MATH 401: Lecture 23 (11/04/2025)

Today: $\int$ or $\frac{d}{dx}$ of $\{f_n(x)\}$

4.3 Integrating and Differentiating Sequences

We consider more general questions about properties of $\{f_n(x)\}$ that are "preserved" under convergence. In particular, given $\{f_n(x)\} \xrightarrow[\text{pointwise}]{\text{uniform or}} f(x)$, we consider the following questions:

1. Does $\left\{\int_a^b f_n(x)dx\right\} \xrightarrow[\text{pointwise}]{\text{uniform or}} \int_a^b f(x)dx$?

2. Does $\{f_n'(x)\} \xrightarrow[\text{pointwise}]{\text{uniform or}} f'(x)$?

Problem 3, LSIRA pg 91 Let $f_n : [0,1] \to \mathbb{R}$ be $f_n(x) = nx(1-x^2)^n$.

Show that $f_n(x) \to 0 \ \forall x \in [0,1]$ as $n \to \infty$, but $\int_0^1 f_n(x)dx \to \frac{1}{2}$.

This is a problem directly from your Calculus I & II classes!

For $x \in [0,1]$, we ask $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n = ?$

We consider the end points first:

$$x=0 \Rightarrow f_n(0) = n\cdot 0\cdot(1-0^2)^n = 0.$$

$$x=1 \Rightarrow f_n(1) = n\cdot 1\cdot(1-1^2)^n = 0.$$

For $x \in (0,1)$, $(1-x^2) \in (0,1) \Rightarrow (1-x^2)^n \to 0$ as $n \to \infty$.

One could guess that the rate of increase on n is smaller than the rate of decrease of $(1-x^2)^n$ as $n \to \infty$, and hence the product $\to 0$. But we can compute the limit exactly!

We use L'Hôpital's rule (L'H):

$$\lim_{n\to\infty} nx(1-x^2)^n = \lim_{n\to\infty} \frac{nx}{(1-x^2)^{-n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{x}{\ln(1-x^2)\,(1-x^2)^{-n}(-1)}$$

$$= \lim_{n\to\infty} \frac{-x\overset{0}{(1-x^2)^n}}{\ln(1-x^2)} = 0.$$

Let's look at the integral now...

$$\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx$$

$$= \left(-\frac{n}{2}\right)\int_0^1 (1-x^2)^n(-2x\,dx)$$

$$= \left(-\frac{n}{2}\right)\int_1^0 u^n\,du = \frac{-n}{2}\left(\frac{u^{n+1}}{n+1}\right)\Bigg|_1^0$$

$$= \frac{n}{2(n+1)}$$

take $u = 1-x^2$

$\Rightarrow \frac{du}{dx} = -2x$

$du = -2x\,dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=0$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}.$$

This problem illustrates the situation where $f_n(x)$ converges pointwise to $f(x)$ (=0 here), but $\int f_n(x)\,dx$ does not converge to $\int f(x)\,dx$.

We get "preservation of integrals" when we have uniform convergence!

Proposition 4.3.1 Let $\{f_n(x)\}$ be a sequence of continuous functions that converge uniformly to $f(x)$ on $[a,b]$. Then the functions

$$F_n(x) = \int_a^x f_n(t)\,dt \quad \text{converge uniformly to}$$

$$F(x) = \int_a^x f(t)\,dt \quad \text{on } [a,b]. \longrightarrow \forall x \in [a,b].$$

Given: $\{f_n(x)\}$ converges uniformly to $f(x)$ $\Rightarrow \exists N \in \mathbb{N}$ s.t.

$$|f_n(t) - f(t)| < \frac{\epsilon}{b-a} \quad \forall t \in [a,b].$$

choice of ϵ will become clear below!

$$\Rightarrow |F_n(x) - F(x)| = \left| \int_a^x (f_n(t) - f(t))\,dt \right|$$

$$\leq \int_a^x |f_n(t) - f(t)|\,dt$$

$$< \int_a^x \frac{\epsilon}{b-a}\,dt \leq \int_a^b \frac{\epsilon}{b-a}\,dt = \frac{\epsilon(b-a)}{(b-a)} = \epsilon.$$

$\Rightarrow \{F_n(x)\}$ converges uniformly to $F(x)$ on $[a,b]$. $\square$

It is often helpful to state the above result with variable lower limits for the integrals (rather than a).

Corollary 4.3.2 Let $\{f_n(x)\}$ be a sequence of continuous functions that converge uniformly to $f(x)$ on $[a,b]$. Then the functions

$$F_n(x) = \int_{x_0}^{x} f_n(t)\,dt \quad \text{converge uniformly to}$$

$$F(x) = \int_{x_0}^{x} f(t)\,dt \quad \text{for any } x_0 \in [a,b], \text{ and } \forall x \in [a,b].$$

We now state these results in terms of series. Just as we did with sequences, we would like "nice" properties of individual functions in the series carry over to the limit...

Reformulation in terms of Series

Def A series of functions $\sum_{n=0}^{\infty} v_n(x)$ **converges pointwise** to $f(x)$ on a set I if the sequence $\{S_N(x)\}$ of partial sums $S_N(x) = \sum_{n=0}^{N} v_n(x)$ converge to $f(x)$ $\forall x \in I$. Similarly, the series $\sum_{n=0}^{\infty} v_n(x)$ **converges uniformly** to $f(x)$ on I if $\{S_N(x)\}$ converges uniformly to $f(x)$ on I.

Corollary 4.3.3 Let $\{v_n(x)\}$ be a sequence of continuous functions such that the series $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on $[a,b]$.

Then $\forall\, x_0 \in [a,b]$, the series

$$\sum_{n=0}^{\infty} \int_{x_0}^{x} v_n(t)\,dt \quad \text{converges uniformly, and}$$

$$\int_{x_0}^{x} \sum_{n=0}^{\infty} v_n(t)\,dt = \sum_{n=0}^{\infty} \int_{x_0}^{x} v_n(t)\,dt.$$

Key message: We can integrate the series term-by-term! We cannot always interchange the order of $\int$ and $\sum$, but can do so here due to uniform convergence of the input series.

In order to use this result, we need a way to test if a given series converges uniformly. How can we check if $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly?

Recall Proposition 4.2.3, which could be used to test uniform convergence of a sequence.

Proposition 4.3.4 (Weierstrass' M-test) Let $\{v_n(x)\}$ be a sequence of functions $v_n : A \to \mathbb{R}$, and let there exist a convergent series $\sum_{n=0}^{\infty} M_n$ such that $M_n \geq 0$ and $|v_n(x)| \leq M_n$ $\forall n \in \mathbb{N}, \forall x \in A$.

Then $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on A.

show existence of a series of term-wise (upper) bounds!

<u>Problem 1, LSIRA pg 91</u> Show that $\sum_{n=0}^{\infty} \frac{\cos(nx)}{n^2+1}$ converges uniformly on $\mathbb{R}$.

$$|v_n(x)| = \left|\frac{\cos(nx)}{n^2+1}\right| \leq \frac{1}{n^2+1} \leq \frac{1}{n^2} = M_n \quad (n \geq 1).$$

Note that the term for $n=0$ is $\frac{\cos(0)}{1} = 1$, which can be handled separately.

Now, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Why?

Hence the input series converges uniformly by Weierstrass' M-test.

$$\underbrace{S_n}_{\text{partial sums}} = \sum_{k=1}^{n} \frac{1}{k^2}.$$

To show a series converges, show the corresponding sequence of partial sums converges.

Note: $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.

$$\Rightarrow S_n \leq 1 + \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2.$$

$$\Rightarrow \lim_{n\to\infty} S_n = 2.$$

Hence by Corollary 4.3.3, we get that

this observation is not part of the solution to Problem 1 above!

$$\int_0^x \sum_{n=0}^{\infty} \frac{\cos(nt)}{n^2+1}\,dt = \sum_{n=0}^{\infty} \underbrace{\int_0^x \frac{\cos(nt)}{n^2+1}\,dt}_{\left.\frac{\sin(nt)}{n(n^2+1)}\right|_0^x} = \sum_{n=0}^{\infty} \frac{\sin(nx)}{n(n^2+1)}.$$

We now consider the same question on derivatives. If a sequence/series of functions converges (uniformly or pointwise) to a function, do their derivatives also converge to the derivative of the limit function?

We immediately get the answer is no in general!

Example

It can be shown that the sequence $\left\{\frac{\sin nx}{n}\right\}$ converges uniformly to 0, but the sequence of derivatives $\{\cos nx\}$ does not converge.

oscillates between -1 and 1