MATH 529 — Lecture 9 (02/06/2024)

Today:
* subdivision
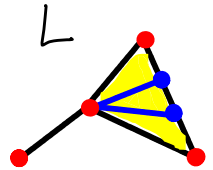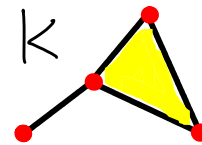* barycentric subdivision
* star, closed star, link

We now define several constructs on simplicial complexes that are simplicial analogues of standard constructs on (continuous) spaces, such as subspaces, open sets, neighborhoods of points, boundary of a neighborhood, etc.

Subdivision

Def A simplicial complex L is a **subdivision** of another complex K if $|K| = |L|$, and $\forall \sigma \in L$, $\sigma \subseteq \tau \in K$.

not $\approx$!

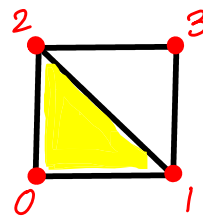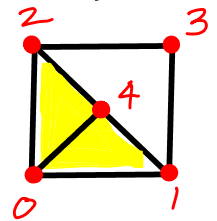In words, every simplex in L is contained in a simplex in K.

Example:

K

L

the 3 triangles are contained in the one triangle in K, for instance.

Here is another example we saw in a different context (in Lecture 6):

$K = K_1$

$L = K_3$

Barycentric Subdivision

One way to create a subdivision of K is by forming the **barycentric subdivision**, denoted $Sd\ K$ (this is a "standard" way to subdivide a complex).

The **barycenter** of a simplex is the centroid of its vertices.

We define (or construct) the barycentric subdivision inductively on the dimension of the simplices.

Inductive construction of $Sd\ K$

> all simplices in K with dimension $\leq j$

**Notation** $K^{(j)} = \{\sigma \in K \mid \dim \leq j\}$ is the j-skeleton of K.

Thus, $K^{(0)}$ is the set of vertices of K.

We start by defining $Sd\ K^{(0)} = K^{(0)}$. (barycenter of a vertex is the vertex itself).
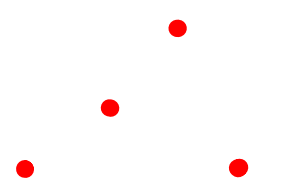
If we have $Sd\ K^{(j-1)}$, we construct $Sd\ K^{(j)}$ by adding the barycenter of every j-simplex as a new vertex, and connecting that vertex to each simplex that subdivides the __boundary__ of that simplex.

$$Bd\ \sigma = \bigcup_{\tau < \sigma} \tau \qquad \partial \left(\begin{array}{c} v_2 \\ v_0 \quad v_1 \end{array} \right) = \begin{array}{c} v_2 \\ v_0 \quad v_1 \end{array}$$
or Bd

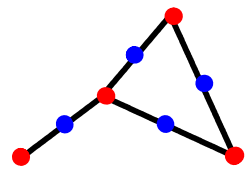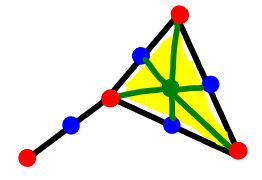We illustrate this construction on the same simplicial complex we used to illustrate subdivisions.
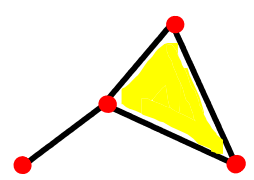
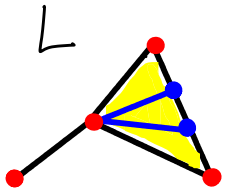$K \qquad\qquad Sd\ K^{(0)} = K^{(0)} \qquad\qquad Sd\ K^{(1)} \qquad\qquad Sd\ K^{(2)} = Sd\ K$

Note that $Sd\, K^{(0)} = K^{(0)}$ as the barycenter of a vertex is the vertex itself. Each edge in K is replaced by 2 edges, while each triangle is replaced by 6 triangles in $Sd\, K$.

L

In the previous example, L is a subdivision of of K, but is not a barycentric subdivision.

In certain applications, we may want not to subdivide certain simplices, e.g., keep a subset of original edges in tact.

We now present definitions on simplicial complexes corresponding to open neighborhoods around points in $\mathbb{R}^d$. We start with some preliminary definitions.

Def A **subcomplex** of K is a simplicial complex L, such that $L \subseteq K$.
A subset that is a simplicial complex by itself

Def The smallest subcomplex containing a subset $L \subseteq K$ is its
closure, $Cl\, L = \{\tau \in K \mid \tau \leq \sigma \in L\}$. → could also use $\overline{L}$
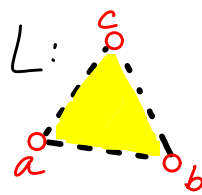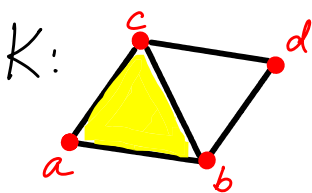
K: L: L is not a subcomplex of K here!

$$Cl\,\{\overline{ac}, d\} = \{\overline{ac}, d, a, c\}. \qquad Cl\,\{\Delta abc\} = \{\Delta abc, \overline{ab}, \overline{ac}, \overline{bc}, a, b, c\}.$$

Notice that $Cl\, L$ is a simplicial complex by itself.

Def For a simplex $\sigma \in K$, we define its **boundary** and **interior** as follows.

$$Bd\, \sigma = \bigcup_{\tau < \sigma} \tau \qquad \text{and} \qquad Int\, \sigma = \sigma \backslash Bd\, \sigma.$$
$$\text{or } \partial\sigma \qquad\qquad\qquad\qquad \text{or } \mathring{\sigma}$$

Def For a vertex $\bar{v}$ of K, the **star** of $\bar{v}$, denoted $St\,\bar{v}$, is

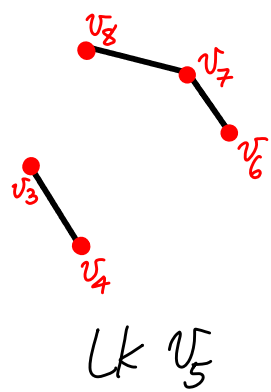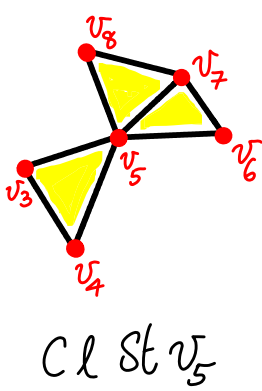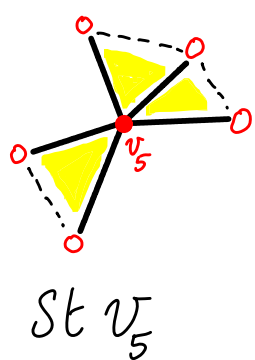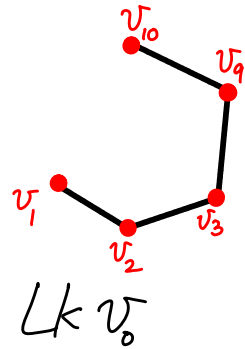$$St\,\bar{v} = \bigcup_{\sigma \succeq \bar{v}} Int\,\sigma.$$

coface

Union of interiors of all simplices that have $\bar{v}$ as a vertex

Also, $Cl\,St\,\bar{v}$ (or $\overline{St\,\bar{v}}$) is the **closed star** of $\bar{v}$

take the closure of $St\,\bar{v}$, i.e., throw in all faces.

The set $Cl\,St\,\bar{v} \setminus St\,\bar{v}$ is the **link** of $\bar{v}$, denoted $Lk\,\bar{v}$.

Here is an example — consider the complex K shown.

$St\,V_0$ $Cl\,St\,V_0$ $Lk\,V_0$

$St\,V_5$ $Cl\,St\,V_5$ $Lk\,V_5$
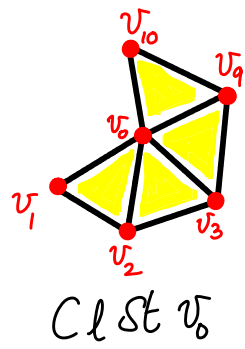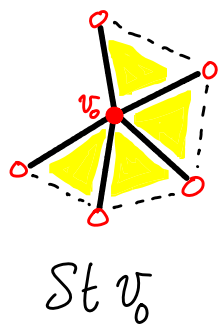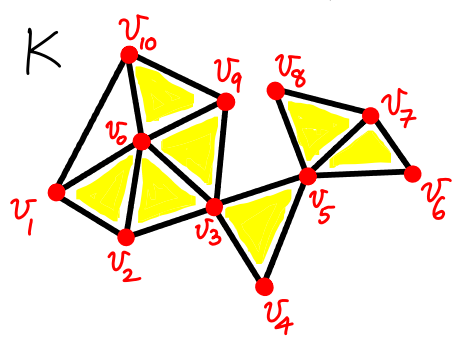
Following these examples, we could make the following observations.

$St\,\bar{v}$ and $Cl\,St\,\bar{v}$ are both <u>path connected</u>, but $Lk\,\bar{v}$ need not be be connected.

↳ *there is a path between any two points in the set*

Also, $Cl\,St\,\bar{v}$ and $Lk\,\bar{v}$ are subcomplexes of K, but $St\,\bar{v}$ is typically not a subcomplex.

An aside on mesh simplification

Links of vertices are employed in mesh simplification — you want a simplicial complex with a smaller number of simplices while preserving topology. A standard operation used in this context is edge contraction, where we replace edge ab, say, with vertex a. We also make associated changes to other simplices connected to edge $\overline{ab}$. Here is an illustration.
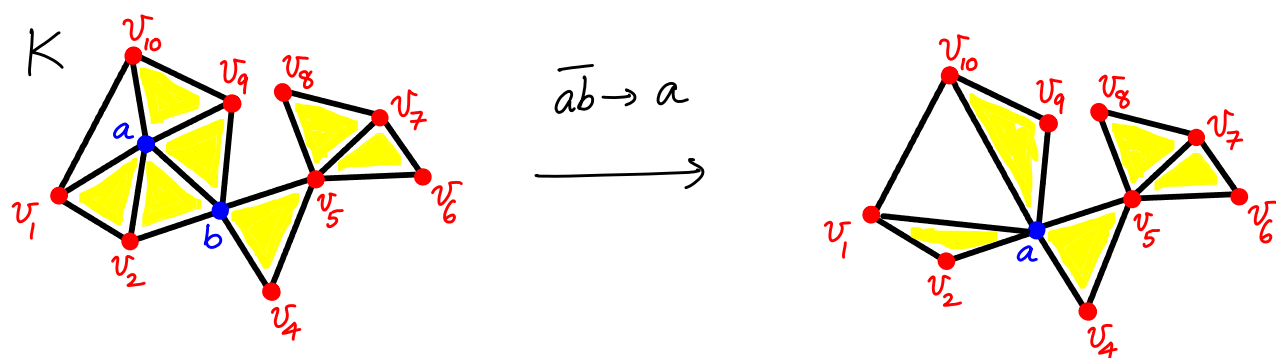
Here, contracting $\overline{ab}$ to a preserves the topology — at least, we do not close any holes in K. But if we were to contract edge $\overline{v_1 v_{10}}$, we will close a hole!

We could define a condition on how the links of a, b, and of $\overline{ab}$ (to be defined next) are related. This is a local condition, and can be checked quickly. If it is satisfied, we can contract $\overline{ab}$ without the fear of closing any holes.

Such operations are critical to efficient computations on simplicial complexes. We will talk about them later in the semester.
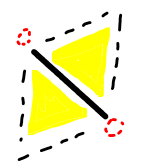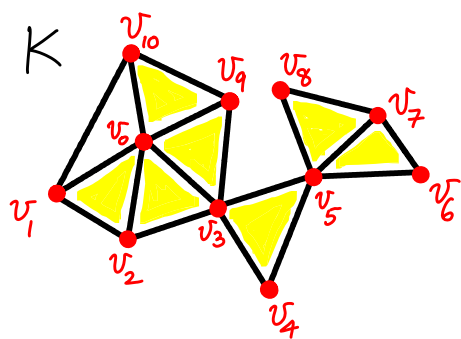
We now extend the definitions of star and link to simplices (of any dimension) and then to collections of simplices.
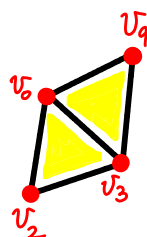
Def For a simplex $\sigma \in K$, we let

$$St\,\sigma = \bigcup_{\tau \geq \sigma} Int\,\tau, \qquad Cl\,St\,\sigma \text{ is its closed star,}$$

and $\quad Lk\,\sigma = \{\bigcup \tau \mid \tau \in Cl\,St\,\sigma,\ \tau \cap \sigma = \phi\}.$

In words, the link of σ is the set of simplices in its closed star which are disjoint from σ.
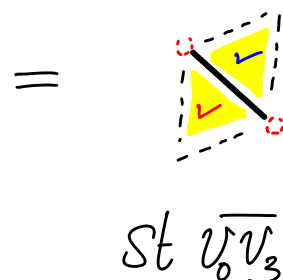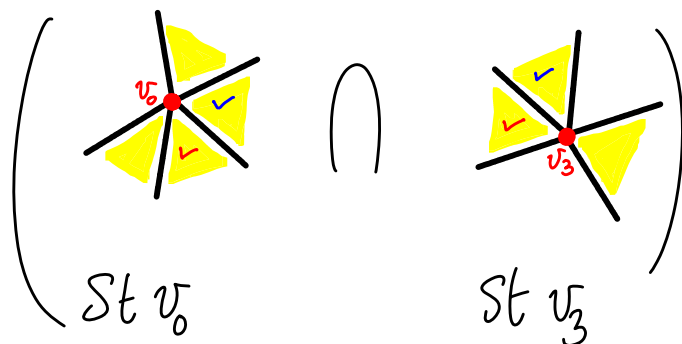

K

$St\,\overline{v_0 v_3}$

$Cl\,St\,\overline{v_0 v_3}$

$Lk\,\overline{v_0 v_3}$

For $\sigma = [v_0, \cdots, v_k]$, $\qquad St\,\sigma = \left(\bigcap_{i=0}^{k} St\,v_i \right)$

For instance, with $\sigma = v_0 v_3$, we get

$$\left(\underset{St\,v_0}{\underbrace{}} \cap \underset{St\,v_3}{\underbrace{}} \right) = \underset{St\,\overline{v_0 v_3}}{\underbrace{}}$$

Def For $X \subseteq K$, we define
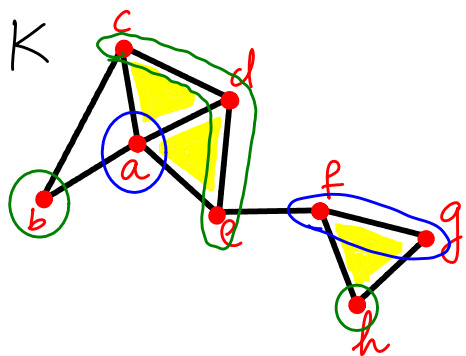
$$St\, X = \{\cup \tau \mid \tau \geq \sigma, \sigma \in X\}.$$

→ Set of cofaces of each simplex in the set X.

$ClSt\, X$ is the closure of $St\, X$. We define

$Lk\, X = $ Set of simplices in $ClSt\, X$ that do not belong to $St(\bar{x})$.

The intuition of the star being the open neighborhood of X and link being the boundary of the closed neighborhood still holds.

K

Let $X = \{a, \overline{fg}\}$. Then we intuitively want

$$Lk\, X = \{b, c, d, e, \overline{cd}, \overline{de}, h\}.$$

Following the definitions, we get

$$St\, X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \triangle acd, \triangle ade, \overline{fg}, \triangle fgh\}.$$

$$ClSt\, X = \{a, \overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \triangle acd, \triangle ade, \overline{fg}, \triangle fgh, \ldots$$
$$b, c, d, e, \overline{cd}, \overline{de}, f, g, h, \overline{fh}, \overline{gh}, \}$$

We will finish this example in the next lecture...