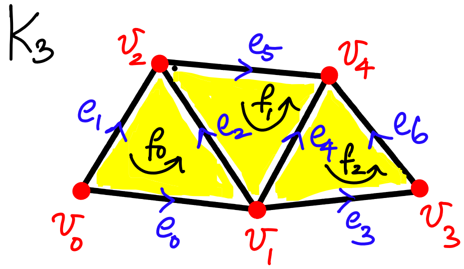
MATH 524 – Lecture 9 (09/19/2023)

Today: * Homology groups of surfaces

Example 3

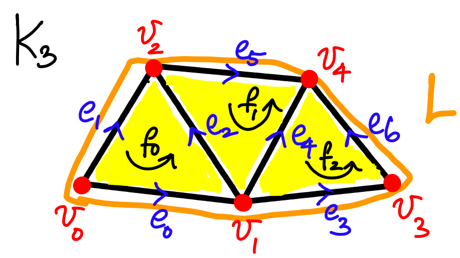


$$\bar{c} = \sum_{i=0}^{6} n_i e_i$$

Let $\bar{c}$ be a general 1-chain on K_3, and let its value on e_2 be c_2, i.e., $\bar{c}(e_2) = c_2$. Then the 1-chain $\bar{c}' = \bar{c} + \partial_2(c_2 f_1)$ has $\bar{c}'(e_2) = 0$. Thus, we have "pushed $\bar{c}$ off e_2". Now, let $\bar{c}'(e_4) = c_4$. Then $\bar{c}'' = \bar{c}' + \partial_2(c_4 f_2)$ has $\bar{c}''(e_4) = 0$. Notice that $\bar{c}''(e_2) = 0$ still, as $e_2 \notin \partial_2 f_2$. So we have pushed $\bar{c}'$ off of e_4.

By combining the two steps, we have pushed $\bar{c}$ off of e_2 and e_4.

So, a 1-chain $\bar{c}$ on K_3 is homologous to $\bar{c}''$ carried by L_3, which is the subcomplex of K_3 made of $\{e_0, e_1, e_3, e_5, e_6\}$. Hence, $\bar{c}$ is a 1-cycle iff $\bar{c}''$ is.



But $\bar{c}''$ is a 1-cycle iff it is a multiple of $e_0 + e_3 + e_6 - e_5 - e_1$.

Hence $Z_1(K_3)$ has rank 1 (a basis is $\{e_0 + e_3 + e_6 - e_5 - e_1\}$).

But notice that this 1-cycle is also a 1-boundary. Precisely, it is $\partial_2(f_0 + f_1 + f_2)$. So $B_1(K_3) = Z_1(K_3)$.

Hence $H_1(K_3) = Z_1(K_3)/B_1(K_3) = 0$.

We also get that $H_p(K_3) = 0$ for $p \geq 2$.

We used the two triangles to push off the general chain to the subcomplex which consists of the boundary of K_3. It's simpler to come up with the criterion for when a chain carried by this subcomplex is a cycle.

Notice that $H_p(K_2) = H_p(K_3)$ for $p \geq 1$, and that $|K_2| \approx |K_3|$. This follows from the fundamental result that the homology groups depend only on the underlying space, and not on the particular simplicial complex chosen.

We could study homology in the "continuous" setting, i.e., without considering simplicial complexes. This homology, termed singular homology, can be shown to be equivalent to simplicial homology.

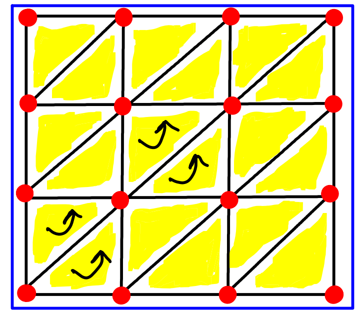
We can apply the techniques illustrated so far to compute the homology groups of more complicated simplicial complexes...

Homology Groups of Surfaces

If K if finite, $C_p(K)$ has finite rank, so does $Z_p(K)$ and $B_p(K)$. Also, $H_p(K)$ is finitely generated, and we can apply the fundamental theorem of finitely generated abelian groups to find the structure of $H_p(K)$. In particular, we want to compute the betti number and torsion coefficients of finite simplicial complexes representing surfaces, e.g., torus, Klein bottle, Möbius strip, etc.

We first formalize the idea of "pushing chains off to the boundary of the simplicial complex".

Lemma 6.1 [M]



Let L be the simplicial complex such that $|L|$ is a rectangle. Let $\operatorname{Bd} L$ be the subcomplex of L representing the edges making up the boundary of the rectangle. Orient each triangle counter clockwise (CCW), and the edges arbitrarily. Then the following results hold.

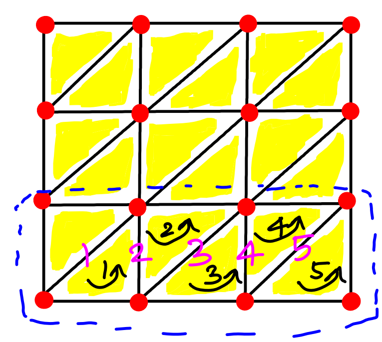
(1) Every 1-cycle of L is homologous to a 1-cycle carried by $\operatorname{Bd} L$.

(2) If $\bar{d}$ is a 2-chain of L, and if $\partial \bar{d}$ is carried by $\operatorname{Bd} L$, then $\bar{d}$ is a multiple of the chain $\sum \sigma_i$, where σ_i are all the triangles.

Proof of Lemma 6.1 [M]

(2) follows because if σ_i and σ_j in $\bar{d}$ have an edge in the middle in common, then $\partial \bar{d}$ has coefficient 0 on that edge. Hence, $\bar{d}$ has the same value on σ_i & σ_j. We can extend this argument to all σ_i's, as $\partial \bar{d}$ is carried by $\operatorname{Bd} L$. giving that $\bar{d}$ has the same coefficient on all of them.

(1) We use the idea of pushing the chain off of edges in the middle (as we did in Example 3).

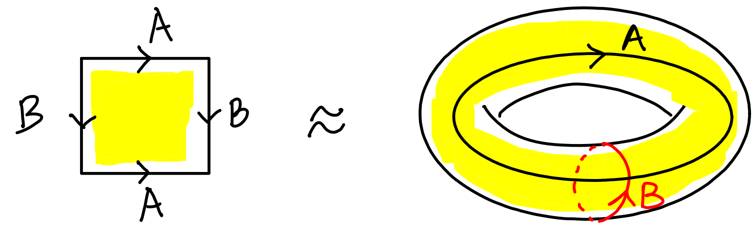


We can use the triangles in the order shown here—from 1 to 5—to push the input chain off of edges marked 1 to 5 (in that order). After these steps, the chain will be pushed on to the edges shown with the dashed outline.

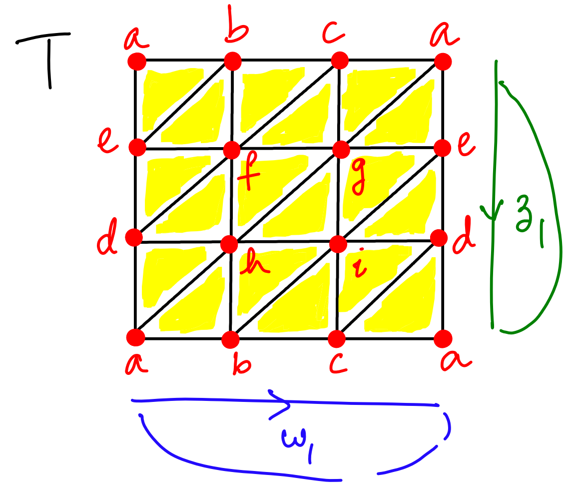
We then repeat the process on the second "horizontal strip" of triangles, and then the top strip. Ultimately, the chain is carried by $\operatorname{Bd} L$. $\square$

Theorem 6.2 [M] $(\mathbb{T}^2)$

(homology of torus)



Let T be the simplicial complex representing L, the rectangle, along with the vertex labels. $|T|$ is the torus.



Then $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $H_2(T) \simeq \mathbb{Z}$.

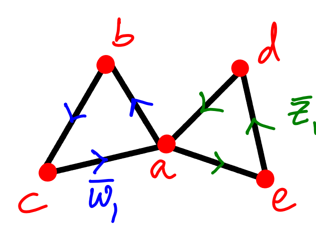
Orient each 2-simplex CCW; let $\bar{\gamma}$ denote their sum (2-chain). Let

$\bar{w}_1 = [a,b] + [b,c] + [c,a]$, and $\bar{z}_1 = [a,e] + [e,d] + [d,a]$.

Then $\bar{\gamma}$ generates $H_2(T)$ and $\{\bar{w}_1, \bar{z}_1\}$ is a basis for $H_1(T)$.

Proof

Let $g : |L| \to |T|$ be the pasting map (labeling), and let $A = g(|\mathrm{Bd}\, L|)$. Then A is homeomorphic to the union of two circles with one point in common (also called a wedge of two circles).



We apply the same "pushing off" arguments as in the proof of the last Lemma. We get the following results.

(1) Every 1-cycle of T is homologous to a 1-cycle carried by A.

(2) If $\bar{d}$ is a 2-chain of T and $\partial \bar{d}$ is carried by A, then $\bar{d}$ is a multiple of $\bar{r}$.

Notice the correspondence of statements (1) and (2) above to those in Lemma 6.1 [M].

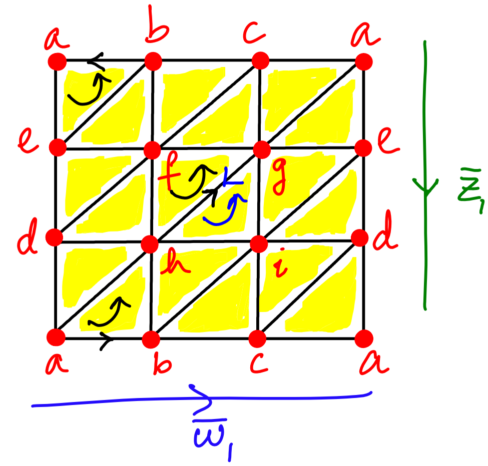
But we get two more results here.

(3) If $\bar{c}$ is a 1-cycle carried by A, then
$\bar{c} = m\bar{w}_1 + n\bar{z}_1$ for $m, n \in \mathbb{Z}$; and

(4) $\partial_2 \bar{r} = 0$.

To see these results, check the orientations induced on each edge by the two triangles it is a face of.

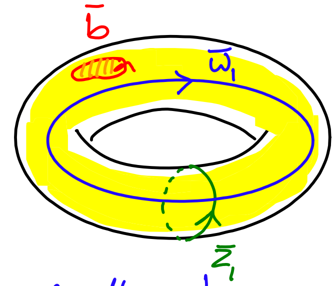
e.g., $[a,b]$ has $+1$ from $\partial_2[abh]$ and -1 from $\partial_2[aeb]$.



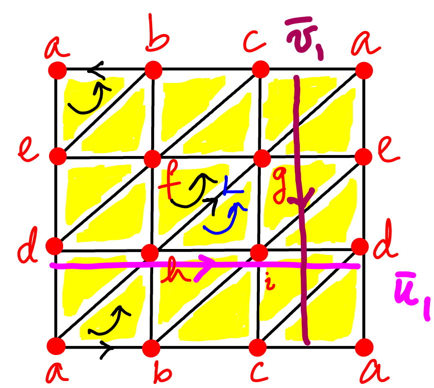
A 1-cycle in A bounds only if it is trivial, as every 1-cycle is $\bar{c} = m\bar{w}_1 + n\bar{z}_1$, and if $\bar{c} = \partial_2 \bar{d}$ then by (2) $\bar{d} = p\bar{r}$, $p \in \mathbb{Z}$. Hence by (4), since $\partial_2 \bar{d} = 0$, we must have $m = n = p = 0$ here.

Hence we can conclude that $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and $\{\bar{w}_1, \bar{z}_1\}$ is a basis.

Imagine elastic bands wrapping around the torus along $\bar{w}_1$ and $\bar{z}_1$ as shown here. These bands cannot be shrunk to a point, while another cycle/band represented by $\bar{b}$ (as shown) can be shrunk to a point. Indeed $\bar{b}$ bounds the patch of surface it encloses.

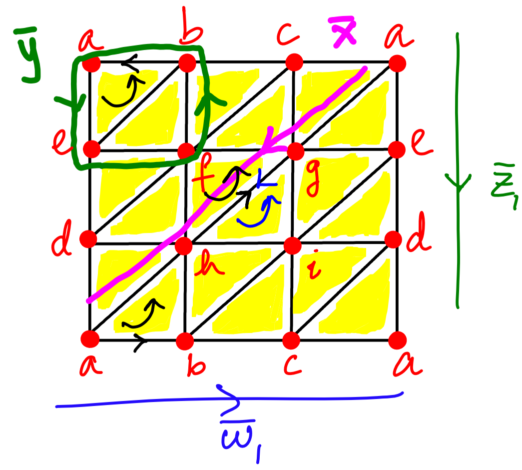


Also, there are many other choices for a basis for $H_1(T)$. For instance, we could use $\{\bar{u}_1, \bar{v}_1\}$, where $\bar{u}_1 = [d,h]+[h,i]+[i,d]$ and $\bar{v}_1 = [c,g]+[g,i]+[i,c]$.

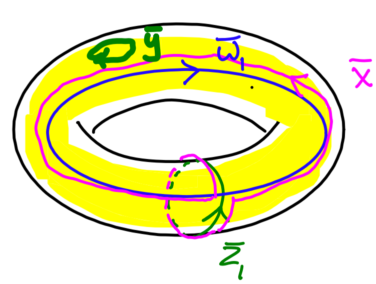


What about the cycle $\bar{x} = [a,g]+[g,h]+[h,a]$?

We can see that $\bar{w}_1 - \bar{z}_1 + \bar{x}$ is the boundary of the 2-chain consisting of all triangles below $\bar{x}$ in the diagram. In other words, $\bar{x}$ is homologous to $\bar{w}_1 + (-1)\bar{z}_1$, or equivalently, we can write $\bar{x}$ as a combination of $\bar{w}_1$ and $\bar{z}_1$.



And, $\bar{y}$ is a boundary — it is the boundary of the 2-chain consisting of triangles aeb and bef. So $\bar{y} \in B_1(T)$, and hence $\notin H_1(T)$.



<u>$H_2(T)$</u>: By (1) and (2), if $\bar{d}$ is a 2-cycle of T, then $\bar{d} = p\bar{r}$, $p \in \mathbb{Z}$. But (4) says $\partial \bar{r} = 0$, and hence every such 2-chain is a 2-cycle. There are no tetrahedra in T, and hence there are no 2-boundaries.

So $H_2(T) = Z_2(T) \simeq \mathbb{Z}$, with $\bar{r}$ being a generator.