MATH 524 — Lecture 5 (09/05/2023)

Today: * Abstract simplicial complexes (ASCs)
* Examples of ASCs

Abstract Simplicial Complexes (ASCs)

Def An abstract simplicial complex (ASC) is a collection $\mathcal{S}$ of finite non-empty sets such that if $A \in \mathcal{S}$, then so is every nonempty subset of A.

Note: $\mathcal{S}$ itself could be infinite, but each $A \in \mathcal{S}$ is finite.

Example: $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ is an ASC.

We specify several more definitions related to ASCs.

Def A (any element of $\mathcal{S}$) is a **simplex** of $\mathcal{S}$. Its **dimension** is given as $\dim(A) = |A| - 1$.
↳ # elements in A, or size of A

The **dimension of the ASC** is defined as follows.

$\dim(\mathcal{S}) =$ largest dimension of any simplex in $\mathcal{S}$, or ∞ if no such largest dimension exists.

The **vertex set** V of $\mathcal{S}$ (or $V(\mathcal{S})$) is the union of all singleton elements (simplices) of $\mathcal{S}$. We do not distinguish between the individual vertices and the singleton sets they represent.

$$v_0 \text{ (vertex)} \equiv \{v_0\} \quad \text{0-simplex of } \mathcal{S}.$$

A subcollection of $\mathcal{S}$ that is a simplicial complex by itself is a **subcomplex** of $\mathcal{S}$.

We can now talk about when two ASCs are "similar".

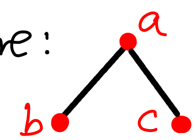
Def Two ASCs $\mathcal{S}$ and $\mathcal{T}$ are **isomorphic** if there exists a bijective correspondence f mapping $V(\mathcal{S})$ to $V(\mathcal{T})$ such that $\{a_0,\dots,a_n\} \in \mathcal{S}$ iff $\{f(a_0),\dots,f(a_n)\} \in \mathcal{T}$.

e.g., With $\mathcal{T} = \{\{d\},\{e\},\{f\},\{d,e\},\{d,f\}\}$, $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.

It turns out the previous notion of simplicial complexes (in $\mathbb{R}^d$) and ASC are directly related.

Def Let K be a (geometric) simplicial complex. Let V be its vertex set. Let $\mathcal{K}$ be the collection of all subsets $\{a_0,\dots,a_n\}$ of V such that $\bar{a}_0,\dots,\bar{a}_n$ span a simplex of K. Then $\mathcal{K}$ is an ASC called the **vertex scheme** of K. Symmetrically, we call K a **geometric realization** of $\mathcal{K}$.

e.g., (continued) $\mathcal{S} = \{\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\}$ has a geometric realization S as shown here:



This complex could be sitting in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

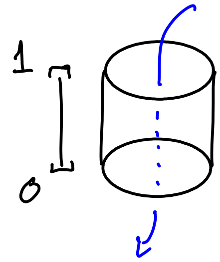
Theorem 3.1 [M] (a) Every ASC $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K.

A version of this result is given as the **geometric realization theorem** which states that every abstract d-complex has a geometric realization in $\mathbb{R}^{2d+1}$.

IDEA: If $\dim(\mathcal{S}) = d$ then let $f: V(\mathcal{S}) \to \mathbb{R}^{2d+1}$ be an injective function whose image is a set of GI points in $\mathbb{R}^{2d+1}$. Specify that for each abstract simplex $\{a_0,\dots,a_n\} \in \mathcal{S}$, $\{f(a_0),\dots,f(a_n)\} \in K$. Then $\mathcal{S}$ is isomorphic to the vertex scheme of K.

Examples of ASCs

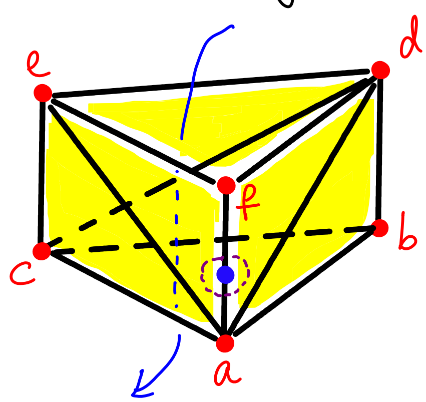
1. Cylinder $\quad S^1 \times I$ (circle; $I \to [0,1]$)



We want to describe a simplicial complex K such that $|K|$ is homeomorphic to the cylinder.

We first describe a geometric simplicial complex K, which could be sitting in $\mathbb{R}^3$, for instance.

K

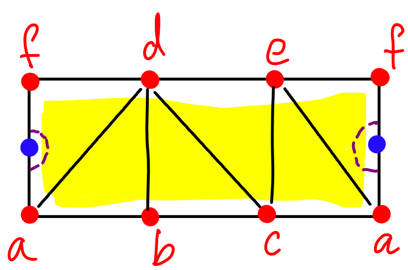


K comprises of the six triangles adf, abd, bcd, cde, ace, and aef.

Indeed, $|K| \approx$ cylinder.

But we now specify an abstract simplicial complex whose underlying space is homeomorphic to the cylinder. We start with a rectangle L, and then assign labels to specific vertices in L. Thus, L along with the labels is the ASC.

L



Notice that both the left and right border edges of L are labeled af going from bottom to top.

We can describe the required map between K and L as follows.

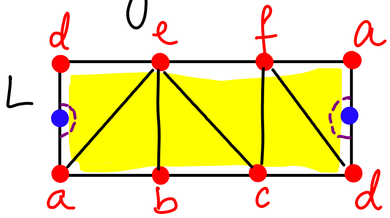
Let $f: K^{(0)} \to L^{(0)}$ is the vertex map that assigns vertices in K the labels in L. We can extend f to a simplicial map $g: |K| \to |L|$. This map g is a "pasting map", or a quotient map.

indeed, we are starting with the rectangular strip (of paper, say) L, and pasting its end edges together (af).

Notice how we can visualize a neighborhood of a point on edge af in K and correspondingly on L.

2. Möbius strip

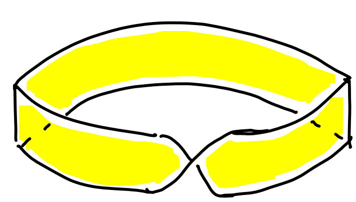
We now start with the rectangular space L and a specific vertex labeling as shown here.



The ASC $\mathcal{S}$ here has 6 triangles ade, abe, bce, cef, cdf, adf, as well as their faces.

We're again gluing the end edges, but now with a "twist."

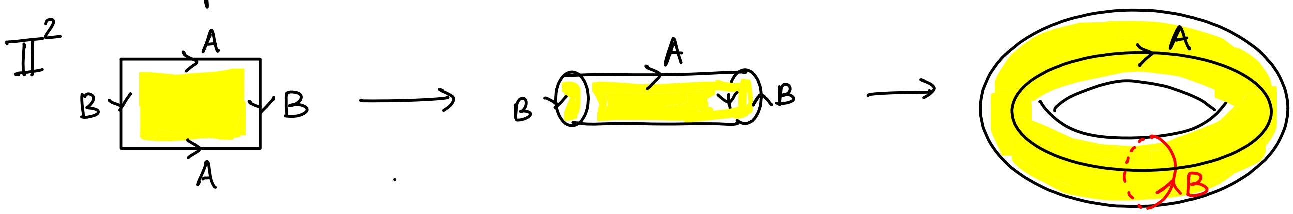
Let K be a geometric realization of $\mathcal{S}$. We can consider a simplicial map $g: |L| \to |K|$, which maps vertices in L to vertices in K. Again, g is a quotient (or "pasting") map that maps the left edge of $|L|$ to the right edge, but with a "twist."

$L \xrightarrow{g}$ 

Notice that we do want a homeomorphism from $|L|$ to K, and just a vertex map is not enough. But of course, the vertex map is naturally (linearly) extended to the desired map from $|L|$ to $|K|$.

\mathbb(T)^2 in LaTeX!

3. Torus ($\mathbb{T}^2$) The quotient space obtained by making identifications on the sides of a rectangle as follows.

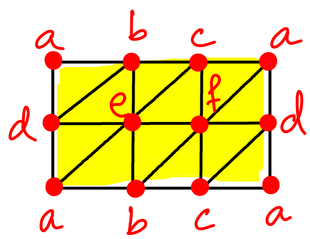


Notice that this is an example of a quotient map defined on a general space, and not on an ASC.
This is the surface of a "donut", and not the solid donut itself.

Now, let us find an ASC K such that $|K| \approx \mathbb{T}^2$.

Let's start with a rectangular space as before, and assign labels that could work. Here is a first try.

K_1:

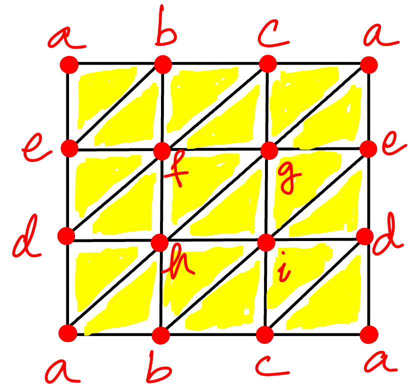


Is $|K_1| \approx \mathbb{T}^2$? No!
We are doing too much gluing!

Notice that $\overline{ad}$ is part of 4 triangles ade, adb, adc, adf, for instance.
The gluings specified above glue only two edges together at a time.
With this gluing, edge ad is part of four triangles, i.e., we get a "fan" of four "flaps" meeting at ad. But notice that there are no such 4-way junctions in the torus.

We need to "spread out" more!

K_2:



We can show that $|K_2| \approx \mathbb{T}^2$. See [M] for details, but on a complex similar to K_2.

$$|K_2| \approx \mathbb{T}^2$$

Every edge is face of exactly two triangles.