MATH 401 : Lecture 29 (12/04/2025)

Today: * Inequalities

Inequalities (will not be tested directly in the final).

Cauchy-Schwarz Inequality (CSI) in $\mathbb{R}^m$

$$|\langle \bar{x},\bar{y}\rangle| = |\bar{x}^T\bar{y}| \leq \|\bar{x}\|\,\|\bar{y}\| \quad \forall\, \bar{x},\bar{y} \in \mathbb{R}^m$$

Equality holds iff $\bar{x} = t\bar{y}$ for $t \in \mathbb{R}$.

Case 1 If $\bar{x}=\bar{0}$ or $\bar{y}=\bar{0}$ then $\langle \bar{x},\bar{y}\rangle = 0 = \underbrace{\|\bar{x}\|\cdot\|\bar{y}\|}_{\text{one of them is } 0}$.

Case 2 Let $\|\bar{x}\|=\|\bar{y}\|=1$ (i.e., both are unit vectors).

For any $\bar{x}\in\mathbb{R}^m$, $\langle \bar{x},\bar{x}\rangle = \|\bar{x}\|^2 \geq 0$.

$$\langle \bar{x}-\bar{y}, \bar{x}-\bar{y}\rangle \geq 0$$

$$\Rightarrow \underbrace{\langle \bar{x},\bar{x}\rangle}_{=\|\bar{x}\|^2=1} + \underbrace{\langle \bar{y},\bar{y}\rangle}_{=\|\bar{y}\|^2=1} - 2\langle \bar{x},\bar{y}\rangle \geq 0$$

$$\Rightarrow 2\langle \bar{x},\bar{y}\rangle \leq 2 \quad \Rightarrow \langle \bar{x},\bar{y}\rangle \leq 1.$$

Also, $\langle \bar{x}+\bar{y}, \bar{x}+\bar{y}\rangle \geq 0$

$$\Rightarrow \underbrace{\langle \bar{x},\bar{x}\rangle}_{=1} + \underbrace{\langle \bar{y},\bar{y}\rangle}_{=1} + 2\langle \bar{x},\bar{y}\rangle \geq 0$$

$$\Rightarrow 2+2\langle \bar{x},\bar{y}\rangle \geq 0 \quad \Rightarrow -\langle \bar{x},\bar{y}\rangle \leq 1$$

$$\Rightarrow |\langle \bar{x},\bar{y}\rangle| \leq 1.$$

Equality iff $\bar{x} = t\bar{y}$

$|\langle \bar{x},\bar{y}\rangle| = 1 \Rightarrow \langle \bar{x},\bar{y}\rangle = \pm 1$

If $\langle \bar{x},\bar{y}\rangle = 1$ then $\langle \bar{x}-\bar{y}, \bar{x}-\bar{y}\rangle = 0 \Rightarrow \bar{x}=\bar{y}$

If $\langle \bar{x},\bar{y}\rangle = -1$ then $\langle \bar{x}+\bar{y}, \bar{x}+\bar{y}\rangle = 0 \Rightarrow \bar{x}=-\bar{y}$.

Hence $|\langle \bar{x},\bar{y}\rangle| = \|\bar{x}\|\,\|\bar{y}\|$ iff $\bar{x} = \pm\bar{y}$.

<u>Case 3</u> Assume $\bar{x}, \bar{y} \neq \bar{0}$ (and not necessarily unit vectors).

Take $\bar{u} = \dfrac{\bar{x}}{\|\bar{x}\|}$, $\bar{v} = \dfrac{\bar{y}}{\|\bar{y}\|}$.

By result of Case 2, $|\langle \bar{u}, \bar{v} \rangle| \leq 1$

$$\Rightarrow \left| \left\langle \frac{\bar{x}}{\|\bar{x}\|}, \frac{\bar{y}}{\|\bar{y}\|} \right\rangle \right| = \left| \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\| \, \|\bar{y}\|} \right| \leq 1$$

$$\Rightarrow \quad |\langle \bar{x}, \bar{y} \rangle| \leq \|\bar{x}\| \, \|\bar{y}\|.$$

<u>Equality</u>

Assume $\langle \bar{x}, \bar{y} \rangle = \|\bar{x}\| \, \|\bar{y}\|$

$$\Rightarrow \left\langle \frac{\bar{x}}{\|\bar{x}\|}, \frac{\bar{y}}{\|\bar{y}\|} \right\rangle = 1 \iff \frac{\bar{x}}{\|\bar{x}\|} = \frac{\bar{y}}{\|\bar{y}\|} \iff \bar{y} = \frac{\|\bar{y}\|}{\|\bar{x}\|} \bar{x}.$$

Other case is similar... $\square$

<u>Triangle Inequality ($\triangle$) in $\mathbb{R}^m$</u>

$$\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$$

<u>Proof</u>

$$\begin{aligned} (\|\bar{x} + \bar{y}\|)^2 &= \langle \bar{x} + \bar{y}, \bar{x} + \bar{y} \rangle \\ &= \underset{\|}{\langle \bar{x}, \bar{x} \rangle} + \underset{\|}{\langle \bar{y}, \bar{y} \rangle} + 2 \langle \bar{x}, \bar{y} \rangle \quad \leq \rightarrow \text{by CSI} \\ &\leq \|\bar{x}\|^2 + \|\bar{y}\|^2 + 2\|\bar{x}\| \|\bar{y}\| \\ &= (\|\bar{x}\| + \|\bar{y}\|)^2 \end{aligned}$$

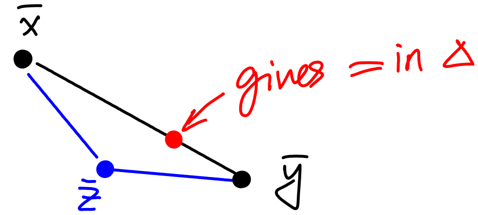
$|\langle \bar{x}, \bar{y} \rangle| \leq \|\bar{x}\| \|\bar{y}\|$
$\Rightarrow \langle \bar{x}, \bar{y} \rangle \leq \|\bar{x}\| \|\bar{y}\|$
as well

As $\| \, \| \geq 0$, taking square root gives ($\triangle$).

Equality in $(\triangle)$

$$\|\bar{x}-\bar{y}\| \le \|\bar{x}-\bar{z}\| + \|\bar{z}-\bar{y}\| \quad (\triangle)$$

This is the form of $(\triangle)$ we used a lot!



Def (Convex combination) For $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^m$, $\bar{z}$ is a **convex combination** of $\bar{x}$ and $\bar{y}$ iff there exists a $0 \le \lambda \le 1$ such that $\bar{z} = \lambda \bar{x} + (1-\lambda)\bar{y}$.

Theorem $(\triangle)$ is satisfied as equality iff $\bar{z}$ is a convex combination of $\bar{x}$ and $\bar{y}$.

$(\Rightarrow)$ Let $\bar{z} = \lambda \bar{x} + (1-\lambda)\bar{y}$, $\lambda \in [0,1]$.

$$\begin{aligned}
\Rightarrow\ & \|\bar{x}-\bar{z}\| + \|\bar{z}-\bar{y}\| \\
&= \|\bar{x} - (\lambda\bar{x} + (1-\lambda)\bar{y})\| + \|\lambda\bar{x} + (1-\lambda)\bar{y} - \bar{y}\| \\
&= \|(1-\lambda)(\bar{x}-\bar{y})\| + \|\lambda(\bar{x}-\bar{y})\| \\
&= (1-\lambda+\lambda)\|\bar{x}-\bar{y}\| = \|\bar{x}-\bar{y}\|.
\end{aligned}$$

$(\Leftarrow)$ Let $\|\bar{x}-\bar{y}\| = \|\bar{x}-\bar{z}\| + \|\bar{z}-\bar{y}\|$

$$\|\bar{a}+\bar{b}\| = \|\bar{a}\| + \|\bar{b}\| \iff \bar{a} = t\bar{b},\ t \ge 0$$

as $\langle \bar{a}, \bar{b} \rangle = \|\bar{a}\|\|\bar{b}\|\underbrace{\cos\theta}_{=1 \iff \theta = 0.}$

$$\begin{aligned}
&\Rightarrow \bar{x} - \bar{z} = t(\bar{z} - \bar{y}) \text{ for } t \ge 0 \\
&\Rightarrow (1+t)\bar{z} = \bar{x} + t\bar{y} \\
&\Rightarrow \bar{z} = \underbrace{\frac{1}{1+t}}_{\lambda}\bar{x} + \underbrace{\frac{t}{1+t}}_{1-\lambda}\bar{y} \\
&\Rightarrow \bar{z} = \lambda\bar{x} + (1-\lambda)\bar{y} \text{ for } \lambda = \frac{1}{1+t} \in [0,1].
\end{aligned}$$

$\square$

Young's Inequality

(Y) $\quad xy \leq \dfrac{x^p}{p} + \dfrac{y^q}{q}$ for $x, y \geq 0$, $p > 1$, q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Equality holds iff $x^p = y^q$.

$\dfrac{1}{q} = 1 - \dfrac{1}{p} = \dfrac{p-1}{p}$

$\Rightarrow q = \dfrac{p}{p-1} > 1$ as well.

and $q(p-1) = p$.

let $f(x) = \dfrac{x^p}{p} + \dfrac{y^q}{q} - xy$ for y fixed, and $x > 0$.

result holds trivially when $x = 0$.

$f(x) \geq 0$ gives (Y).

$f'(x) = \dfrac{p x^{p-1}}{p} - y = 0 \Rightarrow x_0 = (y)^{\frac{1}{p-1}}$ is the critical point.

$f''(x) = (p-1) x^{p-2} > 0$ (as $p > 1$, and $x > 0$).

$\Rightarrow f''(x_0) > 0 \Rightarrow x_0$ is a minimum.

$$f(x_0) = \frac{(x_0)^p}{p} + \frac{(x_0)^{q(p-1)}}{q} - x_0 (x_0)^{(p-1)} \qquad \Bigg| \quad q(p-1) = p$$

$$= (x_0)^p \left(\frac{1}{p} + \frac{1}{q} - 1 \right) = 0.$$

So, $f(x_0) = 0$ is the global minimum of $f(x)$.

$\Rightarrow f(x) \geq 0 = f(x_0)$, as x_0 is the minimum. $\Rightarrow$ (Y).

Equality holds iff $\left(x = y^{\frac{1}{p-1}} \right)^p \Rightarrow x^p = y^q$.

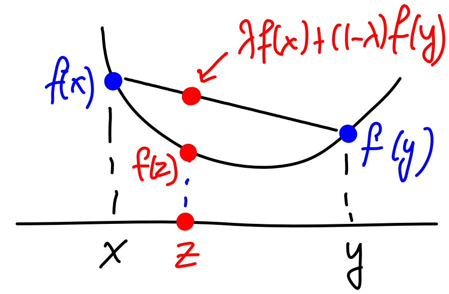
$\square$

Convex Functions

Def $f: \mathbb{R}^n \to \mathbb{R}$ is strongly **convex** if

$$f(\lambda \bar{x} + (1-\lambda)\bar{y}) \leq \lambda f(\bar{x}) + (1-\lambda) f(\bar{y}) \quad \forall \bar{x}, \bar{y} \in \mathbb{R}^n,\ \lambda \in [0,1].$$

The (graph of the) function lies below the line segment connecting end points.



Lemma Let $f: \mathbb{R} \to \mathbb{R}$ be a convex function. For $a<b<c$ in $\mathbb{R}$, show that $f(a-b+c) \leq f(a) - f(b) + f(c)$.

$$b \in (a,c) \Rightarrow b = \lambda a + (1-\lambda)c \text{ for } \lambda \in (0,1)$$

$$f \text{ is convex} \Rightarrow f(b) \leq \lambda f(a) + (1-\lambda) f(c). \quad \text{——— (1)}$$

Also, $a - b + c = a - (\lambda a + (1-\lambda)c) + c$
$= (1-\lambda)a + \lambda c$

Again, as f is convex, we get

$$f(a-b+c) \leq (1-\lambda) f(a) + \lambda f(c). \quad \text{——— (2)}$$

$$(1)+(2) \Rightarrow f(a-b+c) + f(b) \leq f(a) + f(c).$$

$$\Rightarrow f(a-b+c) \leq f(a) - f(b) + f(c).$$