MATH 529: Lecture 1 (01/13/2026)

Today: * syllabus, logistics
* two motivating applications

Call me Bala!

Introduction to **Computational** topology → focus for this course

This course will be offered completely electronically:
- scribes will be posted as "course notes"; videos will also be posted.
- assignments to be turned in electronically (you could submit scanned versions of handwritten assignments).
- web page has all the docs/info.

Topology "Topo" → place or space } in Greek
"logos" → study or talk

Topology talks about how space is connected.

topology
- point set topology (open/closed, connected, ...)
- algebraic topology (groups, addition, basis, ...)

We will concentrate on algebraic topology.

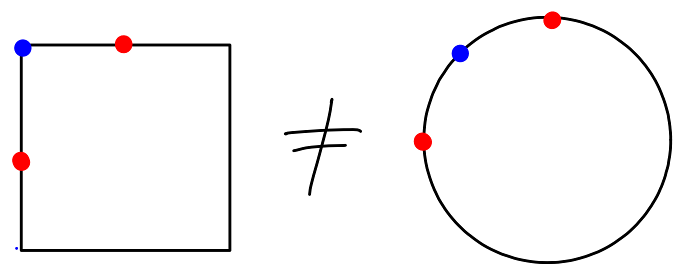
Computational topology

→ combine efficient algorithms and data structures with results from topology to analyze real-life data.

Let's start with an intuition for what we mean by connectivity of spaces.

An Example

According to geometry, the square and circle are not equal.

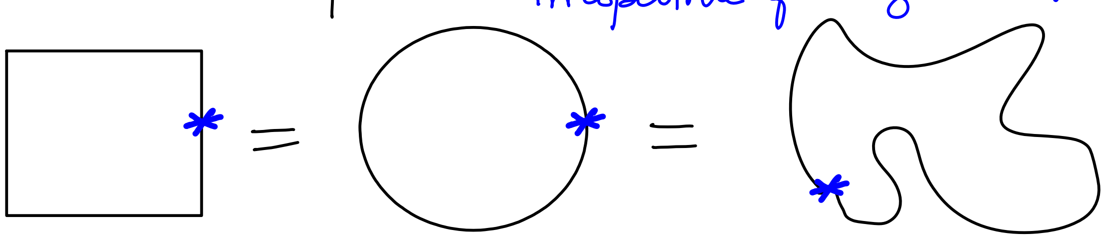


size (length, here) is critical in geometry, but not so much in topology.

But topology says they are same as far as how they are connected!

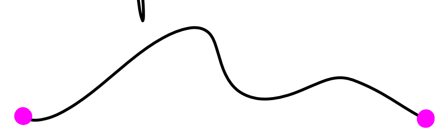
For instance, take a string, and tie a knot to make a loop.

We want to study connectivity irrespective of size here!



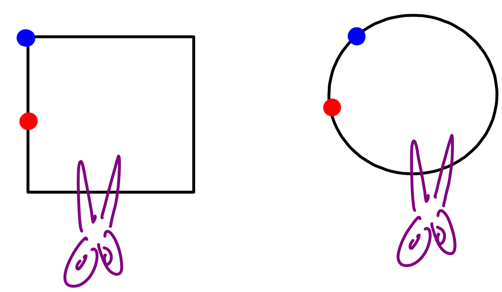
All these objects are same, i.e., they are connected the same way.

But, if you did not tie the knot, the loose string (open thread) differs from any of the above tied loops in connectivity.

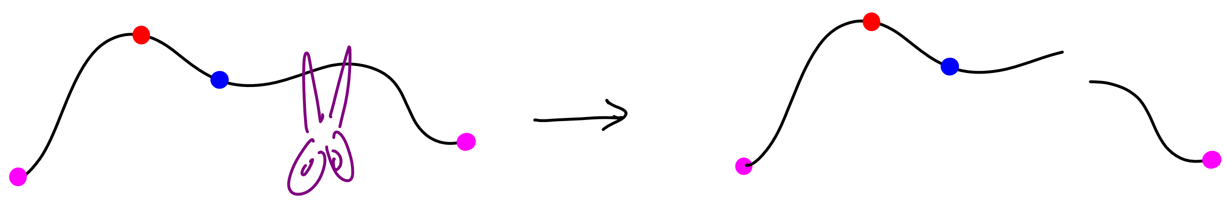


Note that the two end points have different "neighborhoods."

Here is another way to understand connectivity. Consider cutting the string (tied into a loop) once. Such a cut leaves the string in one piece, i.e., connected.

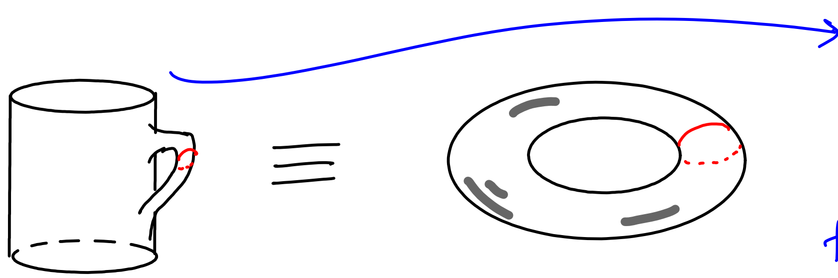


But cutting the open thread once leaves two pieces, i.e., it is disconnected.



two pieces

A popular quote: "A topologist cannot distinguish the coffee cup from a donut!"



the coffee cup, if made of clay, could be deformed to look like the donut, without changing its connectivity.

A more practical example: how we are able to read (recognize) letters of the alphabet in different fonts.

A **A** $\mathcal{A}$ $\neq$ B **B** $\mathcal{B}$ $\neq$ C C **C**

Two Illustrations of Computational Topology

1. Patient antibiotic trajectories

https://doi.org/10.1145/3307339.3342143

? How do agents and doses affect length of stay?

Number of hospitals	25
Number of hospital unit-categories	9
Number of distinct patient-admission records	349, 610
Number of adult patients	334, 207
Number of male patients	148, 540
Number of female patients	201, 052
Average LOS per admission	7 days
Longest LOS → length of stay	405 days
Number of antibacterials used	66
Most used antibacterial	Vancomycin
Average DOT per admission	6
Number of agent ranks	4
Most used agent rank	rank 3

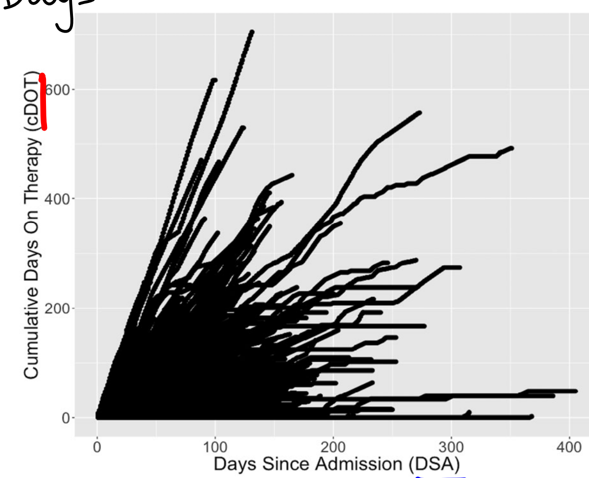
→ Days On Therapy

If a patient gets one dose of an agent (antibiotic) that is counted as 1 Day On Therapy (DOT).

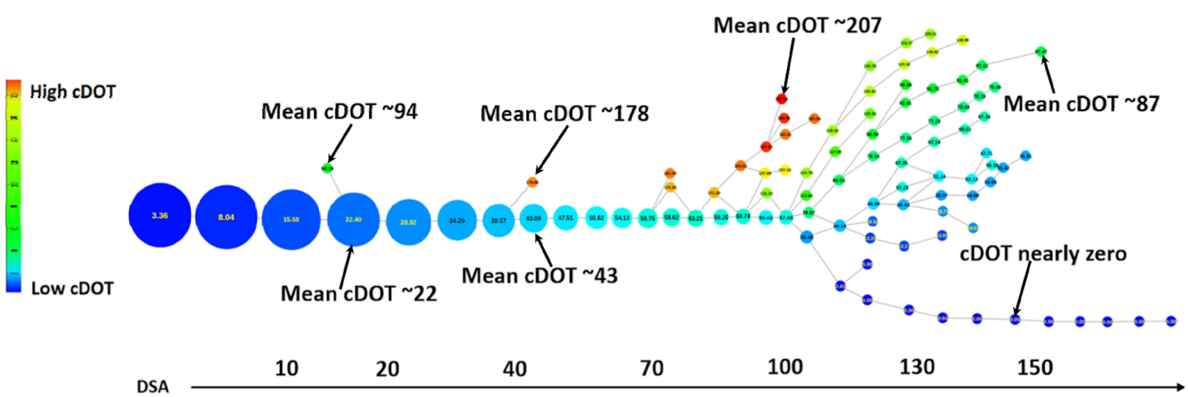
cDOT : cumulative Days On Therapy

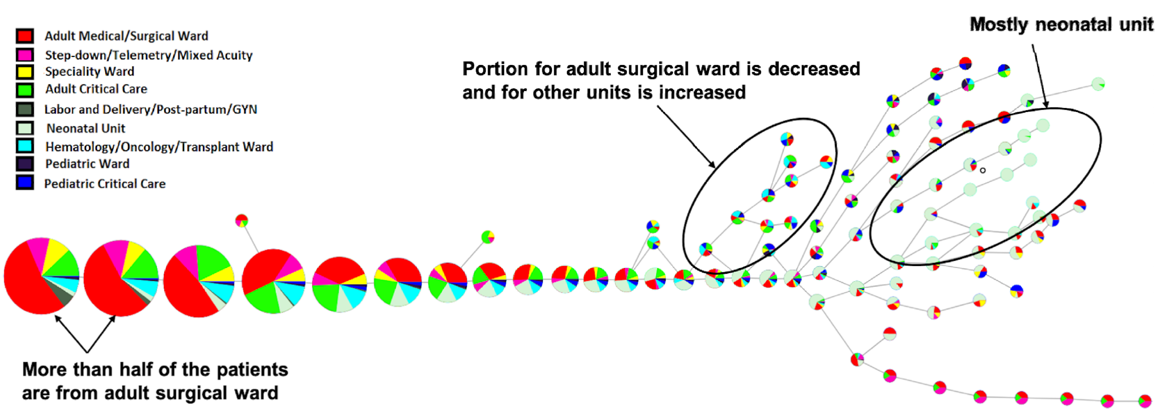
DSA : Days Since Admission.

This plot by itself is not very informative or insightful (even if we were to use color...)



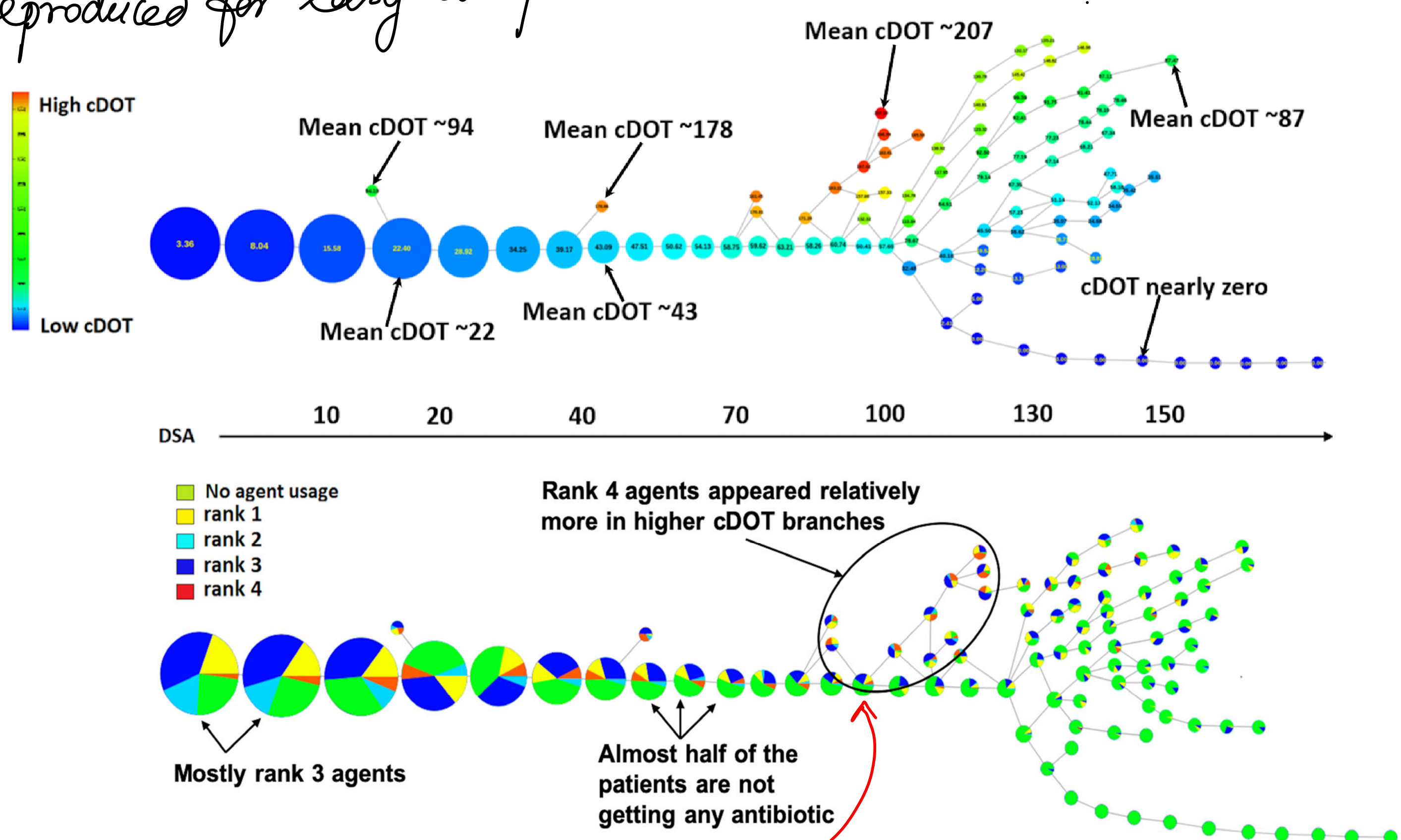
Here are two versions of a Mapper representation of the same data:





Each node represents a cluster of patient trajectories. A lot of the patients got low cDOTs and they had small(er) DSAs – as captured by the big clusters on the left. As the second Mapper shows, many of these patients were treated in the adult surgical ward — one of the most common types of admissions to hospitals.

Here is another version of the Mapper showing ranks (1–4, 4 is strongest) of the antibiotics. The first mapper (using cDOT) is reproduced for easy comparison.

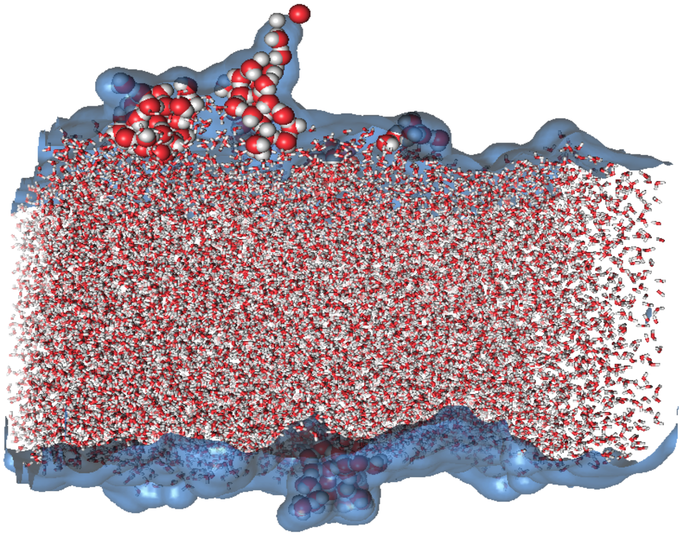


The high cDOT + high rank sub-branches had more patients in the other (higher risk) wards. Similarly, the much higher DSA group (120+) on the right end with relatively smaller cDOT values turned out to be patients in neonatal ICUs.

Note that these nontrivial subgroups are identified in an unsupervised manner — no learning is involved!

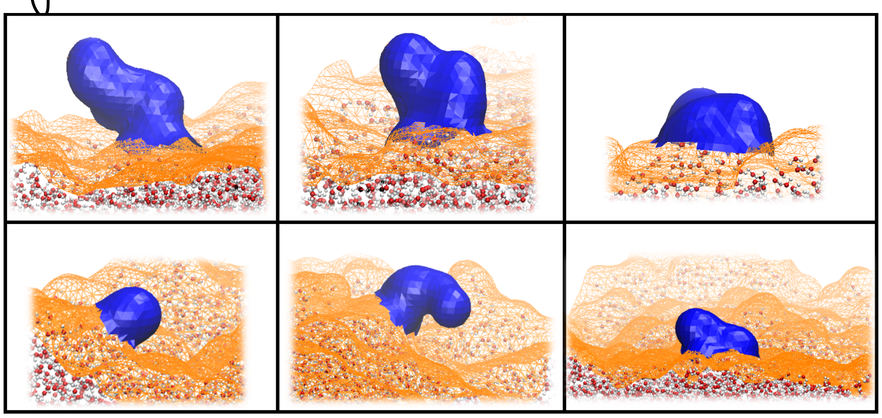
2. Interface features in Chemistry

https://doi.org/10.1021/acs.jctc.0c00260 (https://doi.org/10.26434/chemrxiv.11988048.v1) → preprint



An interface surface separates a water layer from a hexane (organic) layer. When a reagent is added, the reaction is initiated and the water molecules escape to the hexane layer through finger-like features in the interface called "protrusions". These features were identified manually (by observation!)

The goal was to identify and characterize protrusions using geometric measure theory and computational topology.



Which of these six features do you think are protrusions?

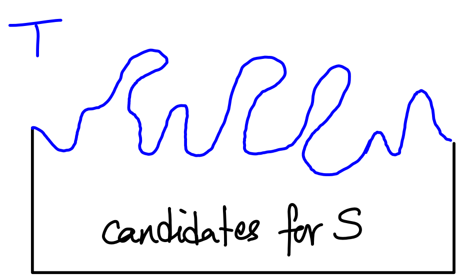
It is not easy to guess!

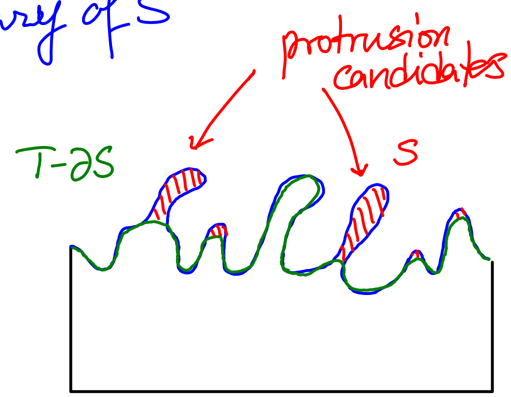
We use the notion of multiscale flat norm of surface T:

$$F_\lambda(T) = \min_{S} \{ \text{Area}(T - \partial S) + \lambda \, \text{Volume}(S) \}, \quad \lambda \geq 0 \to \text{scale parameter}$$

$S \to$ 3D volume, $\partial S \to$ boundary of S

Illustration in 2D:

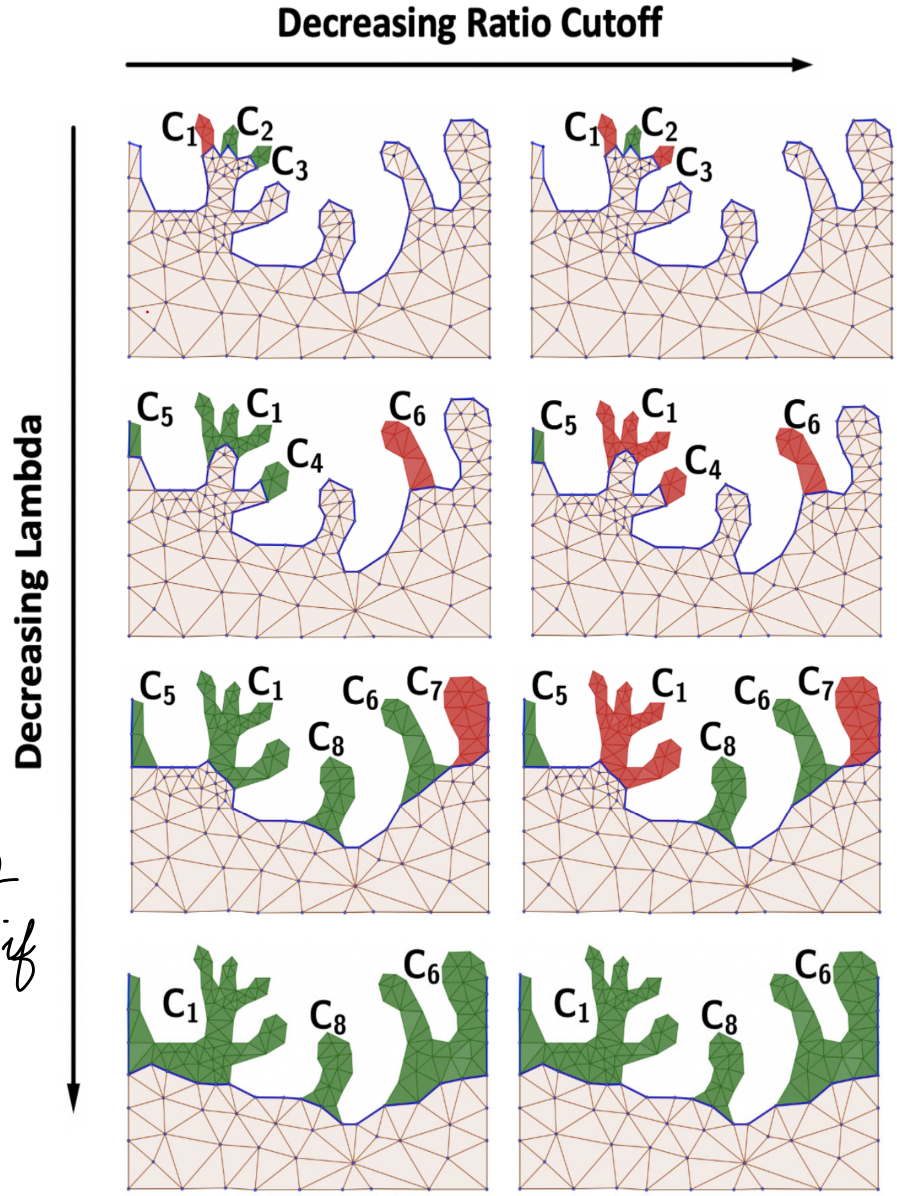




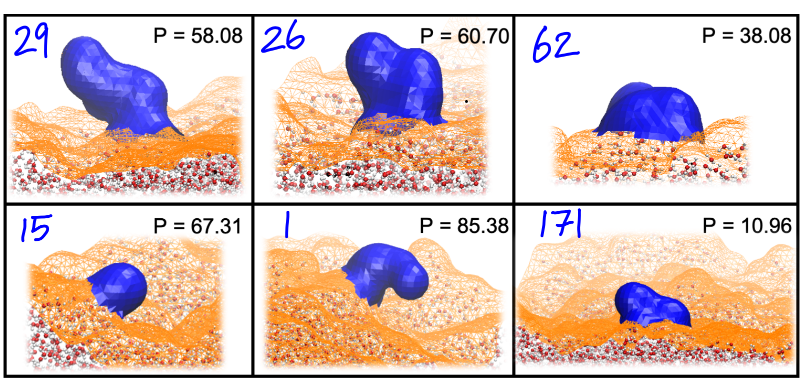
We keep track of connected components in S as $\lambda \downarrow$. We relabel them and also keep track of merging behavior.

We also track the ratio $\frac{Vol(C)}{Vol(B(1/\lambda))}$ for each component C, where $B(1/\lambda)$ is the ball with radius $1/\lambda$ and $Vol(C)$ is the volume of component C.

A component C is "alive" at ratio cutoff r and scale λ if $\frac{Vol(C)}{Vol(B(1/\lambda))} > r$.



The longer a component is alive, the more likely it is to be a protrusion.



It turned out that all six of these features were protrusions! The probabilities (as %'s) along with their ranks among 195 candidate features (lower rank $\Rightarrow$ more likely to be a protrusion) are shown here.

While this example is described in 3D, the underlying concepts are more general, and in fact generate certain key fundamental questions in geometric measure theory (GMT).

In fact, when we talk about applied algebraic topology, the application could be to pure mathematics! We will talk about this aspect toward the end of the semester.

Note that we are showing a discrete version of the surface—in the form of a triangular mesh. Indeed, we need to discretize continuous spaces to perform computations!

Here is a notion of connectivity in the "discrete setting":

The neighborhood, i.e., the set of nearby points, of the two • points are different – they each have only one neighbor, while the • points all have two neighbors each.

MATH 529: Lecture 2 (01/15/2026)

Today: * topology, open/closed sets
* homeomorphism, examples

We define topology as a mathematical method to define and study how a space is connected.

Notation For a set X, we denote by 2^X the power set of X, which is the set of all subsets of X.

Def A **topology** on a set X is a subset $\mathcal{T}$ of 2^X such that the following conditions hold.

1. $A_1, A_2 \in \mathcal{T} \Rightarrow A_1 \cap A_2 \in \mathcal{T}$ (finite intersections)

2. $\{A_j \mid j \in J\} \in \mathcal{T} \Rightarrow \bigcup_{j \in J} A_j \in \mathcal{T}$ (infinite unions or finite)

J: index set, infinite or finite

3. $\emptyset, X \in \mathcal{T}$ ($\emptyset$: empty set)

$(X, \mathcal{T})$ is a topological space, denoted $\mathbb{X}$ ($\mathcal{T}$ is understood from context).

$A \in \mathcal{T}$ is an **open set** of $\mathbb{X}$.

The complement of A, i.e., $X - A$ (or $X \setminus A$ "set minus") is a **closed set**.

Some sets can be both open and closed at the same time, e.g., $\emptyset, X$ are both open and closed in any topology.

We typically specify a topology by specifying its open sets.

interior $\mathring{A}$ of $A \subseteq \mathbb{X}$: $\mathring{A} = \overset{\text{union}}{\bigcup}$ (open sets contained in A)

closure $\bar{A}$ of $A \subseteq \mathbb{X}$: $\bar{A} = \overset{\text{intersection}}{\bigcap}$ (closed sets containing A).

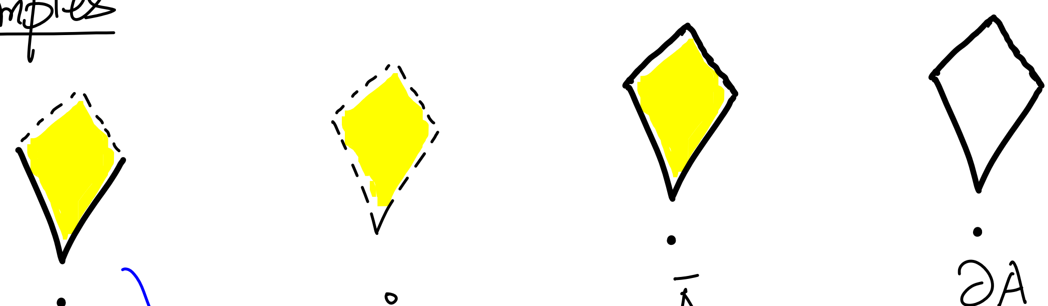
→ minimal closed set that contains A.

boundary ∂A of $A \subseteq \mathbb{X}$: $\partial A = \bar{A} - \mathring{A}$.

$\partial A = \{$points in A that intersect both $\bar{A}$ and $\overline{(X-A)}\}$.

Examples

1.



$A \subseteq \mathbb{X}$ $\quad$ $\mathring{A}$ $\quad$ $\bar{A}$ $\quad$ ∂A

→ A consists of the "half-open" rhombus and a separate point.

2. A discrete example. Let $X = \{a, b, c\}$.

We can define different topologies on X.

Let $T_1 = \{\phi, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ and

$T_2 = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$.

Under T_1, $\{a,b\}$ is open, its complement $\{c\}$ is closed. With $A = \{a,b\}$, $\mathring{A} = \bigcup\{\phi, \{b\}, \{a,b\}\} = \{a,b\} = A$.

We can specify other topologies on X, e.g., $T_3 = 2^X$, where each set in T is both open and closed. But $T_4 = \{\phi, \{a\}, \{b\}, \{a,b,c\}\}$ is not a topology, as, e.g., $\{a\} \cup \{b\} = \{a,b\} \notin T_4$.

Neighborhood Let $\mathbb{X} = (X, T)$. A neighborhood of $x \in X$ is any $A \in T$ such that $x \in \mathring{A}$.

More generally, some books define a neighborhood as any set that includes, i.e., contains as a subset, an open set which contains X. Under this definition, the neighborhood could be a closed set (or neither open nor closed).

Now that we have defined topology, we consider the natural next question of comparing two spaces — how do we say two given spaces have the "same topology"? We introduce the notion of homeomorphism as a (strong) notion of topological similarity.

Homeomorphism

In geometry, we can study transformations that preserve "shape" of a rigid body, e.g., rotation and translation. These transformations "do not change" the geometry of the body.

In topology, we permit more types of transformations — e.g., stretch, shrink, expand, twist, etc., as long as you do not cut one piece into two or more, or join two pieces into one, or poke a hole in your object. All such permitted transformations "preserve topology".

A series of such permitted transformations that preserve topology constitute a homeomorphism. And two spaces are topologically "similar" if such "nice" functions exist from one space to the other and also back. We define what we mean by "nice" here.

We start with some background and definitions on functions.

Def Let A, B be sets. A function $f: A \to B$ is a rule that assigns exactly one $b \in B$ for every $a \in A$.

dom f: domain of $f = A$, cod f: codomain of $f = B$

im f: image of $f = \{b \in B \mid f(a) = b$ for some $a \in A\} = \{f(a) \mid a \in A\}$.

im f is also called the range of f. Note that im $f \subseteq$ cod f.

"for every"

$f: A \to B$ is 1-to-1 or **injective** if $\forall b \in B$, there exists **at most** one $a \in A$ with $f(a) = b$.

can be none

$f: A \to B$ is onto or **surjective** if $\forall b \in B$, there exists **at least** one $a \in A$ with $f(a) = b$.

can be more

If f is both injective and surjective, we say that f is **bijective**, or that f is a bijection.

Def A function $f: X \to Y$ is **continuous** if for every open set $B \subseteq Y$, $f^{-1}(B)$ is open in X. "takes" open sets to open sets.

A continuous function is also called a **map**.

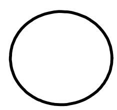
Def A **homeomorphism** $f: X \to Y$ is a bijective function such that both f and f^{-1} are continuous.

We say X is homeomorphic to Y, or $X \approx Y$.

We also say that X and Y have the same topological type.

Examples

It's often easier to argue why two spaces are not homeomorphic — we just identify one (or more) place(s) where things don't work.

1.  $\not\approx$ 

"not homeomorphic"

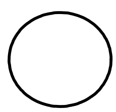
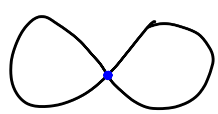
0 1

$[0,1]$ S^1

1-sphere, or circle

We would need a map that assigns both end points of $[0,1]$ to a single point in S^1.

But the inverse of any map that takes both end points of $[0,1]$ to one point in S^1 is not bijective.

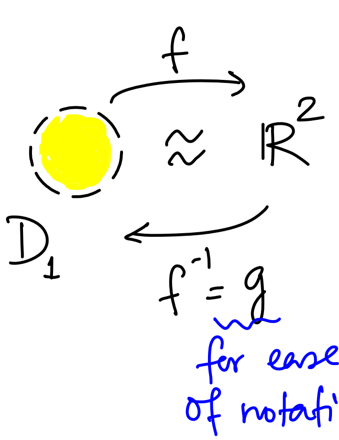
2.  $\not\approx$ 

S^1 Figure 8

The crossing point in ∞ ($\times$) cannot be mapped to a corresponding point in S^1.

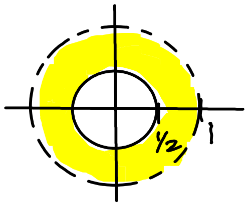
Also, we could map S^1 to one of the two circles in figure-8, but not both.

On the other hand, two show that two spaces are homeomorphic, we need to specify the maps f and f^{-1}!

3. 

$D_1 = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1\}$ → open unit disc

Intuitively, we can shrink all of $\mathbb{R}^2$ into D_1. Similarly, we can stretch D_1 to fill all of $\mathbb{R}^2$.



$$g(\bar{x}) = \frac{\bar{x}}{1 + \|\bar{x}\|_2} \qquad g : \mathbb{R}^2 \to D$$

Euclidean norm

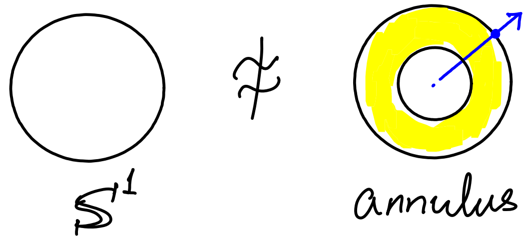
g maps all of D_1 (in $\mathbb{R}^2$) to fit within $D_{1/2} = \{\bar{x} \in \mathbb{R} \mid \|\bar{x}\| < \tfrac{1}{2}\}$, and then fits all of $\mathbb{R}^2$ outside D_1 within the half open annulus with radii $\tfrac{1}{2}$ and 1.

The continuous function going from D_1 to $\mathbb{R}^2$ can be similarly defined:

$f : D \to \mathbb{R}^2$ where $f(\bar{x}) = \dfrac{\bar{x}}{1 - \|\bar{x}\|}$. f is an "infinite stretch".

Note that points $\bar{x}$ in D that are close to the edge, i.e., have $\|\bar{x}\|$ close to 1, are mapped so as to fill up the entire $\mathbb{R}^2$ outside $\bar{D}$. We stretch the open disc so as to fill the entire plane, and hence it is called an infinite stretch.

Usually, we try to define the continuous maps f and f^{-1} to show that two spaces $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic. At the same time, the intuition (geometric when possible) is also important to grasp. On the other hand, to show that $\mathbb{X} \not\cong \mathbb{Y}$, it is often sufficient to identify subset(s) that create the obstructions, e.g., the $\times$ in figure-8 v/s S^1.

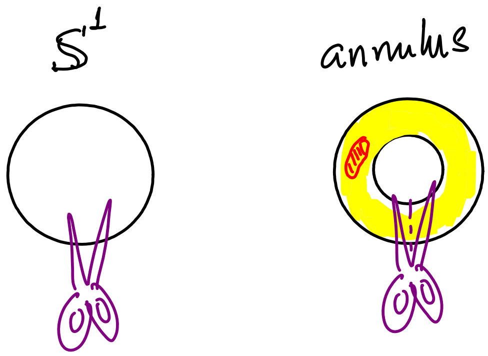
4. 

Both these spaces have the shape of a "hole".

We could shrink the annulus so that it reduces to the circle. The corresponding function maps every point on the annulus radially onto the outer circle, for instance. But we cannot uniquely map the circle back to the annulus — would need to "map" each point on the circle to (infinitely) many points on the thick strip of the annulus.

Another observation highlights the neighborhoods of points in the circle and the annulus. Every point on the circle has open neighborhoods that look like the number line ($\mathbb{R}^1$). On the other hand, points in the annulus have neighborhoods that look like the open disc ($\mathbb{R}^2$) or open half disc (the points on the boundary).

Intuitively, the annulus is 2-dimensional, while $\mathbb{S}^1$ is one-dimensional.



Notice that the two spaces behave the same way under a "cut" as we had been talking about earlier with the string.

In particular, a straight cut along one "edge" of either space would leave them both connected. At the same time, one could "carve out" a 2D disc ( ) from the annulus, but not from the circle.

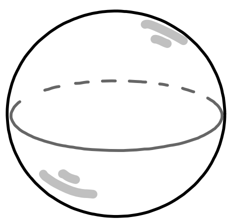
If we "relax" our definition of topological similarity, the two spaces would be considered the same — they both look like a hole, after all. Indeed we will see that checking for homeomorphism is difficult (both theoretically and computationally). We'll work with looser concepts of topological similarity later on — homology!

MATH 529: Lecture 3 (01/20/2026)

Today: * 1 more example of homeomorphism
* manifolds

Examples of homeomorphism (continued...)

5. sphere $\not\approx$ $\mathbb{R}^2$
$\mathbb{S}^2$ (2-sphere)



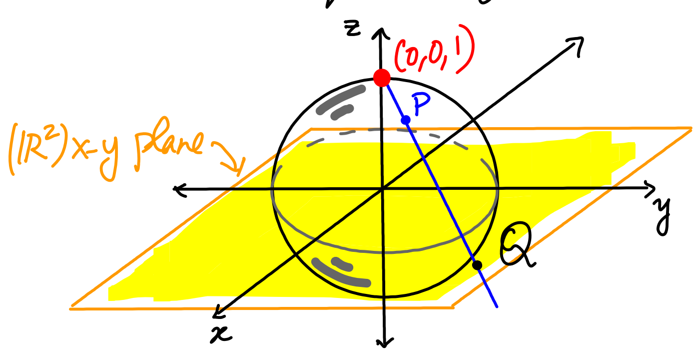
Here is an observation: the sphere encloses a 3D pocket (or void), while $\mathbb{R}^2$ does not do so.

Such enclosed voids are the 3D analogues of holes (which are 2D).

But $\mathbb{R}^2 \cup \{\infty\} \approx \mathbb{S}^2$ → By stereographic projection, which is used to map out the surface of the earth on to a (planar) map, for instance.

"point at infinity"

Recall the equation for $\mathbb{S}^2$: $x^2+y^2+z^2=1$.



If you poke a hole in the sphere, you can spread it out on the 2D plane, like a pierced balloon.

The equation of the line connecting $(0,0,1)$ and $P(x,y,z)$ is given by $\bar{x} = (0,0,1) + t\,(x-0, y-0, z-1)$, $t \in \mathbb{R}$.
This line intersects the x-y plane at Q, which has $z=0$. Hence we get $t(z-1)+1=0 \Rightarrow t = \frac{1}{1-z}$.
Thus, Q is $\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$.

$P(x,y,z)$ on $\mathbb{S}^2$ projected from $(0,0,1)$ → North pole, to $\mathbb{R}^2$ is

$$Q\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

This formula is valid for all points on $\mathbb{S}^2$, except the north pole $(0,0,1)$.

The lower hemisphere of $\mathbb{S}^2$ gets mapped to D (unit disc), and the upper hemisphere gets mapped to the rest of $\mathbb{R}^2$.

According to a topologist, "a sphere is nothing but the plane with a point added at infinity"!

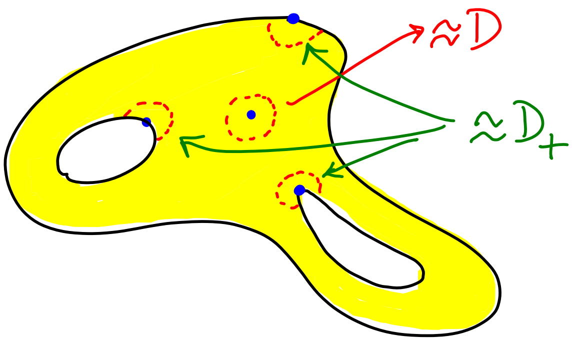
Note that every point on $\mathbb{S}^2$ has a neighborhood that looks like $\mathbb{R}^2$, i.e., "it feels locally Euclidean". Such objects are called **manifolds** and are among the most commonly studied spaces in computational topology.

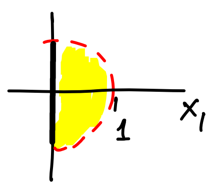
Def A topological space $\mathbb{M}$ is a **2-manifold** if all points in $\mathbb{M}$ lie in open discs, i.e., every point has a neighborhood $\approx D = \{\bar{x} \in \mathbb{R}^2 \mid \|x\| < 1\}$.

e.g., $\mathbb{S}^2$, $\mathbb{R}^2$. these are 2-manifolds without boundary.

A **2-manifold with boundary** is a topological space $\mathbb{M}$ whose every point has a neighborhood homeomorphic to D or to $D_+ = \{\bar{x} \in \mathbb{R}^2 \mid \|\bar{x}\| < 1, x_1 \geq 0\}$ (but not both), and there exist some points of the latter type.

1st entry



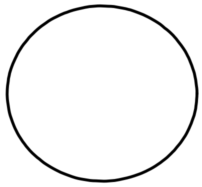
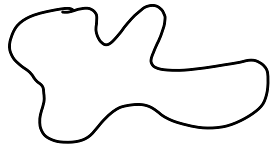
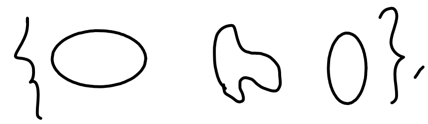


points on the boundary have neighborhoods that are half discs.

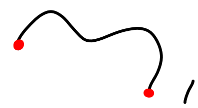
Def The **boundary** of a 2-manifold with boundary $\mathbb{M}$ is the set of points in $\mathbb{M}$ that have neighborhoods homeomorphic to the half disc.

(The definition of ∂A used for sets A is equivalent to this definition).

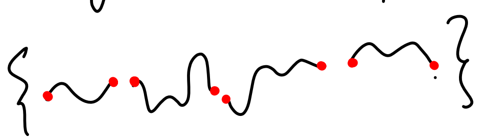
Notice that the 1-manifold is just the circle ($\mathbb{S}^1$),

 or  or ,

a collection of disjoint circles.

A 1-manifold with boundary: , or 

only one end point is included here!



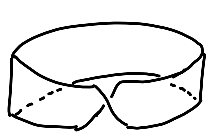
boundary is indeed the set of end points.

0-manifold: Any collection of distinct points (discrete set of points).

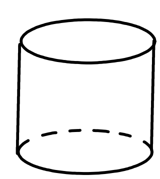
2-manifolds, also called surfaces, are a well studied class of spaces (objects), both from the theoretical as well as applied points of view. We will present several details of the properties of 2-manifolds first. To define and study d-manifolds for $d \geq 2$, we will need a few more definitions and concepts from analysis/point set topology.

By default, we assume a manifold (w/ or w/o boundary) is connected.

A 2-manifold (for that matter, d-manifold for $d \geq 2$) is orientable or non-orientable.



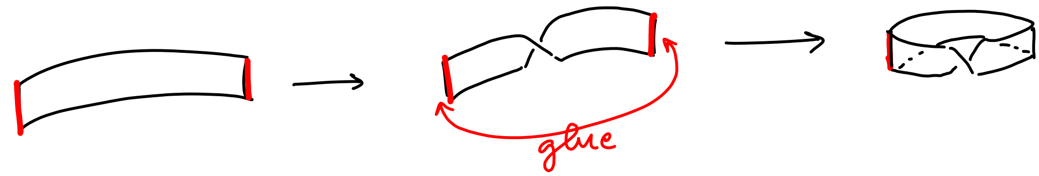
Möbius strip is non-orientable

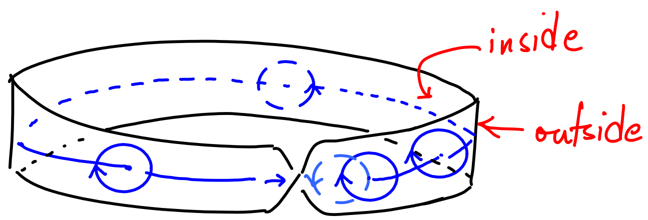


cylinder is orientable

The Möbius strip has only one "side," while the cylinder has two "sides" — inside and outside.

We obtain the Möbius strip by taking a rectangular strip of paper, and gluing its short edges together after twisting the strip once.





there are two "sides" at each point (locally) on the Möbius strip— front/back or inside/outside.

Consider sliding an oriented loop, or a clock along the surface of the Möbius strip. Look at the path followed by the center of the clock. Once the center comes back to where it started, its orientation is reversed (as it goes over the "twist").

The path traced by the center of the clock here is hence an **orientation reversing** closed curve. If the orientation is not reversed this way, the curve is said to be **orientation preserving**.

Def If all closed curves in a 2-manifold (with or without boundary) are orientation preserving, then the 2-manifold is **orientable**, else it is **nonorientable**.

$\mathbb{R}^2$, S^{1^2}, torus, etc.

Möbius strip, Klein bottle, etc.

Classification of Manifolds

Enumerate all possible manifolds upto homeomorphisms for each dimension.

We mention the classification for 0- and 1-dimensional manifolds, but will come back to give some more definitions to finish the discussion for 2-manifolds.

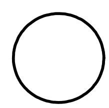
each point has to have a neighborhood $\approx \mathbb{R}^0$, i.e., a point.

<u>0-manifolds</u>: a discrete space, e.g., $\mathbb{Z}^2$ — all points with integer coordinates in $\mathbb{R}^2$.

Notice that $2\mathbb{Z}^2$, all points with even integer coordinates in $\mathbb{R}^2$ is homeomorphic to $\mathbb{Z}^2$.

We will define "compact" soon...

<u>1-manifolds</u>
connected

- compact: $\approx S^1$, 
- compact with boundary: $\approx [0,1]$
  closed unit interval
- non-compact: $\approx \mathbb{R}^1$ or $(0,1)$
- non-compact with boundary $\approx \mathbb{R}^1_+$ or $[0,1)$
  $\mathbb{R}^1_+$: $x \geq 0$
  $[0,1)$: half-open unit interval

if a 1-manifold is not a single connected space, each connected subspace has one of these structures.

MATH 529: Lecture 4 (01/22/2026)

Today: * d-manifolds in general
* Classification of 2-manifolds

We will now introduce some concepts which we will use to define d-manifolds in general (in particular, for $d \geq 2$).

Def A **cover** of $A \subseteq X$ is a family $\{C_j \mid j \in J\}$ in 2^X such that $A \subseteq \bigcup_{j \in J} C_j$.

↳ index set

An **open cover** is a cover made of open sets.

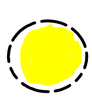
index set is a subset

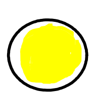
A **subcover** of A is a cover $\{C_k \mid k \in K\}$ such that $K \subseteq J$.

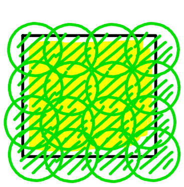
Def A set A is **compact** if every open cover of A has a finite subcover. Correspondingly, a topological space $\mathbb{A} \subseteq \mathbb{X}$ is compact if every open cover of $\mathbb{A}$ has a finite subcover.

Note: In $\mathbb{R}^d$, closed + bounded $\Longleftrightarrow$ compact.

e.g., $\mathbb{S}^2$ is compact, but $\mathbb{R}^2$ is not.

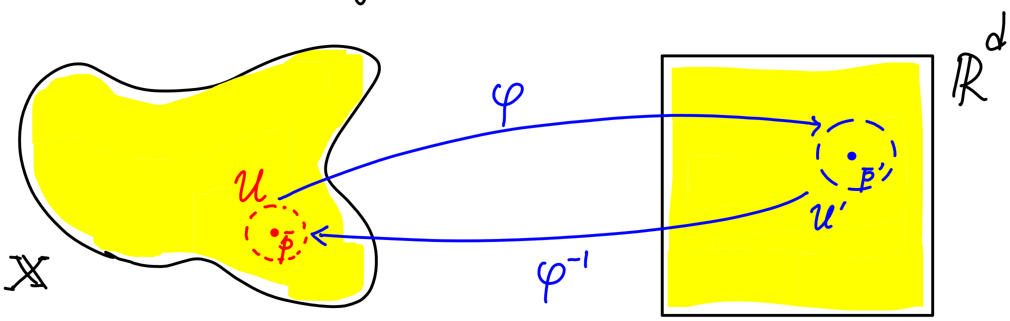
 D_1 is not compact.

 $\overline{D}_1$ (closed disc) is a compact 2-manifold with boundary



Example of finite subcover: consider a unit square, which is a subset of $\mathbb{R}^2$. Consider open discs of radius $1/4$ centered at each rational point within the square (denoted here as). There are infinitely many such discs, which together cover the square. But a finite subset of those discs also covers the square.

Def A **chart** at $p \in \mathbb{X}$ is a homeomorphism $\varphi : \mathcal{U} \to \mathbb{R}^d$ for $\mathcal{U} \in \mathbb{X}$ an open set containing p.
The **dimension** of the chart is d.



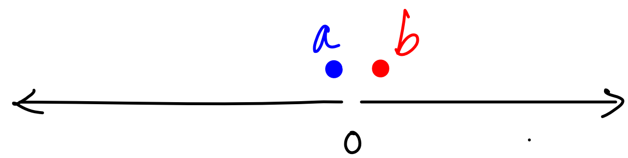
Def (Hausdorff) A topological space $\mathbb{X}$ is **Hausdorff** if $\forall x, y \in X$, $x \neq y$, there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of x, y, respectively, such that $\mathcal{U} \cap \mathcal{V} = \emptyset$.

neighborhoods → an open set containing the point in question

e.g., $\mathbb{R}^2$.

Example of a non-Hausdorff space: $X = L \cup \{a, b\}$ where $L = \mathbb{R} \setminus \{0\}$

a and b are both used in place of the origin. Open sets are the usual open intervals in $\mathbb{R}$.



and such that

$$L \cup \{a\} \approx \mathbb{R} \text{ and } L \cup \{b\} \approx \mathbb{R}.$$

But every pair of open sets $\mathcal{U}$ and $\mathcal{V}$ containing a and b, respectively, intersect!

Def A topological space is **completely separable** (or second countable) if it has a countable basis, i.e., it has a countable collection of open sets such that every open set can be written as a union of open sets from this collection (basis).

think $\mathbb{Z}$, integers, as opposed to $\mathbb{R}$, which is uncountable

e.g., $\mathbb{R}$ is completely separable — it can be shown that open intervals with rational lengths centered at only rational points works as a countable basis.

A space that is not completely separable: take uncountably many copies of $[0,1)$, e.g., with the 0 of $[0,1)$ anchored at all irrational points — called the long line or Alexandroff line.

Def (Manifold) d-dimensional manifold A completely separable, Hausdorff space $\mathbb{X}$ is a **d-manifold** if there exists a d-dimensional chart at every $\bar{x} \in \mathbb{X}$, i.e., $\bar{x}$ has a neighborhood homeomorphic to $\mathbb{R}^d$.

$\mathbb{X}$ is a d-manifold with boundary if every $\bar{x} \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$ or $\mathbb{H}^d = \{\bar{x} \in \mathbb{R}^d \mid x_1 \geq 0\}$ (d-dimensional half space).

The boundary of $\mathbb{X}$, denoted by $\partial\mathbb{X}$, is the set of $\bar{x} \in \mathbb{X}$ with a neighborhood homeomorphic to $\mathbb{H}^d$.

The **dimension** of the manifold is d here.

Notice the correspondence between the definition of 2-manifolds introduced previously, and the general definition here. The main condition is the existence of neighborhoods $\approx \mathbb{R}^d$ around each point.

Def (Embedding) An **embedding** of $\mathbb{X}$ in $\mathbb{Y}$ is a map $g: \mathbb{X} \to \mathbb{Y}$ whose restriction to $g(\mathbb{X})$ is a homeomorphism.

Manifolds are manifolds irrespective of their embedding! $\mathbb{S}^2$ is a 2-manifold even if it is not sitting in $\mathbb{R}^3$.

We will introduce alternative representations of manifolds to highlight this point. In fact, in many cases, we can study the manifold easily using such representations.

Classification of Manifolds (continued)

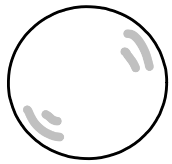
Enumerate all possible manifolds of a given dimension upto homeomorphism. We already listed the classifications for 0- and 1-dimensional manifolds.

We now consider the case of compact, connected, closed 2-manifolds. We first list the "basic building blocks", so to speak, which include the 2-sphere, torus, Möbius strip, and the real projective plane. We can build larger 2-manifolds by gluing these building blocks together.

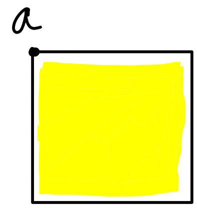
2-Manifolds (We consider compact 2-manifolds)

First, let us study several typical 2-manifolds, some of which we already saw previously.

1. $\mathbb{S}^2$ 2-sphere



$\{\bar{x} \in \mathbb{R}^3 \mid \|\bar{x}\|_2 = 1\}$

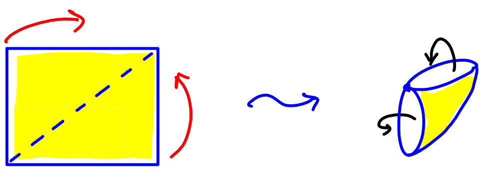


"identify" all points on boundary with the point a.

This is a "diagram" of $\mathbb{S}^2$.

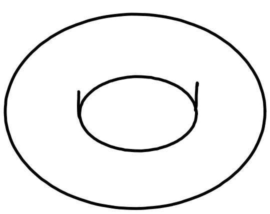
both are 2-spheres!

Start with a square sheet of paper and glue its all its edges together to make a sphere.



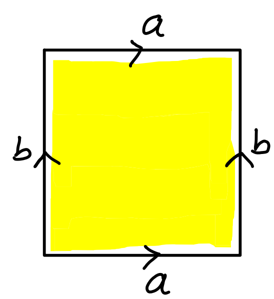
Arrows capture how edges are glued – with or without twist.

2.



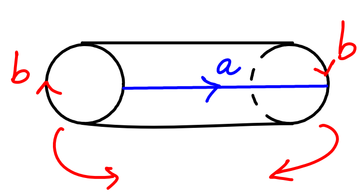
$\mathbb{T}^2$ torus

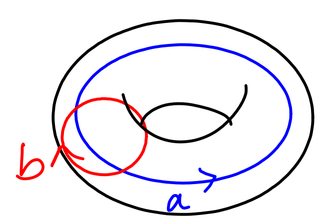
$\approx$



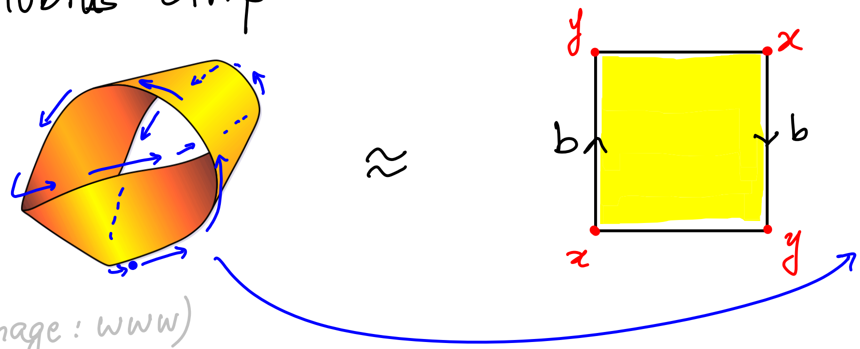
we could mathematically study this representation

Imagine folding a rectangular sheet of paper first into an open cylinder, and then gluing its end circles to form a torus.





3. Möbius strip → 2-manifold with boundary



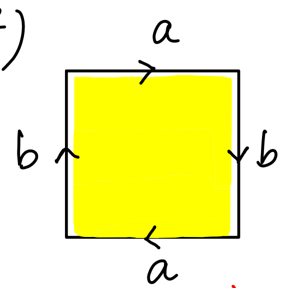
(image: www)

Notice that we can traverse the "edge" of the Möbius strip in one go — it's one big circle!

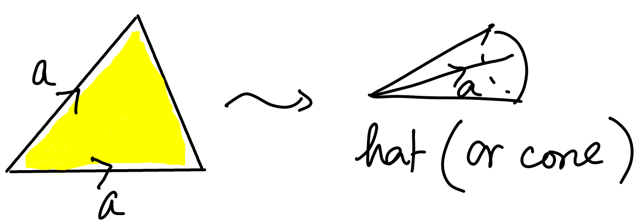
Notice that we are not identifying the horizontal edges. So they remain as boundaries. At the same time all four edges are identified pairwise in the case of the torus. Indeed, the Möbius strip is a manifold with boundary, while the torus is a manifold (without boundary).

cannot be embedded in $\mathbb{R}^3$!

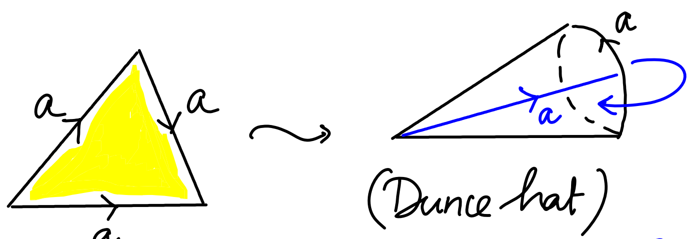
4. (Real) Projective plane ($\mathbb{R}P^2$) (also, Dunce hat)



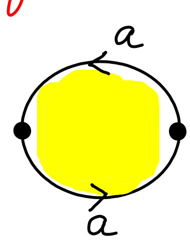
identify the free edges of Möbius strip in an opposing sense.



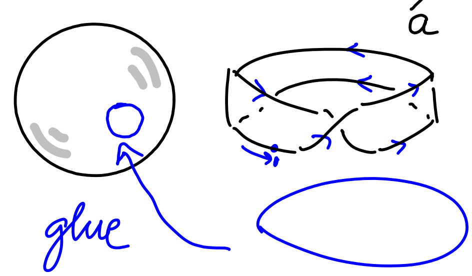
Same as gluing the boundary of a disc to the boundary of a Möbius strip.



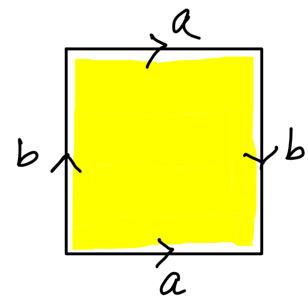
Here's another representation



Yet another way to make $\mathbb{R}P^2$: Cut an open disc out of S^2 and glue a Möbius strip along the edge left by the cut.

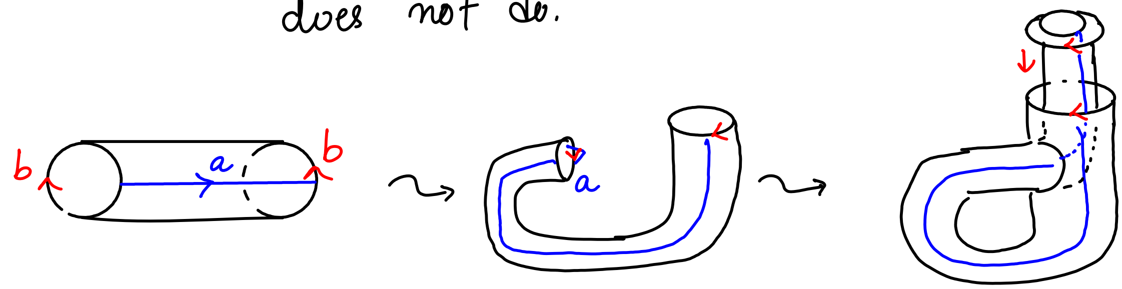


5. Klein bottle ($\mathbb{K}^2$)



Identify free edges of the Möbius strip in the same direction.

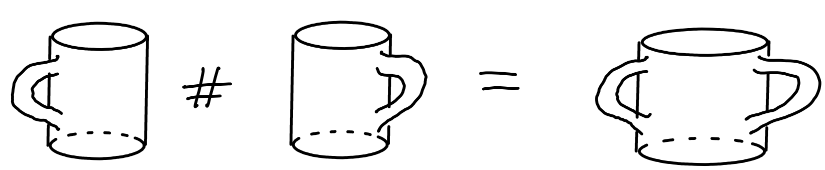
An "immersion" in 3D:
→ allows self intersection, which an embedding does not do.



We get $\mathbb{K}^2$ also by gluing together two Möbius strips along their boundary circles. Or, cut two discs out of $\mathbb{S}^2$ and glue a Möbius strip each along the edges of both cuts.

Note that $\mathbb{S}^2$ and $\mathbb{T}^2$ are orientable manifolds, while the Möbius strip, $\mathbb{RP}^2$, and $\mathbb{K}^2$ are non-orientable manifolds (with or without boundary).

We can obtain more general 2-manifolds by "gluing" these basic shapes together. For example, we can connect two coffee cups to get one coffee cup with two handles!



We can do this kind of "gluing" to join manifolds in any dimension (as long as the manifolds being joined have the same dimension). This ing is formally termed connected sum.

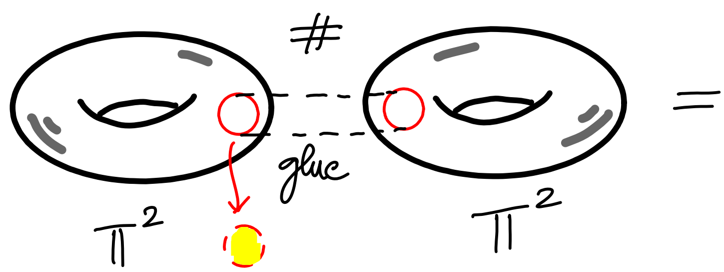
Def (Connected sum). Let M_1, M_2 be d-manifolds. The **connected sum** of these d-manifolds is another d-manifold defined as follows.

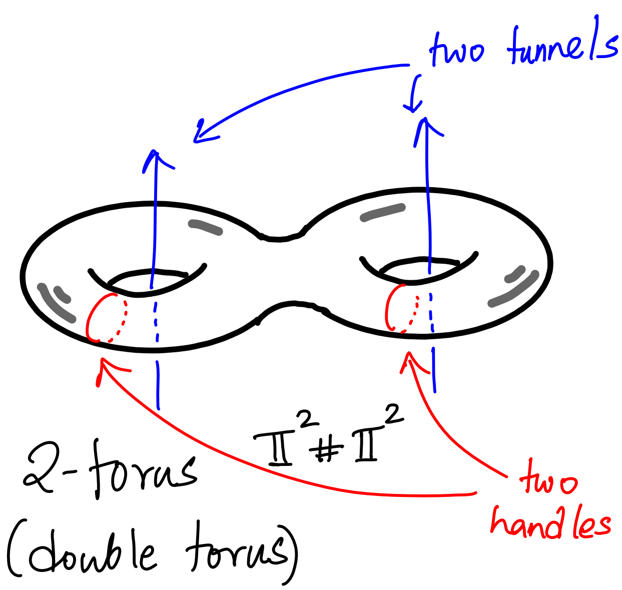
$$M_1 \# M_2 = (M_1 - D_1^d) \underset{\partial \bar{D}_1^d \cong \partial \bar{D}_2^d}{\bigcup} (M_2 - D_2^d)$$

↳ identified by homeomorphism

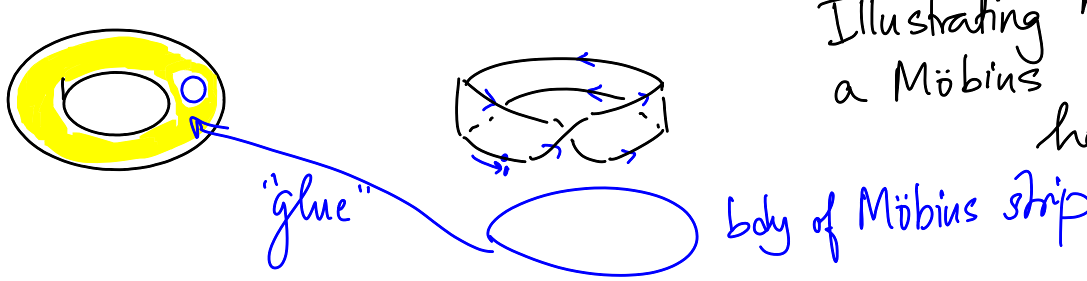
D_1^d, D_2^d are d-dimensional open discs in M_1, M_2 respectively.

Here is an illustration:





Remove open dics from both tori, and "glue" along the boundaries of these circular holes.



Illustrating how to glue a Möbius strip to a hole in a torus.

MATH 529: Lecture 5 (01/27/2026)

Today: * classification of 2-manifolds
* simplices and simplicial complexes
* ~~abstract simplicial complexes~~ didn't get to it ...

Classification of Compact, connected 2-manifolds

Result Every compact, connected 2-manifold is homeomorphic to $\mathbb{S}^2$, or a connected sum of copies of $\mathbb{T}^2$, or a connected sum of copies of $\mathbb{RP}^2$.
If a 2-manifold is not connected, each component has this structure.

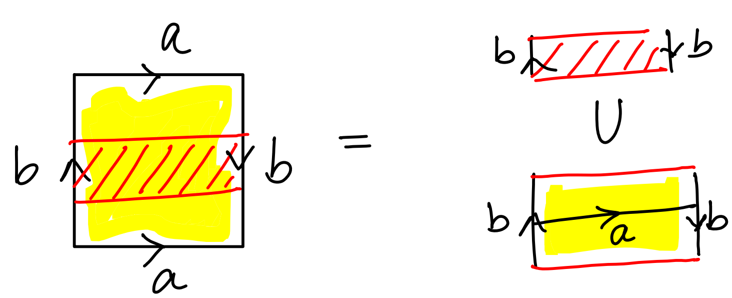
Examples

1. $\mathbb{S}^2 \# \mathbb{RP}^2 \approx \mathbb{RP}^2$

   In fact, $\mathbb{S}^2 \# \mathbb{M} \approx \mathbb{M}$ for any 2-manifold $\mathbb{M}$. (you just close back the open disc cut out from $\mathbb{M}$!)

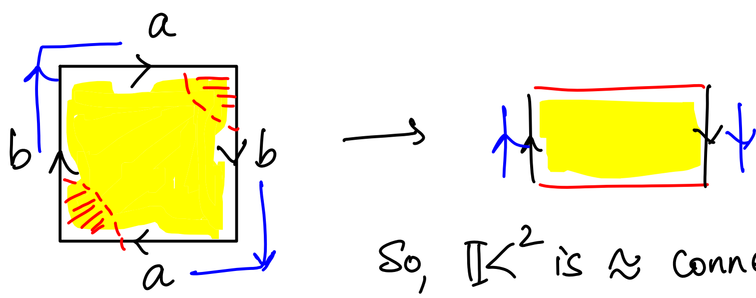
2. $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx \mathbb{K}^2$ Here's a "proof" by pictures:

$\mathbb{K}^2$:



Connected sum of 2 Möbius strips!

And $\mathbb{RP}^2 - D^2$: (open 2-disc)



Mobius strip

So, $\mathbb{K}^2$ is $\approx$ connected sum of 2 $\mathbb{RP}^2$'s.

Also, we get $\mathbb{RP}^2$ as $\mathbb{S}^2 \# M$, for a Möbius strip M.

3. $\underset{\text{orientable}}{\mathbb{T}^2} \# \underset{\text{nonorientable}}{\mathbb{RP}^2} \approx \#(\mathbb{RP}^2)^3$

Once you join a non-orientable 2-manifold, the result stays non-orientable.

As one would expect, the corresponding result for 3-manifolds is much more complicated. Thurston's geometrization conjecture states that all compact 3-manifolds can be canonically decomposed into submanifolds that have geometric structure. This conjecture implies the famous Poincaré conjecture, which states that every compact, simply connected 3-manifold is homeomorphic to $\mathbb{S}^3$, the 3-sphere.

(Informally, an object is simply connected if there are no "holes" passing through the object).

Perelman presented a proof of the Poincaré conjecture, almost 100 years after it was originally proposed (in 1904; Perelman's proof appeared in 2003). The corresponding result for n-manifolds with $n \geq 4$ turns out to be easier to prove, informally because of the "increased geometric freedom" one can afford in higher dimensions.

The main concepts used in Perelman's proof can be used to provide a proof for Thurston's geometrization conjecture (Perelman presented such a proof in 2003, along with his proof of the Poincaré conjecture).

Simplices

While we can study sample 2-manifolds as is, we cannot do computations on them. For this purpose, we need a discretized version of the spaces in question, which could be stored and handled naturally by a computer. Can we, for instance, use some sort of "counting arguments" to distinguish S^2 from $\mathbb{T}^2$?

We introduce the concept of simplicial complexes in this context, and use concepts from combinatorial algebraic topology. The idea is that we can handle the combinatorics using efficient algorithms. Similarly, there are efficient data structures that can be used to work with simplicial complexes modeling the spaces. We also want to separate the topology from the geometry of the object.

we use the geometry in many applications.

We first introduce some definitions.

Def (Combinations) Let $S = \{\bar{p}_0, \dots, \bar{p}_k\} \subseteq \mathbb{R}^d$. A

linear combination of $\bar{p}_i$ is $\bar{x} = \sum_{i=0}^{k} \lambda_i \bar{p}_i$, $\lambda_i \in \mathbb{R}\ \forall i$.

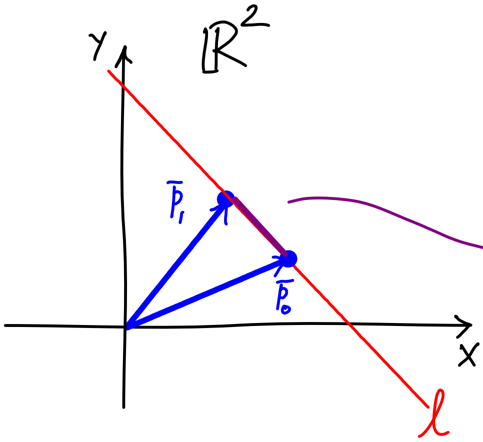
If $\sum_{i=0}^{k} \lambda_i = 1$, then $\bar{x}$ is an **affine combination** of $\bar{p}_i$'s.

In addition, if $\lambda_i \geq 0\ \forall i$, $\bar{x}$ is a **convex combination** of $\bar{p}_i$'s.

The set of all convex combinations of elements in S is the **convex hull** of S, denoted as

$$\operatorname{conv}(S) = \left\{ \sum_{i=0}^{k} \lambda_i \bar{p}_i \,\middle|\, \lambda_i \geq 0\ \forall i,\ \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

Illustration in 2D



set of all linear combinations $= \mathbb{R}^2$

set of all affine combinations $= \ell$, the line through $\bar{p}_0, \bar{p}_1$

$\operatorname{conv}(\{\bar{p}_0, \bar{p}_1\})$ = line segment connecting $\bar{p}_0, \bar{p}_1$.

$\bar{p}_0, \bar{p}_1$ are not parallel here.

Def (Independence) S with $|S| \geqslant 2$ is **linearly (affinely) independent** if no point in S is a linear (affine) combination of other points in S.

We denote linearly independent in short as LI, and affinely independent as AI.

$|S|=1$ case: $\{\bar{p}_0\}$ is LI if $\bar{p}_0 \neq \bar{0}$ (zero vector), but $\{\bar{p}_0\}$ is AI for all $\bar{p}_0$ (even if $\bar{p}_0 = \bar{0}$).

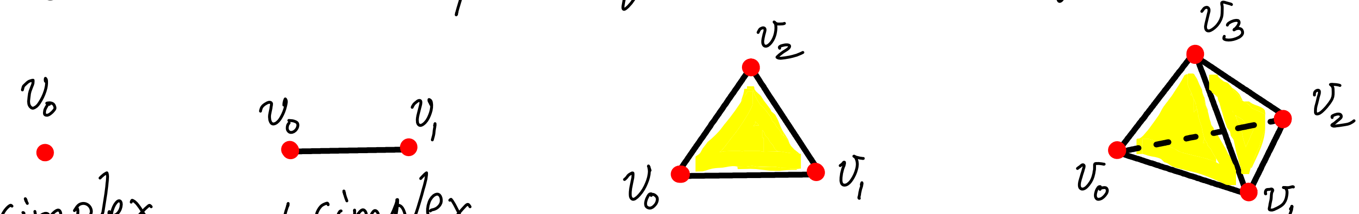
For example, 3 points in $\mathbb{R}^2$ are AI as long as they are not collinear.

We will use convex hulls of AI points as our building blocks — called simplices.

Def (simplex) The convex hull of $(k+1)$ affinely independent points $S = \{\bar{v}_0, \dots, \bar{v}_k\}$ is a **k-simplex**. The **dimension** of the simplex is k, and $\bar{v}_j$'s are the **vertices** of the k-simplex.

vertices, and hence $\bar{v}_i$, rather than $\bar{P}_i$.

Here are the k-simplices for small values of k: $0 \le k \le 3$.



0-simplex (point) | 1-simplex (line segment) or edge | 2-simplex (triangle) | 3-simplex (tetrahedron)

Notice that the k-simplex is homeomorphic to the closed k-ball, i.e., $\bar{B}_k = \{\bar{x} \in \mathbb{R}^k \mid \|\bar{x}\| \le 1\}$. "=" gives S^{k-1}, the $(k-1)$-sphere.

Indeed, the boundary of the k-ball is the $(k-1)$-sphere, e.g., 2-ball (or 2-disc) has the circle (1-sphere) as the boundary.

Each p-simplex is made of lower dimensional simplices, i.e., k-simplices with $k \le p$. Thus, $\Delta v_0 v_1 v_2$ contains vertices v_0, v_1, v_2, edges $\overline{v_0 v_1}, \overline{v_1 v_2}, \overline{v_0 v_2}$, and $\Delta v_0 v_1 v_2$ itself.

Def (face/coface). Let σ be the k-simplex defined on $S=\{\bar{v}_0, \bar{v}_1, \dots, \bar{v}_k\}$. A simplex τ defined on a subset T of S, $|T| \neq \phi$, is a **face** of σ, and σ is a **coface** of τ. The notation is $\tau \leq \sigma$, $\sigma \geq \tau$ (some books use $\tau \preceq \sigma$, $\sigma \succeq \tau$ \succeq, \preceq in LaTeX.

Thus, $\overline{v_0 v_1}$ is a face of $\Delta v_0 v_1 v_2$. So are v_0, v_1, v_2, $\overline{v_1 v_2}$, and $\overline{v_0 v_2}$.

A simplex is always a face of itself, i.e., $\sigma \leq \sigma$ and $\sigma \geq \sigma$.

We can attach simplices together "nicely" to form bigger objects called simplicial complexes.

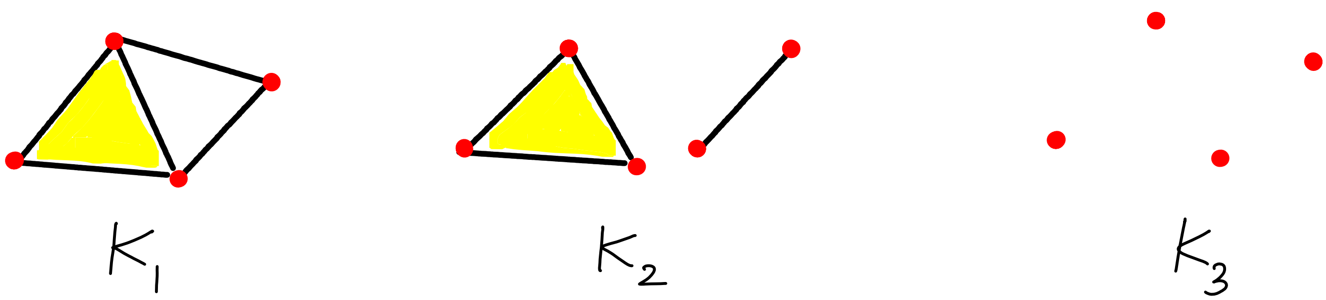
Def A ***simplicial complex*** K is a set of simplices such that

1. $\sigma \in K$, $\tau \leq \sigma \Rightarrow \tau \in K$; and — Every face of a simplex in K is also in K.
2. $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$ when $\sigma \cap \sigma' \neq \phi$. — In particular, the non-empty intersection of two simplices in K is a face of both of them, and hence in K as well.

The above definition holds in the case of both finite and nonfinite K. In the latter case, K has infinitely many simplices satisfying the two conditions. But we will usually limit our attention in this course to finite simplicial complexes, unless mentioned otherwise.

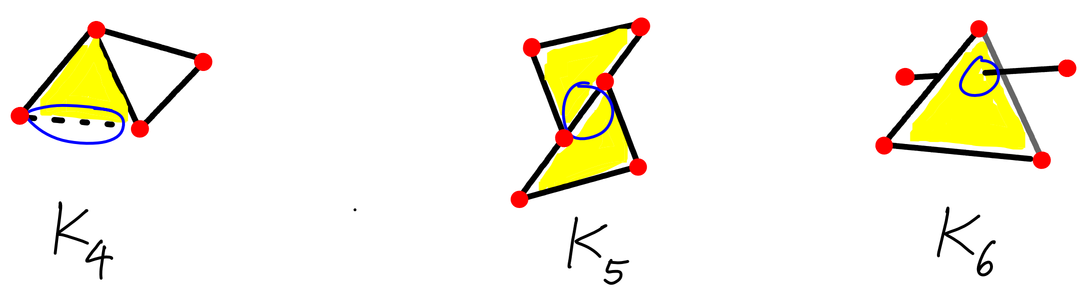
Note that a simplex is the convex hull of a **finite** number of affinely independent points. So, we never talk about "infinite-dimensional" simplices. We could talk about each of these points sitting in infinite-dimensional space, but we restrict our attention to $\mathbb{R}^d$ for finite d.

<u>Examples</u> Here are some simplicial complexes:



In particular, notice that a simplicial complex need not consist of just one connected component.

Here are some collections that are not simplicial complexes:



K_4 violates Condition 1, as one of the faces of the triangle in K_4 is not in the collection. K_5 and K_6 violate Condition 2, as the intersection of two simplices is not a face of either simplex in both cases.

Def The dimension of a simplicial complex is the same as that of the highest dimensional simplex in it, i.e.,

$$\dim K = \max \{\dim \sigma \mid \sigma \in K\}.$$

In the previous examples, $\dim(K_1) = 2$, $\dim(K_2) = 2$, and $\dim(K_3) = 0$.

Earlier we have been talking about continuous surfaces, e.g., the 2-sphere, torus, etc. And now we are talking about simplicial complexes as discrete objects. How do we "reconcile" the two notions? Indeed, we can formally define the "space" modeled by a simplicial complex. Later on, we will talk about a simplicial complex "triangulating", say, a torus, when this "space" is homeomorphic to $\mathbb{T}^2$.

Def The **underlying space** of a simplicial complex K the space made of all simplices in K together with the topology inherited from the ambient Euclidean space. We denote the underlying space of the simplicial complex K by $|K|$. Thus,

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

$|K|$ is also called the polyhedron (or polytope) of K.

$A \subseteq |K|$ is closed in $|K|$ iff $A \cap \sigma$ is closed $\forall \sigma \in K$.

MATH 529 : Lecture 6 (01/29/2026)

Today: * Abstract simplicial complexes (ASCs)
* geometric realization
* examples of ASCs

Def An **abstract simplicial complex** (ASC) is a collection $\mathcal{S}$ of finite non-empty sets such that if $A \in \mathcal{S}$, and $B \subseteq A$ with $B \neq \emptyset$, then $B \in \mathcal{S}$.

Note that the condition specified in the above definition of an abstract simplicial complex is equivalent to the first condition in the definition of a (regular) simplicial complex, which says that every face of a simplex in the complex is also in the complex.

The second intersection condition is trivially satisfied in the case of abstract simplicial complexes. The intersection of two sets is indeed a subset of both sets.

$\mathcal{S}$ itself can be finite or infinite, but each $A \subseteq \mathcal{S}$ is a finite set.

The sets in $\mathcal{S}$ are called the **simplices** of $\mathcal{S}$. The dimension of a simplex $A \in \mathcal{S}$ is $\dim(A) = |A| - 1$.

cardinality (# entries) of A

Note the correspondence of the above definition to the definition of simplices in the usual sense. Recall that a k-simplex is the convex hull of $(k+1)$ affinely independent points, which are its vertices. We maintain this correspondence by defining $\dim A = |A| - 1$ for any set $A \in \mathcal{S}$.

The **dimension** of $\mathcal{S}$ is $\dim(\mathcal{S}) = \max\{\dim(A) \mid A \in \mathcal{S}\}$.

In the definition of $\mathcal{S}$, we do assume that all $A \in \mathcal{S}$ are finite sets. And there exists a maximum dimensional simplex in $\mathcal{S}$. which is finite

The singleton sets in $\mathcal{S}$ are called its **vertices**, and is denoted by $\text{Vert}(\mathcal{S})$.

Again note the correspondence of these singleton sets to the vertices (0-simplices) in (geometric) simplicial complexes.

Here is an example of an abstract simplicial complex.

$$\mathcal{S} = \{\underbrace{\{0\}, \{1\}, \{2\}, \{3\}}_{\text{vertices}}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\}\}.$$

We can indeed check that the condition on inclusion of subsets is satisfied. For instance, consider the set $\{0,1,2\}$.

$$\{0,1,2\} \in \mathcal{S} \Rightarrow \text{need } \underset{\text{trivial}}{\{0,1,2\}}, \underbrace{\{0,1\}, \{0,2\}, \{1,2\}, \{0\}, \{1\}, \{2\}}_{\text{indeed!}} \in \mathcal{S}.$$

$\dim \mathcal{S} = 2$.

Given any (geometric) simplicial complex K, we can create an abstract simplicial complex $\mathcal{S}$ by taking just the sets of vertices in each simplex of K (and ignoring the geometry). $\mathcal{S}$ here is called the **vertex scheme** of K.

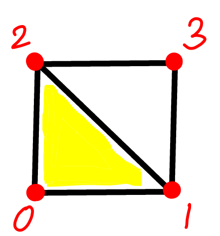
Symmetrically, K is a **geometric realization** of $\mathcal{S}$.
there could be other geometric realizations

Let's consider the ASC we saw previously:

$$\mathcal{S} = \{ \underbrace{\{0\}, \{1\}, \{2\}, \{3\}}_{\text{vertices}}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\} \}.$$

K shown here is a geometric realization of $\mathcal{S}$.

K



K can sit in $\mathbb{R}^2$

We have a vertex corresponding to each singleton set (i.e., vertex) in $\mathcal{S}$, an edge corresponding to each doublet, and a triangle for the triplet.

But this is just one geometric realization. In particular, if we were to specify the (x,y) coordinates of the vertices, we can imagine other realizations, e.g., by translating this one. We could also have a realization in $\mathbb{R}^3$, for instance.

It turns out that every ASC has a geometric realization.

Theorem (Geometric realization theorem) Every abstract simplicial complex $\mathcal{S}$ with $\dim \mathcal{S} = d$ has a geometric realization in $\mathbb{R}^{2d+1}$.

Idea of the Proof

We map vertices of $\mathcal{S}$ injectively to points in $\mathbb{R}^{2d+1}$, say, $f: \mathrm{Vert}(\mathcal{S}) \to \mathbb{R}^{2d+1}$. Why $2d+1$? (set of vertices in $\mathcal{S}$)
We use the fact that $2d+2$ or fewer points in $\mathbb{R}^{2d+1}$ that are in **general position** are affinely independent (AI).

Def $(d+1)$ points in $\mathbb{R}^d$ are in **general position** if no hyperplane contains more than d of those points.

The idea is that the points do not satisfy any more linear relationships than they must. For instance 3 points in $\mathbb{R}^2$ that are not collinear are in general position.

Recall that a d-simplex is the convex hull of $(d+1)$ AI points. We need to make sure that we will have "enough freedom", i.e., affine independence, among the mapped vertices so that we can map all the simplices in $\mathcal{S}$ to corresponding simplices in the geometric simplicial complex.

Consider $A, B \in \mathcal{S}$. Since $\dim(\mathcal{S}) = d$, $|A|, |B| \leq d+1$.

Hence, $|A \cup B| = |A| + |B| - |A \cap B| \leq d+1 + d+1 = 2d+2$.

Hence by going to $\mathbb{R}^{2d+1}$ and choosing points there in general position, we can ensure that (up to) $2d+2$ points are AI.

$\Rightarrow$ Any convex combination $\bar{x}$ in $A \cup B$ is unique.
$\Rightarrow$ $\bar{x} \in A$ and $\bar{x} \in B \iff \bar{x} \in A \cap B$.
↳ ensures the second requirement of nonempty intersections of two simplices being their faces.

We could often find geometric realizations in $\mathbb{R}^{d'}$ for d' smaller than $2d+1$.

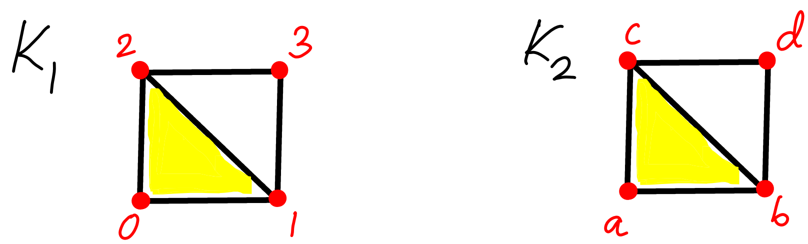
$\square$

How do we compare abstract simplicial complexes? We had previously defined the concept of homeomorphism to study when two topological spaces are "similar". We now define corresponding notions for simplicial complexes. Recall that $\mathrm{Vert}(\mathcal{S})$ represents the vertex set of the ASC $\mathcal{S}$.

Def Two abstract simplicial complexes $\mathcal{S}_1$ and $\mathcal{S}_2$ are **isomorphic** if there exists a bijection $\varphi : \mathrm{Vert}(\mathcal{S}_1) \to \mathrm{Vert}(\mathcal{S}_2)$ such that $A \in \mathcal{S}_1$ iff $\varphi(A) \in \mathcal{S}_2$. φ is an isomorphism between $\mathcal{S}_1$ and $\mathcal{S}_2$. We write $\mathcal{S}_1 \approx \mathcal{S}_2$ here.

simplex

In this setting, every simplex in $\mathcal{S}_1$ has a unique corresponding simplex in $\mathcal{S}_2$.



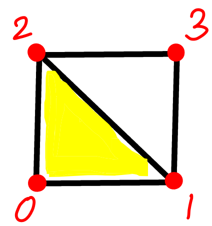
vertex schemes of K_1 and K_2 are isomorphic

Notice that the similarity here is defined for **abstract** simplicial complexes. Hence, K_1 and K_2 above need not be sitting in the same space. Still, they are isomorphic as ASCs. We make this notion precise in the following theorem.

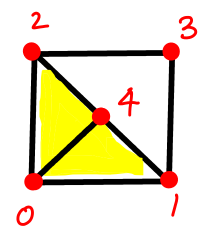
Theorem Two (geometric) simplicial complexes K_1 and K_2 are isomorphic, or **simplicially homeomorphic**, iff their vertex schemes $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic as abstract simplicial complexes. We denote this fact by $K_1 \cong K_2$, which implies $|K_1| \approx |K_2|$, and $\mathcal{S}_1 \approx \mathcal{S}_2$.

The implication might not go the other way, though. For instance, $K_1 \cong K_2$ above. Now consider K_3 as shown below.

K_1



K_3



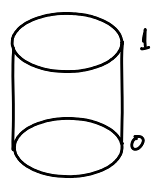
Notice that $K_1 \ncong K_3$, even though $|K_1| \approx |K_3|$. In fact, their underlying spaces could very well be identical!

From the computational point of view, while simplicial complexes that are "smaller", i.e., have a smaller number of simplices while modeling the same topological space are usually preferred. At the same time, geometry might dictate that we need a large number of simplices to capture the complexity.

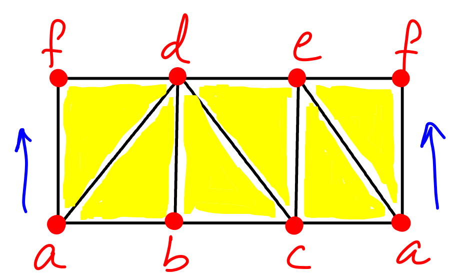
How do we use ASCs? We illustrate several examples

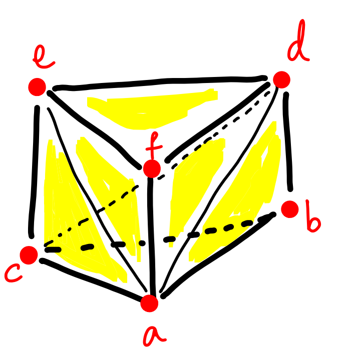
1. cylinder $S^1 \times [0,1]$

geometric representation!



L





K

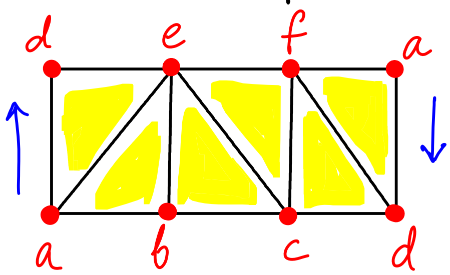
The underlying space of L appears to be rectangle, but notice how the vertex labels identify the left and right edges (both are af).

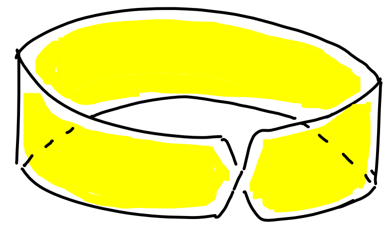
$$L = \{ \{a,b,d\}, \{a,d,f\}, \{b,c,d\}, \{c,d,e\}, \{a,c,e\}, \{a,e,f\}, \text{ and all nonempty subsets} \}$$

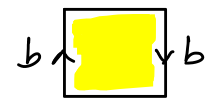
K is one geometric realization of L here.

2. Möbius strip

We start with the ASC L here

L 

$\rightsquigarrow$ K 



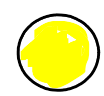
The left and right vertical edges are identified, after a "twist". Like in Example 1, the underlying space is a rectangle, but vertex labels are different.

L represents the Möbius strip, i.e., K is a geometric realization of L.

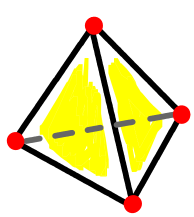
We could also specify L abstractly:

$L = \{ \{a,d,e\}, \{a,b,e\}, \{b,c,e\}, \{c,e,f\}, \{c,d,f\}, \{a,d,f\}$, and all nonempty subsets of these triplets $\}$.

The abstract simplicial complexes shown above consist of triangles – indeed, they are triangulations. In general, triangulations consist of triangles (in 2D), and simplices in general as we formalize below.

Notice that each k-simplex $\approx$ k-ball (closed). 2-ball is the closed 2-disc. 

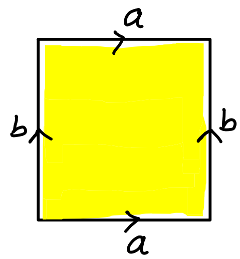
Def (Triangulation). A **triangulation** of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.

Example: 

surface of a tetrahedron (triangles and their faces) is a triangulation of the 2-sphere S^2.

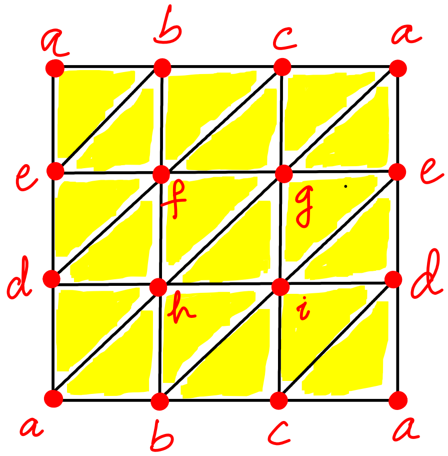
A triangulation is a piecewise linear representation of the topological space.

3. <u>Torus</u> ($\mathbb{T}^2$)



$\longrightarrow$ this diagram is not a triangulation!

K:

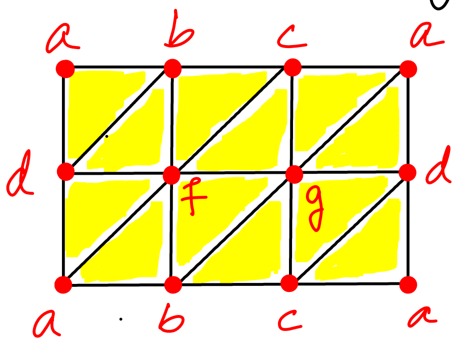


Why so many triangles?

This triangulation K of the torus has 18 (different) triangles. One might wonder if we could produce a triangulation using a much smaller number of triangles.

Consider the following candidate triangulation K':

K'



Is K' a triangulation of $\mathbb{T}^2$? No! For instance, consider edge $\overline{ad}$. It is a face of four triangles: adb, adf, adc, adg.

Hence, points on $\overline{ad}$ do not have neighborhoods homeomorphic to $\mathbb{R}^2$. We're doing "too much gluing" here.

Q. What is the minimum number of triangles needed to produce a triangulation of $\mathbb{T}^2$?

A. We need at least 14 triangles.

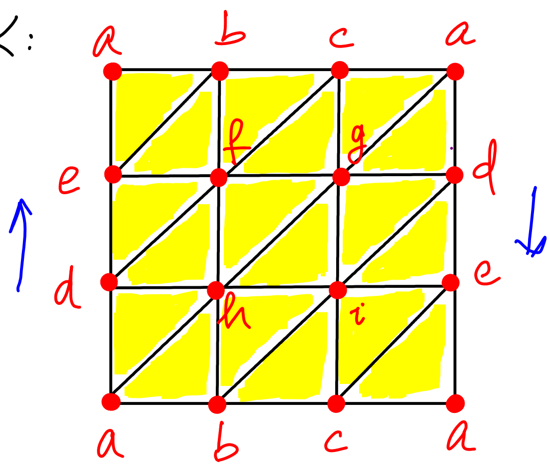
<u>**Rule:**</u> In a triangulation of a 2-manifold (with boundary), each edge must be part of (one or) two triangles.
Edges that are part of only one triangle each form the boundary of the 2-manifold.

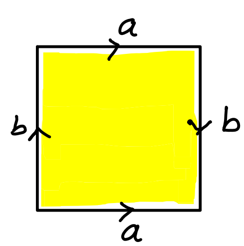
Recall that every point on a 2-manifold has a neighborhood homeomorphic to an open disc. The corresponding requirement for edges becomes that each edge has to be part of exactly two triangles. Similarly, points on the boundary of a 2-manifold have neighborhoods homeomorphic to half discs. Edges that are part of single triangles are indeed the boundary edges in the triangulation. The result extends to d-dimensions — every (d-1)-simplex is the face of one or two d-simplices.

The above rule could be used to check if a given simplicial complex is the triangulation of a manifold or not. But, be warned that satisfying this rule **alone** is not enough to identify the given simplicial complex as the triangulation of a specific manifold, e.g., the torus.

4. Klein bottle ($\mathbb{K}^2$)

K:





Same rule applies here — each edge is shared by two triangles exactly.

For example, edge $\overline{ae}$ is part of triangles Δabe and Δace.

We talked about distinguishing spaces that are not homeomorphic. In practice, spaces or objects are usually represented by triangulations. Checking for homeomorphisms between triangulations is not easy. How can we distinguish two topological spaces computationally? One option is to use an invariant.