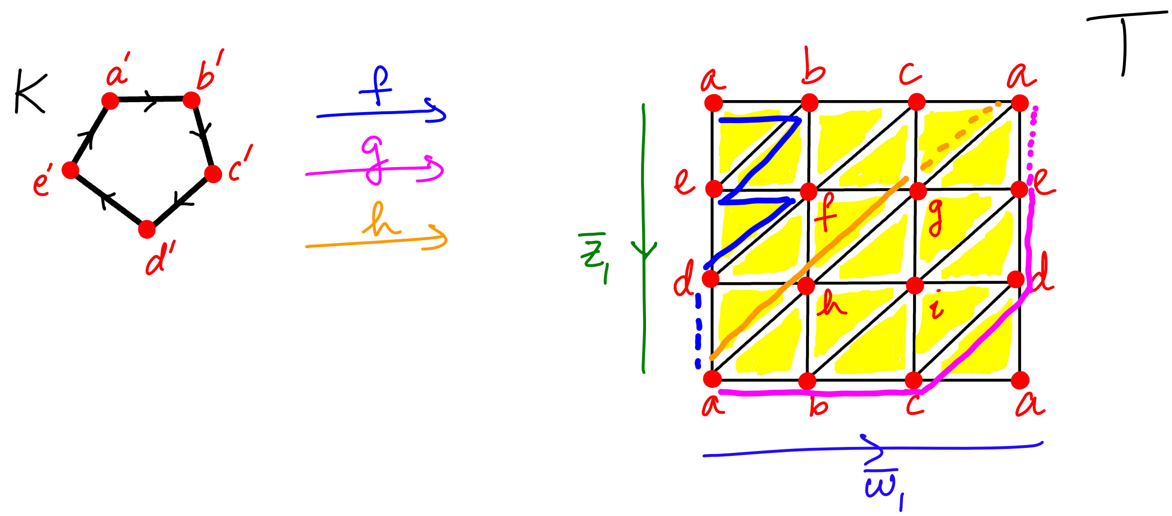
As described previously, with $\bar{w}_1 = [a,b] + [b,c] + [c,a]$ and $\bar{z}_1 = [a,e] + [e,d] + [d,a]$, $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and $\{\bar{w}_1, \bar{z}_1\}$ is a basis.
We consider three different simplicial maps $f, g, h : K \to T$, described by the maps for each vertex in K.

$f: a' \to a$, $b' \to b$, $c' \to e$, $d' \to f$, $e' \to d$

$g: a' \to a$, $b' \to b$, $c' \to c$, $d' \to d$, $e' \to e$

$h: a' \to a$, $b' \to h$, $c' \to h$, $d' \to g$, $e' \to g$

We can visualize the three maps as follows.

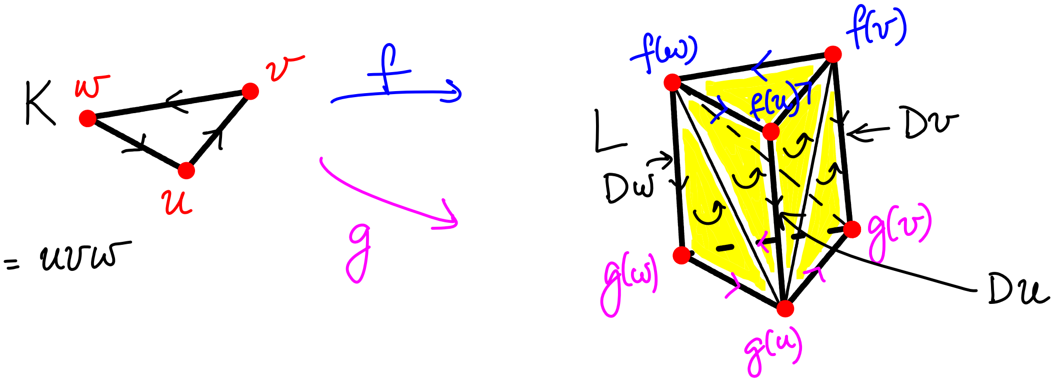


We can check that $f_\#(\bar{z}) \sim z_1$, $g_\#(\bar{z}) \sim \bar{w}_1 - \bar{z}_1$, and $h_\#(\bar{z}) \sim \bar{w}_1 - \bar{z}_1$. Hence g_* and h_* are equal as homomorphisms of the first homology group.
It can be checked that g_* and h_* are equal as homomorphisms of the 0-dimensional homology groups as well.

When can this observation hold in general?

Given simplicial maps $f, g: K \to L$, we want to find conditions under which $f_\#(z) \sim g_\#(z)$ $\forall z \in Z_p(K)$. Thus we want to find a $(p+1)$-chain Dz of L such that $f_\#(z) - g_\#(z) = \partial Dz$.

Here's an example where we can find Dz straightforwardly.



$z = uvw$

L consists of 6 triangles such that $|L|$ is the cylinder. K is made of 3 edges forming a cycle, which we term z. f and g are two simplicial maps which map z to the top and bottom cycles, respectively, in L. The triangles in L can be oriented consistently, e.g., CCW when looking from outside.

Here, Dz can be chosen to be the 2-chain made of the 6 triangles in the middle.

But for a different pair of maps f' and g', it might not be as straightforward to identify Dz in all cases.

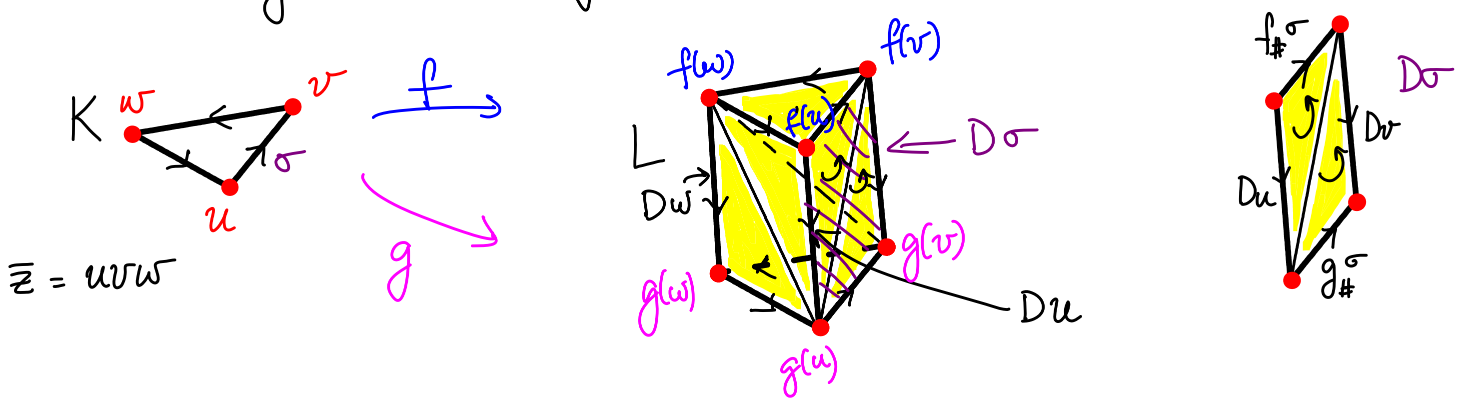
When can we find Dz easily for all cycles z? Could we specify some sufficient conditions for the existence of Dz?

MATH 524: Lecture 14 (10/02/2025)

Today: * chain homotopy
* star condition

We continue with the example where we could identify $D\bar{z}$. But now we identify $D\sigma$ for elementary chains $\sigma \in K$, starting with vertices and proceeding to higher dimensional simplices. Our goal is to identify some sort of formula that $D\bar{c}$ should satisfy for a general p-chain $\bar{c} \in C_p(K)$.

For vertex $v \in K^{(0)}$, define Dv to be the edge in L connecting $f(v)$ and $g(v)$.



For edge uv, with $\sigma = uv$, $D\sigma$ is the sum of the two triangles between $(f(u), f(v))$ and $(g(u), g(v))$.

Notice that we get

$$\partial(D\sigma) = g_\#(\sigma) - Dv - f_\#(\sigma) + Du.$$

In other words, we have $\partial(D\sigma) = g_\#(\sigma) - f_\#(\sigma) - D(\underbrace{\partial\sigma}_{v-u})$.

This example in fact suggests the form that $D\sigma$ should satisfy in general. We want $D(\partial\sigma) + \partial(D\sigma) = g_\#(\sigma) - f_\#(\sigma)$.

We define the existence of such a $(p+1)$-chain for each p-simplex as the required sufficient condition in general for all p.

Def Let $f, g : K \to L$ be simplicial maps. Suppose that for all p, there is a homomorphism $D : C_p(K) \to C_{p+1}(L)$ which satisfies

$$\partial D + D \partial = g_\# - f_\# .$$

Then D is said to be a **chain homotopy** between $f_\#$ and $g_\#$.

Intuitively, the images of each p-simplex σ under f and g are "close" to each other if there is a chain homotopy. Notice that the requirement is specified for all dimensions.

We could be more precise in writing the equation by including subscripts of dimension $(p, p+1)$ and simplicial complexes (K and L). We express the maps in detail as follows.

$$\begin{tikzcd}
 & C_{p+1}(L) \arrow[d, "(\partial_{p+1})_L"] \\
C_p(K) \arrow[ur, "D_p"] \arrow[r, shift left, "(f_\#)_p"] \arrow[r, shift right, "(g_\#)_p"'] \arrow[d, "(\partial_p)_K"'] & C_p(L) \\
C_{p-1}(K) \arrow[ur, "D_{p-1}"'] &
\end{tikzcd}$$

The detailed relation we want is the following:

$$(\partial_{p+1})_L D_p + D_{p-1} (\partial_p)_K = (g_\#)_p - (f_\#)_p .$$

But we usually will write $\partial D + D \partial = g_\# - f_\#$, for brevity.

The following theorem describes why we want to study chain homotopies.

Theorem 12.4 [M] If there is a chain homotopy between $f_\#$ and $g_\#$, then the induced homomorphisms f_* and g_*, for both reduced and absolute homology, are equal.

Proof If $\bar{z} \in Z_p(K)$, then

$$g_\#(\bar{z}) - f_\#(\bar{z}) = \partial D\bar{z} + D\partial\bar{z} = \partial D\bar{z} + 0.$$

So, $g_\#(\bar{z}) \sim f_\#(\bar{z})$, and hence $g_*(\{\bar{z}\}) = f_*(\{\bar{z}\})$.

We now give a sufficient condition for existence of a chain homotopy.

Def Two simplicial maps $f, g : K \to L$ are said to be **contiguous** if for every simplex $\sigma = (v_0 \cdots v_p)$ of K, the points $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$ span a simplex τ of L.

Note: 1. $0 \le \dim(\tau) \le 2p+1$.
2. $f(\sigma)$ and $g(\sigma)$ are both faces of a (possibly) larger simplex τ of L.
i.e, $f(\sigma)$ and $g(\sigma)$ are "close" to each other

Theorem 12.5 [M] If $f, g : K \to L$ are contiguous simplicial maps, then a chain homotopy exists between $f_\#$ and $g_\#$.

Proof (outline; see [M] for details)

For $\sigma = v_0, \dots, v_p$ of K, let $L(\sigma)$ be the subcomplex of L made of the simplex spanned by $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$, and all its faces. We should have the following results.

(1) $L(\sigma)$ is nonempty, $\tilde{H}_i(L(\sigma)) = 0 \ \forall i$.

(2) If τ is a face of σ, then $L(\tau) \subset L(\sigma)$.

(3) For every oriented simplex σ, $f_\#(\sigma)$ and $g_\#(\sigma)$ are both carried by $L(\sigma)$.

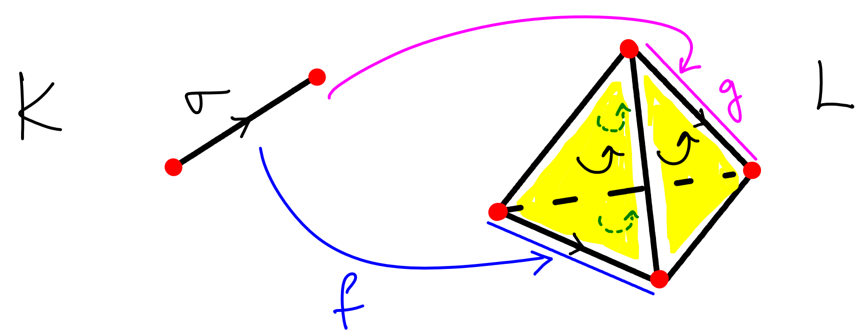
We now show that $D\sigma$ exists for each p-simplex σ using induction on p.

$p=0$ Notice that $\epsilon\big(g_\#(v) - f_\#(v)\big) = 1 - 1 = 0$. Hence $g_\#(v) - f_\#(v) \in \tilde{H}_0(L(v))$. But $\tilde{H}_0(L(v)) = 0$, so we can choose a 1-chain Dv of L carried by $L(v)$ such that

$$\partial(Dv) = g_\#(v) - f_\#(v).$$

(See [M] for the induction step going from $p-1$ to p). □

Notice that the theorem guarantees the existence of some $D\sigma$ for each σ — the choice may not be unique. Indeed, consider the case where a 1-simplex σ gets mapped by f and g to two opposite edges of a tetrahedron. Then there are two choices for $D\sigma$ — the two triangles of the tetrahedron visible in front, or the other two tetrahedron lying behind.



Application to relative homology

Def Let $K_0 \subseteq K$ and $L_0 \subseteq L$ be subcomplexes. Let $f, g : (K, K_0) \to (L, L_0)$ be two simplicial maps. We say f and g are **contiguous as maps of pairs** if for every simplex $\sigma = v_0 \cdots v_p$ of K, the points $f(v_0), \dots, f(v_p)$, $g(v_0), \dots, g(v_p)$ span a simplex of L, and if σ is contained in K_0, then they span a simplex of L_0.

With maps that are contiguous as maps of pairs, we can extend the concept of chain homotopy to the case of relative homology, and how equal homomorphisms are induced on relative homology groups.

Theorem 12.6 [M] Let $f, g : (K, K_0) \to (L, L_0)$ be contiguous as maps of pairs. Then there exists a homomorphism $D : C_p(K, K_0) \to C_{p+1}(L, L_0)$ for all p such that

$$\partial D + D\partial = g_\# - f_\#.$$

Thus, f_* and g_* are equal as maps of the relative homology groups.

See [M] for proof details.

The main point is to notice that D maps $C_p(K_0)$ to $C_{p+1}(L_0)$.

Topological Invariance of Homology Groups

Want to show: $H_p(K)$ depends only on $|K|$, and not on the specific choice of K.

Method: We showed that a simplicial map $f : |K| \to |L|$ induces a homomorphism f_* of the homology groups. We want to argue that an arbitrary continuous map $h : |K| \to |L|$ can be approximated by a simplicial map f, and then argue that the induced homomorphism depends only on h, and not on the particular approximation chosen.

Simplicial Approximation

We present the concept of approximation in the context of simplicial complexes. Rather than specifying an error of approximation as is the practice in some other fields of mathematics, we present a condition defined using star of the vertices.

Def Let $h : |K| \to |L|$ be a continuous map. We say h satisfies the **star condition** relative to (or w.r.t.) K and L if for every vertex $v \in K^{(0)}$, there exists a vertex $w \in L^{(0)}$ such that

$$h(\mathrm{St}\, v) \subset \mathrm{St}\, w.$$

Lemma 14.1 [M] Let $h : |K| \to |L|$ satisfy the star condition relative to K and L. Choose $f : K^{(0)} \to L^{(0)}$ such that $\forall\, v \in K^{(0)}$, $h(\mathrm{St}\, v) \subset \mathrm{St}\, \underbrace{f(v)}_{w}$.

(a) For $\sigma \in K$, choose $\bar{x} \in \mathrm{Int}\, \sigma$ and $\tau \in L$ such that $h(\bar{x}) \in \mathrm{Int}\, \tau$. Then f maps each vertex of σ to a vertex of τ.

(b) f may be extended to a simplicial map of K into L, which we also call f.

(c) If $g : K \to L$ is another simplicial map such that $h(\mathrm{St}\, v) \subset \mathrm{St}\, (g(v))$ $\forall\, v \in K^{(0)}$, then f and g are contiguous.

Proof

(a) Let $\sigma = v_0 \cdots v_p$. Then $\bar{x} \in \mathrm{St}\, v_i \ \forall i$. So

$$h(\bar{x}) \in h(\mathrm{St}\, v_i) \subset \mathrm{St}\, f(v_i) \quad \forall i.$$

So, $h(\bar{x})$ has positive barycentric coordinates w.r.t. each vertex $f(v_i)$, $i = 0, \ldots, p$. These vertices must form a subset of the vertex set of τ.

(b) Straightforward.

(c) Since $h(\bar{x}) \in h(\mathrm{St}\, v_i) \subset \mathrm{St}(g(v_i)) \ \forall i$, the vertices $g(v_0), \ldots, g(v_p)$ must also be vertices of τ. Thus $f(v_0), \ldots, f(v_p), g(v_0), \ldots, g(v_p)$ span a face of τ. □

We define the concept of simplicial approximation using the star condition.
More in the next lecture...

MATH 524: Lecture 15 (10/07/2025)

Today: * simplicial approximation
* subdivision

Recall $h: |K| \to |L|$ satisfies star condition: $h(\mathrm{St}\, v) \subset \mathrm{St}\, w$

Def Let $h: |K| \to |L|$ be a continuous map. If $f: K \to L$ is a simplicial map such that $h(\mathrm{St}\, v) \subset \mathrm{St}\, f(v)$ $\forall\, v \in K^{(0)}$, then f is called a **simplicial approximation** to h.

Intuitively, f is "close to" h in the following sense: given $\bar{x} \in |K|$, $\exists$ a simplex τ of L s.t. $h(\bar{x}), f(\bar{x}) \in \tau$. We formalize this concept now.

Lemma 14.2 [M] Let $f: K \to L$ be a simplicial approximation to $h: |K| \to |L|$. Given $\bar{x} \in |K|$, there exists a simplex $\tau \in L$ such that $h(\bar{x}) \in \mathrm{Int}\, \tau$, $f(\bar{x}) \in \tau$.

Proof Follows from Lemma 14.1 (a).

We can also compose simplicial approximations to get a simplicial approximation for the composition of continuous maps.

Theorem 14.3 [M] Let $h: |K| \to |L|$ and $k: |L| \to |M|$ have simplicial approximations $f: K \to L$ and $g: L \to M$, respectively. Then $g \circ f$ is a simplicial approximation to $k \circ h$.

Proof 1. $g \circ f$ is a simplicial map.

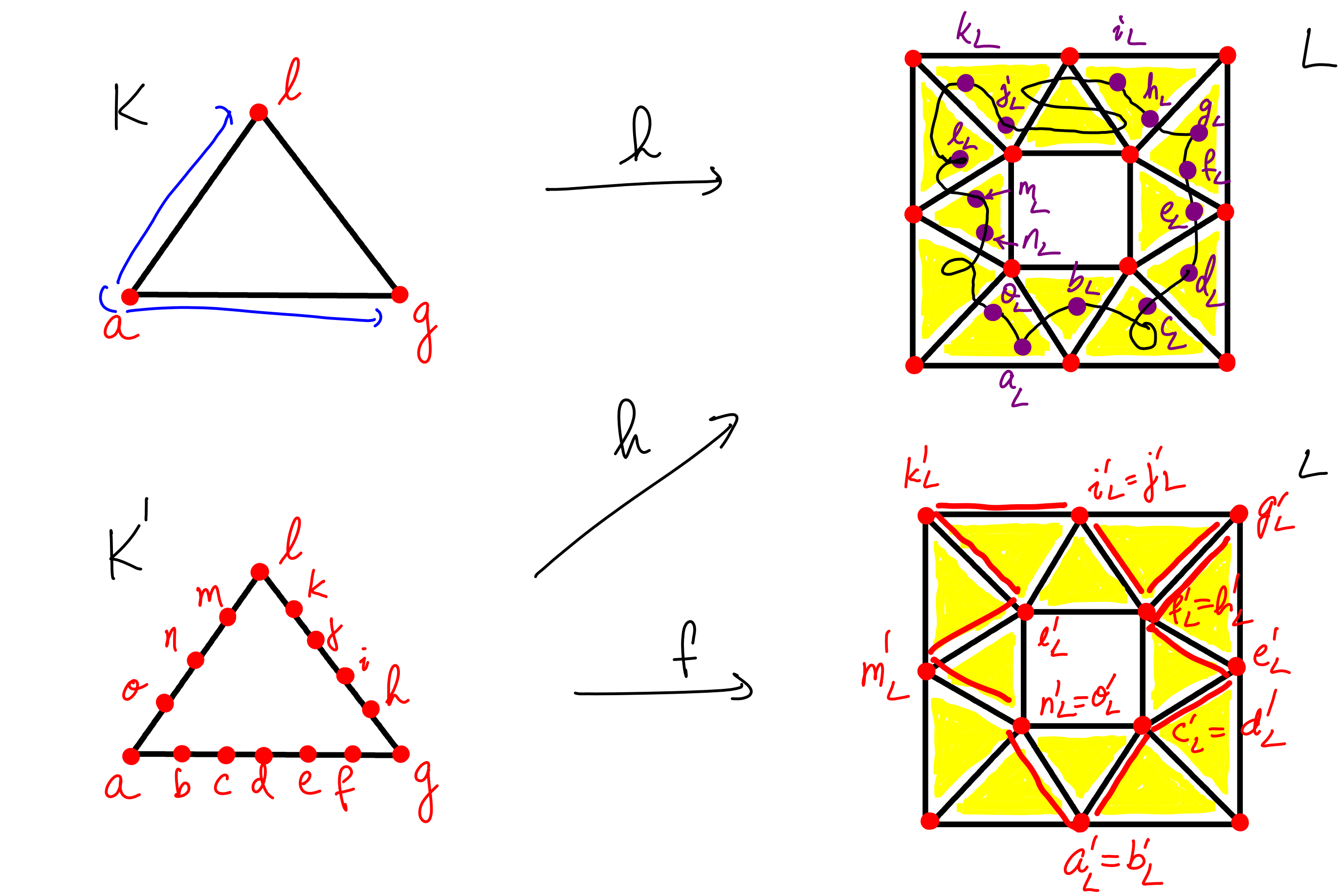
2. If $v \in K^{(0)}$, then $h(\mathrm{St}\, v) \subset \mathrm{St}\, f(v)$, as f is a simplicial approximation to h. Hence

$k(h(\mathrm{St}\, v)) \subset k(\mathrm{St}\, f(v)) \subset \mathrm{St}\,(g(f(v)))$,

as g is a simplicial approximation to k. $\square$

Example

$h(st(a,K)) \not\subset st(v,L)$ for any $v \in L^{(0)}$.



We consider K to be the 1-complex made of 3 1-simplices, and L to be the 2-complex that models an annulus. Let $h: |K| \to |L|$ map all of $|K|$ to the loop on $|L|$ as shown. We also consider a "refinement" of K by adding several more vertices to obtain K' such that $|K| = |K'|$. Hence, h applies without change to K'.

It is clear that h does not satisfy the star condition relative to K and L. Indeed, notice that $St(a,K) = K - \{\overline{lg}, l, g\}$, and there is no vertex in L such that $h(St(a,K))$ is a subset of its star in L.

But h does satisfy the star condition relative to K' and L. So h has a simplicial approximation $f: K' \to L$, and one such approximation is shown.

If $h: |K| \to |L|$ satisfies the star condition relative to K and L, there exists a well defined homomorphism

$$h_* : H_p(K) \to H_p(L) \quad \text{for all } p$$

obtained by setting $h_* = f_*$, where f is a simplicial approximation to h.

Not surprisingly, we can extend the star condition to the level of relative homology.

Lemma 14.4 [M] Let $h: |K| \to |L|$ satisfy the star condition relative to K & L, and suppose h maps $|K_0|$ into $|L_0|$.

(a) Any simplicial approximation $f: K \to L$ to h also maps $|K_0|$ into $|L_0|$. Also, the restriction of f to K_0 is a simplicial approximation to the restriction of h to $|K_0|$.

(b) Any two simplicial approximations f and g to h are contiguous as maps of pairs.

Subdivision

We had seen in the example that $h: |K| \to |L|$ did not satisfy the star condition relative to K and L, but it did relative to K' and L, where K' is a "finer" or "refined" version of K. We formalize this idea now, and talk about subdivisions.

We first formally define a subdivision. We then introduce barycentric subdivision as a "canonical" subdivision.

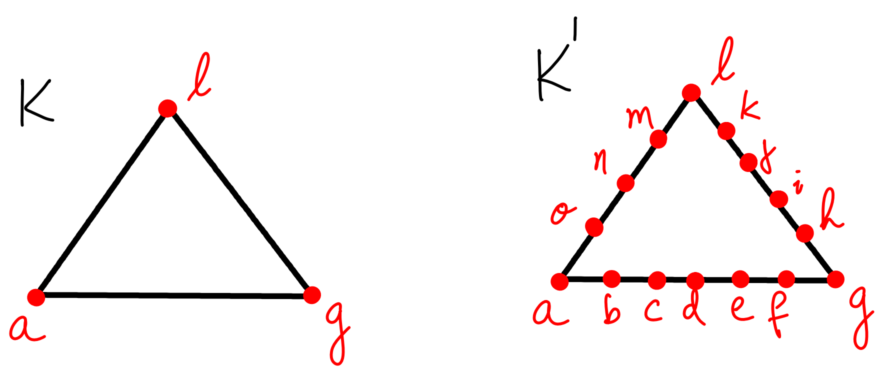
Def Let K be a geometric complex in $\mathbb{R}^d$. A complex K' is said to be a **subdivision** of K if

1. each simplex of K' is contained in a simplex of K, and
2. each simplex of K is the union of **finitely** many simplices of K'.

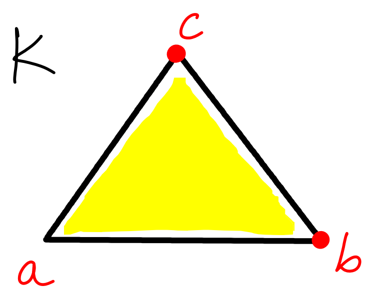
The conditions $\Rightarrow$ $|K|$ and $|K'|$ are equal as sets. The finiteness condition in 2. guarantees that $|K|$ and $|K'|$ are equal as topological spaces.

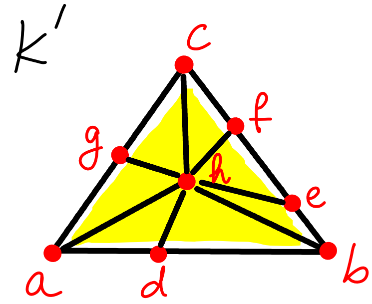
Examples

1.



2.





In 1 and 2 above, K' is a subdivision of K.

3. K : $[0,1]$ (1-simplex and its vertices)

K' : $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ $\forall n \in \mathbb{Z}_{>0}$, and their vertices, and the vertex 0.

$|K| = |K'|$ as sets, but they are not equal as topological spaces, as the finiteness requirement in Condition 2 is violated. Hence K' is not a subdivision of K.

We get some results directly from the definition of subdivision.

<u>Properties</u>

1. If K'' is a subdivision of K', and K' is a subdivision of K, then K'' is a subdivision of K.

2. If K' is a subdivision of K, and $K_0 \subset K$ is a subcomplex, then the collection of simplices of K' that lie in $|K_0|$ is automatically a subdivision of K_0. We call this subdivision the subdivision of K_0 induced by K'.

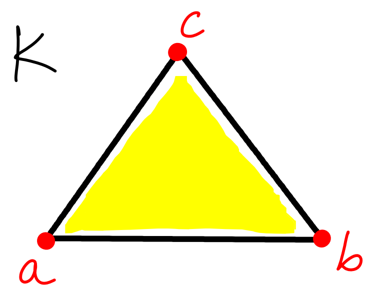
Subdivision satisfy a sort of "star condition", as the following lemma describes.

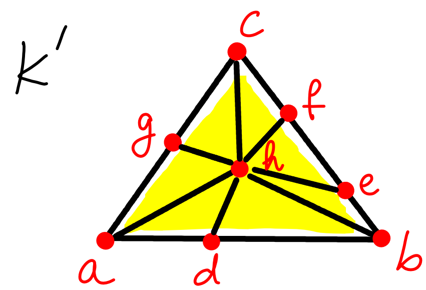
Lemma 15.1 [M] Let K' be a subdivision of K. Then for every $\bar{w} \in K'^{(0)}$, there exists a vertex $\bar{v} \in K^{(0)}$ such that

$$St(\bar{w}, K') \subset St(\bar{v}, K).$$

Indeed, if σ is a simplex in K s.t. $\bar{w} \in \operatorname{Int}\sigma$, then this inclusion holds precisely when $\bar{v}$ is a vertex of σ.

Example





Here, $St(h, K') \subset St(a, K)$, for instance.

Proof ($\Rightarrow$) (Straightforward). $\bar{w} \in St(\bar{w}, K')$ by definition. Hence by the given inclusion, $\bar{w}$ belongs to some open simplex of K, which has $\bar{v}$ as a vertex.

($\Leftarrow$) Let $\bar{w} \in \operatorname{Int}\sigma$, and $\bar{v}$ be a vertex of σ. Then we show that

$$|K| - St(\bar{v}, K) \subset |K| - St(\bar{w}, K')$$

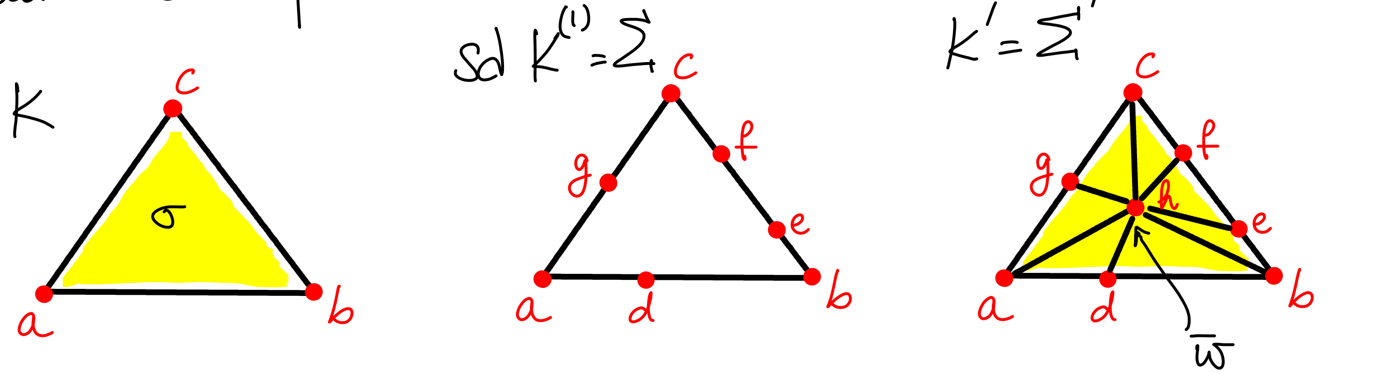
Notice that $|K| - St(\bar{v}, K)$ is the union of all simplices in K that do not have $\bar{v}$ as a vertex. This is also a collection of simplices τ in K'. No such τ can have $\bar{w}$ as a vertex, as $\bar{w} \in \operatorname{Int}\sigma \subset St(\bar{v}, K)$. Hence any such τ lies in $|K| - St(\bar{w}, K')$. $\square$

MATH 524: Lecture 16 (10/09/2025)

Today: * cone of K with vertex $\bar{w}$
* barycentric subdivision

We now consider ideas for how to construct subdivisions in general — one approach is to do it in increasing dimensions of the skeleton of the complex.

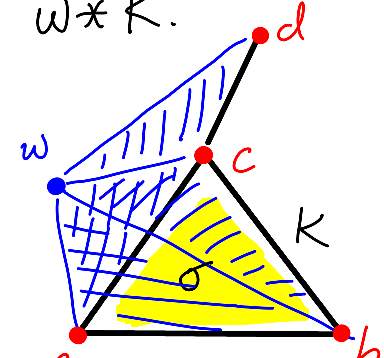
Back to Example 2:



We can extend the subdivision Σ_1 of $K^{(1)}$ to that of $K^{(2)} = K$ by forming the **cone** $\bar{w} * \Sigma_1$, where $\bar{w}$ is any interior point of σ (here K is σ and its faces). In general, we can extend the subdivision Σ of $K^{(p)}$ to that of $K^{(p+1)}$ by forming the cone $\bar{w} * \Sigma$, where $\bar{w}$ is an interior point of the $(p+1)$-simplex σ.

Def Let K be a simplicial complex in $\mathbb{R}^d$, and $\bar{w} \in \mathbb{R}^d$ is a point such that each ray emanating from $\bar{w}$ intersects $|K|$ in at most one point. Then the **cone of K with vertex $\bar{w}$** is the collection of all simplices of the form $\bar{w}\bar{a}_0 \cdots \bar{a}_p$, where $\bar{a}_0 \cdots \bar{a}_p$ is a simplex of K, along with all faces of such simplices. We denote this collection as $\bar{w} * K$.

Example: Consider K to be the 2-complex shown — Δabc, edge $\overline{cd}$, and faces. Let $\bar{w}$ be a point lying "above" K. The cone $\bar{w} * K$ has the tetrahedron $wabc$, triangle wcd, and faces.



Note

1. $\bar{w} * K$ is indeed a well-defined simplicial complex, and has K as a subcomplex. We refer to K as the **base of the cone** $\bar{w} * K$.

2. $\dim(\bar{w} * K) = \dim(K) + 1$, as the ray intersection condition requires that $\bar{w} \notin \text{plane}(\sigma)\ \forall \sigma \in K$.

Back to example 2: Let the new subdivision of K be called Σ'. Then Σ' is obtained by "**starring** Σ from $\bar{w}$." → or "coning Σ from $\bar{w}$."

We can define the subdivision of K in an inductive fashion, going up one dimension at each step. We need a basic result first.

Lemma 15.2[M] If K is a complex, the intersection of any collection of subcomplexes of K is a subcomplex of K. Conversely, if $\{K_\alpha\}$ is a collection of complexes in $\mathbb{R}^d$ and the intersection $|K_\alpha| \cap |K_\beta|$ is the polytope of a complex that is a subcomplex of both K_α and K_β for all α, β, then $\bigcup_\alpha K_\alpha$ is a complex.

We will use this lemma to justify how we define the subdivision in an inductive (or iterative) fashion. In particular, we star from one point within each simplex to the subdivision of its boundary.

Def Let K be a complex. Suppose L_p is the subdivision of $K^{(p)}$. Let σ be a $(p+1)$-simplex of K. $\mathrm{Bd}\,\sigma$ is a polytope of a subcomplex of $K^{(p)}$, and hence of L_p; we denote the latter by L_σ. For $\bar{w}_\sigma \in \mathrm{Int}\,\sigma$, the cone $\bar{w}_\sigma * L_\sigma$ is a complex whose underlying space is σ. We define L_{p+1} to be the union of L_p and the cones $\bar{w}_\sigma * L_\sigma$ as σ ranges over all $(p+1)$-simplices of K. L_{p+1} is the subdivision of $K^{(p+1)}$ obtained by **starring L_p from the points $\bar{w}_\sigma$.**

For the above definition to be correct, we need to verify that L_{p+1} is indeed a simplicial complex. To this end, we note the following facts.

(1) $|\bar{w}_\sigma * L_\sigma| \cap |L_p| = \mathrm{Bd}\,\sigma$ is the polytope of the subcomplex L_σ of both $\bar{w}_\sigma * L_\sigma$ and L_p.

(2) If τ is another $(p+1)$-simplex of K, then $|\bar{w}_\sigma * L_\sigma|$ and $|\bar{w}_\tau * L_\tau|$ intersect in the simplex $\sigma \cap \tau$ of K, which is the polytope of a subcomplex of L_p, and hence of both L_σ and L_τ. Hence it follows from Lemma 15.2 that L_{p+1} is a simplicial complex.

How do we choose the point $\bar{w}_\sigma$ for each σ. While there are (infinitely) many choices, we can use a "canonical" choice.

Def The **barycenter** of $\sigma = v_0 \cdots v_p$ is defined to be the point

$$\hat{\sigma} = \sum_{i=0}^{p} \frac{1}{(p+1)} v_i .$$

$\hat{\sigma}$ is the point of Int σ all of whose barycentric coordinates with respect to the vertices of σ are equal.

σ: 0-simplex $\Rightarrow \hat{\sigma} = \sigma$

1-simplex $\Rightarrow \hat{\sigma}$: midpoint

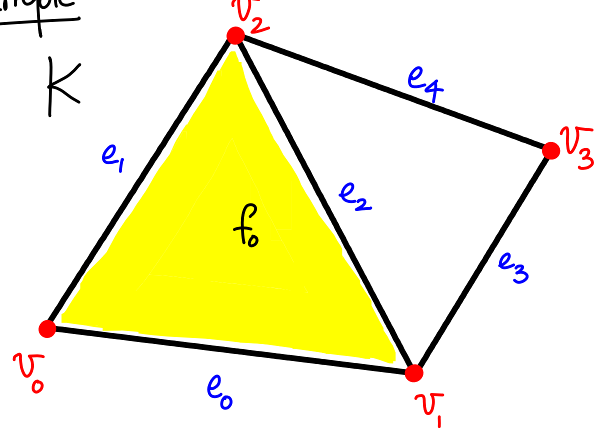
p-simplex $\Rightarrow \hat{\sigma}$: centroid of σ. $p \geq 1$

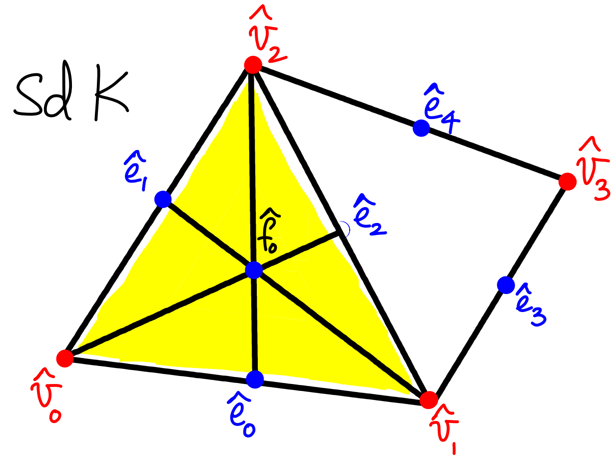
We star from the barycenters to construct the barycentric subdivision.

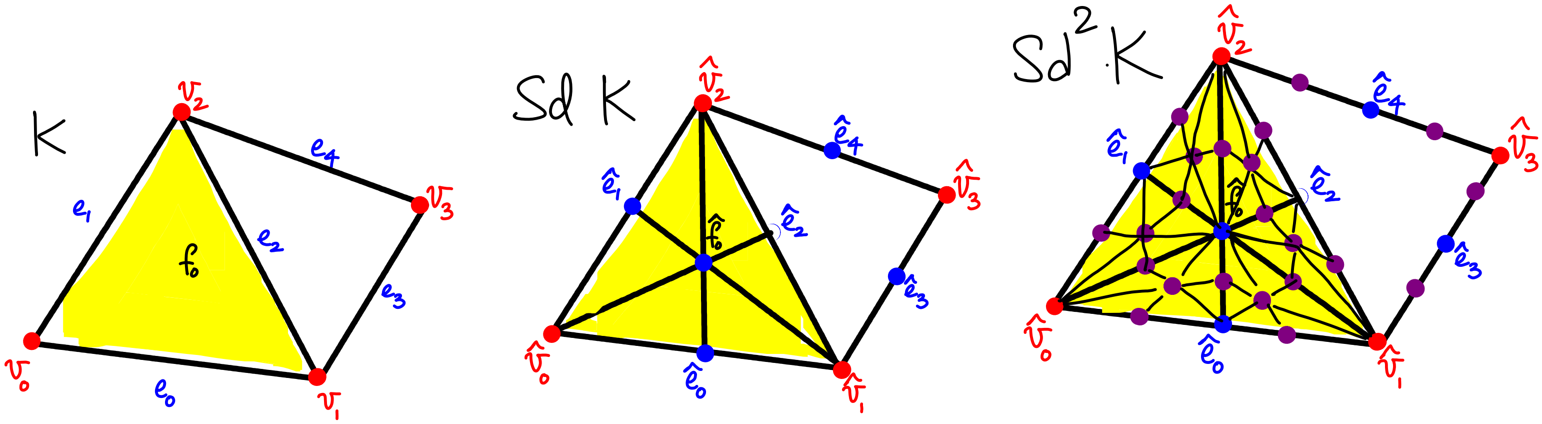
Def Let K be a simplicial complex. Let $L_0 = K^{(0)}$. In general, L_p is the subdivision of the p-skeleton of K. Let L_{p+1} be the subdivision of $K^{(p+1)}$ obtained by starring L_p from the barycenters of all (p+1)-simplices of K. By Lemma 15.2, the union of the complexes L_p is a subdivision of K. This is the **first barycentric subdivision** of K, denoted **Sd K**.

The first barycentric subdivision of Sd K, denoted Sd(Sd K) or $\mathrm{Sd}^2 K$, is the second barycentric subdivision of K. Similarly, we define $\mathrm{Sd}^r K$, the r^{th} barycentric subdivision for any integer $r \geq 0$, with $\mathrm{Sd}^0 K = K$.

Example





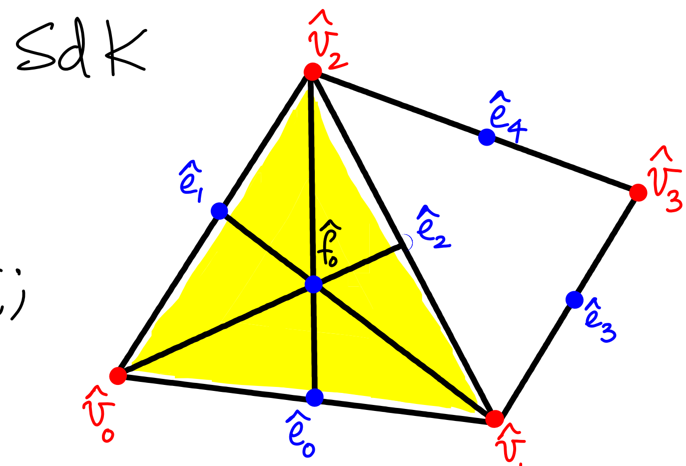


Explicit Description of the simplices in Sd K

Notation $\sigma_1 > \sigma_2$ means σ_2 is a proper face of σ_1, or equivalently, σ_1 is a proper coface of σ_2.

Lemma 15.3 [M] Sd K is the collection of simplices of the form $\hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_p$ where $\sigma_1 > \sigma_2 > \cdots > \sigma_p$.

Illustration



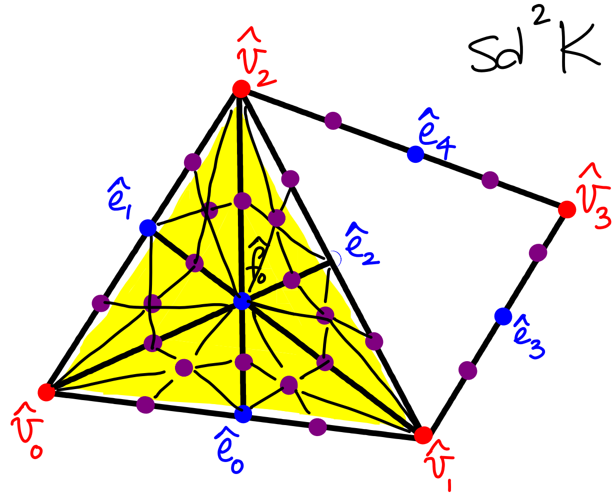
The edges in Sd K are of the form $\hat{e}_j \hat{v}_i$ where $e_j > v_i$; or of the form $\hat{f}_0 \hat{e}_j$ where $f_0 > e_j$. Similarly, the triangles in Sd K are of the form $\hat{f}_0 \hat{e}_j \hat{v}_i$ where $f_0 > e_j > v_i$.

<u>Proof</u> (by induction)

True for $K^{(0)}$ (as $\hat{v} = v \ \forall v \in K^{(0)}$).

Now suppose each simplex of Sd K lying in $|K^{(p)}|$ is of this form. Let τ be a simplex of Sd K lying in $|K^{(p+1)}|$, but not in $|K^{(p)}|$. Then τ belongs to one of the complexes $\hat{\sigma} * L_\sigma$, where σ is a $(p+1)$-simplex of K, and L_σ is the first barycentric subdivision of the complex made of the proper faces of σ. By induction, each simplex of L_σ is of the form $\hat{\sigma}_0 \hat{\sigma}_1 \cdots \hat{\sigma}_p$. Then τ must be of the form $\hat{\sigma}\hat{\sigma}_0 \cdots \hat{\sigma}_p$. $\square$

Notice that the simplices in $Sd^2 K$ are much "smaller" than the simplices in Sd K. This observation is formalized in the following theorem.



Theorem 15.4 [M] Given a finite complex K, a metric for |K|, and $\varepsilon > 0$, there exists an r such that each simplex in $Sd^r K$ has diameter less than ε.

Def For a subset S of a metric space (X, d), its **diameter** is $\text{diam}(S) = \sup\{d(\bar{x}, \bar{y}) \mid \bar{x}, \bar{y} \in S\}$.

↳ the metric of the metric space.

See [M] for the proof.

MATH 524: Lecture 17 (10/14/2025)

Today: * Simplicial approximation
* exact sequences

Simplicial Approximation

We now talk about how to use subdivision to find a simplicial approximation of any continuous function $h: |K| \to |L|$.

Recall: A simplicial approximation of a continuous map $h: |K| \to |L|$ by a simplicial map $f: K \to L$ satisfies

$$h(\operatorname{St} v) \subset \operatorname{St}(f(v)) \ \forall\, v \in K^{(0)}.$$

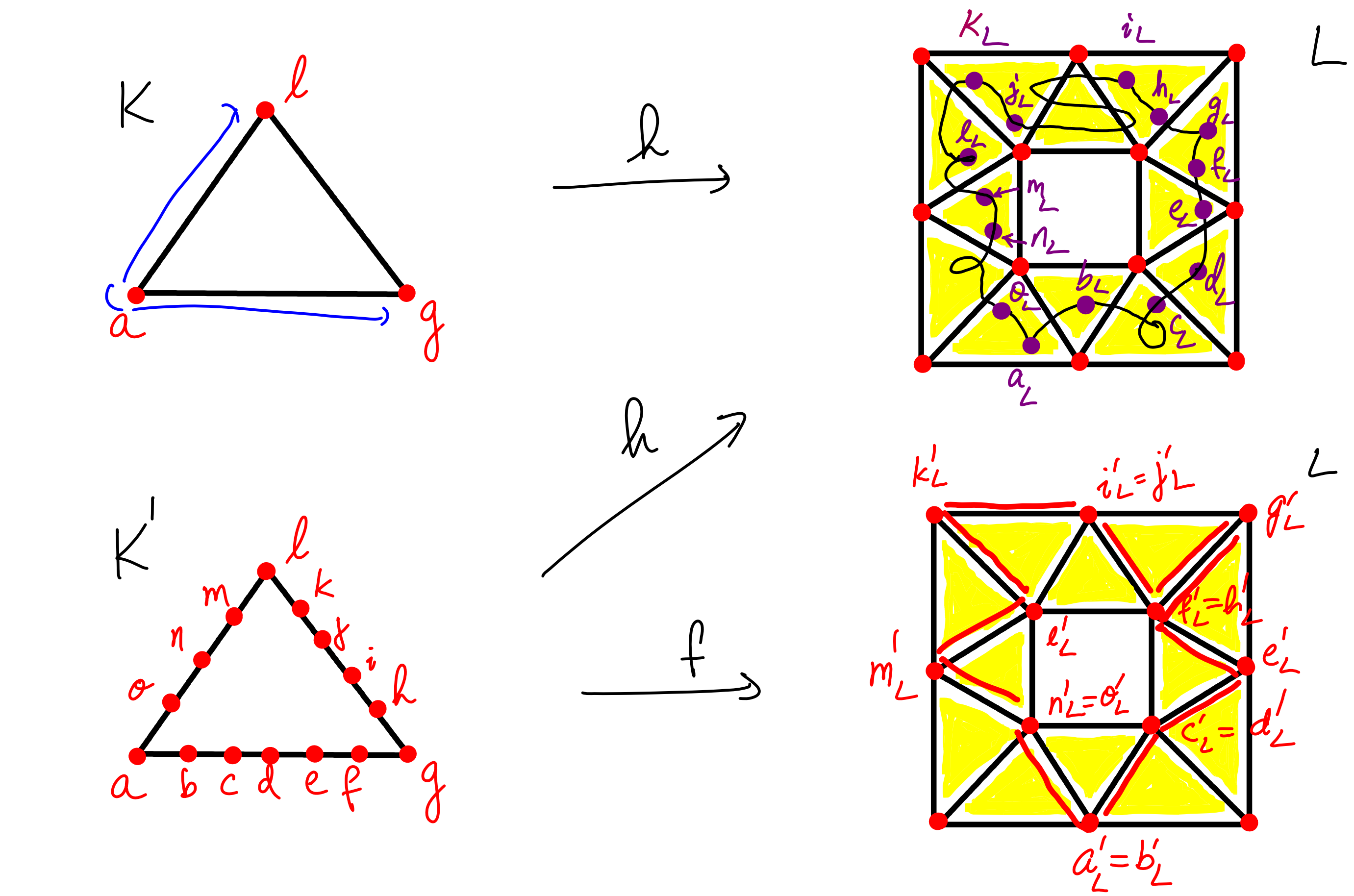
We had also seen that homomorphisms $f_\#$ associated with simplicial maps $f: K \to L$ induce isomorphisms at the homology level. Our ultimate goal is to argue that the homology groups are determined by the underlying spaces, rather than specific choices of the complex.

We now look at the next step toward that goal - to show that we can always approximate a continuous map by a simplicial map once we have a fine enough subdivision of the original complex.

The result for the case when K is finite is quite accessible compared to that when K is infinite. We will discuss only the finite case in detail here.

Theorem 16.1 [M] **(The finite simplicial approximation theorem)**
Let K and L be complexes, and let K be finite. Given a continuous map $h: |K| \to |L|$, there is an r such that h has a simplicial approximation $f: \operatorname{Sd}^r K \to L$.

Here is an illustration we already saw in lecture 15.



Of course, K' is not a barycentric subdivision of K here. But the example illustrates the result nonetheless. The key idea is that by subdividing K enough, we could approximate h by a simplicial map from K' to L.

Even in the illustration shown, one could argue that f misses the detail in h in some places, e.g., between i_L and j_L where h looks like "S", while f looks like "•". But one could consider a finer subdivision K'' of K' where we have some more vertices between i and j. The image under f could be closer to h in that case.

In this context, the barycentric subdivision is just one kind of subdivision we could use. At the same time, its nice structure makes it convenient to devise proofs of results. On the other hand, $Sd^r K$ might produce "bad" simplices, e.g., triangles that are too skinny. There are other classes of subdivision where the triangles produced are "round" (while still holding a small diameter).

<u>Proof</u> Cover $|K|$ by open sets $h^{-1}(St\,\overline{w})$, as $\overline{w}$ ranges over $L^{(0)}$. Let this covering be called $\mathcal{A}$. Then $\mathcal{A}$ is an open covering of the compact metric space $|K|$. So, there exists a number λ such that any set of $|K|$ with diameter less than λ lies in one of the elements of $\mathcal{A}$. This number is called a **Lebesgue number** for $\mathcal{A}$.

Here is the standard argument for why a Lebesgue number should exist in this case.

Suppose there does not exist a Lebesgue number for $\mathcal{A}$. Then we can choose a sequence C_n of sets such that $\mathrm{diam}(C_n) < \frac{1}{n}$, but C_n does not lie in any element of $\mathcal{A}$. Choose $\overline{x}_n \in C_n$. By compactness, some subsequence $\{\overline{x}_{n_i}\}$ converges, to say, $\overline{x}$. Now, $\overline{x} \in A$ for some $A \in \mathcal{A}$. As A is open, it contains C_{n_i} for i sufficiently large — a contradiction.

Back to the main proof now...

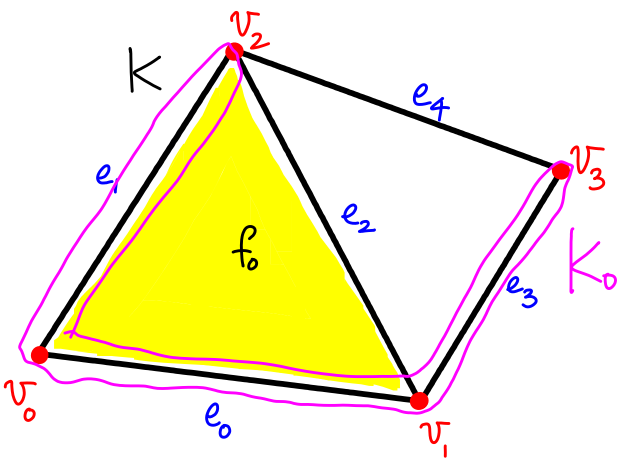
Choose r s.t. each simplex σ in $Sd^r K$ has $\operatorname{diam}(\sigma) < \lambda/2$.
Then each $St\,\bar{v}$ for $\bar{v} \in (Sd^r K)^{(0)}$ has diameter $< \lambda$. So, it $(St\,\bar{v})$ lies in one of the sets $h^{-1}(St\,\bar{w})$. So, $h : |K| \to |L|$ satisfies the star condition relative to $Sd^r K$ and L, and hence a simplicial approximation exists. $\square$

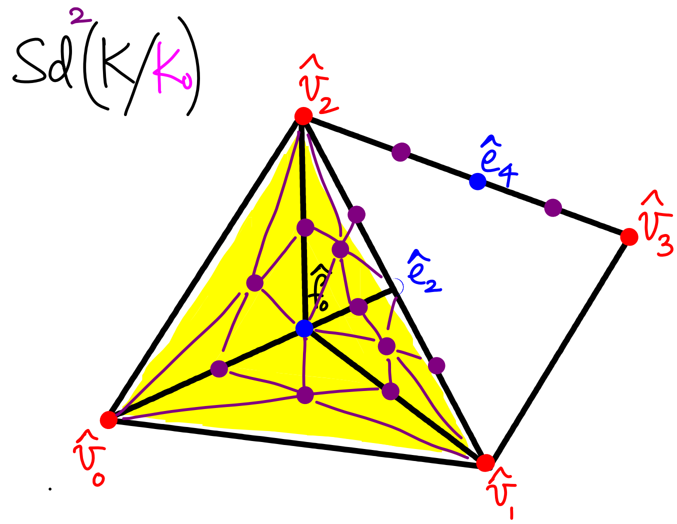
Extending the simplicial approximation theorem to the case when K is not finite $(h : |K| \to |L|)$ is much more involved.
We introduce a key technique related to subdivision used in this process. In particular, the default barycentric subdivision will not work.

Subdividing K while keeping K_0 (a subcomplex) fixed

Def Here is a sequence of subdivisions of skeletons of K.
Let $J_0 = K^{(0)}$. In general, J_p is a subdivision of $K^{(p)}$, and each simplex τ of K_0 with $\dim(\tau) \le p$ belongs to J_p. Define J_{p+1} to be the union of J_p, all $\tau \in K_0$ with $\dim(\tau) = p+1$, and the cones $\hat{\sigma} * J_\sigma$ as σ ranges over all $(p+1)$-simplices of K **not in** K_0. Here J_σ is a subcomplex of J_p whose polytope is $Bd\,\sigma$.
The union of all complexes J_p $\forall p$ is a subdivision of K, denoted $Sd\,(K/K_0)$, and is called the **first barycentric subdivision of K, holding K_0 fixed.**

We define $Sd^r(K/K_0)$ similarly: $Sd^2(K/K_0) = Sd\left(Sd(K/K_0)/K_0\right)$, for instance.





K_0: {edges e_0, e_1, e_3, and all v_j}.

$Sd(K/K_0)$ and $Sd^2(K/K_0)$

We finish by listing the main result. See [M] for proof.

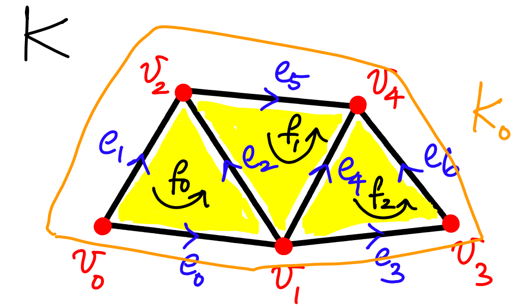
Theorem 16.5 [M] (**The general simplicial approximation theorem**)

Let K and L be complexes, and let $h: |K| \to |L|$ be a continuous map. Then there exists a subdivision K' of K such that h has a simplicial approximation $f: K' \to L$.

That's all we will cover in this subtopic. Next we move on to an important algebraic technique — exact sequences.

Exact Sequences

What are the relationships between $H_p(K, K_0)$, $H_p(K)$, and $H_p(K_0)$?



Example (same as example 3 in lecture 9)

Here, $H_2(K, K_0) \simeq \mathbb{Z}$. $\bar{r} = \sum_{i=0}^{2} \bar{f}_i$ is a generator.

Also, $H_1(K_0) \simeq \mathbb{Z}$, $\{\bar{z}\}$ is a basis, where $\bar{z} = \bar{e}_0 + \bar{e}_3 + \bar{e}_6 - \bar{e}_5 - \bar{e}_1$.

So $H_2(K, K_0) \simeq H_1(K_0)$.

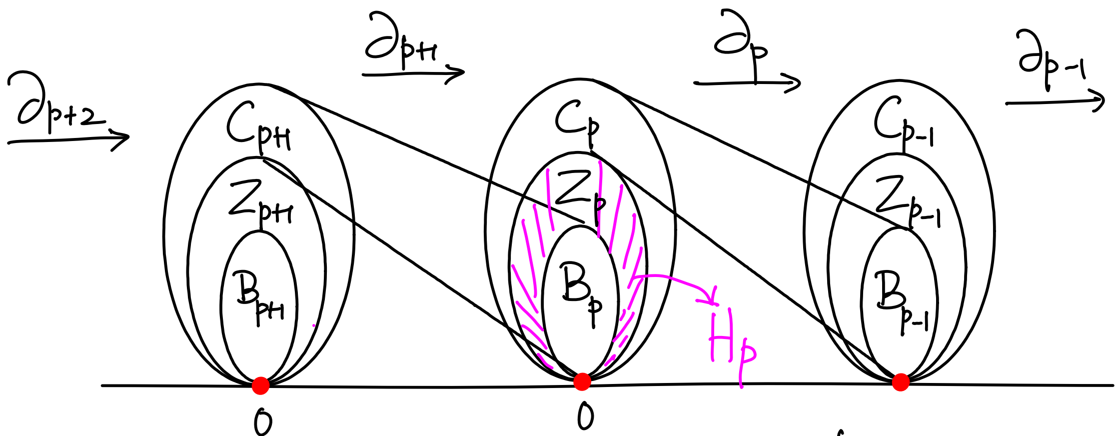
It turns out that $H_2(K, K_0) \simeq H_1(K_0)$ here is not a mere coincidence. To present the general result, we first need to introduce the algebraic machinery of exact sequences – of objects (think groups, rings, etc.) and maps (homomorphisms) between them.

Def Consider a sequence (finite or infinite) of groups and homomorphisms

$$\cdots \xrightarrow{\phi_{i-2}} A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \rightarrow \cdots$$

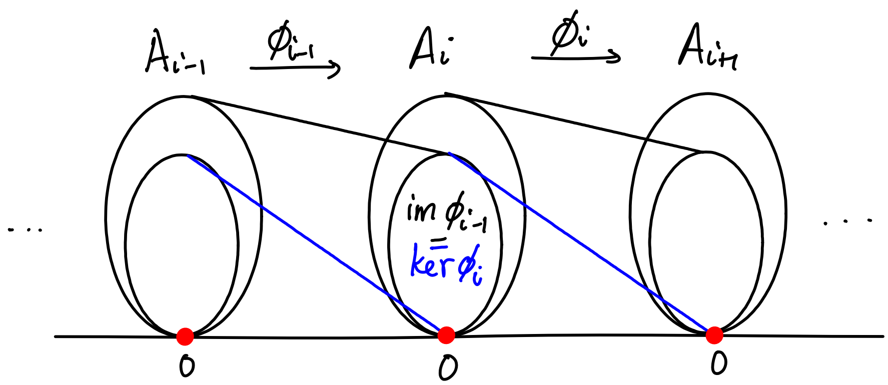
This sequence is **exact** at A_i if image ϕ_{i-1} = kernel ϕ_i. If it is exact everywhere, it is an **exact sequence**. Exactness is not defined at the first and last group of the sequence, if they exist.

The sequence we have seen already, of chain groups and boundary homomorphisms, is **not** exact!



The indices are decreasing left to right here, but that is not an issue. Indeed, notice that $\operatorname{im}\partial_{p+1} = B_p \neq \ker\partial_p = Z_p$.

Here is the picture of exact sequences that we want.



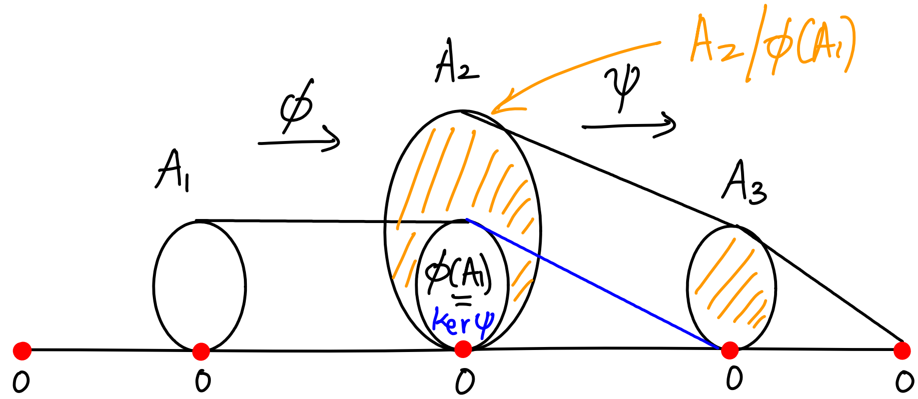
Several results on exact sequences (with abelian groups)

$0 \to$ denotes the trivial group

1. $A_1 \xrightarrow{\phi} A_2 \longrightarrow 0$ is exact iff ϕ is an epimorphism (surjective/onto).

2. $0 \longrightarrow A_1 \xrightarrow{\phi} A_2$ is exact iff ϕ is a monomorphism (injective/1-to-1).

3. Suppose the sequence $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ is exact. Such a sequence is called a **short exact sequence** (SES).



Then $A_2/\phi(A_1) = \text{cok}\,\psi$ is isomorphic to A_3; this isomorphism is induced by ψ. Conversely, if $\psi : A \to B$ is an epimorphism with $\ker\psi = K$, then the sequence

$$0 \to K \xrightarrow{i} A \xrightarrow{\psi} B \to 0$$

is exact, where i is inclusion.

4. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\phi} A_3 \xrightarrow{\beta} A_4$ is exact. Then the following statements are equivalent.
   (i) α is an epimorphism.
   (ii) β is a monomorphism.
   (iii) ϕ is the zero homomorphism.

5. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \to A_3 \to A_4 \xrightarrow{\beta} A_5$ is exact. Then so is the induced sequence $0 \to \text{cok}\,\alpha \to A_3 \to \ker\beta \to 0$.

It will be instructive to draw diagrams similar to the base case for each of these results!

Def Consider two sequences of groups and homomorphisms having the same index set.

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] & A_{i+1} \arrow[r, "\phi_{i+1}"] \arrow[d, "\alpha_{i+1}"] & \cdots \\
\cdots \arrow[r] & B_i \arrow[r, "\psi_i"] & B_{i+1} \arrow[r, "\psi_{i+1}"] & \cdots
\end{tikzcd}$$

A **homomorphism of the first sequence into the second** is a family of homomorphisms $\alpha_i : A_i \to B_i$ such that each square of maps

$$\begin{tikzcd}
A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] \arrow[dr, phantom, "\square"] & A_{i+1} \arrow[d, "\alpha_{i+1}"] \\
B_i \arrow[r, "\psi_i"] & B_{i+1}
\end{tikzcd}$$

commutes, i.e., $\alpha_{i+1} \circ \phi_i = \psi_i \circ \alpha_i$.

If is an isomorphism of sequences if each α_i is an isomorphism.

MATH 524: Lecture 18 (10/16/2025)

Today: * chain complex and chain map
* connecting homomorphism

Recall: homomorphism of one sequence into a second..

We had studied simplicial maps (from K to L), and associated homomorphisms between the chain groups (and its subgroups) in both complexes. Indeed, that set up illustrates the above definition. At the same time, it turns out we could study such collections of groups and homomorphisms in a much more general setting — and not necessarily on a simplicial complex.

Example: chain maps between chain complexes, which we define now.

Def A **chain complex** $\mathcal{C}$. is a family $\{C_p, \partial_p\}$ of abelian groups C_p and homomorphisms ∂_p
$\partial_p : C_p \to C_{p-1}$ such that $\partial_p \circ \partial_{p+1} = 0 \ \forall p$.

The group $H_p(\mathcal{C}) = \ker \partial_p / \operatorname{im} \partial_{p+1}$ is the p-th **homology group** of the chain complex $\mathcal{C}$.

Notice that the chain, cycle, boundary, and homology groups, along with boundary homomorphisms does indeed fit this framework — and hence the overloading of notation! At the same time, chain complexes could be much more general! We do need $\partial_p \circ \partial_{p+1} = 0 \ \forall p$ in the general setting.

Now consider two chain complexes $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C_p', \partial_p'\}$. We can define a family of homomorphisms from C_p to C_p' with additional requirements on "connecting" them to ∂_p and ∂_p' as follows. We define $\phi_p : C_p \to C_p'$ to be the homomorphism from the p^{th} abelian group of $\mathcal{C}$ to the p^{th} abelian group of $\mathcal{C}'$, for each p.

ϕ_p should be such that each "square" in the diagram commutes.

ϕ satisfies $\quad \partial_p' \circ \phi_p = \phi_{p-1} \circ \partial_p \quad \forall\, p.$

$$
\begin{tikzcd}
\mathcal{C} & {} \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[d, "\phi_{p+1}"] & C_p \arrow[r, "\partial_p"] \arrow[d, "\phi_p"'] & C_{p-1} \arrow[r] \arrow[d, "\phi_{p-1}"] & \cdots \\
\mathcal{C}' & {} \arrow[r] & C_{p+1}' \arrow[r, "\partial_{p+1}'"] & C_p' \arrow[r, "\partial_p'"] & C_{p-1}' \arrow[r] & \cdots
\end{tikzcd}
$$

$$\partial_p' \circ \phi_p = \phi_{p-1} \circ \partial_p$$

Then entire family of homomorphisms, $\phi_p \ \forall\, p$, is referred to as a **chain map** from $\mathcal{C}$ to $\mathcal{C}'$ — $\phi : \mathcal{C} \to \mathcal{C}'$.

Recall how we talked about simplicial maps inducing homomorphisms at the homology level. We get the same result in the general setting as well.

A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ induces a homomorphism

$$(\phi_*)_p : H_p(\mathcal{C}) \longrightarrow H_p(\mathcal{C}').$$

We now introduce one more concept related to short exact sequences.

Def Consider a short exact sequence

$$0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0.$$

This sequence is said to **split** if the group $\phi(A_1)$ is a direct summand in A_2. So the sequence becomes

$$0 \to A_1 \xrightarrow{\phi} \phi(A_1) \oplus B \xrightarrow{\psi} A_3 \to 0.$$

where ϕ defines an isomorphism of A_1 with $\phi(A_1)$, and ψ defines an isomorphism of B with A_3.

We end by stating two results on short exact sequences that split. See [M] for details and proofs.

Theorem 23.1 [M] Let $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ be exact. Then the following statements are equivalent.

(1) The sequence splits.

(2) There exists a map $p : A_2 \to A_1$ such that $p \circ \phi = i_{A_1}$

identity in A_1

(3) There exists a map $j : A_3 \to A_2$ such that $\psi \circ j = i_{A_3}$.

identity in A_3

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, shift left, "\phi"] & A_2 \arrow[l, shift left, "p"] \arrow[r, shift left, "\psi"] & A_3 \arrow[l, shift left, "j"] \arrow[r] & 0 \end{tikzcd}$$

Corollary 23.2 [M]. Let $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ be exact. If A_3 is free abelian, the sequence splits.

Exact homology sequence of a pair K, K_0

Goal: Connect $H_p(K,K_0)$, $H_p(K)$, $H_p(K_0)$

We first need to define a homomorphism connecting $H_p(K,K_0)$ and $H_{p-1}(K_0)$

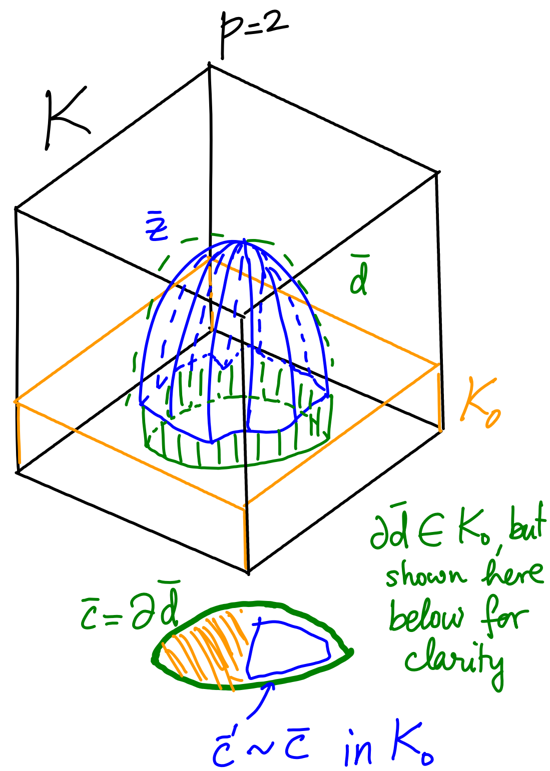
$$\partial_* : H_p(K,K_0) \longrightarrow H_{p-1}(K_0)$$

We call this homomorphism the **homology boundary homomorphism** or the **connecting homomorphism**.

Consider a cycle $\bar{z} \in C_p(K,K_0)$.

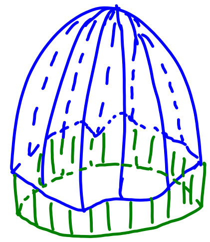
We consider the class $\{\bar{z}\}$ as the coset modulo $C_p(K_0)$ of a p-chain $\bar{d}$ of K such that $\partial\bar{d}$ is carried by K_0. Notice that $\partial\bar{d}$ is automatically a $(p-1)$-cycle of K_0. We define

$$\partial_* \{\bar{z}\} = \{\partial\bar{d}\}$$

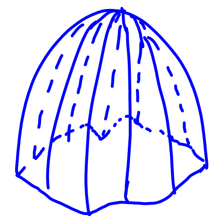


We detail the algebraic construction/definition in this fashion.

$\bar{d}$



$\bar{z}$

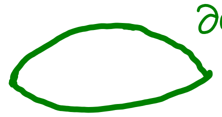


$$\begin{tikzcd}
 & C_p(K) \arrow[r, "\pi_\#"] \arrow[d, "\partial"] & C_p(K,K_0) \\
C_{p-1}(K_0) \arrow[r, "i_\#"] & C_{p-1}(K) &
\end{tikzcd}$$

$\pi_\#(\bar{d}) = \bar{z}$

$\bar{c}$

$(p-1)$-cycle



$\partial\bar{d} \in K_0$

$\{\bar{c}\} =: \partial_*\{\bar{z}\}$

$i : K_0 \to K$ and $\pi : (K,\phi) \to (K,K_0)$ are inclusions.

$i_\#$ is an inclusion, $\pi_\#$ is projection of $C_p(K)$ onto $C_p(K)/C_p(K_0)$.

So we define $\partial_*\{\bar{z}\}$ by a "zig-zag" process.

Def A **long exact sequence** is an exact sequence whose index set is $\mathbb{Z}$. So the sequence is infinite in both directions. It could begin or end with an infinite string of trivial groups.

Theorem 23.3 [M] (The exact homology sequence of a pair)

Let K_0 is a subcomplex of K. There is a long exact sequence

$$\cdots \to H_p(K_0) \xrightarrow{i_*} H_p(K) \xrightarrow{\pi_*} H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

where $i: K_0 \to K$ and $\pi: (K, \phi) \to (K, K_0)$ are inclusions and ∂_* is the connecting homomorphism. There exists a similar long exact sequence in reduced homology.

$$\cdots \to \tilde{H}_p(K_0) \xrightarrow{i_*} \tilde{H}_p(K) \xrightarrow{\pi_*} H_p(K, K_0) \xrightarrow{\partial_*} \tilde{H}_{p-1}(K_0) \longrightarrow \cdots$$

It turns out $\tilde{H}_p(K, K_0) = H_p(K, K_0)$ as long as $K_0 \neq \phi$. Essentially, relative homology groups are already reduced.

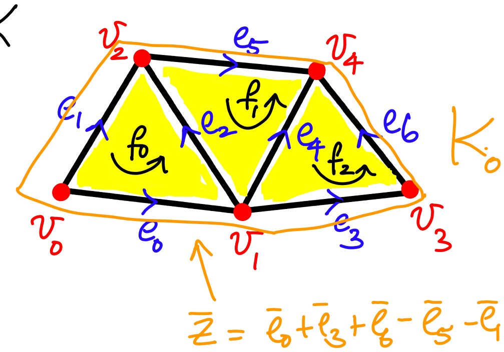
One direct use of the above result is in figuring out the structure of $H_p(K, K_0)$ when the structures of $H_p(K)$ and $H_p(K_0)$ are known. In many cases, the latter homology groups could be characterized more easily, and hence could be used in conjunction with this exact homology sequence to identify $H_p(K, K_0)$.

We apply this result to a few examples.

1. We had seen that ← in Lecture 12

$H_2(K, K_0) \simeq \mathbb{Z}$ with $\bar{r} = \sum_{i=0}^{2} f_i$ being a generator.

K



Also, $H_1(K_0) \simeq \mathbb{Z}$ with $\bar{z}$ being a generator.

Notice that $\partial \bar{r} = \bar{z}$. In this case $\partial_*: H_2(K, K_0) \to H_1(K_0)$ is an isomorphism. We could reach the same conclusion using the exact sequence result. A portion of the long exact sequence is

$$\underbrace{H_2(K)}_{=0} \longrightarrow H_2(K, K_0) \xrightarrow{\partial_*} H_1(K_0) \longrightarrow \underbrace{H_1(K)}_{=0}.$$

$H_2(K)$ and $H_1(K)$ are both trivial, and hence ∂_* is both a monomorphism and an epimorphism, i.e., it's an isomorphism.

There are no 2-cycles to start with. → Notice that any 1-cycle in K is also a 1-boundary. More intuitively, K has no holes.

→ Recall results 1 and 2 from Lecture 18 on exact sequences!

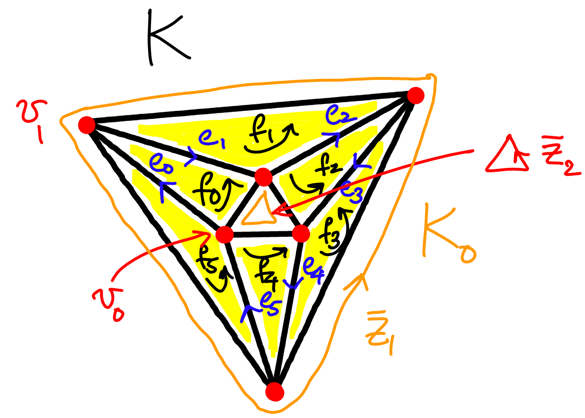
MATH 524: Lecture 19 (10/21/2025)

Today: * exact sequences of chain complexes
* zigzag lemma, diagram chasing

Recall: Exact homology sequence of a pair K, K_0.

$$\cdots \to H_p(K_0) \xrightarrow{i_*} H_p(K) \xrightarrow{\pi_*} H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

2. Consider the annulus we saw in Lecture 12.



$H_2(K, K_0) = ?$ $\quad H_1(K, K_0) = ?$

Consider reduced homology (for $\tilde{H}_0(K_0)$).

K_0 consists of the outer and inner perimeters, both oriented CCW.

Recall that with $\bar{r} = \sum_{i=0}^{5} \bar{f}_i$, $\partial \bar{r} = \bar{z}_1 - \bar{z}_2$.

Also, $\partial_1 \bar{e}_0 = v_1 - v_0$.

We consider the relevant portion of the exact homology sequence:

$$H_2(K) \xrightarrow{0} H_2(K, K_0) \xrightarrow{(\partial_*)_2} H_1(K_0) \xrightarrow{(i_*)_1} H_1(K) \xrightarrow{(\pi_*)_1} H_1(K, K_0) \xrightarrow{(\partial_*)_1} \tilde{H}_0(K_0) \xrightarrow{(i_*)_0} \tilde{H}_0(K)$$

$$0 \longrightarrow \ ? \ \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \ ? \ \longrightarrow \mathbb{Z} \longrightarrow 0$$

Under the terms: $\mathbb{Z}$ (under the first ?); $\{\bar{z}_1\}, \{\bar{z}_2\}$ (under $\mathbb{Z} \oplus \mathbb{Z}$); $\{\bar{z}_1\}$ or $\{\bar{z}_2\}$ (under $\mathbb{Z}$); $\mathbb{Z}$ (under the second ?); $\{v_1 - v_0\}$ (under the last $\mathbb{Z}$).

If $i : K_0 \to K$ is inclusion, i_* maps both $\{\bar{z}_1\}$ and $\{\bar{z}_2\}$ to, say, $\{\bar{z}_1\}$.
So $(i_*)_1$ is an epimorphism, and $\ker (i_*)_1 \simeq \mathbb{Z}$, and it is generated by $\{\bar{z}_1\} - \{\bar{z}_2\}$. Hence, we get that $(\pi_*)_1$ is the zero homomorphism.
Equivalently, notice that any $\bar{z} \in H_1(K)$ is homologous to $\bar{z}_1$ (or $\bar{z}_2$), so is projected out by π_* in $H_1(K, K_0)$.

So, we have

$$\xrightarrow{\;0\;} H_1(K, K_0) \xrightarrow{(\partial_*)_1} \underset{\mathbb{Z}}{\tilde{H}_0(K_0)} \xrightarrow{\;0\;} 0$$

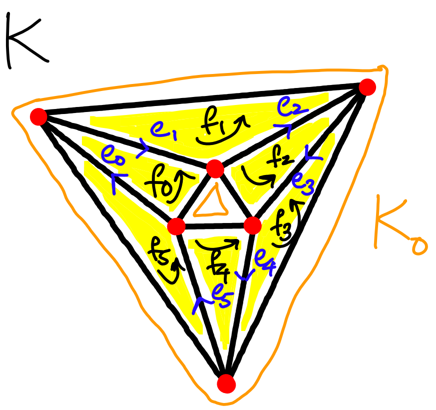
$\Rightarrow (\partial_*)_1$ is an isomorphism, so $H_1(K, K_0) \simeq \mathbb{Z}$.
It is generated by, e.g., $\{\bar{e}_0\}$ with $\partial \bar{e}_0 = v_1 - v_0$.

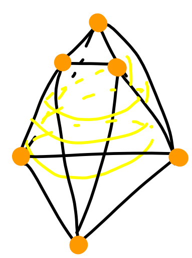
Again, by applying results 1 and 2 from Lecture 19 on exact sequences here, we notice $(\partial_*)_1$ is both an epimorphism and a monomorphism.

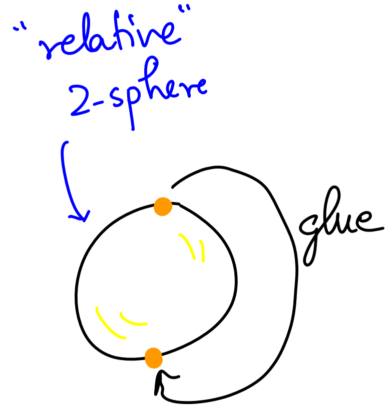
We also get $\operatorname{im}(\partial_*)_2 = \ker (i_*)_1$ and $(\partial_*)_2 : H_2(K, K_0) \longrightarrow \ker (i_*)_1$ is an isomorphism. Hence $H_2(K, K_0) \simeq \mathbb{Z}$. It is generated by $\bar{r} = \sum_{i=0}^{5} \bar{f}_i$, as $\partial_2 \bar{r} = \bar{z}_1 - \bar{z}_2$, which in turn generates $\ker (i_*)_1$, as we noted previously.

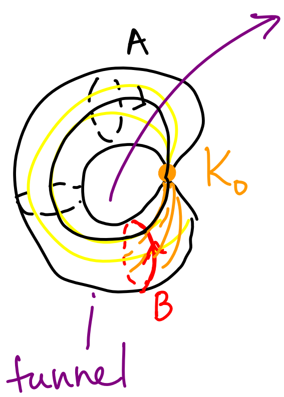
While the ~~for~~ formal method works, it also helps to think intuitively, how the complex looks after "shrinking" all of K_0 to a point.

Think about shrinking both $\bar{z}_1$ and $\bar{z}_2$ (which comprise K_0) to a point each, and then "gluing" these two points.

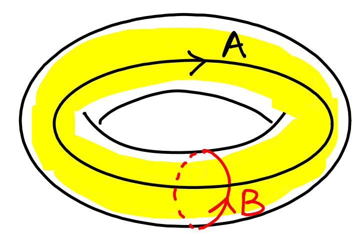








← This object is the pinched torus. Compare this object to the torus.



Notice that while the tunnel loop (A) still exists, the handle loop (B) is now a boundary - it bounds the two chain from the pinched point (representing K_0) to B (looks like a cap). Hence, $H_1(K, K_0) \simeq \mathbb{Z}$.

Also, there is still one enclosed space, or void, and hence $H_2(K, K_0) \simeq \mathbb{Z}$ as well here.

Recall: chain complexes and chain maps

We had introduced the (far more) general concept of chain complexes and chain maps between them. A chain complex $\mathcal{C}$ consists of a set of objects (groups, for instance) with maps (homomorphisms) between them that satisfy the condition that composition of consecutive maps is trivial (i.e., zero).

We have $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$, with $\partial'_p \circ \partial'_{p+1} = 0$. A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ is a family of homomorphisms that commutes with ∂_p, ∂'_p, i.e.,

$$\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p \quad \forall\, p.$$

Each "square" commutes:

$$\begin{tikzcd} {} \arrow[r, "\partial_p"] \arrow[d, "\phi_p"'] & {} \arrow[d, "\phi_{p-1}"] \\ {} \arrow[r, "\partial_p"] & {} \end{tikzcd}$$

So, cycle (boundaries) in $\mathcal{C}$ get mapped to cycles (boundaries) in $\mathcal{C}'$, and ϕ induces a homomorphism of the homology groups

$$(\phi_*)_p : H_p(\mathcal{C}) \longrightarrow H_p(\mathcal{C}').$$

Notice that we can define $Z'_p = \ker \partial'_p$, $B'_p = \operatorname{im} \partial'_{p+1}$, and $H'_p = Z'_p / B'_p$ for $\mathcal{C}'$.

We present the result on existence of long exact sequences given a family of short exact sequences in the general setting of chain complexes.

Notation $\mathcal{C}, \mathcal{D}, \mathcal{E}$: chain complexes

$$\mathcal{C} = \{C_p, \partial_C\}, \quad \mathcal{D} = \{D_p, \partial_D\}, \quad \mathcal{E} = \{E_p, \partial_E\}$$

groups in the chain complexes

homomorphisms for each chain complex

We will supress listings of subscripts to avoid clutter.

Def Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be chain complexes and 0 denote the trivial chain complex whose groups vanish in each dimension. Let $\phi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{E}$ be chain maps. We say the sequence $\mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E}$ is **exact** at $\mathcal{D}$ if $\ker \psi_p = \operatorname{im} \phi_p$ $\forall p$, i.e., if the sequence $C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p$ is exact $\forall p$.

We say the sequence $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$ is a **short exact sequence of chain complexes** if in each dimension p, the sequence

$$0 \longrightarrow C_p \xrightarrow{\phi} D_p \xrightarrow{\psi} E_p \longrightarrow 0 \quad \text{is an}$$

exact sequence of groups.

Example Let $K_0 \subseteq K$ be a subcomplex of simplicial complex K. Then the sequence

$$0 \longrightarrow \mathcal{C}(K_0) \xrightarrow{i} \mathcal{C}(K) \xrightarrow{\pi} \mathcal{C}(K, K_0) \longrightarrow 0$$

is exact, as $C_p(K, K_0) = C_p(K)/C_p(K_0)$ by definition.

We have $\ker \pi_p = \operatorname{im} i_p \ \forall p$.

Here $\mathcal{C}(K) = \{C_p(K), \partial_p\}$, $\mathcal{C}(K_0) = \{C_p(K_0), \partial_p\}$, and so on. Notice that we directly get the following results:

$\mathcal{C}(K_0) \longrightarrow \mathcal{C}(K)$ is injective and
$\mathcal{C}(K) \longrightarrow \mathcal{C}(K, K_0)$ is surjective.

We can construct/define connecting homomorphisms using which we can connect such short exact sequences of chain complexes to build long exact sequences of chain complexes. Recall the result from the previous lecture about long exact sequences for homology groups of a pair (K, K_0) — we will see that this result follows as a direct instance of the more general result specified on chain complexes and chain maps. We first state the general result, and come back to the above example to illustrate the same.

Lemma 24.1 [M] (The **zig-zag lemma**) or (Snake lemma).

Suppose one is given chain complexes $\mathcal{C} = \{C_p, \partial_C\}$, $\mathcal{D} = \{D_p, \partial_D\}$, and $\mathcal{E} = \{E_p, \partial_E\}$, and chain maps ϕ, ψ such that the sequence $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$ is exact. Then there is a **long exact homology sequence**

$$\cdots H_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \xrightarrow{\phi_*} H_{p-1}(\mathcal{D}) \to \cdots$$

where ∂_* is the **connecting homomorphism** and is induced by the boundary operator in $\mathcal{D}$ (∂_D).

<u>Back to the example</u> on long exact sequence of homology.

We just saw that the sequence

$$0 \longrightarrow \mathcal{C}(K_0) \xrightarrow{i} \mathcal{C}(K) \xrightarrow{\pi} \mathcal{C}(K, K_0) \longrightarrow 0$$

is exact. The exactness in the middle follows from the fact that a chain of K is carried by K_0 iff it is zero in $C(K, K_0)$.

So Lemma 24.1 implies the existence of a long exact homology sequence of pair (K, K_0):

$$\cdots \to H_p(K_0) \longrightarrow H_p(K) \longrightarrow H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

Proof (sketch).

Main step: define connecting homomorphism ∂_*. We illustrate the technique of "diagram chasing" here – it's applied in more general settings (and not just to simplicial complexes).

$$
\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] & D_{p+1} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] & E_{p+1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_p \arrow[r, "\phi"] \arrow[d, "\partial_C"'] & D_p \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\square_0"] & E_p \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\square_1"] & D_{p-1} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] & E_{p-1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "\phi"] & D_{p-2} \arrow[r, "\psi"] & E_{p-2} \arrow[r] & 0
\end{tikzcd}
$$

d_p e_p c_{p-1} $\partial_D d_p$ $\partial_C c_{p-1}$

$\square_0$: $\psi \partial_D = \partial_E \psi$

$\square_i$: squares are indexed in the order in which they're used in the proof.

Step 1: (defining ∂_*). Given a cycle e_p in E_p, since ψ is surjective, we can choose $d_p \in D_p$ such that $\psi(d_p) = e_p$. Since $\square_0$ commutes, the element $\partial_D d_p$ of D_p lies in $\ker \psi$, as

$$\psi(\partial_D d_p) = \partial_E(\psi(d_p)) = \partial_E(\underset{\text{cycle}}{e_p}) = 0.$$

Therefore, there exists $c_{p-1} \in C_{p-1}$ such that $\phi(c_{p-1}) = \partial_D d_p$ as $\ker \psi = \operatorname{im} \phi$. Since ϕ is injective, c_{p-1} is unique here.

Further, c_{p-1} is a cycle here, since

$$\phi(\partial_C c_{p-1}) = \partial_D \phi(c_{p-1}) = \partial_D(\partial_D d_p) = 0,$$

as $\square_1$ commutes. Again, since ϕ is injective, $\partial_C c_{p-1} = 0$.

We define $\partial_* \{e_p\} = \{c_{p-1}\}$, where $\{\cdot\}$ means "homology class of."

We'll present the rest of the proof in the next lecture...

MATH 524: Lecture 20 (10/23/2025)

Today: * more on zig-zag lemma
* "stacking" sequences of chain complexes

Recall: proof of zigzag lemma... Step 1: Define $\partial_*\{e_p\} = \{c_{p-1}\}$.

Step 2 Show ∂_* is well defined — independent of the choice of $e_p \in \ker \partial_E$ and choice of c_{p-1} from $\{c_{p-1}\}$.

Recall that we defined ∂_* on homology classes — $\partial_*\{e_p\} = \{c_{p-1}\}$ for cycle $e_p \in E_p$ and corresponding cycle $c_{p-1} \in C_{p-1}$. We want to now show that this definition is independent of the choice of e_p and c_{p-1}. To this end, we start with cycles e_p, e_p' in E_p $(e_p, e_p' \in \ker \partial_E : E_p \to E_{p-1})$. We assume that $e_p \sim e_p'$ (homologous), and then argue that $c_{p-1} \sim c_{p-1}'$.

Given $e_p \sim e_p'$, we can find $e_{p+1} \in E_{p+1}$ such that $e_p - e_p' = \partial_E e_{p+1}$ (by definition of homology). Using the upper portion of the diagram, we argue that we can find $c_p \in C_p$ such that $c_{p-1} - c_{p-1}' = \partial_C c_p$.

$$\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] & \overset{d_{p+1}}{D_{p+1}} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\square_2"] & \overset{e_{p+1}}{E_{p+1}} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & \overset{c_p}{C_p} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\square_3"] & \overset{d_p,\, d_p'}{D_p} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\square_0"] & \overset{e_p \sim e_p'}{E_p} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & \overset{c_{p-1}' \sim c_{p-1}}{C_{p-1}} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\square_1"] & \overset{\partial_D d_p}{D_{p-1}} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] & E_{p-1} \arrow[r] \arrow[d, "\partial_E"] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "\phi"] & D_{p-2} \arrow[r, "\psi"] & E_{p-2} \arrow[r] & 0
\end{tikzcd}$$

$\square_i$: squares are indexed in the order in which they're used in the proof.

ψ is surjective. So choose d_p, d_p' such that $\psi(d_p) = e_p$ and $\psi(d_p') = e_p'$. Using the same arguments in Step 1, choose c_{p-1} and c_{p-1}' such that $\phi(c_{p-1}) = \partial_D d_p$ and $\phi(c_{p-1}') = \partial_D d_p'$.

Suppose $e_p - e_p' = \partial_E e_{p+1}$. $\overbrace{\text{Choose } d_{p+1} \in D_{p+1}}^{\text{recall that } \psi \text{ is surjective}}$ such that $\psi(d_{p+1}) = e_{p+1}$. Notice that

as $\square_2$ commutes, $\psi\partial_D = \partial_E\psi$

$$\begin{aligned}
\psi(d_p - d_p' - \partial_D d_{p+1}) &= e_p - e_p' - \partial_E \psi(d_{p+1}) \\
&= e_p - e_p' - \partial_E e_{p+1} = 0.
\end{aligned}$$

So $d_p - d_p' - \partial_D d_{p+1} \in \ker \psi : D_p \to E_p$. By exactness, it should also be in $\operatorname{im} \phi : C_p \to D_p$.

So we can choose $c_p \in C_p$ such that $\phi(c_p) = d_p - d_p' - \partial_D d_{p+1}$.

So $\phi(\partial_C c_p) = \partial_D \phi(c_p)$ as $\square_3$ commutes, $\phi \partial_C = \partial_D \phi$

$$= \partial_D(d_p - d_p' - \partial_D d_{p+1}) = \phi(c_{p-1} - c_{p-1}').$$

But ϕ is injective, so $\partial_C c_p = c_{p-1} - c_{p-1}'$. So $c_{p-1} \sim c_{p-1}'$.

We need to show also that ∂_* is indeed a homomorphism. Notice that $\psi(d_p + d_{p'}) = e_p + e_{p'}$ and $\phi(c_{p-1} + c_{p-1}') = \partial_D(d_p + d_{p'})$. So $\partial_* \{e_p + e_p'\} = \{c_{p-1} + c_{p-1}'\}$ by definition, and the latter part equals $\partial_* \{e_p\} + \partial_* \{e_p'\}$.

Thus, $\partial_* \{e_p + e_p'\} = \partial_* \{e_p\} + \partial_* \{e_p'\}$, showing ∂_* is a homomorphism.

<u>Steps 3, 4, 5</u> Prove exactness at $H_p(\mathcal{D})$, $H_p(\mathcal{E})$, and $H_{p-1}(\mathcal{C})$.

See [M] for details. □

Notice how we zig-zag down and to the left to go from e_p to c_{p-1} in the process of defining $\partial_* \{e_p\}$. Hence the name "zig-zag" or "snake" lemma.

It turns out we can extend this type of results on existence of long exact sequences with connecting homomorphisms to pairs (or more) of exact sequences of chain complexes.

Theorem 24.2 [M] Suppose we are given a commutative diagram

internal zig-zag (in "top floor")

$$\begin{tikzcd}
0 \arrow[r] & \overset{c_{p-1}}{\mathcal{C}} \arrow[r, "\phi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\boxed{}_2"] & \overset{d_p}{\mathcal{D}} \arrow[r, "\psi"] \arrow[d, "\beta"] \arrow[dr, phantom, "\boxed{}_1"] & \overset{e_p}{\mathcal{E}_e} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \underset{c'_{p-1}}{\mathcal{C}'} \arrow[r, "\phi'"] & \underset{d'_p}{\mathcal{D}'} \arrow[r, "\psi'"] & \underset{e'_p}{\mathcal{E}'_e} \arrow[r] & 0
\end{tikzcd}$$

internal zig-zag (in "bottom floor")

where horizontal sequences are exact sequences of chain complexes, and α, β, γ are chain maps. Then the following diagram commutes as well:

$$\begin{tikzcd}
\cdots & H_p(\overset{c_{p-1}}{\mathcal{C}}) \arrow[r, "\phi_*"] \arrow[d, "\alpha_*"'] & H_p(\overset{d_p}{\mathcal{D}}) \arrow[r, "\psi_*"] \arrow[d, "\beta_*"'] & H_p(\overset{e_p}{\mathcal{E}_e}) \arrow[r, "\partial_*"] \arrow[d, "\gamma_*"'] & H_{p-1}(\mathcal{C}) \arrow[r] \arrow[d, "\alpha_*"'] & \cdots \\
\cdots & H_p(\mathcal{C}') \arrow[r, "\phi'_*"] & H_p(\mathcal{D}') \arrow[r, "\psi'_*"] & H_p(\overset{e'_p}{\mathcal{E}'_e}) \arrow[r, "\partial'_*"] & H_{p-1}(\mathcal{C}') \arrow[r] & \cdots
\end{tikzcd}$$

Notice that each "level" here, e.g., $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E}_e \to 0$, represents a collection of groups and homomorphisms as we have seen previously. We have exactness within this substructure, and similarly within the $\mathcal{C}', \mathcal{D}', \mathcal{E}'_e$ substructure. α, β, γ are chain maps connecting corresponding parts of the two substructures.

Proof Commutativity of first and second squares is immediate, as it holds at the chain level. Commutativity of the last (3^{rd}) square involves the definition of ∂_* and ∂_*'. Given $\{e_p\} \in H_p(\mathcal{E})$, choose d_p such that $\psi(d_p) = e_p$, and choose c_{p-1} such that $\phi(c_{p-1}) = \partial_D d_p$. Then $\partial_*\{e_p\} = \{c_{p-1}\}$ by definition.

Notice that we are not explicitly displaying this "internal" zig-zag in the picture above. Now we want to consider the corresponding images under γ, β, and α, and show that the structure is "preserved".

Let $e_p' = \gamma(e_p)$; we want to show $\partial_*'\{e_p'\} = \alpha_*\{c_{p-1}\}$.

Intuitively, this result follows because each step in the definition of ∂_* commutes.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\boxed{}_2"] & \mathcal{D} \arrow[r, "\psi"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\boxed{}_1"] & \mathcal{E} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \mathcal{C}' \arrow[r, "\phi'"] & \mathcal{D}' \arrow[r, "\psi'"] & \mathcal{E}' \arrow[r] & 0
\end{tikzcd}$$

$\beta(d_p)$ is a suitable pullback for e_p', as $\boxed{}_1$ commutes: $\psi'\beta(d_p) = \gamma\psi(d_p) = \gamma(e_p) = e_p'$. Similarly, $\alpha(c_{p-1})$ is a suitable pullback for $\partial_D'\beta(d_p)$, since $\boxed{}_2$ commutes: $\phi'\alpha(c_{p-1}) = \beta\phi(c_{p-1}) = \beta(\partial_D d_p) = \partial_D'(\beta(d_p))$.

$\Longrightarrow \partial_*'\{e_p'\} = \{\alpha(c_{p-1})\}$ by definition. $\square$

Here is another result in the same flavor.

Lemma 24.3 [M] (**The Steenrod five lemma**) Suppose we are given the commutative diagram of abelian groups and homomorphisms

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

where the horizontal sequences are exact. If f_1, f_2, f_4, f_5 are all isomorphisms, then so is f_3.

You'll get a chance to prove this lemma in homework ☺!

Application to relative homology: see Lemma 24.4 and Theorem 24.5 in [M].

Meyer-Vietoris Sequences

We use the zig-zag lemma to derive another long exact sequence to compute homology groups. It relates the homology of two given spaces to that of their union and their intersection. The overarching theme is once again the "easy" or "efficient" identification or computation of homology groups.

Theorem 25.1 [M] Let K be a complex, and $K', K'' \subseteq K$ be subcomplexes such that $K = K' \cup K''$. Let $A = K' \cap K''$. Then there is a long exact sequence

$$\cdots H_p(A) \longrightarrow H_p(K') \oplus H_p(K'') \longrightarrow H_p(K) \xrightarrow{\partial} H_{p-1}(A) \rightarrow \cdots$$

called the Meyer-Vietoris sequence of (K', K''). There exists a similar exact sequence in reduced homology if A is nonempty.

∂ is the connecting homomorphism — notice that ∂ takes us from dimension p to $p-1$.

Notation: The book uses different notation. The one used here is probably more intuitive. We will use $'$ and $''$ as superscripts for all objects related to K' and K'', respectively.

Proof idea: We construct short exact sequences of chain complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K') \oplus \mathcal{C}(K'') \xrightarrow{\psi} \mathcal{C}(K) \longrightarrow 0$$

and apply the zig-zag lemma.

MATH 524: Lecture 21 (10/28/2025)

Today: * Meyer-Vietoris Sequence

Recall: Theorem 25.1: $K', K'' \subseteq K$ with $A = K' \cap K''$ and $K' \cup K'' = K$

Meyer-Vietoris sequence (MVS):

$$\cdots \to H_p(A) \to H_p(K') \oplus H_p(K'') \to H_p(K) \xrightarrow{\partial} H_{p-1}(A) \to \cdots$$

Proof idea: We construct short exact sequences of chain complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K') \oplus \mathcal{C}(K'') \xrightarrow{\psi} \mathcal{C}(K) \longrightarrow 0$$

and apply the zig-zag lemma.

We first define the chain complex in the middle. Its chain group in dimension p in $C_p(K') \oplus C_p(K'')$, and its boundary operator is ∂ is defined by

overload of notation

$$\partial(\bar{c}', \bar{c}'') = (\partial' \bar{c}', \partial'' \bar{c}'')$$

where ∂', ∂'' are the boundary operators in $\mathcal{C}(K'), \mathcal{C}(K'')$, respectively.

Second, we define chain maps ϕ, ψ. Consider inclusion mappings in the following commutative diagram:

$$\begin{tikzcd}
 & K' \arrow[dr, "j'"] & \\
A \arrow[ur, "i'"] \arrow[rr, "k"] \arrow[dr, "i''"'] & & K \\
 & K'' \arrow[ur, "j''"'] &
\end{tikzcd}$$

i', i'': inclusion maps of A into K', K''

j', j'': inclusion maps of K', K'' into K

k: inclusion map of A into K

Define the homomorphisms ϕ and ψ as

$$\phi(\bar{c}) = \left(i'_{\#}(\bar{c}), -i''_{\#}(\bar{c})\right), \quad \text{and}$$

notice the "$-$" here!

$$\psi(\bar{c}', \bar{c}'') = \left(j'_{\#}(\bar{c}') + j''_{\#}(\bar{c}'')\right).$$

We can verify that ϕ and ψ are indeed chain maps. Check for exactness:

ϕ is injective, as both $i'_{\#}$ and $i''_{\#}$ are just inclusions of chains.

Also, ψ is surjective. Given $\bar{c} \in C_p(K)$, let $\bar{c}'$ be its part carried by K', and then $\bar{c} - \bar{c}'$ carried by K'', and we get $\psi(\bar{c}', \bar{c} - \bar{c}') = \bar{c}$ $(= \bar{c}' + \bar{c} - \bar{c}')$.

To confirm exactness at the middle term, note that

$$\psi\phi(\bar{c}) = k_{\#}(\bar{c}) - k_{\#}(\bar{c}) = 0$$

recall the "$-$" in the definition of ϕ!

Conversely, if $\psi(\bar{c}', \bar{c}'') = 0$, then $\bar{c}' = -\bar{c}''$ as chains of K. Since $\bar{c}' \in K'$ and $\bar{c}'' \in K''$, they must be carried by $A = K' \cap K''$ (as $\bar{c}' = -\bar{c}''$). Hence $(\bar{c}', \bar{c}'') = (\bar{c}', -\bar{c}') = \phi(\bar{c}')$ as needed.

The homology for the middle chain complex in dimension p is

$$\frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}} = \frac{\ker \partial'_p \oplus \ker \partial''_p}{\operatorname{im} \partial'_{p+1} \oplus \operatorname{im} \partial''_{p+1}} \simeq H_p(K') \oplus H_p(K'').$$

The Meyer-Vietoris (MV) sequence now follows from the zig-zag lemma. A similar argument can be used to get the Meyer-Vietoris sequence in reduced homology groups (when $A \neq \emptyset$).

More details on structure of $H_p(K)$ $\forall p$:

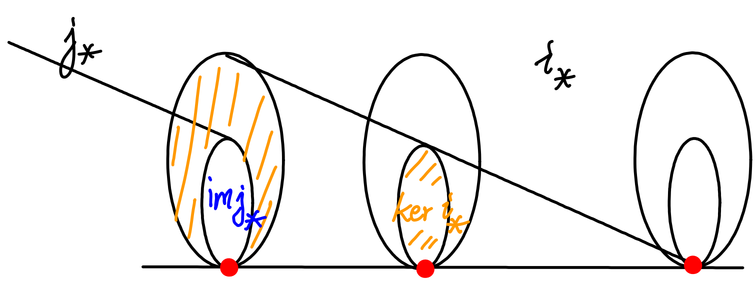
$$\cdots \longrightarrow H_p(A) \xrightarrow{i_*} H_p(K') \oplus H_p(K'') \xrightarrow{j_*} H_p(K) \xrightarrow{\partial_*}$$

$$\hookrightarrow H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(K') \oplus H_{p-1}(K'') \xrightarrow{j_*} H_{p-1}(K) \longrightarrow$$

We write the part of the sequence for each dimension in one level, or "floor". We will come back to this representation later...

Consider the connecting maps now.

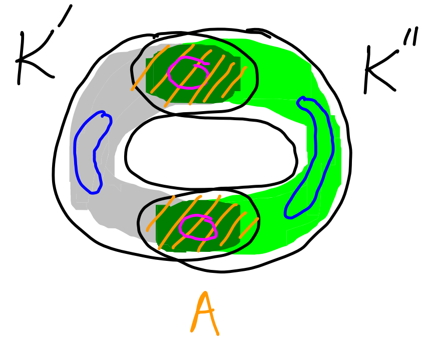
$$\longrightarrow H_p(K') \oplus H_p(K'') \xrightarrow{j_*} H_p(K) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(K') \oplus H_{p-1}(K'') \longrightarrow \cdots$$



Exactness of the Meyer-Vietoris sequence at $H_p(K)$ tells us that this group is a direct sum of the image of
$j_*: H_p(K') \oplus H_p(K'') \longrightarrow H_p(K)$ with the kernel of
$i_*: H_{p-1}(A) \longrightarrow H_{p-1}(K') \oplus H_{p-1}(K'')$.

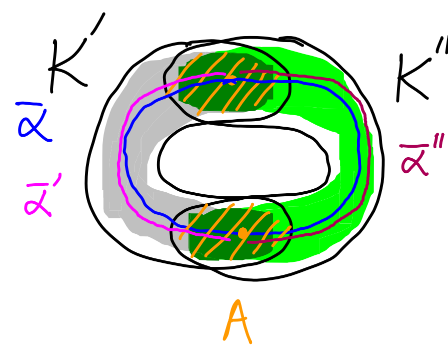
We use exactness at $H_{p-1}(A)$ here.

Hence we can distinguish two types of homology classes in K — one class in $\operatorname{im} j_*$ that lives in K' or K'', and the other one lives in both, e.g., as illustrated here.



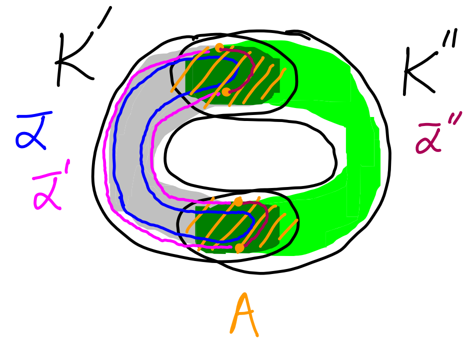
A class in $\ker i_* \equiv (p-1)$-cycle $\bar{\gamma}_{p-1} \in A$ that bounds both in K' and K''. If we write $\bar{\gamma}_{p-1} = \partial \bar{\alpha}_p' = -\partial \bar{\alpha}_p''$ where $\bar{\alpha}_p' \in C_p(K')$ and $\bar{\alpha}_p'' \in C_p(K'')$, then $\bar{\alpha}_p = \bar{\alpha}_p' + \bar{\alpha}_p''$ is a cycle in K which represents the second type of the class.

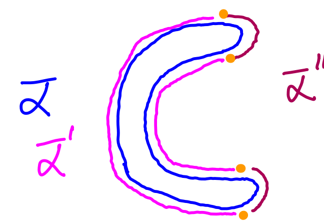
Here is another example. The 1-cycle $\bar{\alpha}$ decomposes into $\bar{\alpha}'$ and $\bar{\alpha}''$. Their boundaries (∂' and ∂'' in K' and K'', respectively) is the 0-chain made of 2 points (with signs reversed) which is a reduced 0-cycle in A.



↳ between K' and K''

What about this 1-cycle $\bar{\alpha}$?
This cycle also represents a homology class of the <u>second</u> type, with one possible decomposition of $\bar{\alpha}$ into $\bar{\alpha}'$ and $\bar{\alpha}''$ illustrated below.

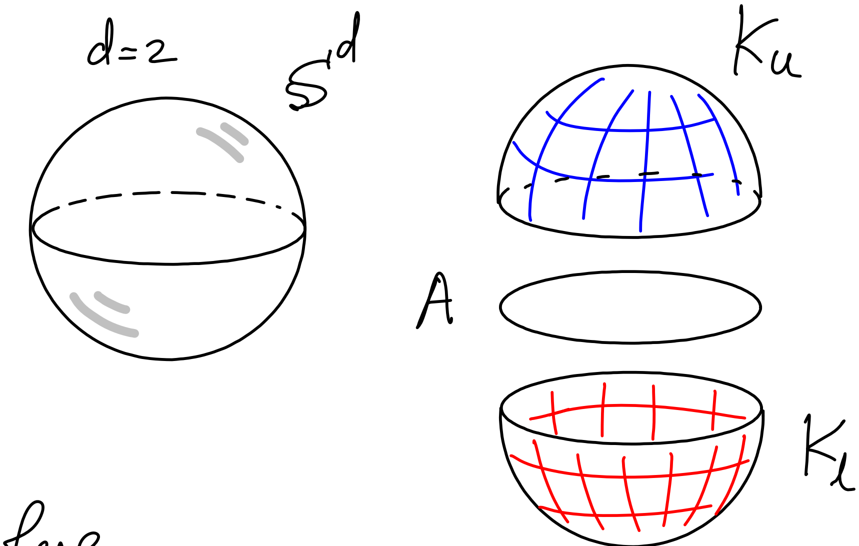




The connecting homo'm ∂_* can be explicitly defined as follows. Consider a cycle $\bar{z} \in K$. We can choose $\bar{c}' \in K'$ and $\bar{c}'' \in K''$ s.t. $\bar{z} = \bar{c}' + \bar{c}''$. $\bar{c}'$ and $\bar{c}''$ need not be cycles themselves, but it must hold that
$\partial \bar{c}' = -\partial \bar{c}''$, as $\partial \bar{z} = \partial(\bar{c}' + \bar{c}'') = 0$.
Also, $\partial \bar{c}'$ and $\partial \bar{c}''$ must both be carried by $A = K' \cap K''$.
We define $\partial_* \{\bar{z}\} = \{\partial \bar{c}'\}$, or $\{-\partial \bar{c}''\}$, equivalently.

Example 1 Homology of $\mathbb{S}^d$ (d-sphere): We want to show:

$$\tilde{H}_p(\mathbb{S}^d) \simeq \mathbb{Z} \text{ if } p=d, \text{ and}$$
$$\tilde{H}_p(\mathbb{S}^d) = 0 \text{ if } p \neq d.$$



We set $\mathbb{S}^d = K_u \cup K_\ell$, where K_u, K_ℓ are the upper and lower hemisphere, respectively.

And $A = K_u \cap K_\ell$ is the equator.

Notice that $K_u, K_\ell \approx \mathbb{B}^d$ (d-disc or d-ball), and $A \approx \mathbb{S}^{d-1}$. Now we compute $\tilde{H}_p(\mathbb{S}^d)$ inductively using the reduced homology MVS.

$$\cdots \underset{A}{\tilde{H}_p(\mathbb{S}^{d-1})} \longrightarrow \underset{0}{\tilde{H}_p(K_u)} \oplus \underset{0}{\tilde{H}_p(K_\ell)} \longrightarrow \underset{K}{\tilde{H}_p(\mathbb{S}^d)} \xrightarrow{\partial_*} \tilde{H}_{p-1}(\mathbb{S}^{d-1}) \longrightarrow \cdots$$

For $d=0$, $\mathbb{S}^d$ is the set of 2 points. Hence

$$\tilde{H}_0(\mathbb{S}^0) \simeq \mathbb{Z}, \quad \tilde{H}_p(\mathbb{S}^0) = 0 \;\; \forall p \neq 0.$$

This result gives the start (or base) of the induction.

For general d, the sequence breaks down into pieces of the form

$$0 \oplus 0 \longrightarrow \tilde{H}_p(\mathbb{S}^d) \longrightarrow \tilde{H}_{p-1}(\mathbb{S}^{d-1}) \longrightarrow 0 \oplus 0,$$

as $\tilde{H}_p(K_u) = 0$ and $\tilde{H}_p(K_\ell) = 0 \;\; \forall p$.

Hence we get an isomorphism $\tilde{H}_p(\mathbb{S}^d) \simeq \tilde{H}_{p-1}(\mathbb{S}^{d-1})$, which along with the inductive step implies that

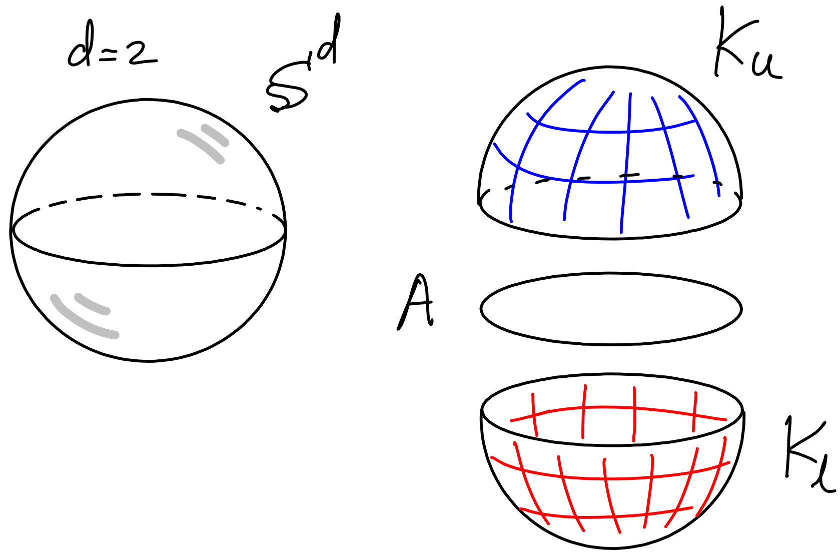
$$\tilde{H}_d(\mathbb{S}^d) \simeq \mathbb{Z} \text{ and } \tilde{H}_p(\mathbb{S}^d) = 0 \;\; \forall p \neq d.$$

The generator for $\tilde{H}_d(\mathbb{S}^d)$ is of the second type, consisting of the union of two d-chains, one each in K_u and K_ℓ, and their intersection generates $\tilde{H}_{d-1}(\mathbb{S}^{d-1})$.

MATH 524: Lecture 22 (10/30/2025)

Today: * Applications of MVS

Recall: Homology of $\mathbb{S}^d$: $\tilde{H}_p(\mathbb{S}^d) \simeq \mathbb{Z}$ if $p=d$, $=0$ o.w.



Let's consider absolute homology now. The MVS is

$$\cdots \to H_p(\mathbb{S}^{d-1}) \xrightarrow{i_*} H_p(K_u) \oplus H_p(K_\ell) \xrightarrow{j_*} H_p(\mathbb{S}^d) \xrightarrow{\partial_*} H_{p-1}(\mathbb{S}^{d-1}) \xrightarrow{i_*} H_{p-1}(K_u) \oplus H_{p-1}(K_\ell) \xrightarrow{j_*} H_{p-1}(\mathbb{S}^d) \to \cdots$$

In the middle, again, we get for $p>0$,

$$0 \oplus 0 \longrightarrow H_p(\mathbb{S}^d) \longrightarrow H_{p-1}(\mathbb{S}^{d-1}) \longrightarrow 0 \oplus 0 .$$

Notice that $H_p(K_u)$ and $H_p(K_\ell)$ are both trivial for $p>0$, as we had with reduced homology groups (as they are both balls). Thus the middle map is an isomorphism. We will use this general result in the induction!

Here are the details for $d=2$ about how we finish. Arguments are similar for more general d.

$$0 \oplus 0 \longrightarrow H_2(\mathbb{S}^2) \longrightarrow \underset{\mathbb{Z}}{H_1(\mathbb{S}^1)} \xrightarrow{i_*} \overset{0}{H_1(K_u)} \oplus \overset{0}{H_1(K_\ell)} \xrightarrow{j_*} H_1(\mathbb{S}^2) \longrightarrow$$

$$\longrightarrow \underset{\mathbb{Z}}{H_0(\mathbb{S}^1)} \xrightarrow{i_*} \underset{\mathbb{Z}}{H_0(K_u)} \oplus \underset{\mathbb{Z}}{H_0(K_\ell)} \xrightarrow{j_*} H_0(\mathbb{S}^2) \longrightarrow 0.$$

single component each

We can look at smaller portions of the sequence to figure out the structure of the homology groups we seek.

First part: $0 \longrightarrow H_2(\mathbb{S}^2) \xrightarrow{\partial_*} \mathbb{Z} \longrightarrow 0$ $\;0 \oplus 0$, to be precise

$\Rightarrow \partial_*$ is an isomorphism. $\Rightarrow H_2(\mathbb{S}^2) \simeq \mathbb{Z}$.

Second part:

$$0 \oplus 0 \to H_1(\mathbb{S}^{1\,2}) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_*} H_0(\mathbb{S}^{1\,2}) \to 0$$

$\uparrow \mathbb{Z}$, by exactness

Let's look at the structure of $i_\#$. Notice that $i_\#$ is injective, and $\ker i_\# = 0$. By exactness, we get $\operatorname{im} \partial_* = \ker i_\# = 0$, which gives that $H_1(\mathbb{S}^{1\,2}) = 0$.

a 0-chain in K_u or K_ℓ corresponds injectively to the 0-chain in $A = \mathbb{S}^{1}$.

Then we could apply induction to get the result:

$H_p(\mathbb{S}^{1\,d}) \simeq \mathbb{Z}$ when $p = d$ or $p = 0$, and

$H_p(\mathbb{S}^{1\,d}) = 0$ otherwise. $\square$

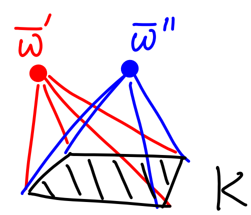
<u>Example 2</u> Homology of the suspension of a simplicial complex.

Def Given a simplicial complex K, let $\bar{w}' * K$ and $\bar{w}'' * K$ be two cones whose polytopes intersect in $|K|$ alone. Then $S(K) = (\bar{w}' * K) \cup (\bar{w}'' * K)$ is a complex called the **suspension** of K. $S(K)$ is uniquely defined up to simplicial isomorphism.

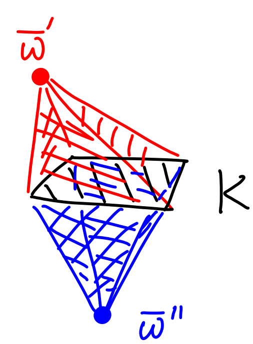
Recall the definition of cone from Lecture 16:

Def Let K be a simplicial complex in $\mathbb{R}^d$, and $\bar{w} \in \mathbb{R}^d$ is a point such that each ray emanating from $\bar{w}$ intersects $|K|$ in at most one point. Then the **cone of K with vertex $\bar{w}$** is the collection of all simplices of the form $\bar{w}a_0 \ldots a_p$ where $\bar{a}_0 \ldots \bar{a}_p$ is a simplex of K, along with all faces of such simplices. We denote this collection as $\bar{w} * K$.

Indeed, the specific choices of $\bar{w}'$ and $\bar{w}''$ are not important, due to the restriction that the two cones intersect only in $|K|$. Thus we do not get the situation shown here, where the two cones intersect outside of $|K|$.



Due to the same intersection condition, it would also follow that $\bar{w}'$ and $\bar{w}''$ are on the "opposite sides" of K. Hence the name suspension is quite appropriate — K is "suspended" in the middle by connections from $\bar{w}'$ and $\bar{w}''$.



We want to study how $H_*(S(K))$ and $H_*(K)$ are related. And we will use the Mayer-Vietoris sequence in a natural way.

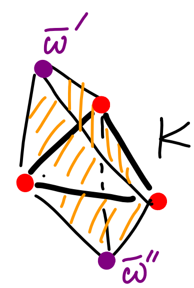
Theorem 25.4 [M] For a simplicial complex K, there is an isomorphism $\tilde{H}_p(S(K)) \longrightarrow \tilde{H}_{p-1}(K)$ $\forall p$.

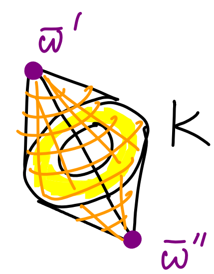
<u>Proof</u> Let $K' = \bar{\omega}' * K$, $K'' = \bar{\omega}'' * K$. Then $K' \cup K'' = S(K)$, and $A = K' \cap K'' = K$. In the reduced homology Meyer-Vietoris sequence, we have

$$\underset{0}{\tilde{H}_p(K')} \oplus \underset{0}{\tilde{H}_p(K'')} \xrightarrow{j_\#} \tilde{H}_p(S(K)) \xrightarrow{\partial_*} \overset{A}{\tilde{H}_{p-1}(K)} \longrightarrow \underset{0}{\tilde{H}_{p-1}(K')} \oplus \underset{0}{\tilde{H}_{p-1}(K'')}$$

Both end terms vanish $(0 \oplus 0)$ as K', K'' are both cones.
Hence the middle map is an isomorphism. $\square$

Here is an example. Let K consist of 3 edges and 3 vertices forming a circle ($\approx S^1$). Then $S(K)$ consists of 6 triangles forming the surface of a sphere. Indeed, $S(K) \approx S^2$ and we do have $H_2(S(K)) \simeq H_1(K) \simeq \mathbb{Z}$. A bit more interesting version of this example has K an annulus. Then both K' and K'' are solid 3D "half cones" with $S(K)$ enclosing a single void in between.



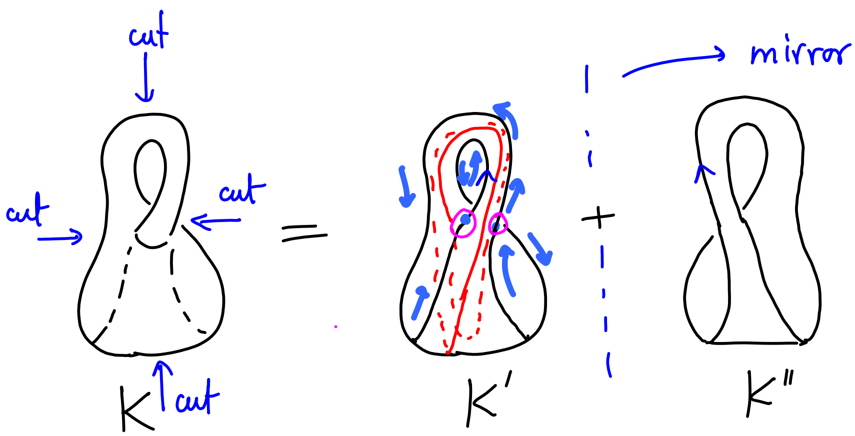


We can naturally talk about $S(S(K))$, which is the suspension of a suspension of K which is also denoted $S^2(K)$.

We could consider $S(K)$ also in the abstract setting.

Example 3 Klein bottle

We now consider the homology of $\mathbb{K}^2$ using its Mayer-Vietoris sequence. Imagine cutting the Klein bottle down the middle into two pieces, both of which are Möbius strips. We denote the original object/space by K, and the two pieces by K' and K''. We get K by gluing K' and K'' along the "cut", i.e., along the edges of the two Möbius strips.

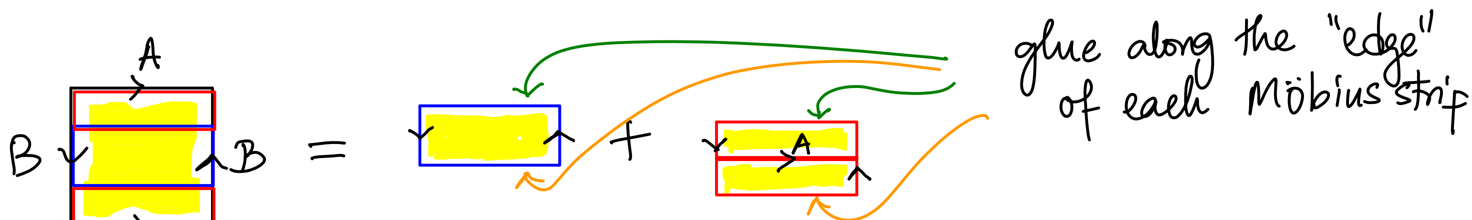


cut K down the middle

Here's a more illustrative picture:

(image: www)

We could also represent the splitting on the square diagram (with pairs of opposite edges identified appropriately).



two Möbius strips

Another way to consider the Klein bottle is to imagine cutting out 2 disks from a 2-sphere, and gluing 2 Möbius strips along the boundaries created by the cuts, which are circles.

Thus we have $\mathbb{K}^2 \approx K = K' \cup K''$; $A = K' \cap K'' \approx S^1$; K', K'' are both Möbius strips.

Let's consider the reduced homology Meyer-Vietoris sequence:

$$0 \longrightarrow \overset{0}{\tilde{H}_2(A)} \xrightarrow{i_\#} \overset{0}{\tilde{H}_2(K')} \oplus \overset{0}{\tilde{H}_2(K'')} \xrightarrow{\partial_\#} \tilde{H}_2(K) \longrightarrow$$

$$\longrightarrow \underset{\mathbb{Z}}{\tilde{H}_1(A)} \xrightarrow{i_\#} \underset{\mathbb{Z}}{\tilde{H}_1(K')} \oplus \underset{\mathbb{Z}}{\tilde{H}_1(K'')} \xrightarrow{\partial_\#} \tilde{H}_1(K) \longrightarrow$$

$$\longrightarrow \underset{0}{\tilde{H}_0(A)}$$

Notice $A \approx S^1$, hence $\tilde{H}_1(A) \simeq \mathbb{Z}$. Similarly, since K' and K'' are both Möbius strips, $\tilde{H}_1(K') \simeq \mathbb{Z}$ and $\tilde{H}_1(K'') \simeq \mathbb{Z}$.

We will finish the argument in the next lecture...

MATH 524: Lecture 23 (11/04/2025)

Today: * MVS of $\mathbb{K}^2$
* MVS of $\mathbb{T}^2$
* categories

Recall: $\mathbb{K}^2 \simeq K = K' \cup K''$, $A = K' \cap K''$

Möbius strips (K', K''); circle (A)

$$0 \to \tilde{H}_2(A) \xrightarrow{i_*} \overset{0}{\tilde{H}_2(K')} \oplus \overset{0}{\tilde{H}_2(K'')} \xrightarrow{j_*} \tilde{H}_2(K)^{?} \to$$

$$\to \underset{\mathbb{Z}}{\tilde{H}_1(A)} \xrightarrow{i_*} \underset{\mathbb{Z}}{\tilde{H}_1(K')} \oplus \underset{\mathbb{Z}}{\tilde{H}_1(K'')} \xrightarrow{j_*} \tilde{H}_1(K)^{?} \to$$

$$\to \underset{0}{\tilde{H}_0(A)}$$

Recall that $H_1(A) = H_1(\mathbb{S}^1) \simeq \mathbb{Z}$; similarly, $H_1(K') \simeq H_1(K'') \simeq \mathbb{Z}$, as they are both Möbius strips. H_2 is trivial in all cases. Also, $\tilde{H}_0(A) = 0$, as it has one component.

First piece: $0 \to \tilde{H}_2(K) \xrightarrow{\partial_*} \underset{A}{\mathbb{Z}} \xrightarrow{i_*} \underset{K'}{\mathbb{Z}} \oplus \underset{K''}{\mathbb{Z}} \to \cdots$

which induces i_*

Let's consider $i_\#$. It's given by $i_\# : 1 \to (2,-2)$. Notice that the "edge" of a Möbius strip wraps twice around its "middle" circle. Also, the two Möbius strips K' and K'' are mirror images, so to speak. In particular, the orientations of their "edges" are opposite. Hence the map is given as $(2,-2)$. We note that $i_\#$ is injective (every cycle in $\tilde{H}_1(K')$ and $\tilde{H}_1(K'')$ correspond uniquely to a cycle in $\tilde{H}_1(A)$, up to homology). Also, we note that $\ker i_* = 0$. So $\operatorname{im} \partial_* = 0$, due to exactness at $\tilde{H}_1(A)$.

Hence $\tilde{H}_2(K) = 0$.

To identify $\tilde{H}_1(K)$, we look at the second piece:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_*} \tilde{H}_1(K) \xrightarrow{\partial_*} 0$$

We can apply Result 3 on exact sequences (Lecture 17) to get that $\dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\text{im } i_*} \simeq \tilde{H}_1(K)$.

3. Suppose the sequence $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ is exact; then $A_2/\phi(A_1) = \text{cok } \psi$ is isomorphic to A_3; this isomorphism is induced by ψ.

First, note that $i_* : 1 \longrightarrow (2,-2)$, i.e., $\text{im } i_* = 2\mathbb{Z}\begin{pmatrix}1\\-1\end{pmatrix}$ or $\mathbb{Z}\begin{pmatrix}2\\-2\end{pmatrix}$.

One basis for $\mathbb{Z} \oplus \mathbb{Z}$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ (as $\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\1\end{pmatrix}$, where $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a natural basis for $\mathbb{Z} \oplus \mathbb{Z}$).

So $\dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\text{im } i_*} \simeq \mathbb{Z} \oplus \mathbb{Z}/_2$.

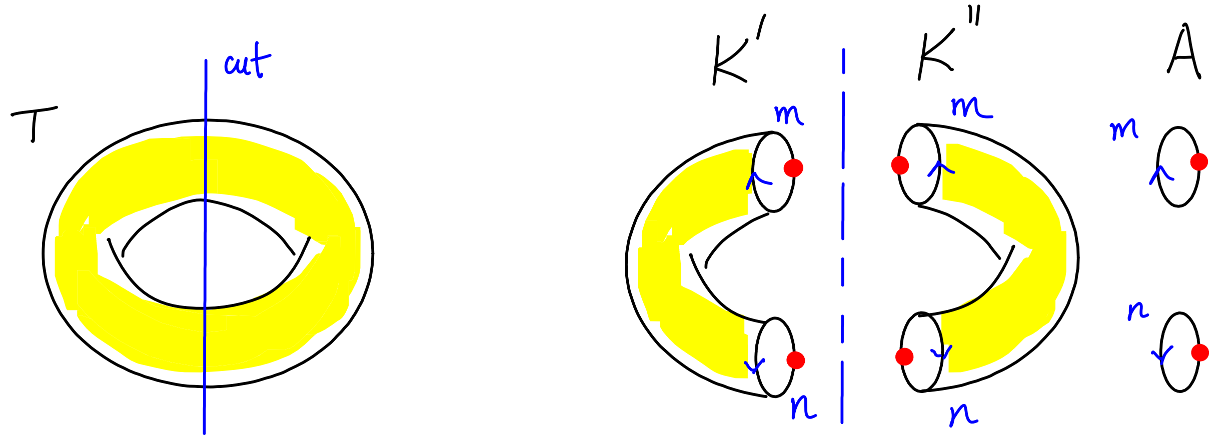
Hence $\tilde{H}_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}/_2$.

Using $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ as the basis is motivated by $\text{im } i_*$ being $(\simeq)$ $2\mathbb{Z}\begin{pmatrix}1\\-1\end{pmatrix}$. With this basis, we can perform the quotienting directly.

Example 4: Torus

We split the torus down the middle into two cylinders whose intersection is the union of two disjoint circles.

We consider the Meyer-Vietoris sequence in absolute homology.



K', K'': cylinders : $H_2(K'')=0$, $H_1(K'')\simeq \mathbb{Z}$, $H_0(K'')\simeq\mathbb{Z}$.

A : two disjoint circles: $H_1(A)\simeq\mathbb{Z}\oplus\mathbb{Z}$, $H_0(A)\simeq\mathbb{Z}\oplus\mathbb{Z}$.

$$0 \longrightarrow H_2(T) \xrightarrow{\partial_*} \overset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A)} \xrightarrow{i_*} \overset{\mathbb{Z}}{H_1(K')} \oplus \overset{\mathbb{Z}}{H_1(K'')} \xrightarrow{j_*} H_1(T) \xrightarrow{\partial_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_0(A)} \xrightarrow{i_*} \underset{\mathbb{Z}}{H_0(K')} \oplus \underset{\mathbb{Z}}{H_0(K'')} \xrightarrow{j_*} \underset{\mathbb{Z}}{H_0(T)}$$

we assume $H_0(T)\simeq\mathbb{Z}$, as it has one component.

First piece:

$$0 \longrightarrow H_2(T) \xrightarrow{\partial_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A)} \xrightarrow{i_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(K') \oplus H_1(K'')}$$

i_* maps (m,n) to $(m-n, -m+n)$. Notice that $\ker i_* \simeq \mathbb{Z}$ (we get $(0,0)$ when $m=n$). By exactness at $H_1(A)$, we get $\operatorname{im} \partial_* = \ker i_* \simeq \mathbb{Z}$. Also, ∂_* is injective (see Rule 2 from Lecture 17). Hence $H_2(T) \simeq \mathbb{Z}$.

a set of generators, not a basis

Notice that $\operatorname{im} i_* \simeq \mathbb{Z}$ $\left(\mathbb{Z}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}\right\}, \text{ but } \begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix}. \right)$

More directly, $m-n$ and $-m+n$ are not independent of each other.

The inclusion homomorphism i_* at level 0 has identical structure to the i_* at level 1. i_* again maps (m,n) to $(m-n, -m+n)$. Consider two points, one each in the 2 circles in A, with multipliers m, n, respectively, and how i_* maps them to K' and K''.

two points, one on either circular boundary of the cylinder, are homologous due to a 1-chain connecting them (on the wall of the cylinder).

Second piece: To identify $H_1(T)$, we consider five groups in the sequence with $H_1(T)$ in the middle.

$$H_1(A) \xrightarrow{i_*} H_1(K') \oplus H_1(K'') \xrightarrow{j_*} H_1(T) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(K') \oplus H_0(K'')$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_*} H_1(T) \xrightarrow{\partial_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z}$$

Use Result 5 on exact sequences (Lee 17):

5. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \to A_3 \to A_4 \xrightarrow{\beta} A_5$ is exact. Then so is the induced sequence $0 \to \operatorname{cok} \alpha \to A_3 \to \ker \beta \to 0$.

So,

$$0 \longrightarrow \operatorname{cok} i_* \longrightarrow H_1(T) \longrightarrow \ker i_* \to 0 \text{ is exact.}$$

$$\operatorname{im} i_* \simeq \mathbb{Z}, \text{ so } \operatorname{cok} i_* \simeq \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} = \mathbb{Z}.$$

$$\Rightarrow \quad 0 \to \mathbb{Z} \to H_1(T) \to \mathbb{Z} \to 0 \text{ is exact.}$$

$$\Rightarrow \quad H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

In the last few lectures, we will give a brief overview of cohomology, which is "dual" to homology. The concepts used to define cohomology are lot more algebraic in nature. We start by introducing the machinery of categories and functors.

Categories and Functors § 28 in [M]

[M] defines three things; which we list here as 1, 2, 3 as well

Def A **category** $\mathcal{C}$ consists of two things:

1. A class (or a collection) of **objects** ($\mathcal{C}_o$); "oh"
2. for every ordered pair (X, Y) with $X, Y \in \mathcal{C}_o$, a set **hom(X,Y)** of **morphisms** f (or arrows).

   One writes $f: X \to Y$ or $X \xrightarrow{f} Y$ for the morphism $f \in \hom(X,Y)$. Here, $X = \mathrm{dom}(f)$, i.e., its domain, and $Y = \mathrm{cod}(f)$, i.e., its codomain.
   The collection of all morphisms is denoted $\mathcal{C}_m$. this is the second "thing".

3. A function, called the **composition of morphisms** is defined for every triple (X, Y, Z) of objects:

$$\hom(X,Y) \times \hom(Y,Z) \longrightarrow \hom(X,Z).$$

The image of the pair (f, g) under composition is defined as $g \circ f$ (or gf).

The second "thing" $\mathcal{C}_m$ must have the compositions defined — the book calls this the third "thing".

In other words, when we have morphisms f and g with $\mathrm{dom}(f) = \mathrm{cod}(g)$, the composition of f and g is gf with its domain as $\mathrm{dom}(f)$ and codomain as $\mathrm{cod}(g)$.

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (X \xrightarrow{gf} Z)$$

The following two properties must be satisfied by the objects.

4. **Axiom 1** (Associativity) The composition of morphisms is associative: If $f \in \hom(W,X)$, $g \in \hom(X,Y)$, $h \in \hom(Y,Z)$, then
$$h\circ(g\circ f) = (h\circ g)\circ f.$$
Another notation: If $f: W \to X$, $g: X \to Y$, $h: Y \to Z$, then $h(gf) = (hg)f$.

5. **Axiom 2** (Existence of identity) some other books use id_X
For every $X \in \mathcal{C}_0$, there exists a morphism $1_X \in \hom(X,X)$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ $\forall f \in \hom(W,X)$ and $g \in \hom(X,Y)$, where W and Y are arbitrary objects.

Notice that 1_X (identity morphism) is unique. Suppose $1_X \circ f = f$ and $g = g \circ 1'_X$ $\forall f \in \hom(W,X)$ and $\forall g \in \hom(X,Y)$.
(we are assuming there exist two identity morphisms $1_X, 1'_X$).
Then, setting $f = 1'_X$ and $g = 1_X$, we get
$$1_X \circ 1'_X = 1'_X \quad \text{and} \quad 1_X = 1_X \circ 1'_X, \quad \text{i.e.,} \quad 1_X = 1'_X.$$

Examples of categories

1. $\bar{1}$: a category with one object $*$ and one morphism 1_*.
2. Top : The category of topological spaces and continuous maps.
3. Grp : The category of groups and group homomorphisms.

MATH 524 : Lecture 24 (11/06/2025)

Today: more on categories and functors

Recall: Category: $\mathcal{C}_o$, $\mathcal{C}_m$, Axiom 1: $h(gf)=(hg)f$; Axiom 2: $1_X \circ f = f$, $f \circ 1_X = f$.

Examples of categories

1. $\bar{1}$: a category with one object $*$ and one morphism 1_*.
2. Top : The category of topological spaces and continuous maps.
3. Grp : The category of groups and group homomorphisms.
4. Set : The category of sets and functions between sets.
5. Simplicial complexes and simplicial maps
6. Chain complexes and chain maps.
7. Short exact sequences and homomorphisms between them.

We introduce one more concept on categories, and then introduce functors as maps between categories.

Def (Inverse) Let $f \in \hom(X,Y)$, and $g, g' \in \hom(Y,X)$.

If $g \circ f = 1_X$, we call g a **left inverse** of f. If $f \circ g' = 1_Y$, we call g' a **right inverse** of f

$$X \underset{g,g'}{\overset{f}{\rightleftarrows}} Y$$

If f has a left inverse g and a right inverse g', then they are equal. Since $(g\circ f)\circ g' = 1_X \circ g' = g'$ and $g \circ (f \circ g') = g \circ 1_Y = g$, and hence by Axiom 1, $g = g'$. This map $g=g'$ is called an **inverse** to f, and it is unique.

If f has an inverse, it is called an **equivalence**.

In example 5. Simplicial complexes and simplicial maps, the equivalences are simplicial homeomorphisms.

Def A (covariant) **functor** F from category $\mathbb{C}$ to category $\mathbb{D}$ is a function assigning to each object X of $\mathbb{C}$ an object $F(X)$ of $\mathbb{D}$, and to each morphism $f: X \to Y$ of $\mathbb{C}$ the morphism $F(f): F(X) \longrightarrow F(Y)$ of $\mathbb{D}$ such that

$$F(1_X) = 1_{F(X)} \quad \forall\, X \in \mathbb{C}_0 \quad \text{and}$$

$$F(g \circ f) = F(g) \circ F(f) \quad \forall\, f, g \in \mathbb{C}_m.$$

Here is a pictorial representation:

$$\left\{\begin{array}{ccc} \mathbb{C} & & X \xrightarrow{f} Y \xrightarrow{g} Z \\ F\big\downarrow & & \\ \mathbb{D} & & F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \end{array}\right.$$

Informally, a functor preserves composition and identities. Also, if f is an equivalence in $\mathbb{C}$, then $F(f)$ is an equivalence in $\mathbb{D}$.

Examples

1. There is a functor $\mathcal{U}: \text{Top} \to \text{Set}$ which assigns to any topological space X its underlying set (i.e., $|X|$ as a set), and to continuous maps their underlying set maps. This functor is called the **forgetful functor**, as it "forgets" the topological structure involved.

Indeed, the functor $\mathcal{U}$ here does not preserve all the "structure"!

2. The correspondence $K \to \mathcal{C}(K)$ and $f \to f_{\#}$ is a functor from the category of simplicial complexes and simplicial maps to the category of chain complexes and chain maps.

3. The zig-zag lemma assigns to each short exact sequence of chain complexes a long exact sequence of homology groups.

The functors we have seen so far "preserve" the arrows. We can also define functors that "reverse" each arrow.

Def A **contravariant functor** G from category $\mathbb{C}$ to category $\mathbb{D}$ is a rule that assigns to each object X of $\mathbb{C}$ an object $G(X)$ of $\mathbb{D}$, and to each morphism $f: X \to Y$ of $\mathbb{C}$ a morphism $G(f): G(Y) \longrightarrow G(X)$ of $\mathbb{D}$, such that

$$G(1_X) = 1_{G(X)} \quad \text{and}$$

$$G(g \circ f) = G(f) \circ G(g).$$

notice that the arrow is reversed here!

We can define the "opposite" category $\mathbb{C}^{op}$ for a given category $\mathbb{C}$ by setting $\mathbb{C}^{op}_o = \mathbb{C}_o$ but the morphisms are reversed. If $f : X \to Y \in \mathbb{C}_m$, then $f^{op} : Y \to X \in \mathbb{C}^{op}_m$.

Composition is defined as: $f^{op} \circ g^{op} = (g \circ f)^{op}$.

Then, a contravariant functor G from $\mathbb{C}$ to $\mathbb{D}$ is a (covariant) functor from $\mathbb{C}^{op}$ to $\mathbb{D}$, or equivalently from $\mathbb{C}$ to $\mathbb{D}^{op}$!

Notice that f^{op} need not be the "inverse" of f, at least not in general. All we require is that the direction of the morphism is reversed. In specific examples, f^{op} could be equal to f^{-1}, though.

More examples of functors

4. Given two categories $\mathbb{C}, \mathbb{D}$, we can define a product category $\mathbb{C} \times \mathbb{D}$, which has as objects the pairs $(C, D) \in \mathbb{C}_o \times \mathbb{D}_o$, and as morphisms from (C, D) to (C', D') the pairs (f, g) with $f : C \to C' \in \mathbb{C}_m$, and $g : D \to D' \in \mathbb{D}_m$, denoted $f \times g$. Then there are the **projection functors** $\pi_{\mathbb{C}} : \mathbb{C} \times \mathbb{D} \to \mathbb{C}$ and $\pi_{\mathbb{D}} : \mathbb{C} \times \mathbb{D} \to \mathbb{D}$.

5. Given functors $F: \mathbb{C} \to \mathbb{D}$ and $G: \mathbb{D} \to \mathbb{E}$, we can define the composition functor $GF: \mathbb{C} \to \mathbb{E}$.
This composition operation is associative (use F, G, H).
There is also the identity functor of every category: $1: \mathbb{C} \to \mathbb{C}$. So we get **Cat**, the category which has a objects categories, and as morphisms the functors.

We introduce one more concept on how to combine multiple functors to define more general objects.

Def Let G, H be two functors from $\mathbb{C}$ to $\mathbb{D}$. A **natural transformation** T from G to H is a rule assigning to each object X of $\mathbb{C}$ a morphism

$$T_X : G(X) \longrightarrow H(X) \text{ of } \mathbb{D}$$

such that the following diagram commutes for every morphism $f: X \to Y$ of $\mathbb{C}$.

$$\begin{tikzcd} G(X) \arrow[r, "T_X"] \arrow[d, "G(f)"] & H(X) \arrow[d, "H(f)"] \\ G(Y) \arrow[r, "T_Y"] & H(Y) \end{tikzcd}$$

If for each X, the morphism T_X is an equivalence in $\mathbb{D}$, then T is called a **natural equivalence** of the functors.

MATH 524: Lecture 25 (11/13/2025)

Today: * Hom functor
* cohomology groups of simplicial complexes

Cohomology

The Hom functor §41 in [M]

Def Let A, G be abelian groups. Then the set $\text{Hom}(A,G)$ of all homomorphisms from A to G becomes an abelian group if we add two homomorphisms by adding their values in G.
$\phi, \psi : A \to G$ are homomorphisms

For $a \in A$, we define $(\phi+\psi)(a) = \phi(a)+\psi(a)$.
The map $\phi+\psi$ is a homomorphism, as
$(\phi+\psi)(0)=0$ and

$$\begin{aligned}(\phi+\psi)(a+b) &= \phi(a+b) + \psi(a+b)\\ &= \phi(a)+\psi(a) + \phi(b)+\psi(b)\\ &= (\phi+\psi)(a) + (\phi+\psi)(b).\end{aligned}$$

The identity element of $\text{Hom}(A,G)$ is the homomorphism mapping A to id_G (or 1_G), the identity element of G.
The inverse of homomorphism $\phi: A \to G$ is the homomorphism that maps a to $-\phi(a)$ $\forall a \in A$.

Example $\text{Hom}(\mathbb{Z}, G)$ is isomorphic to G itself. The isomorphism assigns to the homomorphism $\phi: \mathbb{Z} \to G$ the element $\phi(1)$.
Notice that any homomorphism $\phi: \mathbb{Z} \to G$ is completely determined by $\phi(1)$.

More generally, if A is free-abelian with finite rank and basis $e_1, \dots, e_n$, then $\text{Hom}(A,G)$ is isomorphic to $\underbrace{G \oplus \cdots \oplus G}_{n \text{ copies}}$.

This isomorphism assigns to any homomorphism $\phi : A \to G$ the n-tuple $(\phi(e_1), \dots, \phi(e_n))$.

As the name cohomology suggests, we want define objects that are dual to homology. Indeed, we define homomorphisms from $\mathrm{Hom}(B,G)$ to $\mathrm{Hom}(A,G)$ for given homomorphisms from $A \to B$.

Def A homomorphism $f : A \to B$ gives rise to a **dual homomorphism**

$\mathrm{Hom}(A,G) \xleftarrow{\tilde{f}} \mathrm{Hom}(B,G)$ going in the reverse direction.

The map $\tilde{f}$ assigns to the homomorphism $\phi : B \to G$, the composite $A \xrightarrow{f} B \xrightarrow{\phi} G$. That is, $\tilde{f}(\phi) = \phi \circ f$.

$\tilde{f}$ is indeed a homomorphism, as $\tilde{f}(0) = 0$, and

$$[\tilde{f}(\phi+\psi)](a) = (\phi+\psi)(f(a)) = \phi(f(a)) + \psi(f(a))$$
$$= [\tilde{f}(\phi)](a) + [\tilde{f}(\psi)](a).$$

For a fixed G, the assignment $A \to \mathrm{Hom}(A,G)$ and $f \to \tilde{f}$ defines a contravariant functor from the category of abelian groups and homomorphisms to itself.

Recall: The opposite category: $\mathbb{C}^{op}$.

Given category $\mathbb{C}$, we consider another category $\mathbb{C}^{op}$ with $\mathbb{C}^{op}_0 = \mathbb{C}_0$ (same objects), but with morphisms reversed:

so, if $f: X \to Y \in \mathbb{C}_m$, then $f^{op}: Y \to X \in \mathbb{C}^{op}_m$.

Composition: $f^{op} g^{op} = (gf)^{op}$.

Then, a contravariant functor G from $\mathbb{C}$ to $\mathbb{D}$ is a (covariant) functor from $\mathbb{C}^{op}$ to $\mathbb{D}$, or equivalently from $\mathbb{C}$ to $\mathbb{D}^{op}$.

For, if $i_A : A \to A$ is the identity homomorphism, then $\tilde{i}_A$ is the identity map of $\mathrm{Hom}(A, G)$.

$\tilde{i}_A(\phi) = \phi \circ i_A = \phi$. Hence $\tilde{i}_A$ is the identity map of $\mathrm{Hom}(A,G)$.

Also, if the left diagram commutes, so does the right one.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(A,G) & & \mathrm{Hom}(C,G) \arrow[ll, "\tilde{h}"'] \arrow[dl, "\tilde{g}"] \\ & \mathrm{Hom}(B,G) \arrow[ul, "\tilde{f}"] & \end{tikzcd}$$

For, $\tilde{h}(\phi) = \phi \circ h = \phi \circ (g \circ f)$, as left diagram commutes.

and $\tilde{f}(\tilde{g}(\phi)) = \tilde{f}(\phi \circ g) = (\phi \circ g) \circ f$, which are equal.

We state a few implications of this correspondence. There are many more results listed in the book. We will then use $\mathrm{Hom}(C_p(K), G)$ to define cohomology groups.

Theorem 41.1 [M] Let f be a homomorphism, and $\tilde{f}$ its dual homomorphism.

(a) If f is an isomorphism, so is $\tilde{f}$.

(b) If f is the zero homomorphism, so is $\tilde{f}$.

(c) If f is surjective, then $\tilde{f}$ is injective. So the exactness of $B \xrightarrow{f} C \longrightarrow 0$ implies the exactness of

$$\mathrm{Hom}(B, G) \xleftarrow{\tilde{f}} \mathrm{Hom}(C, G) \longleftarrow 0.$$

__Proof__ (c) f is surjective. Let $\psi \in \mathrm{Hom}(C, G)$ and suppose $\tilde{f}(\psi) = 0 = \psi \circ f$. So $\psi(f(b)) = 0 \ \forall \ b \in B$. Since f is surjective, we get that $\psi(c) = 0 \ \forall \ c \in C$.

Simplicial Cohomology Groups

Def Let K be a simplicial complex, G be an abelian group. The **group of p-dimensional cochains** of K with coefficients in G is the group $C^p(K;G) = \mathrm{Hom}(C_p(K), G)$. The **coboundary operator** δ^p is defined as the dual of the boundary operator $\partial_{p+1} : C_{p+1}(K) \longrightarrow C_p(K)$. Thus

$$C^{p+1}(K;G) \xleftarrow{\delta^p} C^p(K;G).$$

So δ raises dimension by 1. We define $Z^p(K;G) = \ker \delta^p$ and $B^{p+1}(K;G) = \mathrm{im}\, \delta^p$, the groups of p-cocycles and (p+1)-coboundaries with coefficients in G.
We take $G = \mathbb{Z}$ as the default choices.

If $\bar{c}_p$ is a p-chain, $\bar{c}_p \in C_p$ and ϕ^p is a p-cochain, $\phi^p \in C^p$, then the cochain ϕ^p evaluates $\bar{c}_p$ by mapping it to $\mathbb{Z}$. We denote this evaluation by $\phi^p(\bar{c}_p) = \langle \phi^p, \bar{c}_p \rangle$. ← this notation is preferred

We get $\langle \delta\phi^p, \bar{d}_{p+1} \rangle = \langle \phi^p, \partial \bar{d}_{p+1} \rangle$, or more generally,

$$\langle \delta\phi, \bar{c} \rangle = \langle \phi, \partial \bar{c} \rangle.$$

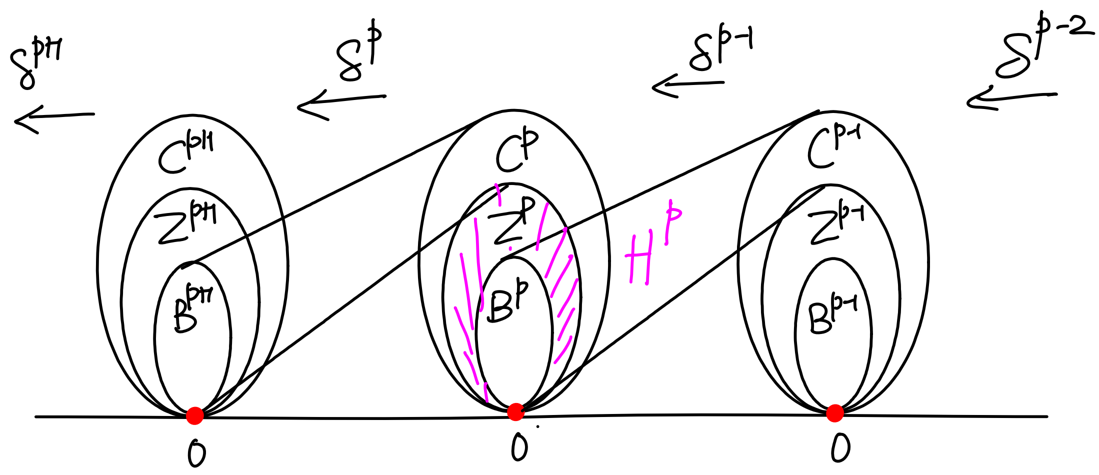
Some intuition: If ϕ evaluates a single edge to 1, and all other edges to 0, then $\delta\phi$ evaluates all triangles that are cofaces of this edge to 1, and all other triangles to 0.

We immediately get that $\delta\delta = 0$, since

$$\langle \delta\delta\phi, \bar{c}\rangle = \langle \delta\phi, \partial\bar{c}\rangle = \langle \phi, \underbrace{\partial\partial\bar{c}}_{=0}\rangle = 0.$$

Similar to $H_p = Z_p/B_p$ in homology, we can define $H^p(K;G) = Z^p(K;G)/B^p(K;G)$, the p-dimensional **cohomology group** of K with coefficients in G.

We get a complementary picture here to that of how $\{C_p, Z_p, B_p, H_p\}$ line up using $\{\partial_p\}$.



Recap:

$$C^p(K;G) = \mathrm{Hom}(C_p(K), G)$$

$$\phi^p(\bar{c}_p) = \langle \phi^p, \bar{c}_p \rangle$$

$$\langle \delta\phi, \bar{c} \rangle = \langle \phi, \partial \bar{c} \rangle, \quad \delta\delta = 0.$$

Elementary cochains

We let σ_α^* be the elementary co-chain (with $G = \mathbb{Z}$) whose value is 1 on basis element σ_α, and 0 on all other basis elements.

If $g \in G$, we let $g_\alpha \sigma_\alpha^*$ denote the cochain whose value is g_α on σ_α, and 0 on all other basis elements. We can write any p-cochain as $\phi^p = \sum g_\alpha \sigma_\alpha^*$ (possibly infinite formal sum).

With this notation, we can write down the coboundary of ϕ^p as

$$\delta\phi^p = \sum g_\alpha (\delta \sigma_\alpha^*). \qquad (*)$$

MATH 524: Lecture 26 (11/18/2025)

Today: * elementary cochains
* computing coboundaries, cohomology

Recall: Elementary cochain: σ_α^* : 1 on σ_α, 0 o.w.

p-cochain $\phi^p = \sum g_\alpha \sigma_\alpha^*$

$$\delta\phi^p = \sum g_\alpha(\delta\sigma_\alpha^*) \qquad \text{———} \quad (*)$$

Let's verify $(*)$: Let τ be a $(p+1)$-simplex, and suppose $\partial\tau = \sum_{i=0}^{p+1} \epsilon_i \sigma_{\alpha_i}$, $\epsilon_i = \pm 1 \ \forall i$.

Then $\langle \delta\phi^p, \tau\rangle = \langle \phi^p, \partial\tau\rangle = \sum_{i=0}^{p+1} \epsilon_i \langle \phi^p, \sigma_{\alpha_i}\rangle$

$= \sum_{i=0}^{p+1} \epsilon_i g_{\alpha_i}$, where g_{α_i} = value of ϕ^p on σ_{α_i}.

Also, $\langle g_\alpha(\delta\sigma_\alpha^*), \tau\rangle = g_\alpha \langle \delta\sigma_\alpha^*, \tau\rangle = g_\alpha \langle \sigma_\alpha^*, \partial\tau\rangle$

$$= \begin{cases} \epsilon_i g_{\alpha_i} & \text{if } \alpha = \alpha_i,\ i = 0, \dots, p+1; \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

So, $(*)$ does hold.

By $(*)$, to compute $\delta\phi^p$, it suffices to compute $\delta\sigma^*$ for each oriented p-simplex σ. But

$$\delta\sigma^* = \sum \epsilon_j \tau_j^*$$

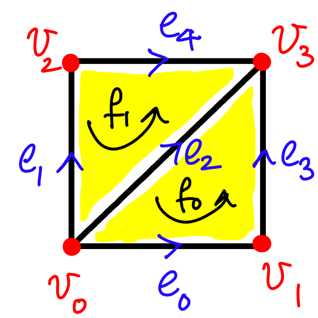
where the sum extends over all $(p+1)$-simplices τ_j that are cofaces of σ, i.e., $\tau_j > \sigma$ (or, τ_j has σ as a face), and $\epsilon_j = \pm 1$ is the sign with which σ appears in the expression for $\partial\tau_j$.

So, we can compute cohomology using elementary cochains. We now explore several examples.

Examples

1. Vertices $\{v_i\}$
   edges $\{e_i\}$
   faces $\{f_i\}$

K



Let's evaluate some cochains, and their coboundaries.

$\delta e_2^* = f_1^* - f_0^*$ → notice $\bar{e}_2$ has $+1$ in $\partial \bar{f}_1$ and -1 in $\partial \bar{f}_0$

$\delta v_3^* = e_2^* + e_3^* + e_4^*$.

Cocycles and coboundaries

Both f_0^* and f_1^* are trivial 2-cocycles (as K has no 3-simplices, so $\delta f_0^* = \delta f_1^* = 0$).

Also, both f_0^* and f_1^* are coboundaries, since

$$\delta e_0^* = f_0^* \quad \text{and} \quad \delta e_1^* = -f_1^*.$$

Also, $\delta e_3^* = f_0^*$ and $\delta e_4^* = -f_1^*$.

The 1-cochain $\phi^1 = e_0^* + e_2^* + e_4^*$ is a 1-cocycle, as

$$\delta \phi^1 = f_0^* + (f_1^* - f_0^*) + -f_1^* = 0.$$

It is also a 1-coboundary, as $\delta(v_1^* + v_3^*) = \phi^1$.

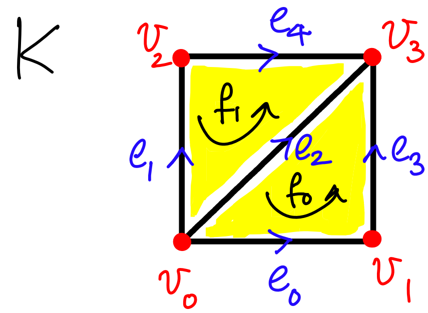
Here are all the 0-coboundaries:

$$\delta v_0^* = -e_0^* - e_1^* - e_2^*$$

$$\delta v_1^* = e_0^* - e_3^*$$

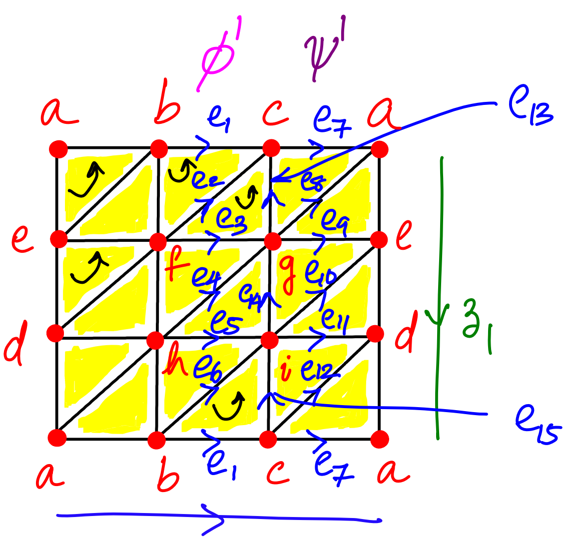
$$\delta v_2^* = e_1^* - e_4^*$$

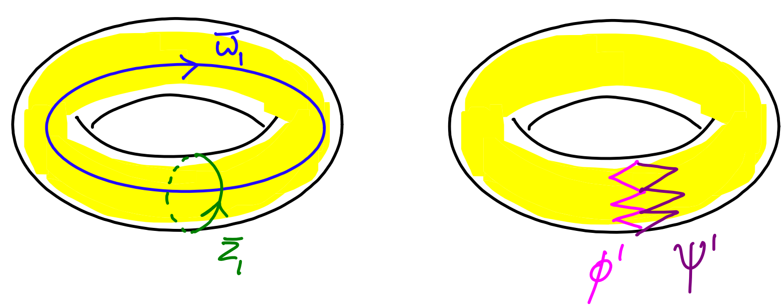
$$\delta v_3^* = e_2^* + e_3^* + e_4^*$$



Hence the 0-cochain $\phi^0 = v_0^* + v_1^* + v_2^* + v_3^*$ is a 0-cocycle (as $\delta\phi^0 = 0$). It cannot be a coboundary, as there are no cochains of dimension -1.

2. Torus





Consider the 1-cochain $\phi^1 = e_1^* + \dots + e_6^*$. It is a 1-cocycle! Each triangle in the middle patch appears with a $+1$ and -1 in the expressions for δe_i^*.

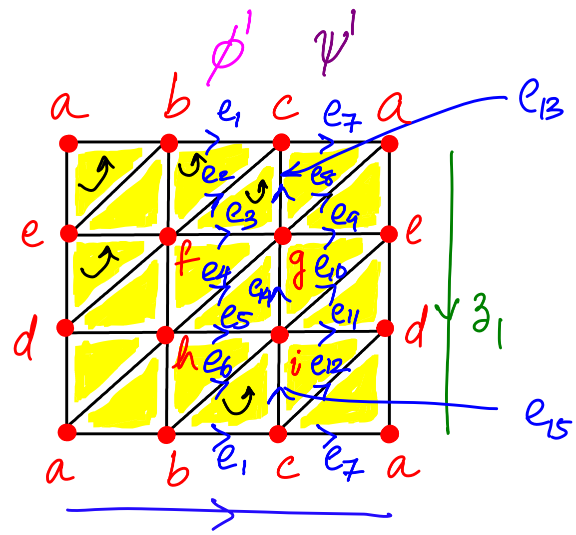
Similarly, $\psi' = e_7^* + \cdots + e_{12}^*$ is also a 1-cocycle, as $\delta\psi' = 0$.

ϕ' and ψ' are **cohomologous**, as $\phi' - \psi' = \delta^0(c^* + g^* + i^*)$.

$$\delta i^* = e_5^* + e_6^* + \cancel{e_{15}^*} - e_{10}^* - e_{11}^* - \cancel{e_{14}^*}$$

$$\delta g^* = e_3^* + e_4^* + \cancel{e_{14}^*} - e_8^* - e_9^* - \cancel{e_{13}^*}$$

$$\delta c^* = e_1^* + e_2^* + \cancel{e_{13}^*} - e_7^* - e_{12}^* - \cancel{e_{15}^*}$$



$$\delta(c^* + g^* + i^*) = \phi' - \psi'.$$

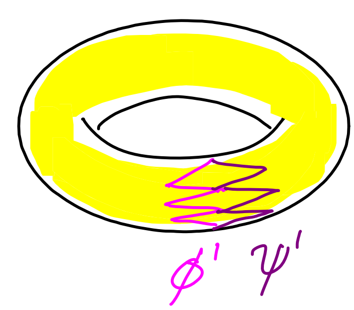
Two cocycles are cohomologous if their difference is the coboundary of a one-dim lower cochain.

We write $\psi' \simeq_c \phi'$ here.

Recall, 2-chains $\bar{c}, \bar{c}'$ are homologous, $\bar{c} \sim \bar{c}'$, if $\bar{c} - \bar{c}' = \partial \bar{d}$.

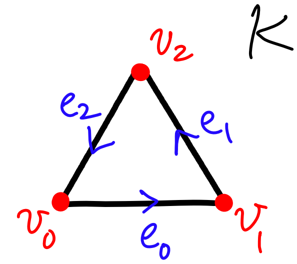
$\simeq_c$: "cohomologous"

We can visualize the cocycles as "picket fences". Two 1-cocycles are cohomologous if the picket fences are attached at the "right" vertices "along the middle".



Example 3

Let's compute cohomology groups of K.



Note that $H_0(K) \simeq \mathbb{Z}$ (one component), and
$H_1(K) \simeq \mathbb{Z}$ (one hole).

The general 0-cochain is $\phi^0 = n_0 v_0^* + n_1 v_1^* + n_2 v_2^*$.
We have $\delta v_0^* = e_2^* - e_0^*$, $\delta v_1^* = e_0^* - e_1^*$, and $\delta v_2^* = e_1^* - e_2^*$.

$$\Rightarrow \delta \phi^0 = \sum_{i=0}^{2} n_i (\delta v_i^*) = (n_1 - n_0) e_0^* + (n_2 - n_1) e_1^* + (n_0 - n_2) e_2^*.$$

Hence ϕ^0 is a 0-cocycle if $\delta \phi^0 = 0$, i.e., when $n_0 = n_1 = n_2 = n$ (say).

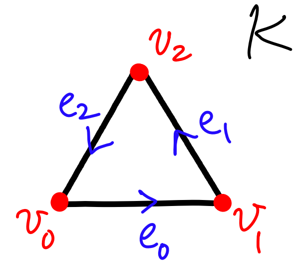
Then $\phi^0 = n \left(\sum_{i=0}^{2} v_i^* \right)$. It is trivially not a coboundary as there are no (-1)-dim. cochains.

$$\Rightarrow H^0(K) \simeq \mathbb{Z}, \text{ and is generated by } \left\{ \sum_{i=0}^{2} v_i^* \right\}.$$

Notice the correspondence of the argument used here to the one used to find the structure of $H_1(K)$ — they're are essentially identical!

Consider the 1-cochain $\psi^1 = \sum_{i=0}^{2} m_i e_i^*$. It is a cocycle (trivially), as there are no 2-cochains. We show that $\psi^1 \underset{c}{\sim}$ some multiple of e_0^*.

We show $e_1^* \underset{c}{\sim} e_0^*$ and $e_2^* \underset{c}{\sim} e_0^*$.



But we get these results from
$\delta v_0^* = e_2^* - e_0^*$ and $\delta v_1^* = e_0^* - e_1^*$.

$\Rightarrow \psi^1 \underset{c}{\sim} m e_0^*$ for some $m \in \mathbb{Z}$, $m \neq 0$.

$m e_0^*$ is not a coboundary unless $m=0$.

Suppose $m e_0^* = \delta\left(\sum_{i=0}^{2} n_i' v_i^*\right) = \sum_{i=0}^{2} n_i' (\delta v_i^*)$

$$= \underset{=0.}{(n_1'-n_0')} e_0^* + \underbrace{(n_2'-n_1')}_{=0} e_1^* + \underbrace{(n_0'-n_2')}_{=0} e_2^* \rightarrow \text{needed.}$$

$\Rightarrow n_0' = n_1' = n_2'$. $\Rightarrow m=0$ if $m e_i^*$ is a coboundary.

Hence we conclude that $H^1(K) \simeq \mathbb{Z}$, and is generated by $\{e_0^*\}$, or by $\{e_1^*\}$ or $\{e_2^*\}$.

Here $H^i(K) \simeq H_i(K)$ $\forall i$ (they are both trivial for $i \geq 2$).

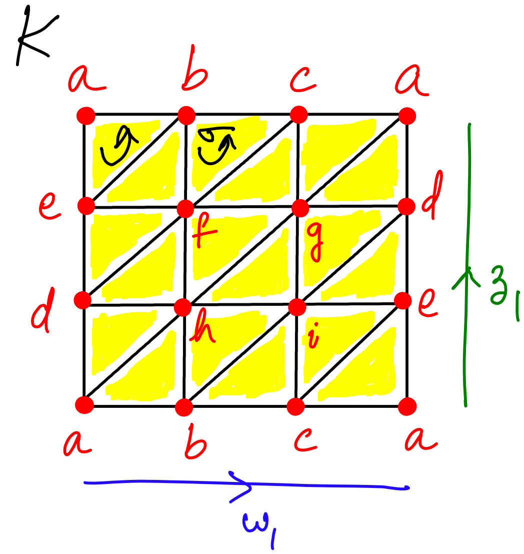
But in general, $H^i(K) \not\simeq H_i(K)$.

Here, $H^1(K) \simeq H_0(K)$ and $H^0(K) \simeq H_1(K)$, actually.

MATH 524: Lecture 27 (11/20/2025)

Today: * Cohomology of $\mathbb{K}^2$
* 0-dimensional cohomology

Example 4 (Klein bottle)



We show that $H^2(K)$ is nontrivial.

Recall, $H_2(K)=0$.

Orient all triangles CCW. Let $\bar{\tau} = \sum_i f_i$ (all elementary 2-chains).

Then, $\bar{\tau}$ is not a 2-cycle.

$$\partial\bar{\tau} = 2\bar{z}_1, \quad \text{where} \quad \bar{z}_1 = [a,e] + [e,d] + [d,a].$$

Let σ be a 2-simplex, [bfc] here. Then σ^* is a 2-cocycle (as there are no 3-simplices). Also, σ^* is not a 2-coboundary.

For, if ϕ^1 is an arbitrary 1-cochain, then

$$\langle \delta\phi^1, \bar{\tau}\rangle = \langle \phi^1, \partial\bar{\tau}\rangle = \langle \phi^1, 2\bar{z}_1\rangle = 2\underbrace{\langle \phi^1, \bar{z}_1\rangle}_{\text{even integer}}.$$

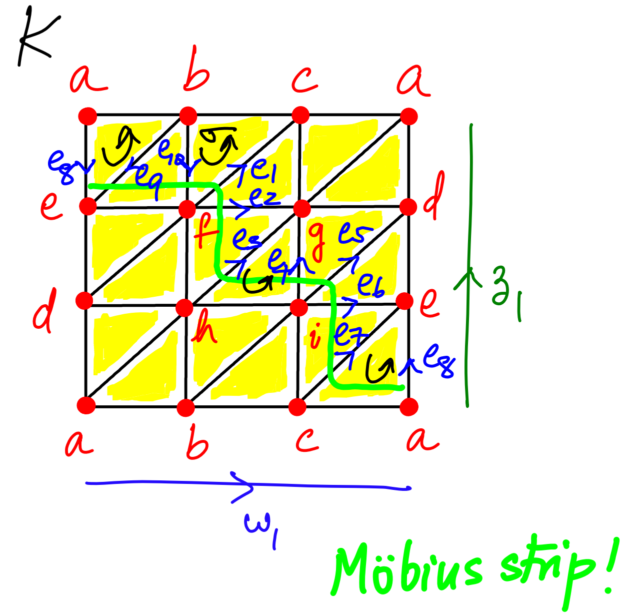
But $\langle \sigma^*, \bar{\tau}\rangle = 1$, which is odd.

$\Rightarrow \sigma^*$ represents a nontrivial member of $H^2(K)$.

In fact, σ^* represents an element of order 2 in $H^2(K)$.

Indeed, for $\psi^1 = \sum_{i=1}^{10} e_i^*$ as shown in the figure,

$$\delta\psi^1 = 2\sigma^*.$$



The CCW orientation of σ agrees with that of both $\bar{e}_1$ and e_{10}. But all other triangles in the "band" appear twice in the expression for $\delta\psi^1$, once with +1 and once with −1 (as part of δe_i^* and δe_{i+1}^* ...).

Thus, for the Klein bottle, $H^2(K;\mathbb{Z}) \not\cong H_2(K)$, which is yet another example where homology and cohomology groups differ in their structure.

In fact, we can show that $H^2(\mathbb{K}^2;\mathbb{Z}) \simeq \mathbb{Z}/_2$.

Zero-dimensional Cohomology

Theorem 42.1 [M] $H^0(K;G)$ is the group of all 0-cochains ϕ^0 such that $\langle \phi^0, v\rangle = \langle \phi^0, w\rangle$ whenever v, w belong to the same component of $|K|$. In particular, if $|K|$ is connected, then $H^0(K) \simeq \mathbb{Z}$, and is generated by the cochain whose value is 1 on each vertex of K.

Proof $H^0(K;G)$ equals the group of 0-cocycles trivially, as there are no (-1)-dimensional simplices. If v, w are vertices that belong to the same component of $|K|$, there exists a 1-chain $\bar{c}$ of K such that $\partial \bar{c} = v - w$. Then, for any 0-cocycle ϕ^0, we have

$$0 = \langle \delta\phi^0, \bar{c}\rangle = \langle \phi^0, \partial\bar{c}\rangle = \langle \phi^0, v\rangle - \langle \phi^0, w\rangle.$$

Conversely, let ϕ^0 be a 0-cochain such that $\langle \phi^0, v\rangle - \langle \phi^0, w\rangle = 0$ whenever v, w lie in the same component of $|K|$. Then for each oriented 1-simplex σ of K,

$$\langle \delta\phi^0, \sigma\rangle = \langle \phi^0, \partial\sigma\rangle = 0.$$

So we conclude that $\delta\phi^0 = 0$. □

In general, $H^0(K) \simeq$ direct product of infinite cyclic groups, one for each component of $|K|$. On the other hand, $H_0(K) \simeq$ direct sum of this collection of groups.