MATH 524 – Lecture 15 (10/10/2023)

Today:
* star condition
* simplicial approximation
* subdivisions

Topological Invariance of Homology Groups

Want to show: $H_p(K)$ depends only on $|K|$, and not on the specific choice of K.

Method: We showed that a simplicial map $f: |K| \to |L|$ induces a homomorphism f_* of the homology groups. We want to argue that an arbitrary continuous map $h: |K| \to |L|$ can be approximated by a simplicial map f, and then argue that the induced homomorphism depends only on h, and not on the particular approximation chosen.

Simplicial Approximation

We present the concept of approximation in the context of simplicial complexes. Rather than specifying an error of approximation as is the practice in some other fields of mathematics, we present a condition defined using star of the vertices.

Def Let $h: |K| \to |L|$ be a continuous map. We say h satisfies the **star condition** relative to (or w.r.t.) K and L if for every vertex $v \in K^{(0)}$, there exists a vertex $w \in L^{(0)}$ such that

$$h(\operatorname{St} v) \subset \operatorname{St} w.$$

Lemma 14.1 [M] Let $h:|K|\to|L|$ satisfy the star condition relative to K and L. Choose $f: K^{(0)}\to L^{(0)}$ such that $\forall v\in K^{(0)}$, $h(\mathrm{St}\, v)\subset \mathrm{St}\, \underbrace{f(v)}_{w}$.

(a) For $\sigma\in K$, choose $\bar{x}\in \mathrm{Int}\,\sigma$ and $\tau\in L$ such that $h(\bar{x})\in \mathrm{Int}\,\tau$. Then f maps each vertex of σ to a vertex of τ.

(b) f may be extended to a simplicial map of K into L, which we also call f.

(c) If $g: K\to L$ is another simplicial map such that $h(\mathrm{St}\, v)\subset \mathrm{St}\,(g(v))$ $\forall v\in K^{(0)}$, then f and g are contiguous.

Proof

(a) Let $\sigma = v_0\cdots v_p$. Then $\bar{x}\in \mathrm{St}\, v_i$ $\forall i$. So

$$h(\bar{x})\in h(\mathrm{St}\, v_i)\subset \mathrm{St}\, f(v_i) \quad \forall i.$$

So, $h(\bar{x})$ has positive barycentric coordinates w.r.t. each vertex $f(v_i)$, $i=0,\dots,p$. These vertices must form a subset of the vertex set of τ.

(b) Straightforward.

(c) Since $h(\bar{x})\in h(\mathrm{St}\, v_i)\subset \mathrm{St}(g(v_i))$ $\forall i$, the vertices $g(v_0),\dots,g(v_p)$ must also be vertices of τ. Thus $f(v_0),\dots,f(v_p), g(v_0),\dots,g(v_p)$ span a face of τ. □

We define the concept of simplicial approximation using the star condition.

Def Let $h: |K| \to |L|$ be a continuous map. If $f: K \to L$ is a simplicial map such that $h(\mathrm{St}\, v) \subset \mathrm{St}\, f(v)$ $\forall\, v \in K^{(0)}$, then f is called a **simplicial approximation** to h.

Intuitively, f is "close to" h in the following sense. Given $\bar{x} \in |K|$, there exists a simplex τ of L such that $h(\bar{x}), f(\bar{x}) \in \tau$. We make this concept formal in a lemma.

Lemma 14.2 [M] Let $f: K \to L$ be a simplicial approximation to $h: |K| \to |L|$. Given $\bar{x} \in |K|$, there exists a simplex $\tau \in L$ such that $h(\bar{x}) \in \mathrm{Int}\, \tau$, $f(\bar{x}) \in \tau$.

Proof Follows from Lemma 14.1 (a).

We can also compose simplicial approximations to get a simplicial approximation for the composition of continuous maps.

Theorem 14.3 [M] Let $h: |K| \to |L|$ and $k: |L| \to |M|$ have simplicial approximations $f: K \to L$ and $g: L \to M$, respectively. Then $g \circ f$ is a simplicial approximation to $k \circ h$.

Proof 1. $g \circ f$ is a simplicial map.

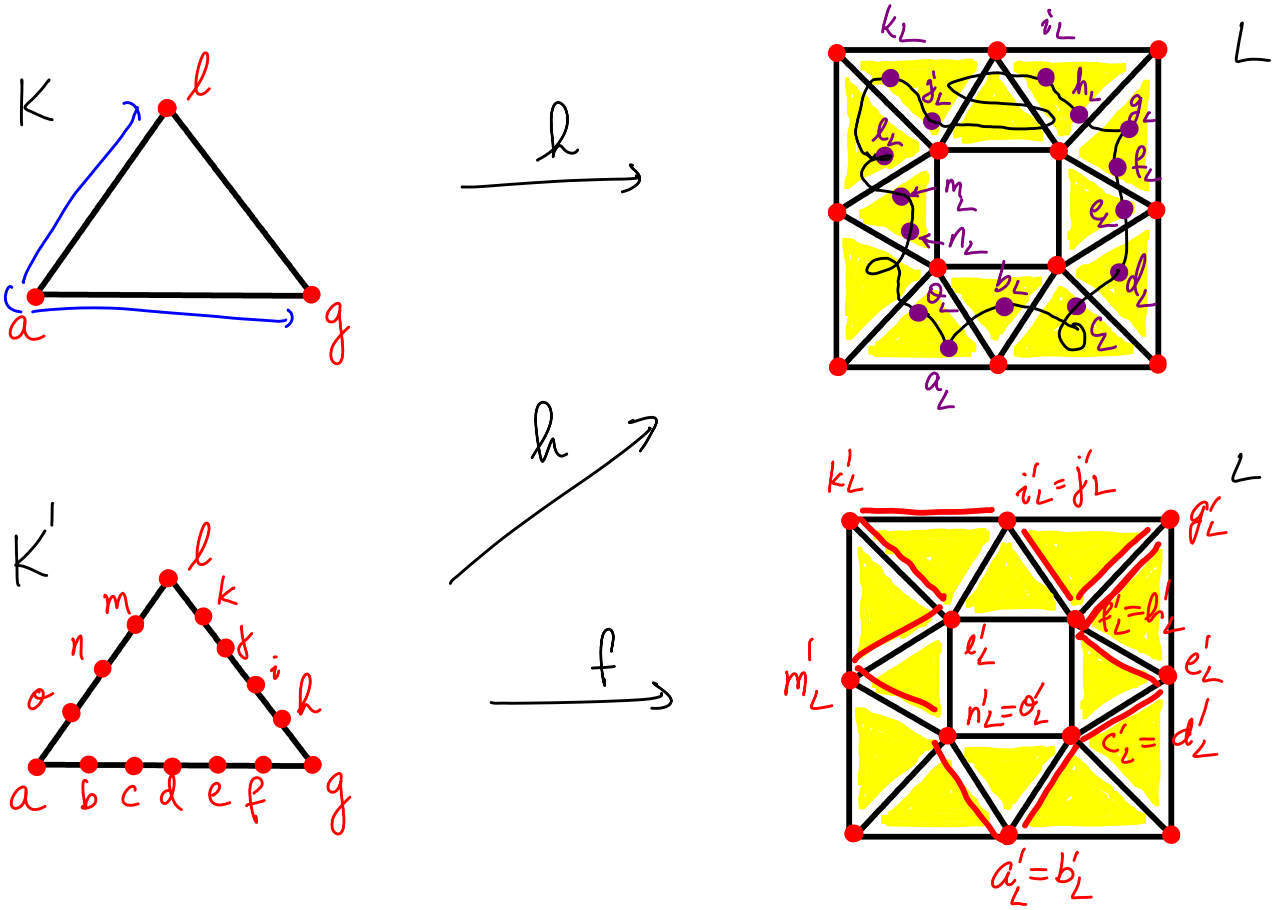
2. If $v \in K^{(0)}$, then $h(\mathrm{St}\, v) \subset \mathrm{St}\, f(v)$, as f is a simplicial approximation to h. Hence

$k(h(\mathrm{St}\, v)) \subset k(\mathrm{St}\, f(v)) \subset \mathrm{St}\,(g(f(v)))$,

as g is a simplicial approximation to k. $\square$

Example

$h(St(a,K)) \not\subset St(v,L)$ for any $v \in L^{(0)}$.



We consider K to be the 1-complex made of 3 1-simplices, and L to be the 2-complex that models an annulus. Let $h: |K| \to |L|$ map all of $|K|$ to the loop on $|L|$ as shown. We also consider a "refinement" of K by adding several more vertices to obtain K' such that $|K| = |K'|$. Hence, h applies without change to K'.

It is clear that h does not satisfy the star condition relative to K and L. Indeed, notice that $St(a,K) = K - \{\overline{lg}, l, g\}$, and there is no vertex in L such that $h(St(a,K))$ is a subset of its star in L.

But h does satisfy the star condition relative to K' and L. So h has a simplicial approximation $f: K' \to L$, and one such approximation is shown.

If $h: |K| \to |L|$ satisfies the star condition relative to K and L, there exists a well defined homomorphism

$$h_*: H_p(K) \to H_p(L) \quad \text{for all } p$$

obtained by setting $h_* = f_*$, where f is a simplicial approximation to h.

Not surprisingly, we can extend the star condition to the level of relative homology.

Lemma 14.4 [M] Let $h: |K| \to |L|$ satisfy the star condition relative to K & L, and suppose h maps $|K_0|$ into $|L_0|$.

(a) Any simplicial approximation $f: K \to L$ to h also maps $|K_0|$ into $|L_0|$. Also, the restriction of f to K_0 is a simplicial approximation to the restriction of h to $|K_0|$.

(b) Any two simplicial approximations f and g to h are contiguous as maps of pairs.

Subdivision

We had seen in the example that $h : |K| \to |L|$ did not satisfy the star condition relative to K and L, but it did relative to K' and L, where K' is a "finer" or "refined" version of K. We formalize this idea now, and talk about subdivisions.

We first formally define a subdivision. We then introduce barycentric subdivision as a "canonical" subdivision.

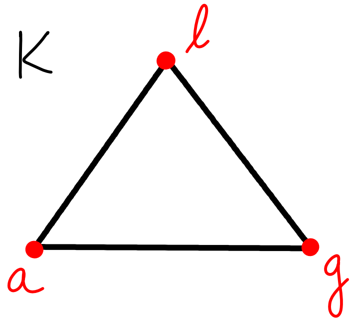
Def Let K be a geometric complex in $\mathbb{R}^d$. A complex K' is said to be a **subdivision** of K if

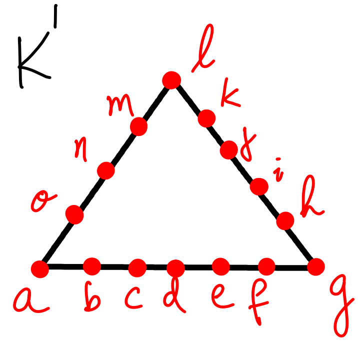
1. each simplex of K' is contained in a simplex of K, and
2. each simplex of K is the union of **finitely** many simplices of K'.

The conditions $\Rightarrow$ $|K|$ and $|K'|$ are equal as sets. The finiteness condition in 2. guarantees that $|K|$ and $|K'|$ are equal as topological spaces.

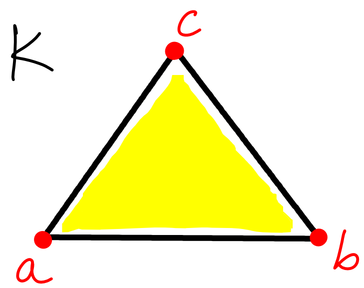
Examples

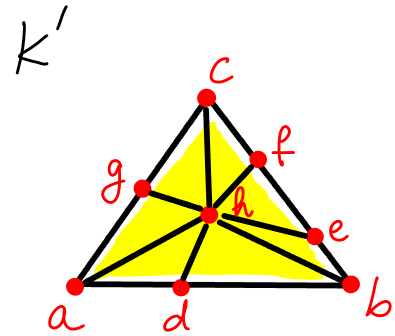
1.





2.





In 1 and 2 above, K' is a subdivision of K.

3. K : $[0,1]$ (1-simplex and its vertices)

K' : $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ $\forall n \in \mathbb{Z}_{>0}$, and their vertices, and the vertex 0.

$|K| = |K'|$ as sets, but they are not equal as topological spaces, as the finiteness requirement in Condition 2 is violated. Hence K' is not a subdivision of K.