MATH 401: Lecture 1 (08/19/2025)

This is Introduction to Analysis I

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Today: * syllabus, logistics → see the course web page for details
* proof techniques
 – contrapositive proof
 – proof by contradiction
 – proof by induction

Book: Lindstrøm: Spaces–An Intro to Real Analysis (LSIRA)

LSIRA 1.1

Logical statements and notation.

If A then B (or $A \xRightarrow{\text{"implies"}} B$)

$A \Rightarrow B$ typically does not mean $B \Rightarrow A$.

e.g., A: p a natural number, is divisible by 6

B: p is divisible by 3.

$A \Rightarrow B$ holds, but $B \not\Rightarrow A$ (B does not imply A),

e.g., $p = 9$.

But if $A \Rightarrow B$ and $B \Rightarrow A$ hold, we say A $\underbrace{\text{if and only if}}_{\text{iff}}$ B, or

$A \Leftrightarrow B$ (or A is equivalent to B).

To prove $A \Leftrightarrow B$, we often prove $A \Rightarrow B$ and $B \Rightarrow A$ ($A \Leftarrow B$) separately.

We start by reviewing certain standard techniques to construct proofs of mathematical statements.

1. Contrapositive Proof

To show $A \Rightarrow B$, equivalently show not $B \Rightarrow$ not A ($\neg B \Rightarrow \neg A$).

"negation" or "not"

"If A happened then B happened"
This statement is equivalent to
"If B did not happen then A did not happen."

LSIRA 1.1 Prob 3. Prove the following Lemma.

Lemma 1 If n is a natural number such that n^2 is divisible by 3, then n is divisible by 3.

This is $A \Rightarrow B$ where A: $3 | n^2$ (n^2 is divisible by 3). → or 3 divides n^2
B: $3 | n$ (n is divisible by 3).

Let's try to prove $A \Rightarrow B$ directly: $n^2 = 3k$ (as $n^2|3$) $\Rightarrow n = \sqrt{3}\sqrt{k}$ (taking square root on both sides)

Hard to conclude that $n|3$ ☹! → Would have to argue $k|3$, which is not obvious!

Let's try proving $\neg B \Rightarrow \neg A$.

$\neg B$: n is not divisible by 3.

$\Rightarrow n = 3p+1$ or $n = 3q+2$, for $p, q \in \mathbb{N}$. (set of natural numbers)

Case 1. $n = 3p+1$

$$\begin{aligned}
\Rightarrow n^2 &= (3p+1)^2 \\
&= 9p^2 + 6p + 1 \\
&= 3(3p^2+2p)+1 \\
&= 3k+1 \text{ for } k = 3p^2+2p
\end{aligned}$$

$\Rightarrow n^2$ is not divisible by 3

Case 2. $n = 3q+2$

$$\begin{aligned}
\Rightarrow n^2 &= (3q+2)^2 \\
&= 9q^2 + 12q + 4 \\
&= 9q^2 + 12q + 3 + 1 \\
&= 3(\underbrace{3q^2+4q+1}_{=k'})+1 \\
&= 3k'+1
\end{aligned}$$

$\Rightarrow n^2$ is not divisible by 3.

Hence we have proved that if n is not divisible by 3, then n^2 is not divisible by 3. Hence, by the contrapositive, we have $n^2|3 \Rightarrow n|3$. □

Should we always try to build a contrapositive proof?
Not necessarily! In cases where $A \Rightarrow B$ could be concluded directly, the contrapositive argument might make life harder! It is one of the different proof approaches that you should be aware of.

2. Proof by Contradiction

Assume opposite of what you want to prove, and end up with a contradiction (or an obviously wrong statement). Hence the original assumption must be wrong, i.e., you have proved the statement.

<u>LSIRA 1.1 Prob 3 (continued)</u> Prove the following Theorem.

<u>Theorem 2</u> $\sqrt{3}$ is irrational.

Assume $\sqrt{3}$ is rational. → the opposite of what you want to prove

$\Rightarrow \left(\sqrt{3} = \frac{p}{q}\right)^2$, $p, q \in \mathbb{N}$ with no common factors. → by definition, any positive rational number can be written in the form p/q as specified.

Let's square both sides, and cross multiply.

$\Rightarrow 3q^2 = p^2 \Rightarrow 3 \mid p^2$ (p^2 is divisible by 3).

Hence by Lemma 1, $3 \mid p$. Let $p = 3k$. ($k \in \mathbb{N}$). Plug $p=3k$ back in:

$\Rightarrow 3q^2 = (3k)^2 = 9k^2$ (divide both sides by 3)

$\Rightarrow q^2 = 3k^2$, i.e., $3 \mid q^2$ (q^2 is divisible by 3).

Again by Lemma 1, $3 \mid q$.

Since we started with the assumption that p and q have no common factors

Thus p and q have a common factor of 3, which is a contradiction.

Hence $\sqrt{3}$ is irrational. $\square$

3. Proof by Induction

To show a statement $P(n)$ holds for all $n \in \mathbb{N}$,

1. show $P(1)$ holds;
2. Assume $P(k)$ holds for some $k \in \mathbb{N}$.
3. Show $P(k+1)$ holds under Assumption 2.

"for all"

Example

Show that $P(n) = 3+5+\cdots+2n+1 = n(n+2)\ \forall n \in \mathbb{N}$.

1. $P(1) = 3 = 1(1+2)$ (so $P(1)$ is true).
2. Assume $P(k) = k(k+2)$ for some $k \in \mathbb{N}$.
3. $P(k+1) = P(k) + 2(k+1)+1 = P(k) + 2k+3$

$$= k(k+2) + 2k+3 \quad \text{by induction assumption.}$$
$$= k(k+2) + k + k+3$$
$$= k(k+3) + k+3$$
$$= (k+1)(k+3) = n(n+2) \text{ for } n = k+1.$$

$\Rightarrow P(n) = n(n+2)\ \forall n \in \mathbb{N}$. $\square$