MATH401: Lecture 10 (09/18/2025)

Today: * Intermediate Value Theorem (IVT)
* Bolzano-Weierstrass (BW) theorem
* Extreme Value theorem (EVT)

Recall: **Proposition 2.1.5** $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x=a$ iff $\lim_{n\to\infty} f(x_n) = f(a)$ for all sequences $\{x_n\}$ that converge to a.

($\Leftarrow$) Contrapositive argument

We assume f not continuous at $x=a$, and show there must exist a sequence $\{x_n\}$ that converges to a, but $\{f(x_n)\}$ does not converge to $f(a)$.

If f is not continuous at $x=a$, we take the converse of what is implied by it being continuous.

$\Rightarrow \exists \epsilon > 0$ s.t. no matter how small you choose $\delta > 0$,

$\exists x$ s.t. $|x-a| < \delta$ but $|f(x) - f(a)| \geq \epsilon$.

x here depends on δ

Also, note that δ can be chosen arbitrarily small.

Pick $\delta = \frac{1}{n}$. $\Rightarrow \exists x_n$ s.t. $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \epsilon$. as f is not continuous at $x=a$.

$\Rightarrow \{x_n\} \to a$ (as $n \to \infty$), but $\{f(x_n)\} \not\to f(a)$. $\square$

This notion of continuity defined in terms of sequences can be quite useful in many contexts, especially when we try to generalize results to higher dimensions.

We now get back to the proof of intermediate value theorem.

Recall:

Theorem 2.3.1 (Intermediate Value Theorem) Assume $f:[a,b]\to\mathbb{R}$ is continuous, and $f(a)$ and $f(b)$ have opposite signs. Then there exists $c\in(a,b)$ such that $f(c)=0$.

Proof Consider $f(a)<0<f(b)$. The other case of $f(a)>0>f(b)$ can be argued similarly.

Let $A=\{x\in[a,b]\mid f(x)<0\}$ and $c=\sup A$. A is bounded (subset of $[a,b]$), hence $\sup A$ exists.

We show $f(c)=0$.

f is continuous and $f(b)>0$. $\Rightarrow c<b$.

$\Rightarrow$ The sequence $x_n=c+\frac{1}{n}\in[a,b]$ $\forall n\geq N$, for sufficiently large N. $c<b$,

$\Rightarrow \{x_n\}\to c$ as $n\to\infty$.

Also, $f(x_n)>0$ $\forall$ such n. as $x_n\notin A$, since $x_n>c$.

By Proposition 2.1.5, as f is continuous, $\lim\limits_{n\to\infty}f(x_n)=f(c)$, and since $f(x_n)>0$ $\forall n$, $f(c)\geq 0$. We can be sure that $f(c)\neq 0$.

On the other hand, by definition of c, consider

$z_n=c-\frac{1}{n}$ for sufficiently large n. when n is large enough, $c-\frac{1}{n}\geq a$.

$\Rightarrow z_n\leq c$ $\forall n$ large enough, and $\{z_n\}\to c$ (as $n\to\infty$).

Also, $z_n\in A\subset[a,b]$ for n large enough. $\Rightarrow f(z_n)<0$.

Again, by Proposition 2.1.5, $f(c)=\lim\limits_{n\to\infty}f(z_n)$ and since $f(z_n)<0$ $\forall n$, we get $f(c)\leq 0$. Hence $f(c)\geq 0$ and $f(c)\leq 0$, i.e., $f(c)=0$. $\square$

Again, we can be sure that $f(c)\neq 0$.

The Intermediate Value Theorem does not hold in $\mathbb{Q}$!

Consider $f(x) = x^2 - 3 \Rightarrow f(0) = -3$ and $f(2) = 1$.

But $\nexists\, x \in [0,2] \cap \mathbb{Q}$ s.t. $f(x) = 0$, as $\sqrt{3} \notin \mathbb{Q}$.

The **Bolzano-Weierstrass** (BW) Theorem

We saw that every Cauchy sequence converges. But what if a sequence is not Cauchy, and hence does not converge? Can we still say something nice about its structure? It turns out yes, when the sequence is bounded! We need the notion of a subsequence first.

Def (Subsequence) Given a sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$, we choose an infinite subset of its terms to form another sequence $\{\bar{y}_k\}$. (Of course, it is interesting only when we do not choose all terms of $\{\bar{x}_n\}$).

$$\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \ldots$$

$$\bar{y}_1 \quad \bar{y}_2 \quad \bar{y}_3 \quad \bar{y}_4 \quad \cdots$$

$\{\bar{y}_k\} \rightarrow$ subsequence of $\{\bar{x}_n\}$.

If $n_1 < n_2 < \cdots < n_k < \cdots$ are indices of terms picked to form a new sequence, then

$$\{\bar{y}_k\} = \{\bar{x}_{n_k}\} \text{ is a } \textbf{subsequence} \text{ of } \{\bar{x}_n\}.$$

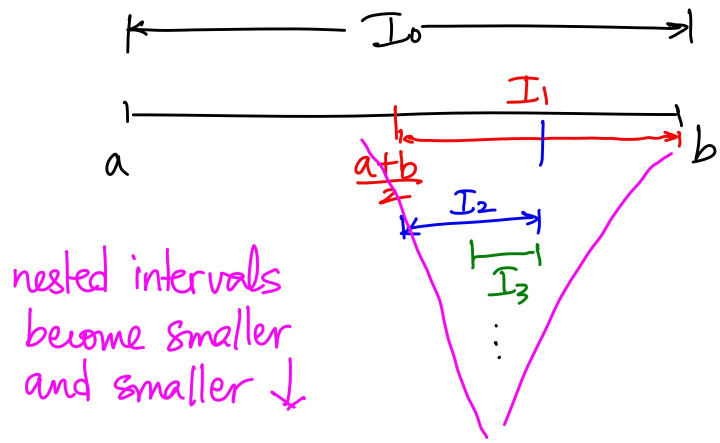
We state and prove BW theorem in $\mathbb{R}$.

Proposition 2.3.2 Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

$\{x_n\}$ is bounded $\Rightarrow \exists\; a \le b$ s.t. $x_n \in [a,b] = I_0 \;\; \forall n$.

We identify a Cauchy subseq.

We argue we can pick smaller and smaller subintervals of I_0, each of which has infinitely many terms of $\{x_n\}$.



Let $I_1 = \left[\frac{a+b}{2}, b\right]$ be such that it has infinitely many terms of $\{x_n\}$.

It could happen that $I_1' = \left[a, \frac{a+b}{2}\right]$ is the one with infinitely many terms, or both I_1 and I_1' have infinitely many terms of $\{x_n\}$. But since $\{x_n\}$ has infinitely many terms, at least one of the two half intervals is guaranteed to have infinitely many terms. We always choose a half interval with infinitely many terms, and continue the process.

In general, I_k is a half interval of I_{k-1} that has infinitely many terms of $\{x_n\}$. Note that I_k is a subinterval of I_{k-1} for each k ($k \geq 1$), and we get a sequence of nested subintervals that are shrinking in size by a factor of $\left(\frac{1}{2}\right)$ in each step.

Since $|I_0| = |[a,b]| = b-a$ is finite, $|I_k| \to 0$ as $k \to \infty$.

We can now specify how to construct the convergent subsequence. Essentially, we pick one term of $\{x_n\}$ from each subinterval I_k as follows.

Let y_1 be the first element of $\{x_n\}$ in I_1. And let y_2 be the first element of $\{x_n\}$ **after** y_1 that is in I_2.

$\vdots$

In general, let y_k be the first element of $\{x_n\}$ **after** y_{k-1} that is in I_k, for $k \geq 1$.

Note that the y_k's are included in nested, shorter and shorter subintervals, and hence are getting closer and closer to each other.

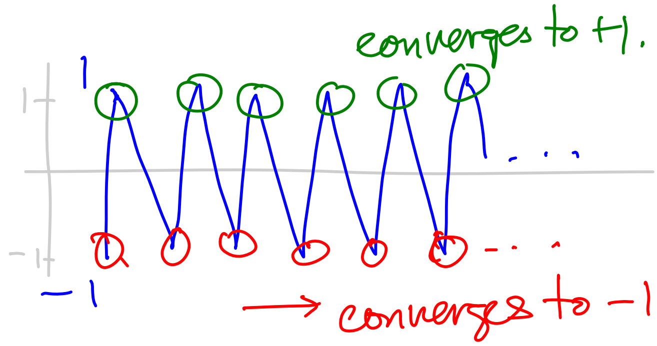
we could make this argument more formal

$\Rightarrow$ $\{y_k\}$ is Cauchy!

$\Rightarrow$ $\{y_k\}$ converges by Proposition 2.2.8 $\square$

Consider a somewhat trivial example. Let $x_n = (-1)^n, n \in \mathbb{N}$.

One can immediately see that $a_n \leq 1 \ \forall n$ and $a_n \geq -1 \ \forall n$ hold, i.e., we can choose $[a,b] = [-1,1]$ in the proof above.



Then $y_k = (-1)^k$ for $k = 2n, n \in \mathbb{N}$ defines a subsequence $\{y_k\} \to 1$, and $z_k = (-1)^k$ for $k = 2n-1$, $n \in \mathbb{N}$ defines a subsequence $\{z_k\} \to -1$.

The BW theorem naturally extends to $\mathbb{R}^m$ — we essentially repeat the above argument one dimension at a time! See LSIRA for details.

We now present two theorems that use the results on sequences to specify properties of "good" (continuous or differentiable) functions defined on the sequences.

The **Extreme Value Theorem** (EVT) in $\mathbb{R}$

Theorem 2.3.4 Let $f:[a,b]\to\mathbb{R}$ be a continuous function on the closed bounded interval $[a,b]$. Then $\exists$ points $c, d \in [a,b]$ such that $f(d) \le f(x) \le f(c)$ $\forall\, x \in [a,b]$.

In words, f has maximum and minimum points in $[a,b]$.

Proof (for maximum) → A similar argument can be made for minimum

Let $M = \sup\{f(x) \mid x \in [a,b]\}$. → We're not sure yet whether M is finite

Choose sequence $\{x_n\}$ in $[a,b]$ such that $f(x) \to M$.

As f is continuous, such a sequence exists. → irrespective of whether M is finite or not

$[a,b]$ is bounded $\Rightarrow$ By BW Theorem, $\{x_n\}$ has a convergent subsequence $\{y_k\}$.

$[a,b]$ is closed $\Rightarrow$ $c = \lim_{k\to\infty} y_k \in [a,b]$.

$\Rightarrow f(y_k) \to M$ by construction. → We chose $\{x_n\}$ so that $f(x_n)\to M$ in the first place.

f is continuous $\Rightarrow$ by Proposition 2.1.5, $f(y_k) \to f(c)$.

$\Rightarrow f(c) = M$, i.e., M is the maximum, and $c \in [a,b]$ is the corresponding maximum point for f. $\square$