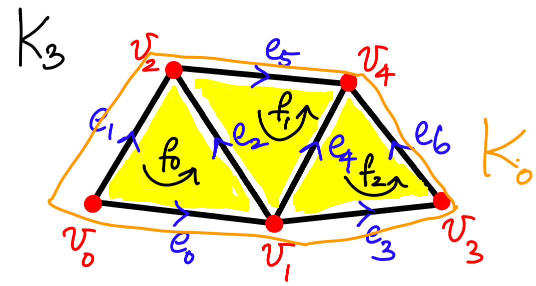
MATH 524 — Lecture 12 (09/28/2023)

Today: * examples of relative homology
* excision theorem

Recall: $K_0 \subset K$, $C_p(K, K_0)$, $H_p(K, K_0)$

Example 2 (Example 3 in Lecture 9)

Let K_0 be the subcomplex consisting of e_0, e_1, e_3, e_5, e_6 and all the vertices. Then we get



$H_2(K_3, K_0) \simeq \mathbb{Z}$, and $\bar{r} = \sum_{i=0}^{2} f_i$ is a generator.

We use the same techniques as before. The triangles are oriented CCW. Then $\bar{r}$, the 2-chain which is the sum of the triangles taken with multipliers of 1 each, has $\partial \bar{r}$ carried by K_0. Hence it is a relative 2-cycle.

There are no tetrahedra in K, and hence there are no 2 boundaries (absolute or relative). Hence $\bar{r}$ generates $H_2(K, K_0)$.

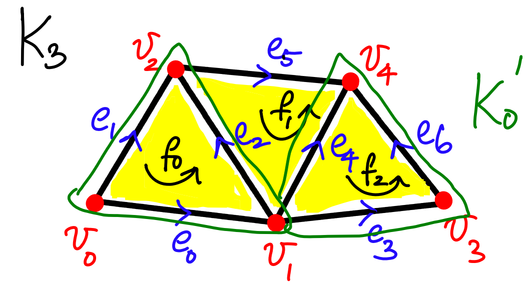
We now consider $H_1(K, K_0)$. Using the same "pushing off edges in the middle" argument as before, we get that any 1-chain in K is homologous to a 1-chain carried by K_0, and hence is a relative 1-cycle that is trivial. In more detail, every 1-chain in K not in K_0 is a relative 1-cycle, and is also a relative 1-boundary since we can find a 2-chain generated by f_1 and f_2 whose difference with this 1-chain is carried by K_0.

Thus, $H_1(K_3, K_0) = 0$, as any 1-chain in K is homologous to a 1-chain carried by K_0.

Here, $H_0(K_3, K_0) = 0$ as well, as $v_i \in K_0 \ \forall i$.

Notice the similarity between Examples 1 (for $p=2$) and 2 – the homology groups are the same. Also, notice that $|K_3|$ and $|K|$ are homeomorphic (both are discs), and $|K_0|$'s are also homeomorphic (to a circle in each case). These examples seem to indicate that relative homology groups are determined by the underlying space, and not by the choice of the simplicial complexes – indeed, this is true in general, but the proof is technical.

Now, consider K_0' as the subcomplex made of $\{e_0, e_1, e_2, e_3, e_4, e_6\}$, and all the vertices.



$H_2(K_3, K_0') \simeq \mathbb{Z} \oplus \mathbb{Z}$ here!

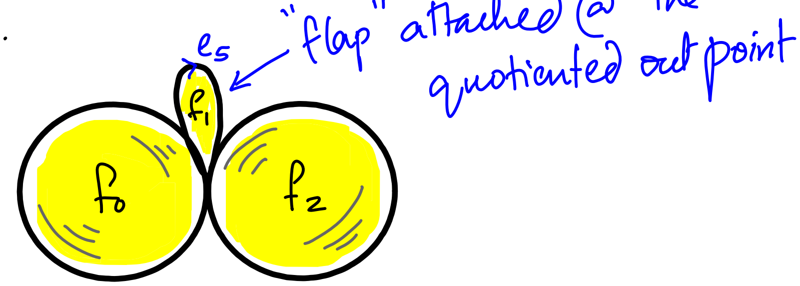
If $\bar{r}' = n_0 \bar{f_0} + n_2 \bar{f_2}$, then $\partial \bar{r}'$ is carried by K_0', and hence is a relative 2-cycle. And $n_0, n_2 \in \mathbb{Z}$ could be chosen arbitrarily. Indeed, $\{\bar{f_0}, \bar{f_1}\}$ is a basis.

elementary chains corresponding to the triangles

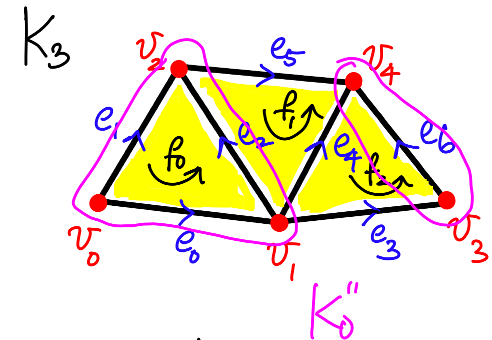
But $H_1(K_3, K_0') = 0$ still. All relative 1-chains are generated by $\{\bar{e}_5\}$, which happens to be a relative 1-cycle as $\partial \bar{e}_5$ is carried by K_0'. But $\bar{e}_5$ is also a relative 1-boundary as $\bar{e}_5 + \partial_2 \bar{f_1}$ is carried by K_0'.

Similarly, $H_0(K_3, K_0') = 0$, as all $v_i \in K_0'$.

Intuitively, one could think of K_3/K_0' as comprised of two spheres touching each other at a point, along with a "flap" (disc) attached to the same point of contact between the spheres.



Now consider K_0'' as shown:

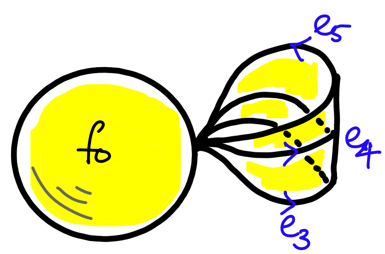


K_0'' : $\{e_0, e_1, e_2, e_6\}$ and all vertices.

We get that $H_2(K_3, K_0'') \simeq \mathbb{Z}$, and $\{\bar{f_0}\}$ is a basis. Notice that $n_1\bar{f_1} + n_2\bar{f_2}$ is not a relative 2-cycle for any $n_1, n_2 \in \mathbb{Z}$, except $n_1 = n_2 = 0$.

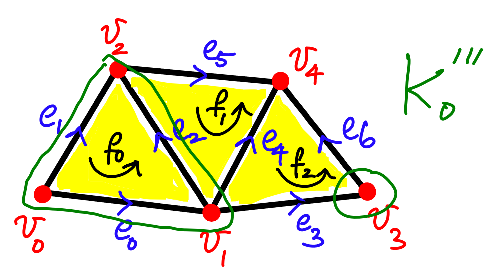
$H_1(K_3, K_0'') \simeq \mathbb{Z}$. We can push off any relative 1-chain in K_3/K_0'' of $\bar{e}_3$ and $\bar{e}_4$, for instance, leaving $\bar{e}_5$ as a generator of $H_1(K_3, K_0'')$.

Intuitively, one could imagine "shrinking" all of $|K_0''|$ to a point, and consider homology of K modulo that point. In this sense, one could think of



Also, notice that different choices of K_0 lead to different $H_p(K, K_0)$ groups.

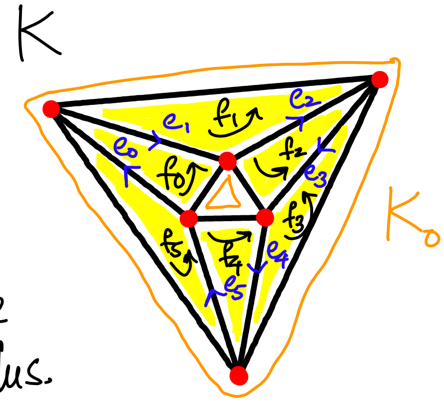
Now consider K_0''' as shown.



What is $H_0(K_3, K_0''') = ?$

Think! Think!

Example 3 (Annulus)



Let K consist of the six triangles $f_0, \dots, f_5$ as shown here, with the triangle in the middle missing. Hence $|K|$ is homeomorphic to the 2D annulus.

Let K_0 consist of all the boundary edges and their vertices, i.e., both the inner and outer circular boundaries.

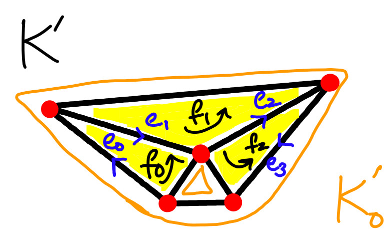
Then $H_2(K, K_0) \simeq \mathbb{Z}$. Notice that $\bar{r} = \sum_{i=0}^{5} f_i$ has $\partial_2 \bar{r}$ carried by K_0. Indeed, $\bar{r}$ generates $H_2(K, K_0)$.

What about $H_1(K, K_0)$? Notice that we can push any relative 1-chain off of $\bar{e}_1$ using $\bar{f}_1$, and then $\bar{e}_2$ using $\bar{f}_2$, and so on, all the way around. But we will be left with $\bar{e}_0$ in this case. Thus, $\{\bar{e}_0\}$ is a relative 1-cycle which is not a relative 1-boundary. Thus, $H_1(K, K_0) \simeq \mathbb{Z}$.

But now consider a modified complex as shown here.

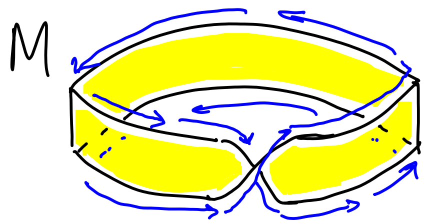
Notice that e_0 is carried by K_0'.

Indeed, $H_1(K', K_0') = 0$ here.



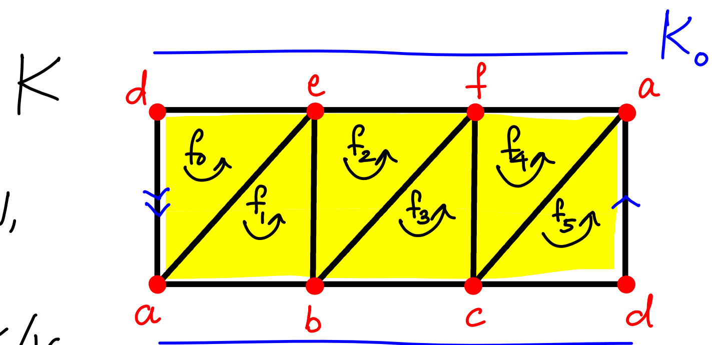
Example 4 Torsion in relative homology groups of Möbius strip:

Recall:



$H_1(M) \simeq \mathbb{Z}$ no torsion! despite the twist.

Let K triangulate the Möbius strip, as shown.
With all triangles oriented ccw, notice that for the 2-chain
$\bar{r} = \sum_{i=0}^{5} f_i$, $\partial_2 \bar{r} = 2(\overline{da})$ in K/K_0.



$\longrightarrow \partial_2 \bar{r}$ is $2\overline{da}$ + edges in K_0

Let K_0 be the "edge" of the Möbius strip, as shown.
Then $H_1(K, K_0) \simeq \mathbb{Z}/_2$, as $2(\overline{da})$ is a relative boundary, but $(\overline{da})$ is not. Of course, $\overline{da}$ is a relative 1-cycle here.

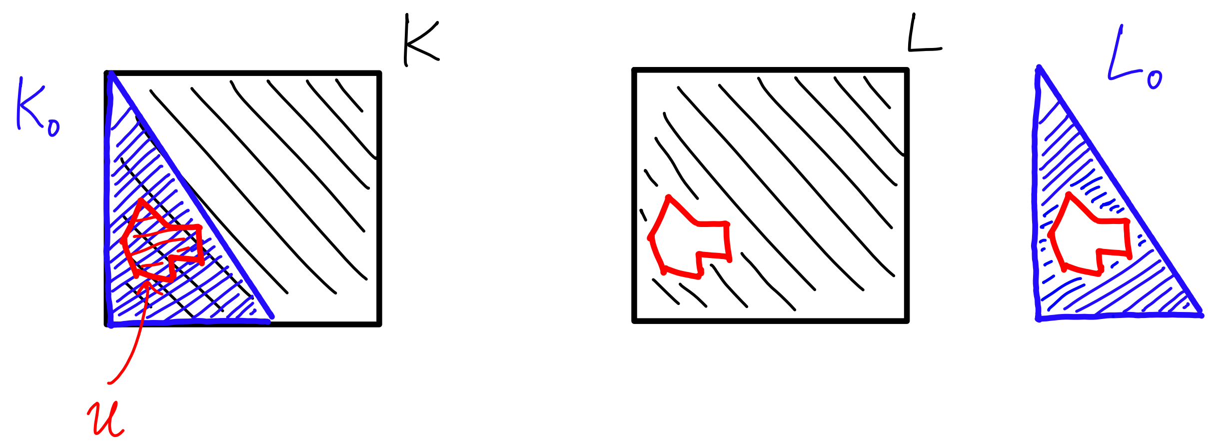
Note that every edge "going across" is a relative 1-cycle here, e.g., $\overline{ae}$, $\overline{bf}$, $\overline{ca}$, etc.

Intuitively, we can "shrink" all of K_0 to a point, and consider $|K|/|K_0|$, after this reduction. This point of view affords some powerful applications/tools. In particular, we could make changes to the interior of K_0, without affecting $H_p(K, K_0)$. We make this notion precise in the following theorem.

Theorem 9.1 [M] (Excision theorem) Let K_0 be a subcomplex of K. Let $\mathcal{U} \subset |K_0|$ (strict subset) be an open set such that $|K| - \mathcal{U}$ is the polytope of a subcomplex L of K and let L_0 be the subcomplex whose polytope is $|K_0| - \mathcal{U}$. Then inclusion induces an isomorphism

$$H_p(L, L_0) \simeq H_p(K, K_0).$$

Here is a schematic illustration. The spaces here are supposed to represent simplicial complexes.



In many cases, L/L_0 is much nicer, or easier to compute with, than K/K_0. In particular, if $\mathcal{U}$ is chosen to be large (but still contained in K_0), L and L_0 might be much simpler than K and K_0. We will encounter applications of the excision theorem later on...