Optimal Cycles and LP/IP

Bala Krishnamoorthy
Washington State U.

Optimal ~~Cycles~~ and LP/IP Chains

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OHCP

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Given p-cycle c in simplicial complex K, find homologous cycle x that is "optimal."

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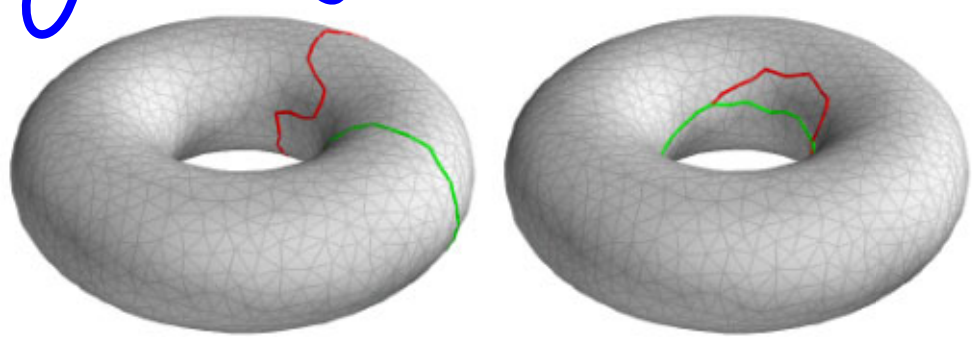

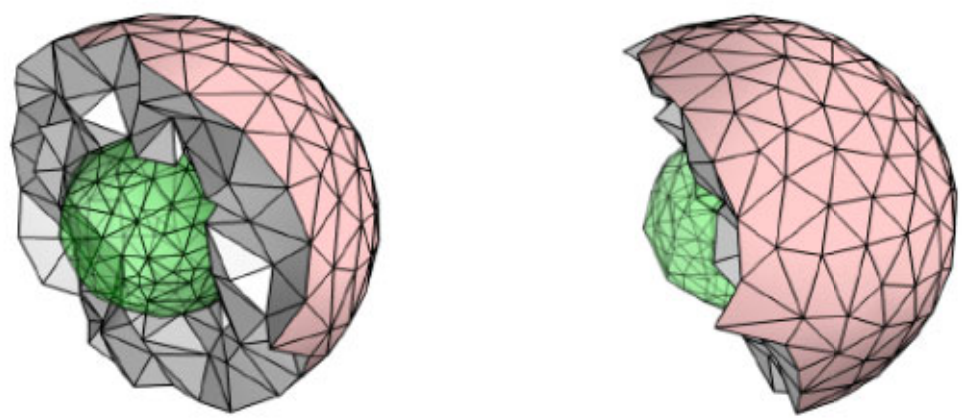

OHCP

The Optimal Homologous Chain Problem:

Given p-chain c in simplicial complex K, find "homologous" cycle x that is "optimal."

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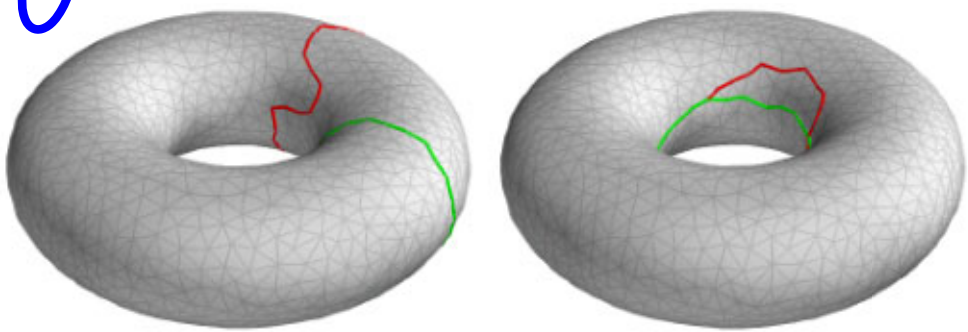

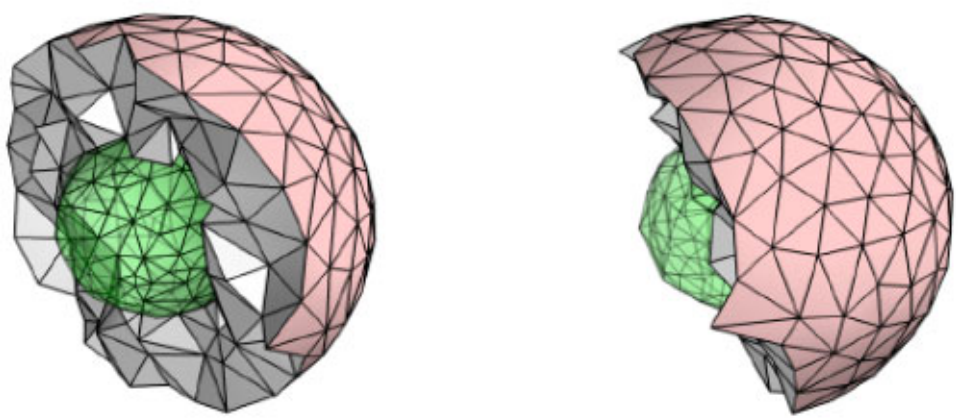

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$$\min \quad w^T x$$

$$\text{s.t.} \quad x = c + [\partial_{p+1}] s$$

$$x \in \mathbb{Z}^m, \; s \in \mathbb{Z}^n$$

OBCP

The Optimal Bounding Chain Problem:

Given p-boundary c in simplicial complex K, find $(p+1)$-chain s bounding c that is "optimal."

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MSFN

Ibrahim, K, Vixie (2013)

Multiscale Simplicial Flat Norm:

Given p-chain c in simplicial complex K, find "homologous" chain x that is "optimal".

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$x = c - \partial_{p+1} s$

$\sum w_i |x_i| + \lambda \sum v_j s_j$

$\lambda \geq 0$

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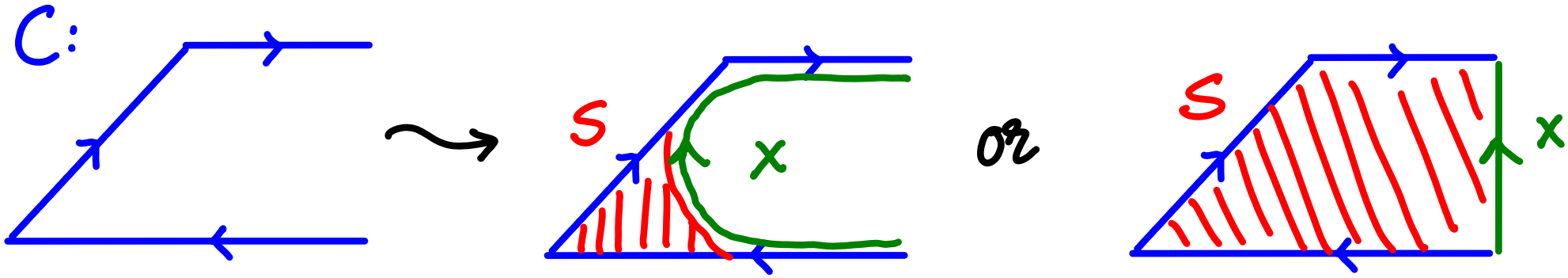
$$\begin{aligned} \min \quad & w^T x + \lambda v^T s \\ \text{s.t.} \quad & x = c - [\partial_{p+1}] s \\ & x \in \mathbb{Z}^m, \; s \in \mathbb{Z}^n \end{aligned}$$

MSFN Example



$X = C - \partial S$

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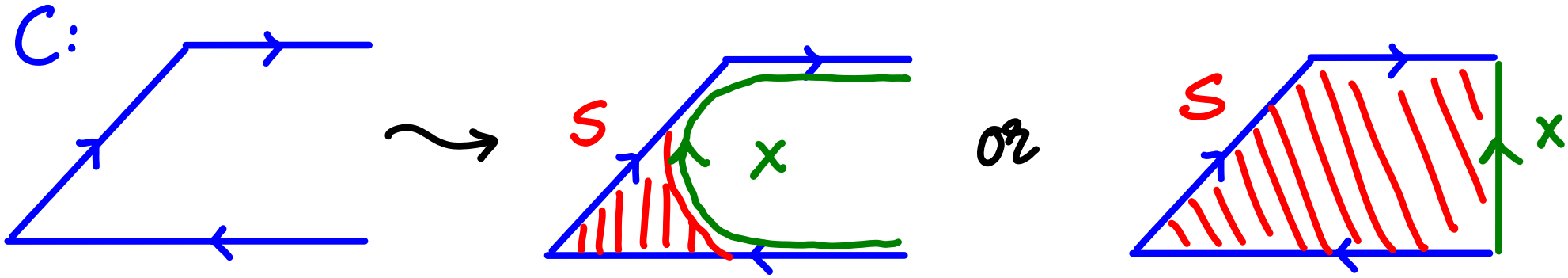

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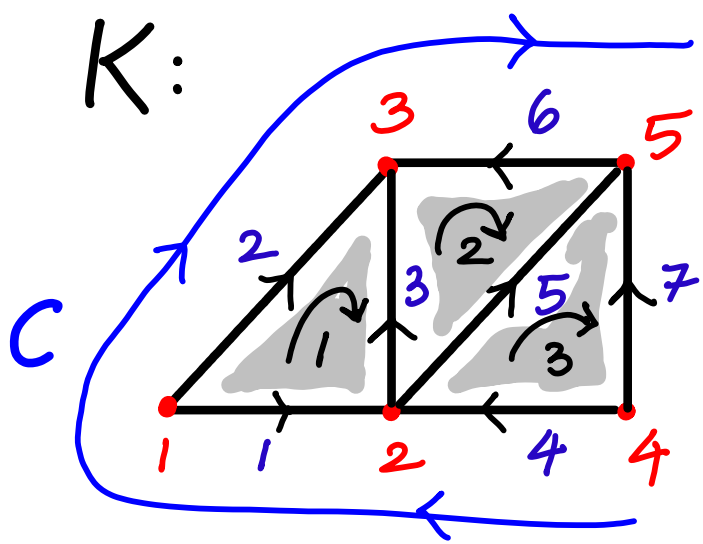
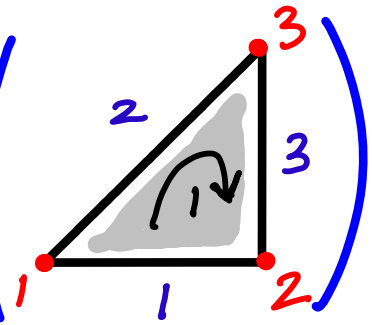
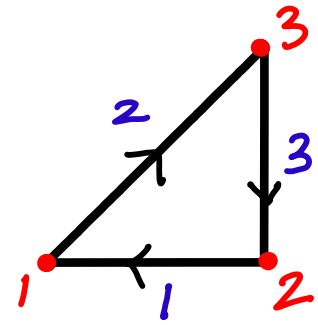


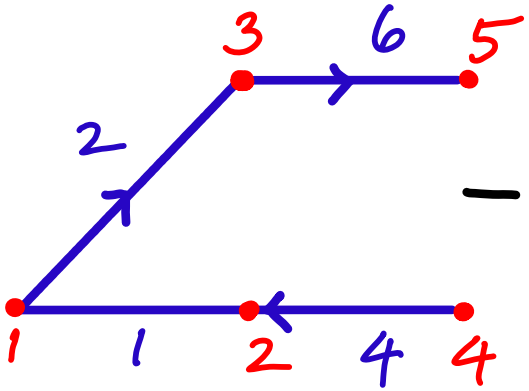
$$w_i = \begin{cases} 1, & i = 1,3,4,6,7 \\ \sqrt{2}, & i = 2,5 \end{cases} \Bigg\} \text{ lengths}$$

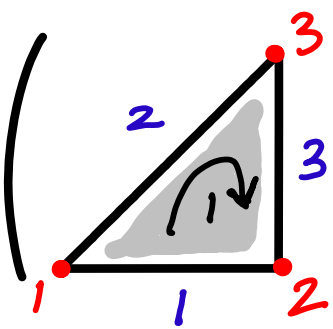
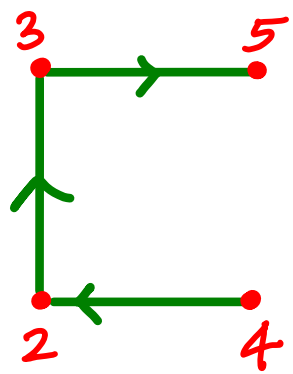
$$v_j = \frac{1}{2},\ j = 1,2,3 \ \Big\} \text{ areas}$$

MSFN Example

boundary

$\partial\Big($  $\Big) =$ 



$- \partial\Big($  $\Big) =$ 

for $\lambda = 2.5$,

$$F_\lambda = 3 + \frac{\lambda}{2}$$

$$= 4.25$$

MSFN Example

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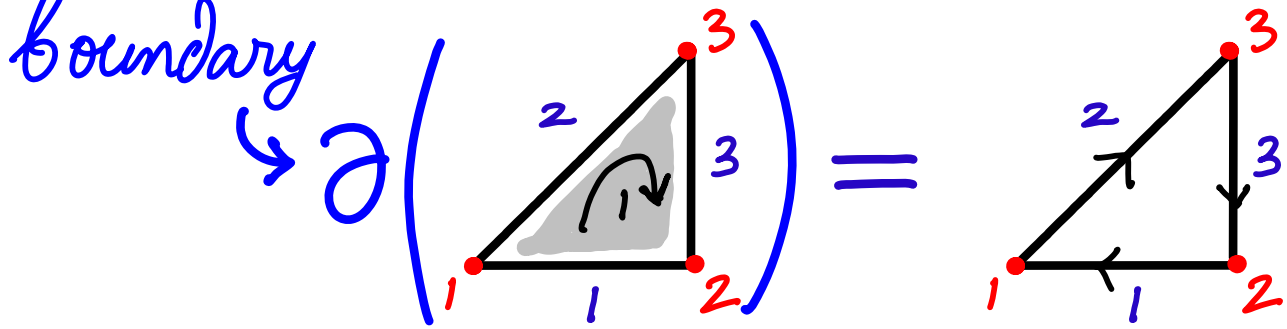
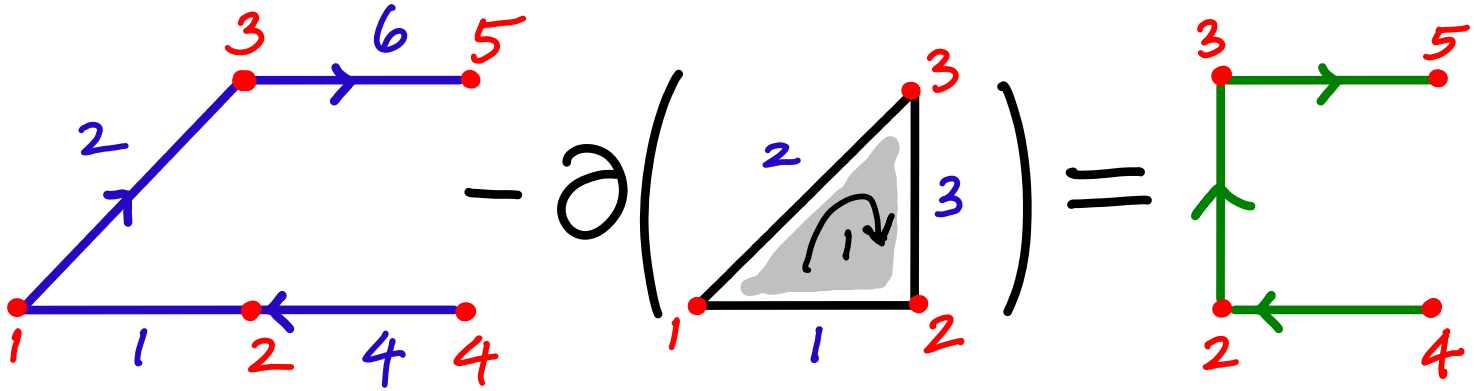
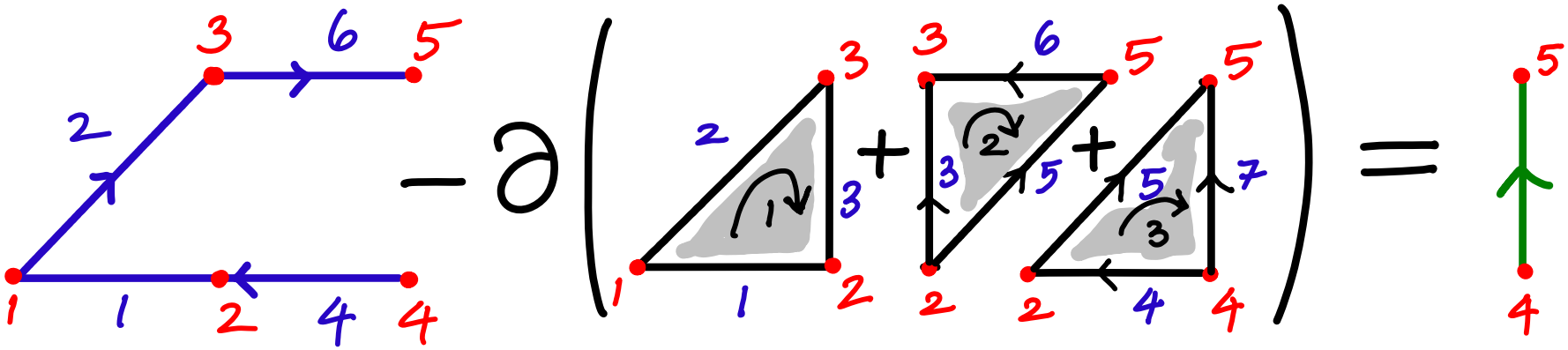




for $\lambda = 2.5$,

$$F_\lambda = 3 + \frac{\lambda}{2} = 4.25$$

for $\lambda = 1$, $F_\lambda = 1 + \frac{3\lambda}{2} = 2.5$

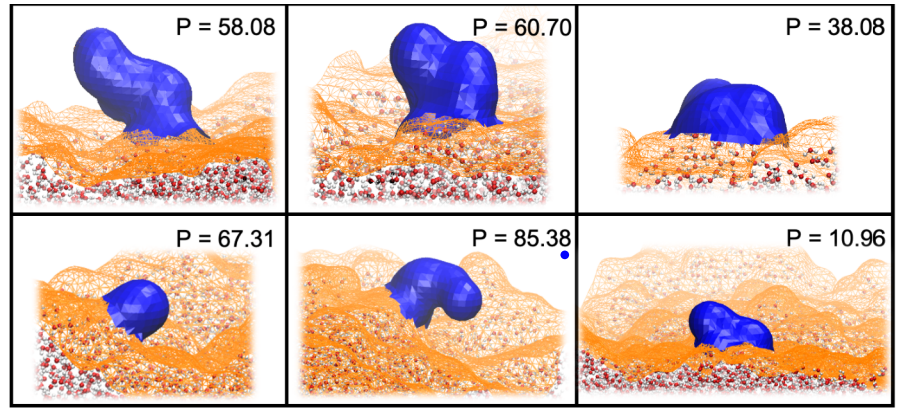


MSFN: Applications

* protrusions in chemistry

E Alvarado, Z Liu, M Servis, BK, A Clark

JCTC 2020



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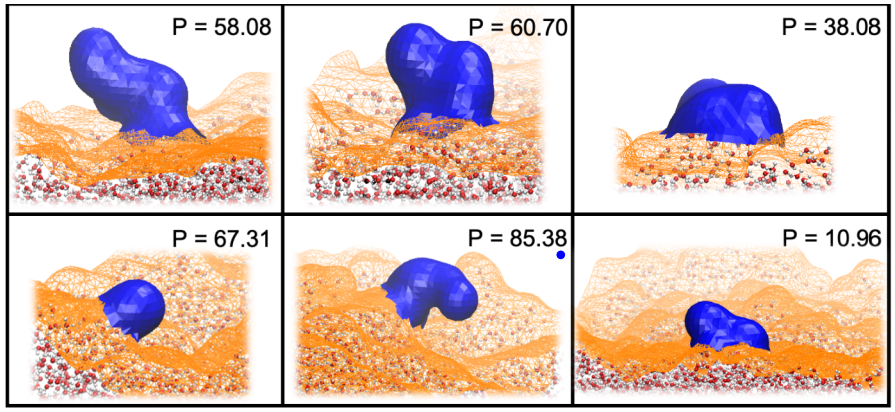
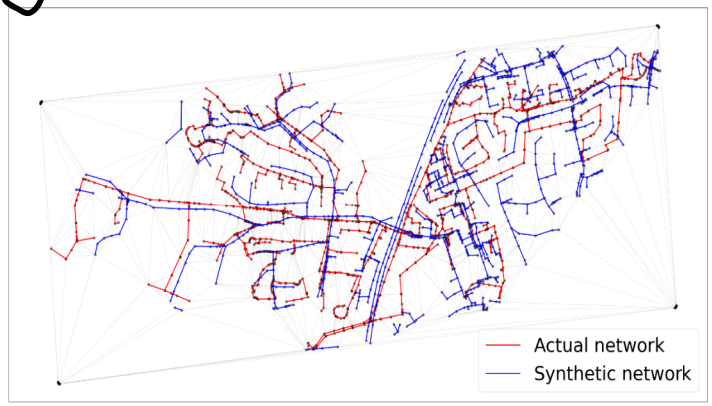


* validating synthetic power networks

K Lyman, R Meyur, BK, M Halappanavar

IJCGA 2025



OHCP under Persistence

* volume-optimal cycles: Obayashi, 2018
* minimal persistent cycles: Dey, Hou, Mandal, 2020 (over $\mathbb{Z}_2$)

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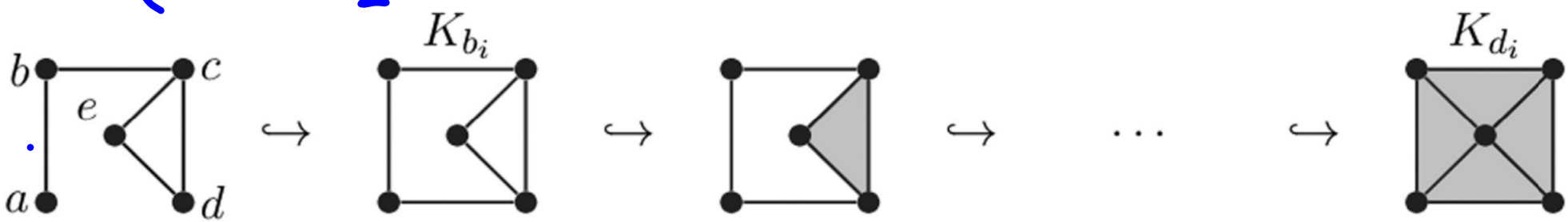

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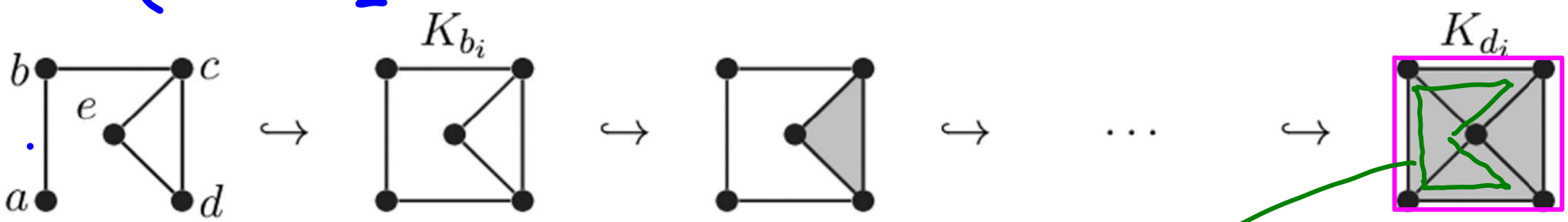

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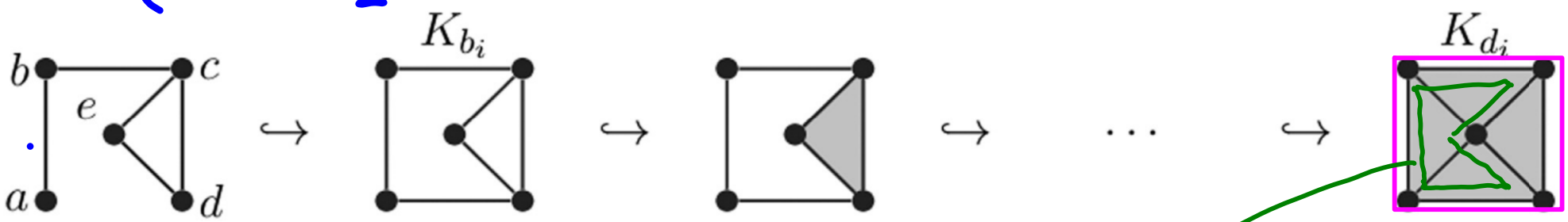

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! Today: OHCP in a single simplicial complex with homology over $\mathbb{Z}$

OHCP as an Integer Program

$$\min_{x,y} \sum_i w_i |x_i| \quad \text{such that} \qquad \text{(OHCP)}$$

$$x = c + [\partial_{p+1}]s, \quad x \in \mathbb{Z}^m, \quad s \in \mathbb{Z}^n \quad (w_i \geq 0)$$

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$$\text{s.t.} \quad x^+ - x^- = c + [\partial_{p+1}](s^+ - s^-) \qquad \text{(IP)}$$

$$x^+, x^-, s^+, s^- \geq 0 \quad x^+, x^- \in \mathbb{Z}^m, \; s^+, s^- \in \mathbb{Z}^n$$

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ignore to get LP

OHCP AND TU OF $[\partial_{p+1}]$

total unimodularity

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OBCP, MSFN as well

When is $[\partial_{p+1}]$ TU?

Dey, Hirani, K, 2010: $[\partial_{p+1}]$ is TU iff K has "no relative torsion."
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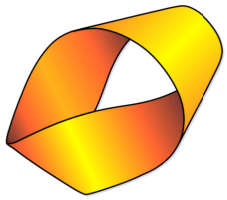

✓ $[\partial_2]$ is TU $\Leftrightarrow$ K has "no Möbius strip"

(DHK '11)

Dual Graph: Max Flow

* Chen & Freedman, 2008 (OHCP, $\mathbb{Z}_2$)
* Dey, Hou, Mandal, 2020 (min. pers. cycle, $\mathbb{Z}_2$)

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! dual graph: tricky when c is not a cycle, MSFN

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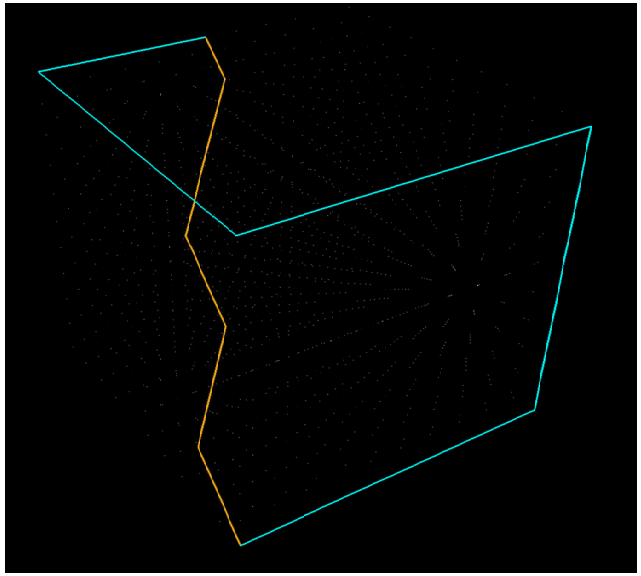
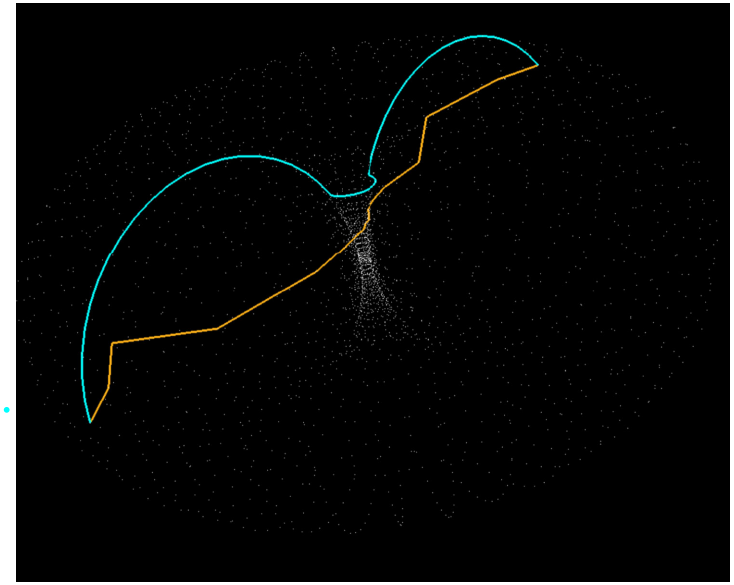
Q: What happens when boundary matrix is not TU?

MSFN in Codim. 2





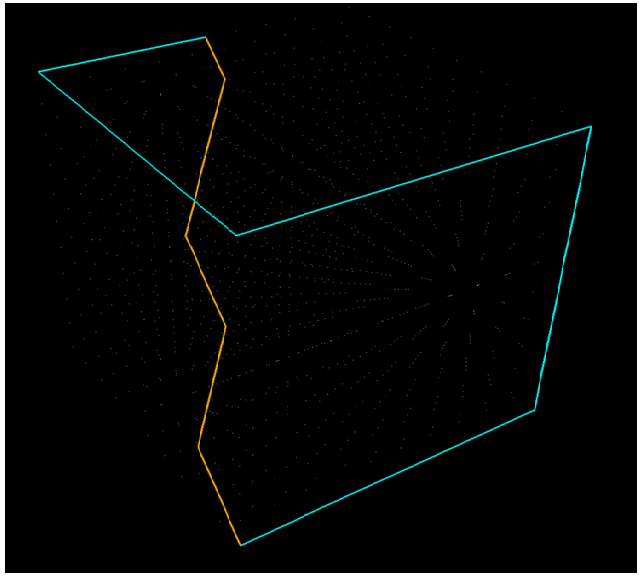
C

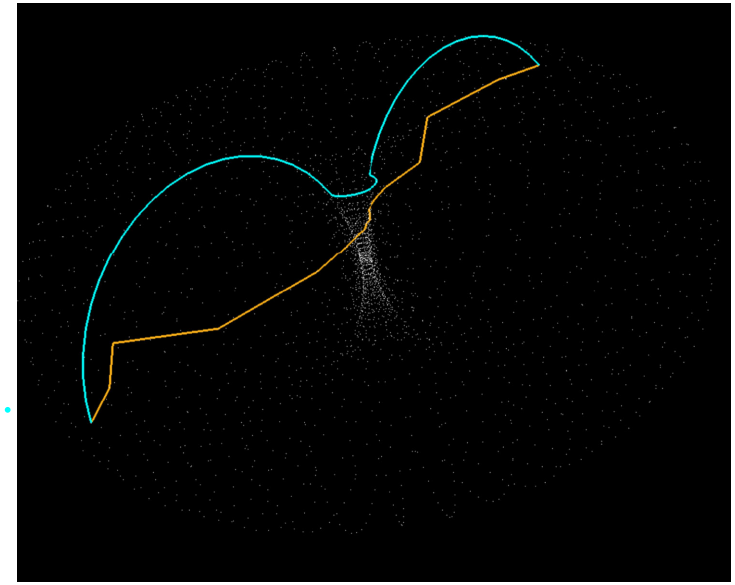
X

K: 2-skeleton of solid cube

2-skeleton of solid torus

MSFN in Codim. 2





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$[\partial_2]$ is not TU, but MSFN LP gives integer soln!

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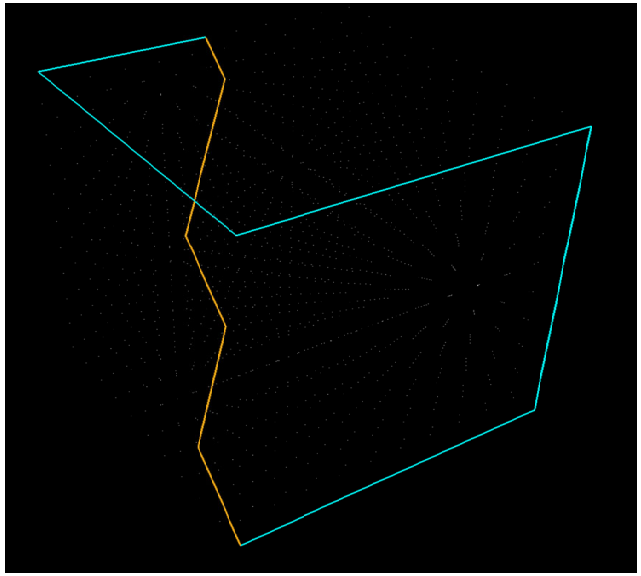

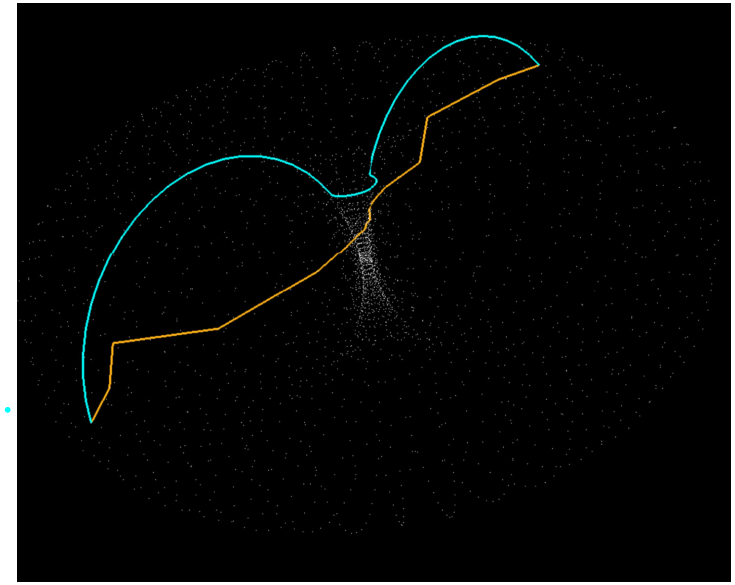

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x

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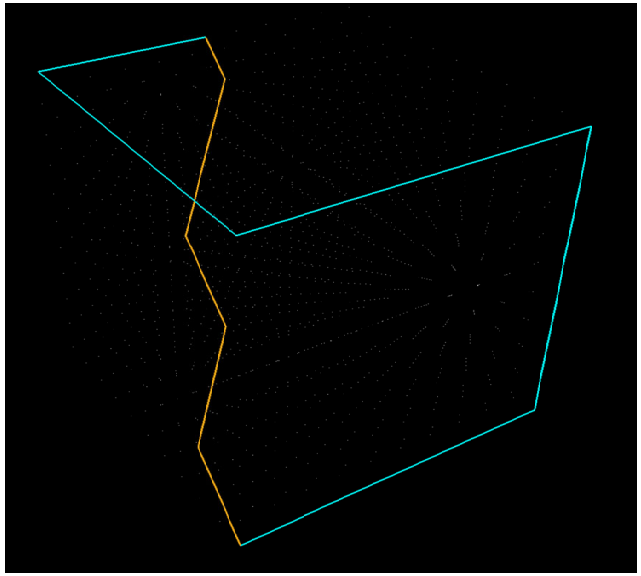
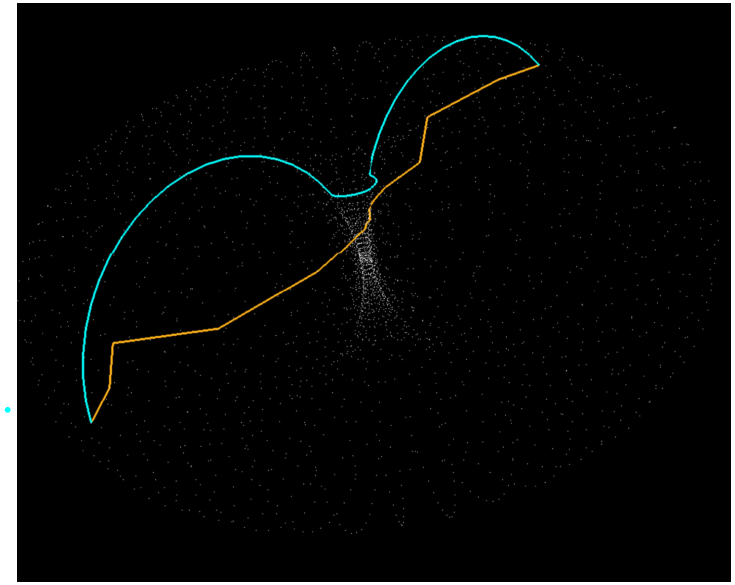
Q1: When/Why does MSFN LP in codim. 2 give integer solutions for free?

MSFN in Codim. 2





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Q1: When/Why does MSFN LP in codim. 2 give integer solutions for free?

— Li et al, 2021: mostly $\{-1,0,1\}$ sol.'s Why?

Non-TU Neutralized K

– K, G Smith, 2018

✓ Define a condition when simplicial complex K is non total-unimodularity neutralized (NTUN)

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✓ property of K (independent of weights on simplices and input chain)

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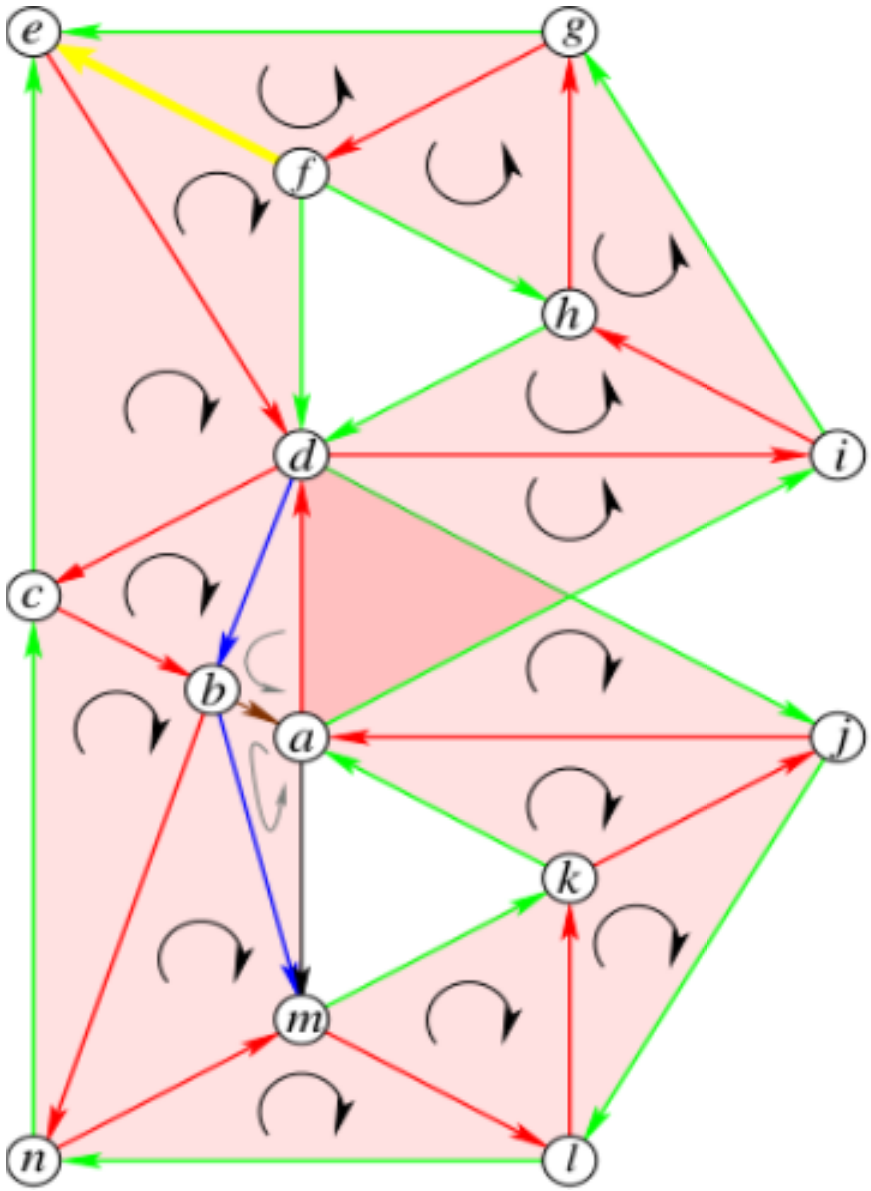
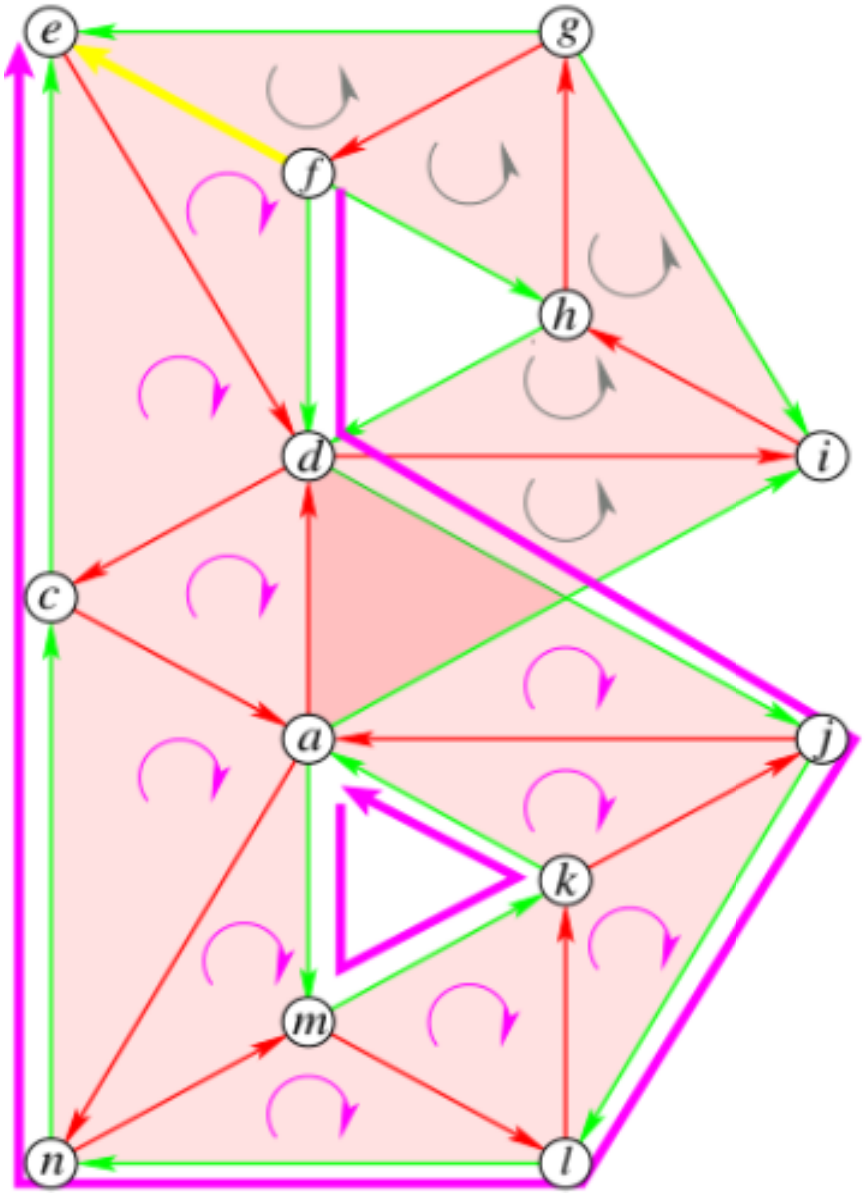
K is non-TU neutralized (NTUN)

TU or Not ?





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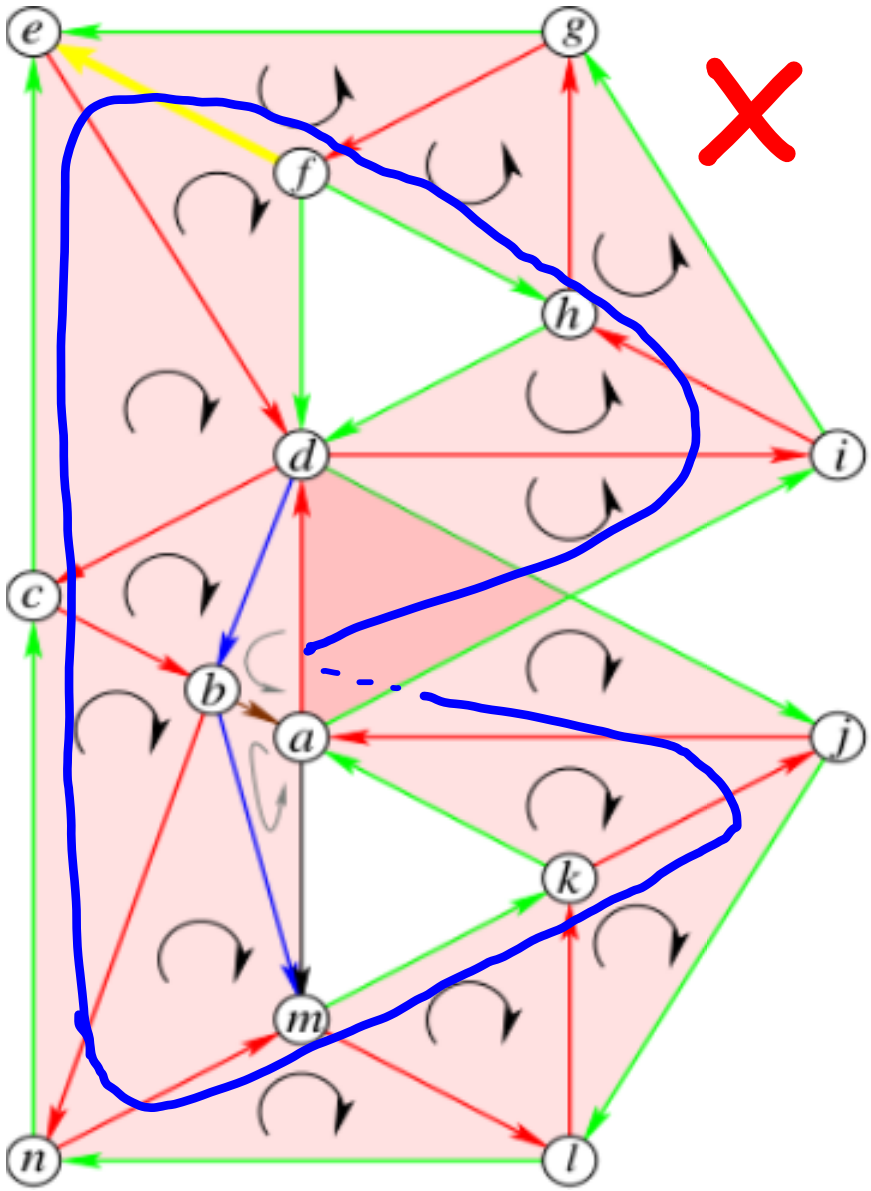
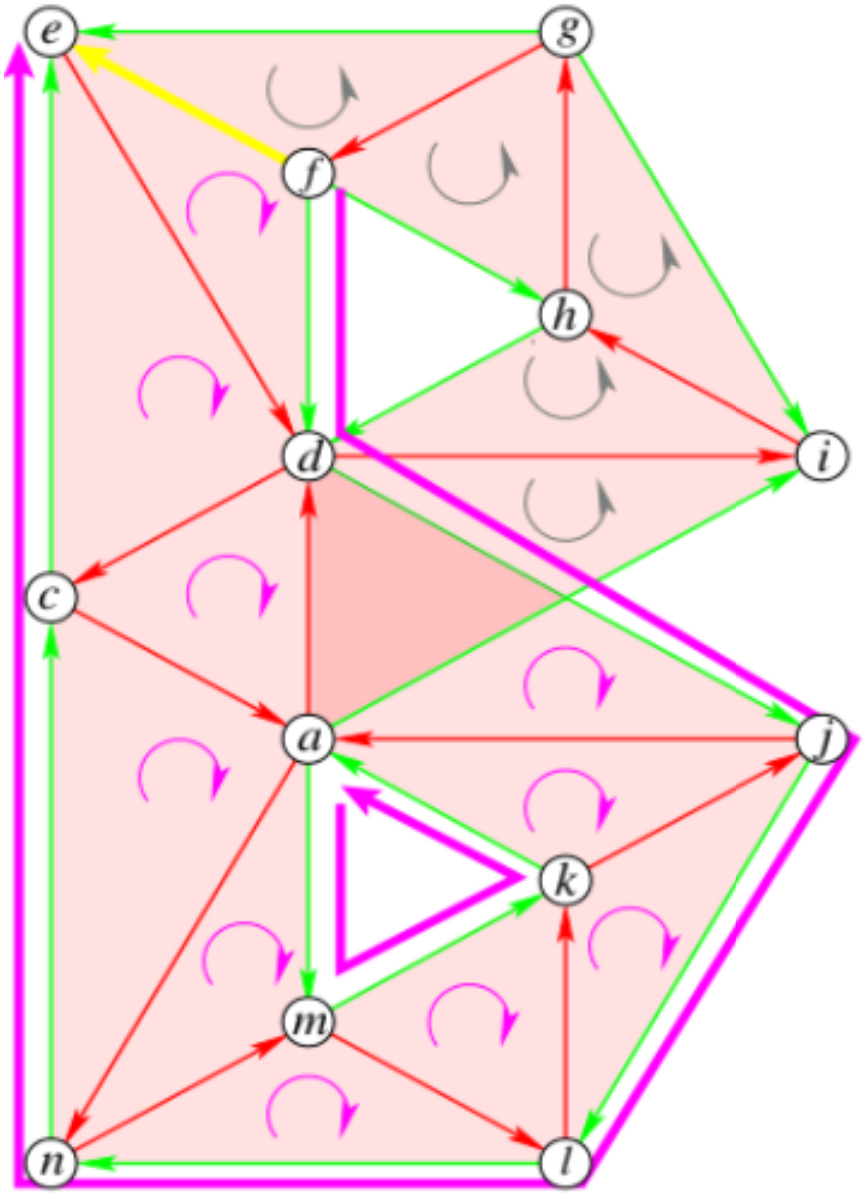


X



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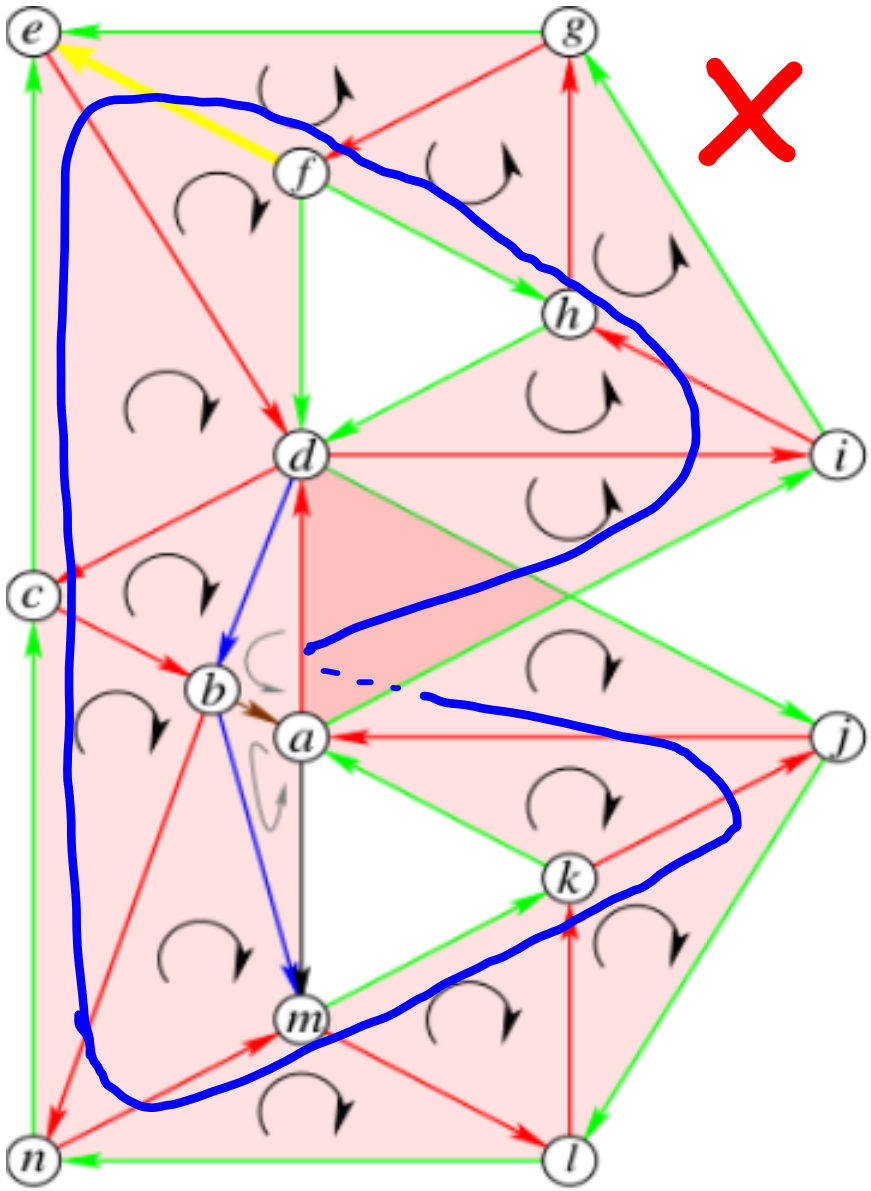
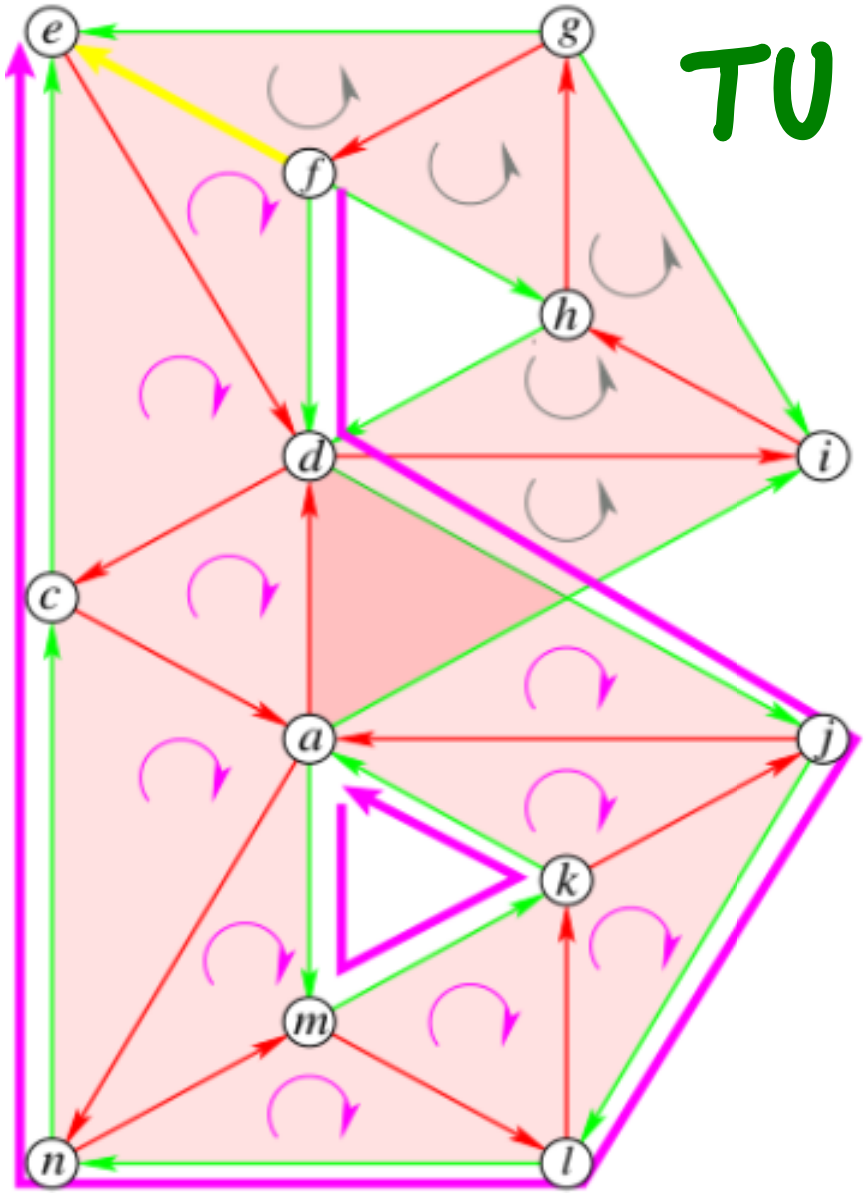
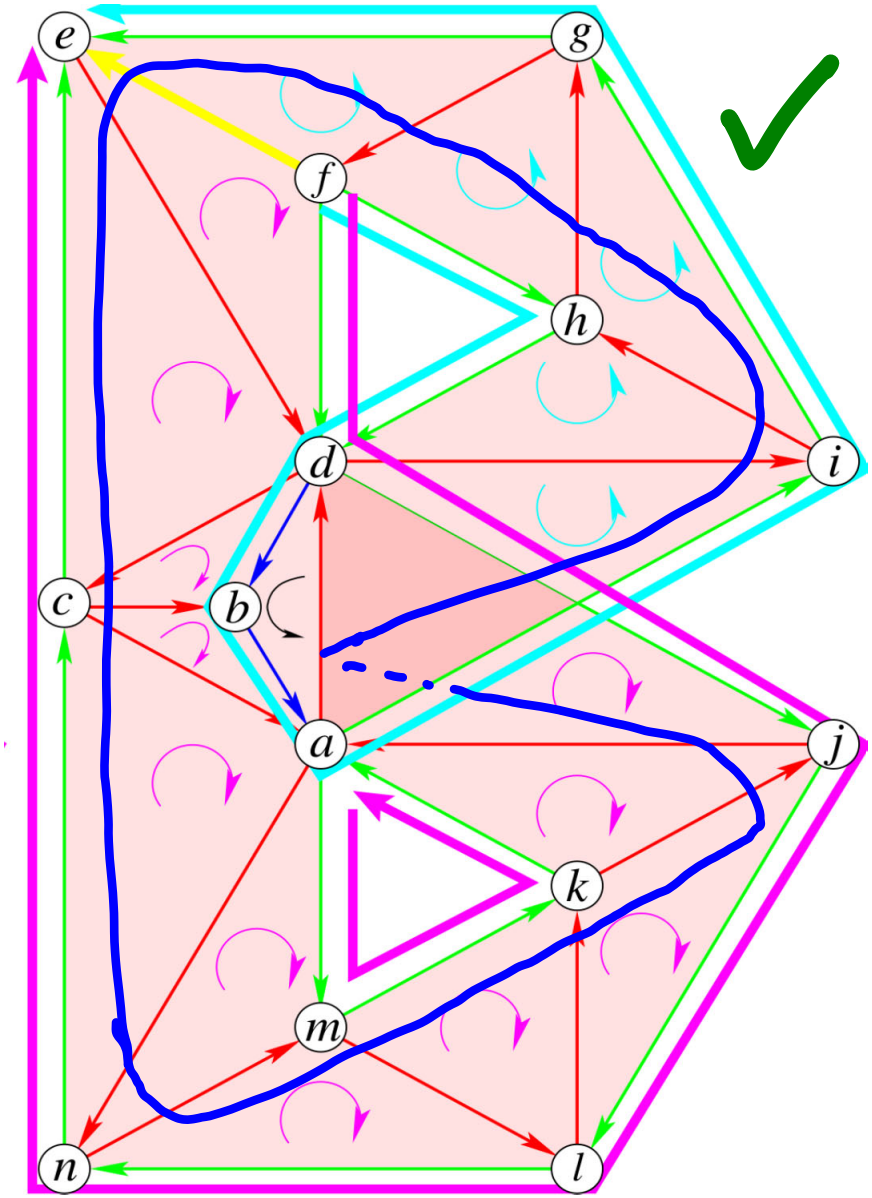


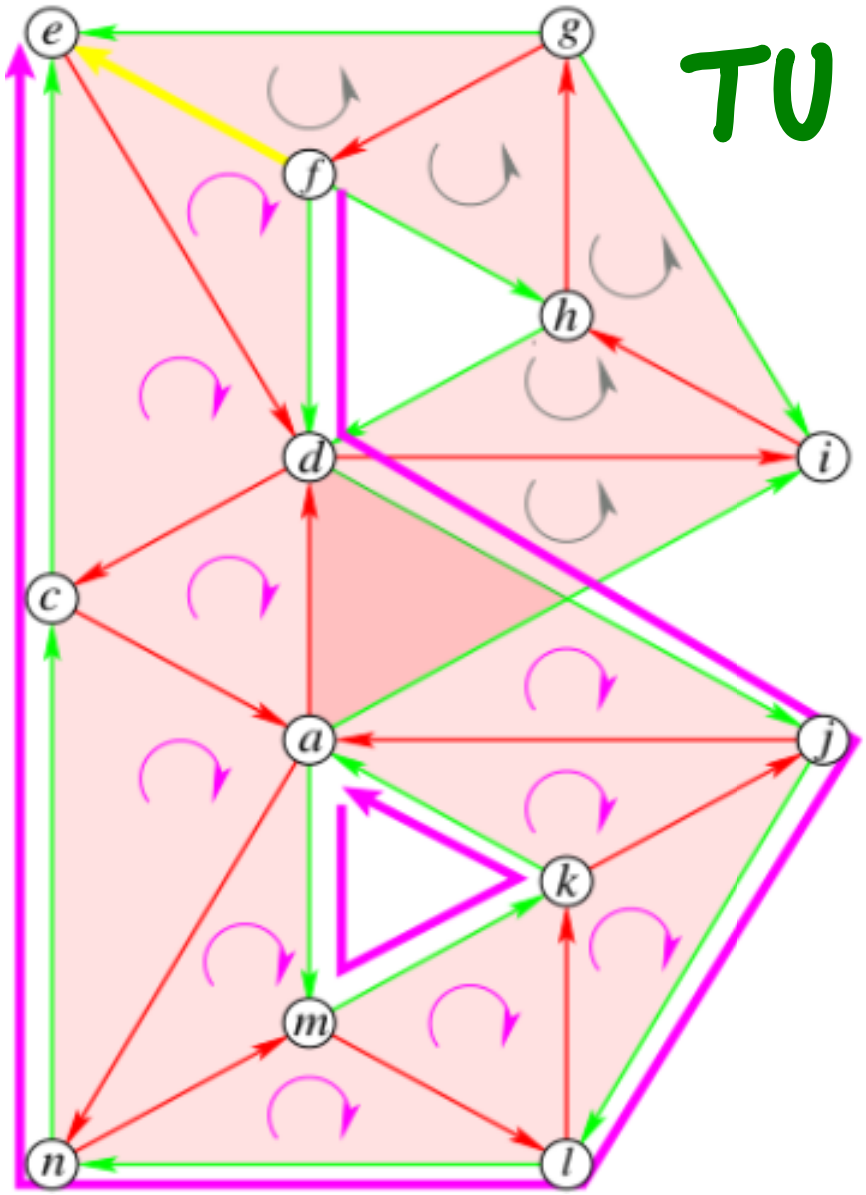
✗



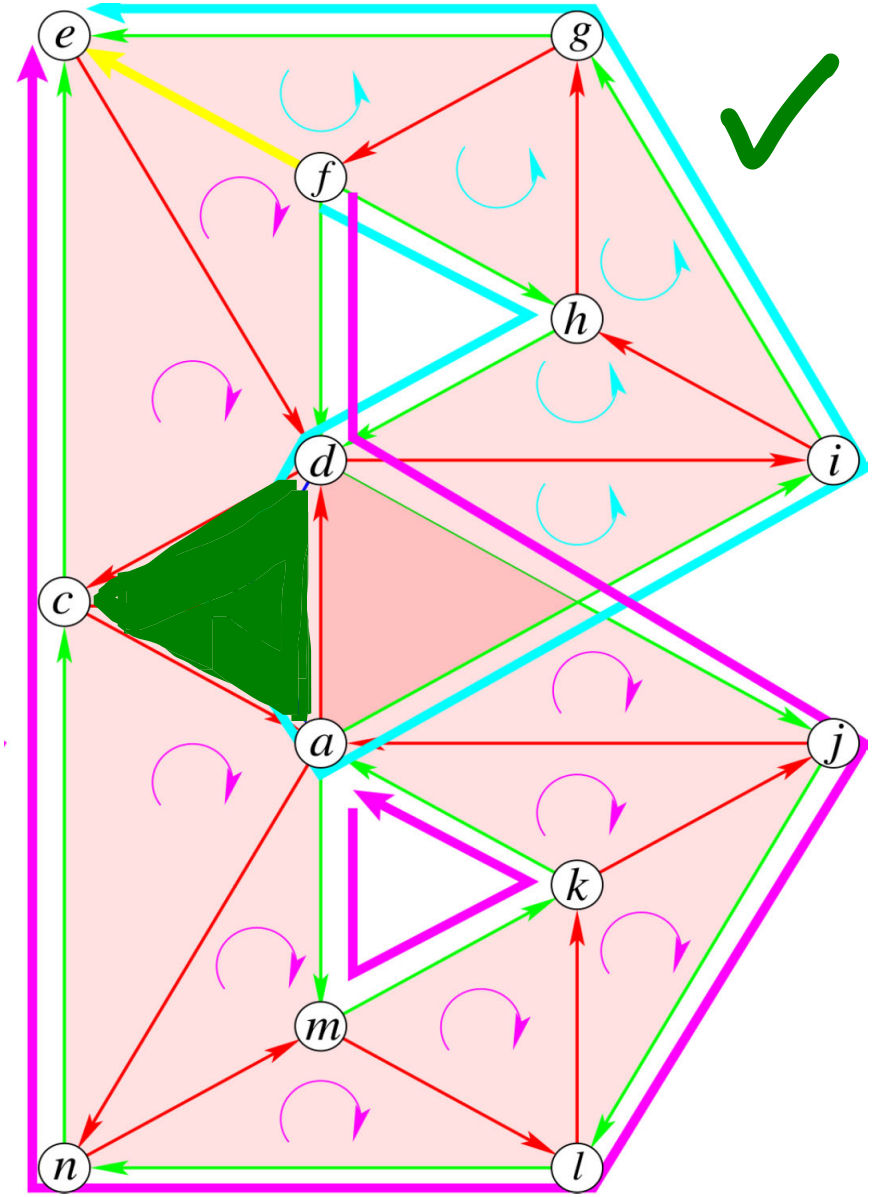
TU ✓

Non-TU Neutralized





Neutralizing Chain



a disc whose boundary consists of an odd number of red (heavy/manifold) edges

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Q2: NTUN result for MSFN ?

Median Shape

Hu, Hudelson, K, Tumurbaatar, Vixie, 2019

* flat distance between T, T': $\mathbb{F}_\lambda(T, T') = \mathbb{F}_\lambda(T - T')$

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* median of $\{T_i\}$:

$$\hat{T} = \arg\min_T \sum_i \mathbb{F}_\lambda(T - T_i)$$

Median Shape : LP

$$\min \begin{bmatrix} & [\ \mathbf{w} \quad \mathbf{w} \quad \lambda\mathbf{v}\ \lambda\mathbf{v}] & [\ \mathbf{w} \quad \mathbf{w} \quad \lambda\mathbf{v} \quad \lambda\mathbf{v}] & \cdots & [\ \mathbf{w} \quad \mathbf{w} \quad \lambda\mathbf{v} \quad \lambda\mathbf{v}] \end{bmatrix} \mathbf{x}$$

s.t.

$$\begin{bmatrix} \begin{bmatrix} I & -I \end{bmatrix} & \begin{bmatrix} -I & I & -B & B \end{bmatrix} & & & \\ \begin{bmatrix} I & -I \end{bmatrix} & & \begin{bmatrix} -I & I & -B & B \end{bmatrix} & & \\ \vdots & & & \ddots & \\ \begin{bmatrix} I & -I \end{bmatrix} & & & & \begin{bmatrix} -I & I & -B & B \end{bmatrix} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_N \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{t}^+ & \mathbf{t}^- & \mathbf{r}_1^+ & \mathbf{r}_1^- & \mathbf{s}_1^+ & \mathbf{s}_1^- & \mathbf{r}_2^+ & \mathbf{r}_2^- & \mathbf{s}_2^+ & \mathbf{s}_2^- & \cdots & \mathbf{r}_N^+ & \mathbf{r}_N^- & \mathbf{s}_N^+ & \mathbf{s}_N^- \end{bmatrix} \geq \mathbf{0}.$$

$B = [\partial_{p+n}]$

Median Shape : LP

$$\min \begin{bmatrix} & [\,\mathbf{w} \quad \mathbf{w} \quad \lambda\mathbf{v} \; \lambda\mathbf{v}] & [\,\mathbf{w} \quad \mathbf{w} \quad \lambda\mathbf{v} \quad \lambda\mathbf{v}] & \cdots & [\,\mathbf{w} \quad \mathbf{w} \quad \lambda\mathbf{v} \quad \lambda\mathbf{v}] \end{bmatrix} \mathbf{x}$$

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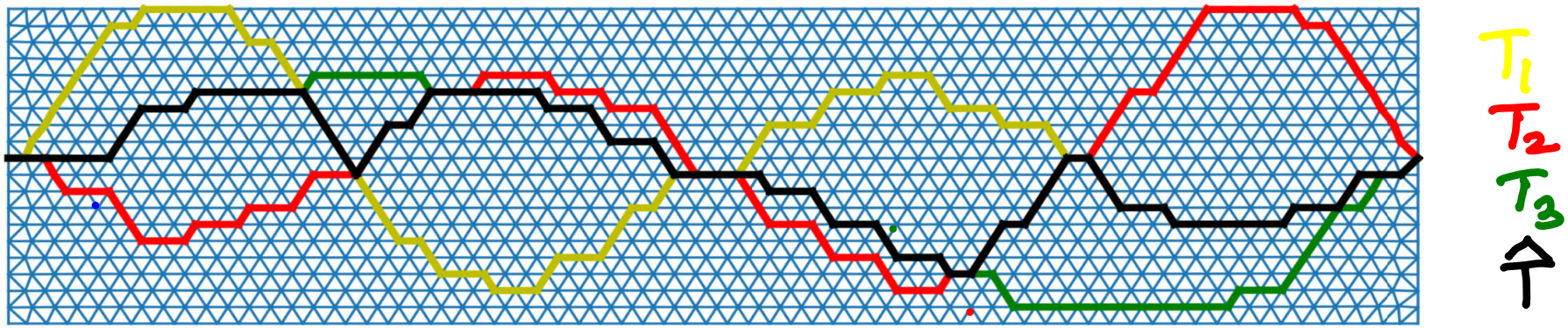
✗ constraint matrix not TU even if $[\partial_{p+1}]$ is

Median Shape: Examples



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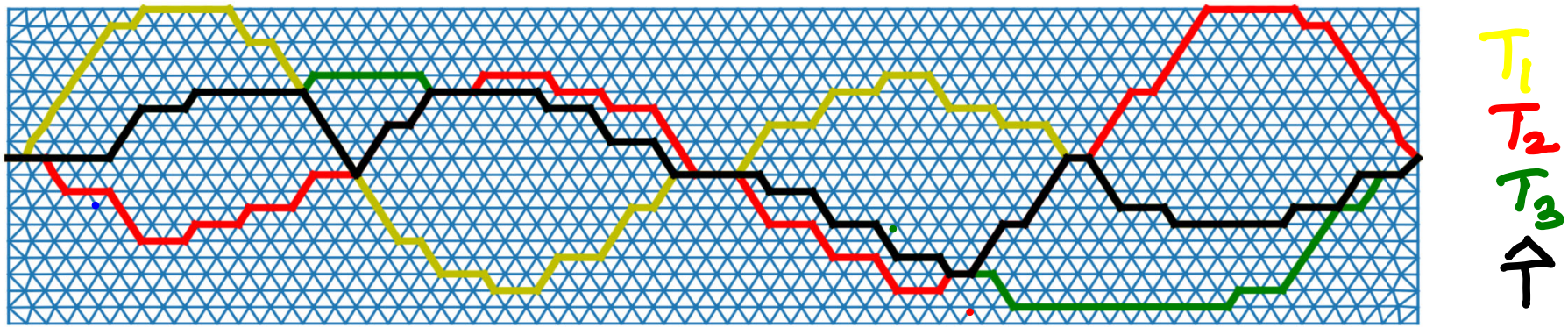

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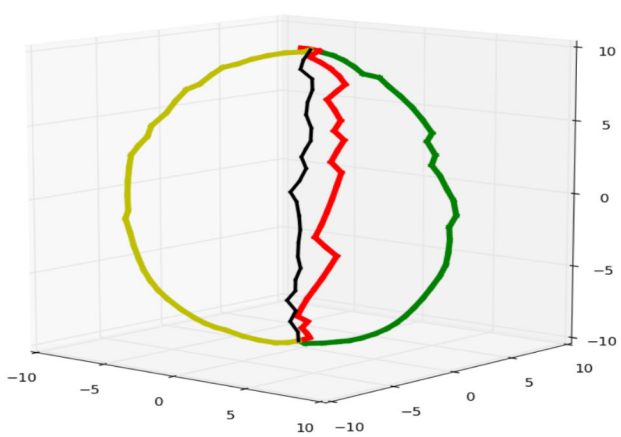

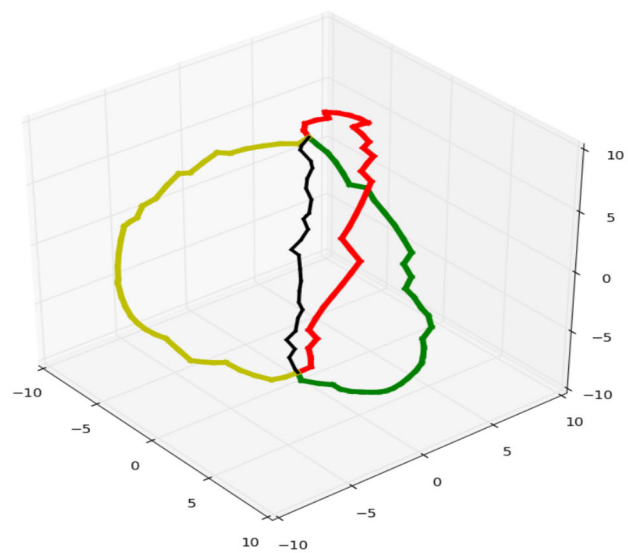

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Median Shape: Questions

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Q4: Network flow (dual graph) model for median shape?

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Thank You!