MATH 524: Lecture 21 (10/28/2025)

Today: * Meyer-Vietoris Sequence

Recall: Theorem 25.1: $K', K'' \subseteq K$ with $A = K' \cap K''$ and $K' \cup K'' = K$

Meyer-Vietoris sequence (MVS):

$$\cdots \to H_p(A) \to H_p(K') \oplus H_p(K'') \to H_p(K) \xrightarrow{\partial} H_{p-1}(A) \to \cdots$$

Proof idea: We construct short exact sequences of chain complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K') \oplus \mathcal{C}(K'') \xrightarrow{\psi} \mathcal{C}(K) \longrightarrow 0$$

and apply the zig-zag lemma.

We first define the chain complex in the middle. Its chain group in dimension p in $C_p(K') \oplus C_p(K'')$, and its boundary operator is ∂ is defined by

overload of notation

$$\partial(\bar{c}', \bar{c}'') = (\partial' \bar{c}', \partial'' \bar{c}'')$$

where ∂', ∂'' are the boundary operators in $\mathcal{C}(K'), \mathcal{C}(K'')$, respectively.

Second, we define chain maps ϕ, ψ. Consider inclusion mappings in the following commutative diagram:

$$\begin{tikzcd}
 & K' \arrow[dr, "j'"] & \\
A \arrow[ur, "i'"] \arrow[rr, "k"] \arrow[dr, "i''"'] & & K \\
 & K'' \arrow[ur, "j''"'] &
\end{tikzcd}$$

i', i'': inclusion maps of A into K', K''

j', j'': inclusion maps of K', K'' into K

k: inclusion map of A into K

Define the homomorphisms ϕ and ψ as

$$\phi(\bar{c}) = \left(i'_{\#}(\bar{c}), -i''_{\#}(\bar{c})\right), \quad \text{and}$$

notice the "$-$" here!

$$\psi(\bar{c}', \bar{c}'') = \left(j'_{\#}(\bar{c}') + j''_{\#}(\bar{c}'')\right).$$

We can verify that ϕ and ψ are indeed chain maps. Check for exactness:

ϕ is injective, as both $i'_{\#}$ and $i''_{\#}$ are just inclusions of chains.

Also, ψ is surjective. Given $\bar{c} \in C_p(K)$, let $\bar{c}'$ be its part carried by K', and then $\bar{c} - \bar{c}'$ carried by K'', and we get $\psi(\bar{c}', \bar{c} - \bar{c}') = \bar{c}$ $(= \bar{c}' + \bar{c} - \bar{c}')$.

To confirm exactness at the middle term, note that

$$\psi\phi(\bar{c}) = k_{\#}(\bar{c}) - k_{\#}(\bar{c}) = 0$$

recall the "$-$" in the definition of ϕ!

Conversely, if $\psi(\bar{c}', \bar{c}'') = 0$, then $\bar{c}' = -\bar{c}''$ as chains of K. Since $\bar{c}' \in K'$ and $\bar{c}'' \in K''$, they must be carried by $A = K' \cap K''$ (as $\bar{c}' = -\bar{c}''$). Hence $(\bar{c}', \bar{c}'') = (\bar{c}', -\bar{c}') = \phi(\bar{c}')$ as needed.

The homology for the middle chain complex in dimension p is

$$\frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}} = \frac{\ker \partial'_p \oplus \ker \partial''_p}{\operatorname{im} \partial'_{p+1} \oplus \operatorname{im} \partial''_{p+1}} \simeq H_p(K') \oplus H_p(K'').$$

The Meyer-Vietoris (MV) sequence now follows from the zig-zag lemma. A similar argument can be used to get the Meyer-Vietoris sequence in reduced homology groups (when $A \neq \emptyset$).

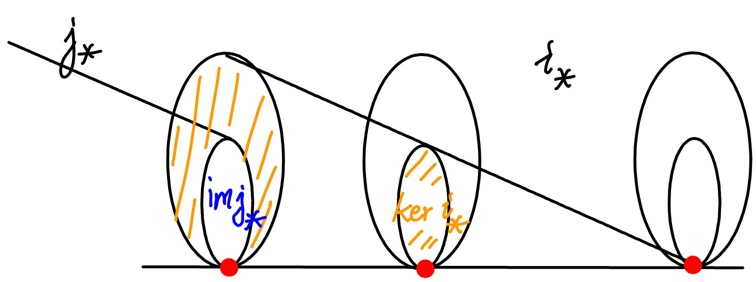
More details on structure of $H_p(K)$ $\forall p$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(A) \arrow[r, "i_*"] & H_p(K') \oplus H_p(K'') \arrow[r, "j_*"] & H_p(K) \arrow[dlll, "\partial_*"'] \\
& H_{p-1}(A) \arrow[r, "i_*"] & H_{p-1}(K') \oplus H_{p-1}(K'') \arrow[r, "j_*"] & H_{p-1}(K) \arrow[r] & {}
\end{tikzcd}$$

We write the part of the sequence for each dimension in one level, or "floor". We will come back to this representation later...

Consider the connecting maps now.

$$\longrightarrow H_p(K') \oplus H_p(K'') \xrightarrow{j_*} H_p(K) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(K') \oplus H_{p-1}(K'') \longrightarrow \cdots$$



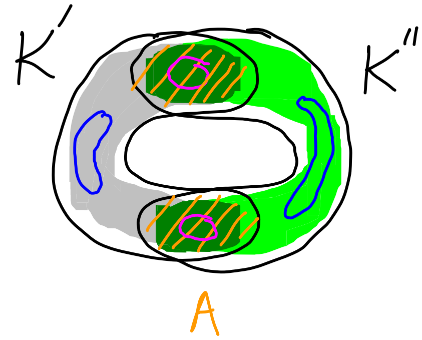
Exactness of the Meyer-Vietoris sequence at $H_p(K)$ tells us that this group is a direct sum of the image of

$j_*: H_p(K') \oplus H_p(K'') \longrightarrow H_p(K)$ with the kernel of

$i_*: H_{p-1}(A) \longrightarrow H_{p-1}(K') \oplus H_{p-1}(K'')$.

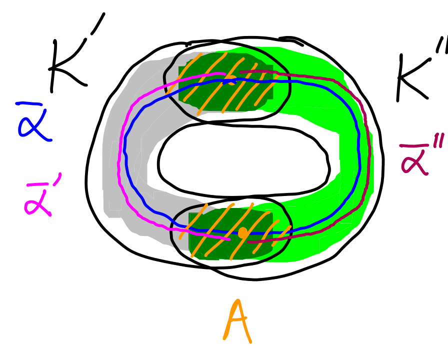
We use exactness at $H_{p-1}(A)$ here.

Hence we can distinguish two types of homology classes in K — one class in $\operatorname{im} j_*$ that lives in K' or K'', and the other one lives in both, e.g., as illustrated here.



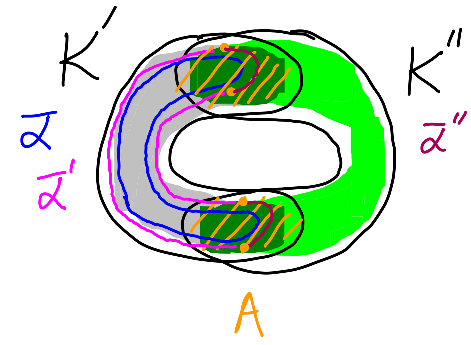
A class in $\ker i_* \equiv (p-1)$-cycle $\bar{\gamma}_{p-1} \in A$ that bounds both in K' and K''. If we write $\bar{\gamma}_{p-1} = \partial \bar{\alpha}_p' = -\partial \bar{\alpha}_p''$ where $\bar{\alpha}_p' \in C_p(K')$ and $\bar{\alpha}_p'' \in C_p(K'')$, then $\bar{\alpha}_p = \bar{\alpha}_p' + \bar{\alpha}_p''$ is a cycle in K which represents the second type of the class.

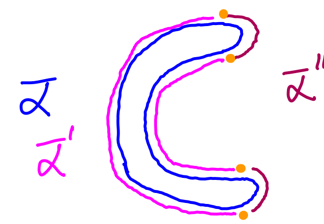
Here is another example. The 1-cycle $\bar{\alpha}$ decomposes into $\bar{\alpha}'$ and $\bar{\alpha}''$. Their boundaries (∂' and ∂'' in K' and K'', respectively) is the 0-chain made of 2 points (with signs reversed) which is a reduced 0-cycle in A.



↳ between K' and K''

What about this 1-cycle $\bar{\alpha}$? This cycle also represents a homology class of the <u>second</u> type, with one possible decomposition of $\bar{\alpha}$ into $\bar{\alpha}'$ and $\bar{\alpha}''$ illustrated below.





The connecting homo'm ∂_* can be explicitly defined as follows. Consider a cycle $\bar{z} \in K$. We can choose $\bar{c}' \in K'$ and $\bar{c}'' \in K''$ s.t. $\bar{z} = \bar{c}' + \bar{c}''$. $\bar{c}'$ and $\bar{c}''$ need not be cycles themselves, but it must hold that

$\partial\bar{c}' = -\partial\bar{c}''$, as $\partial\bar{z} = \partial(\bar{c}' + \bar{c}'') = 0$.

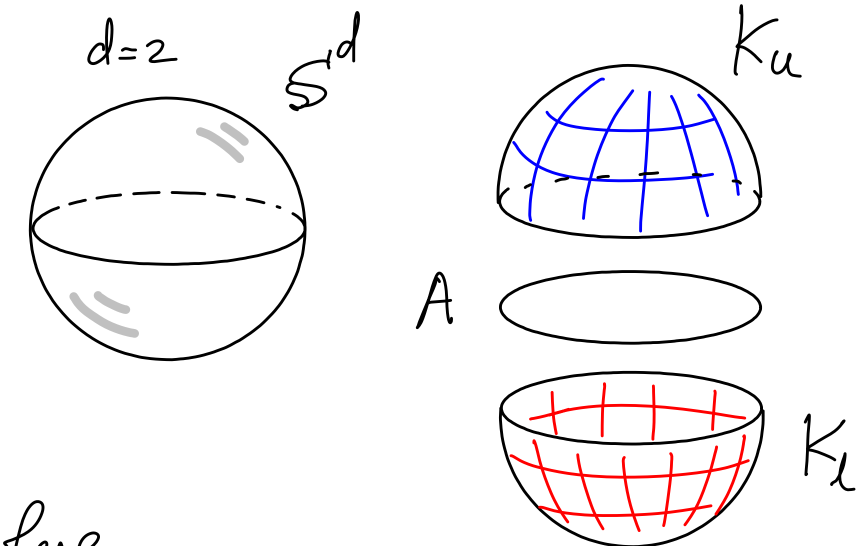
Also, $\partial\bar{c}'$ and $\partial\bar{c}''$ must both be carried by $A = K' \cap K''$.

We define $\partial_*\{\bar{z}\} = \{\partial\bar{c}'\}$, or $\{-\partial\bar{c}''\}$, equivalently.

Example 1 Homology of $\mathbb{S}^d$ (d-sphere): We want to show:

$\tilde{H}_p(\mathbb{S}^d) \simeq \mathbb{Z}$ if $p=d$, and

$\tilde{H}_p(\mathbb{S}^d) = 0$ if $p \neq d$.



We set $\mathbb{S}^d = K_u \cup K_\ell$, where K_u, K_ℓ are the upper and lower hemisphere, respectively.

And $A = K_u \cap K_\ell$ is the equator.

Notice that $K_u, K_\ell \approx \mathbb{B}^d$ (d-disc or d-ball), and $A \approx \mathbb{S}^{d-1}$. Now we compute $\tilde{H}_p(\mathbb{S}^d)$ inductively using the reduced homology MVS.

$$\cdots \ \underset{A}{\tilde{H}_p(\mathbb{S}^{d-1})} \longrightarrow \underset{0}{\tilde{H}_p(K_u)} \oplus \underset{0}{\tilde{H}_p(K_\ell)} \longrightarrow \underset{K}{\tilde{H}_p(\mathbb{S}^d)} \xrightarrow{\partial_*} \tilde{H}_{p-1}(\mathbb{S}^{d-1}) \longrightarrow \cdots$$

For $d=0$, $\mathbb{S}^d$ is the set of 2 points. Hence

$\tilde{H}_0(\mathbb{S}^0) \simeq \mathbb{Z}$, $\tilde{H}_p(\mathbb{S}^0) = 0 \; \forall p \neq 0$. This result gives the start (or base) of the induction.

For general d, the sequence breaks down into pieces of the form

$$0 \oplus 0 \longrightarrow \tilde{H}_p(\mathbb{S}^d) \longrightarrow \tilde{H}_{p-1}(\mathbb{S}^{d-1}) \longrightarrow 0 \oplus 0,$$

as $\tilde{H}_p(K_u) = 0$ and $\tilde{H}_p(K_\ell) = 0 \; \forall p$.

Hence we get an isomorphism $\tilde{H}_p(\mathbb{S}^d) \simeq \tilde{H}_{p-1}(\mathbb{S}^{d-1})$, which along with the inductive step implies that

$\tilde{H}_d(\mathbb{S}^d) \simeq \mathbb{Z}$ and $\tilde{H}_p(\mathbb{S}^d) = 0 \; \forall p \neq d$.

The generator for $\tilde{H}_d(\mathbb{S}^d)$ is of the second type, consisting of the union of two d-chains, one each in K_u and K_ℓ, and their intersection generates $\tilde{H}_{d-1}(\mathbb{S}^{d-1})$.