MATH 524 : Lecture 1 (08/19/2025)

This is Algebraic Topology.
I'm Bala Krishnamoorthy (Call me Bala).

Today:
* syllabus, logistics
* neighborhoods, continuous functions
* topology using neighborhoods
* homeomorphism

I will be teaching computational topology (Math 529) next semester. The two classes - Math 524 and Math 529 will be kept independent. In particular, we will spend nearly no focus on computational aspects in Math 524.

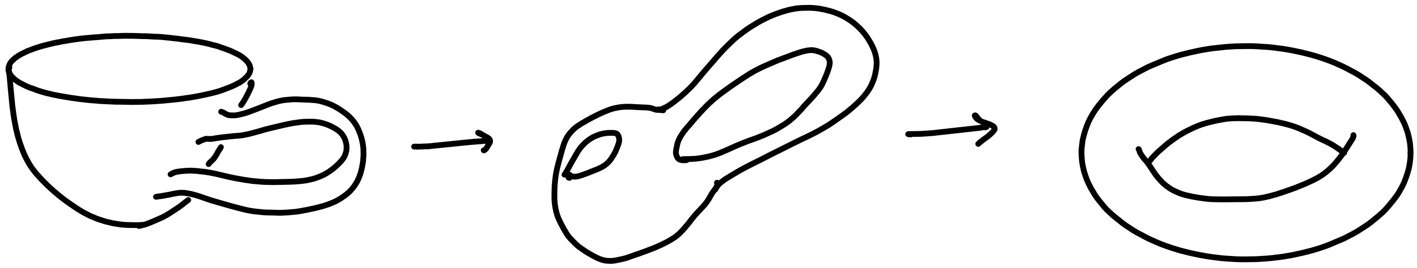
Check the course web page at

https://bala-krishnamoorthy.github.io/Math524.html

All documents, important updates, homework assignments etc. will be posted there. Check the class page frequently.

More about the video assignment to come soon. But you're encouraged to start looking for topics that you might want to make the video on as we proceed in the course.

Topology studies how "space is connected". You might have heard the (true!) joke that the topologist cannot distinguish between a coffee cup and a donut! Indeed, they both are connected the same way.



In algebraic topology, we cast problems on how space is connected as equivalent problems on algebraic objects — groups, rings, etc., and maps between them (homomorphisms).

As a subfield of mathematics, algebraic topology started in late 19th and early 20th century. Poincaré introduced the fundamental group first. Later Betti introduced homology groups, which are much easier to compute (both by hand as well as algorithmically) than the former.

We will spend a lot of time talking about homology groups, and the dual concept of cohomology. We will not be spending much attention on the fundamental group. There are several (equivalent) ways to define homology groups. Perhaps the "nicest" way to do so is using simplicial complexes. We will spend a fair bit on time studying simplicial homology.

We will introduce/refresh background concepts as needed. First, we will talk about continuous functions and topological spaces, defined in terms of neighborhoods.

Continuous functions

We first give the classical ε-δ definition in Euclean spaces.

Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$. f is continuous at $\bar{x} \in \mathbb{R}^n$ if there exists $\delta > 0$ for every $\varepsilon > 0$ such that $\|f(\bar{y}) - f(\bar{x})\| < \varepsilon$ whenever $\|\bar{y} - \bar{x}\| < \delta$ for $\bar{y} \in \mathbb{R}^n$. f is continuous (in all of $\mathbb{R}^n$) if it is so at every $\bar{x} \in \mathbb{R}^n$.

my notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\mu}$, etc., are all vectors – lower case letters with a bar.

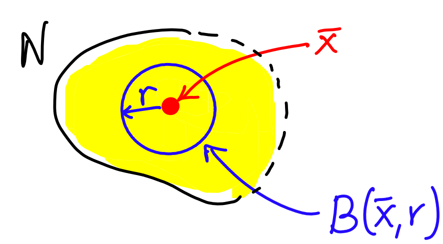
We give an equivalent definition based on neighborhoods.

Def A subset N of $\mathbb{R}^n$ is a **neighborhood** of $\bar{x} \in \mathbb{R}^n$ if for some $r > 0$, the closed ball $B(\bar{x}, r)$ centered at $\bar{x}$ is contained entirely within N.

Notice that neighborhood N can be open or closed.

$$B(\bar{x}, r) = \{ \bar{y} \in \mathbb{R}^n \mid \|\bar{x} - \bar{y}\| \leq r \}$$

closed Ball of radius r centered at $\bar{x}$



Def $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous if given any $\bar{x} \in \mathbb{R}^n$ and a neighborhood N of $f(\bar{x})$ in $\mathbb{R}^m$, $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in $\mathbb{R}^n$.

Now we define what a topological space (or topology) is. We give the definition in terms of neighborhoods first. In most textbooks, you will see the definition given in terms of open sets. Later, we will see that both definitions are equivalent.

Topological space (or topology)

more notation: Upper case letters, e.g., A, B, X, Y, etc, denote sets or matrices.

Def I We are given a set X and a nonempty collection of subsets of X for each $\bar{x} \in X$ called the neighborhoods of $\bar{x}$. This is a topological space if it satisfies the following axioms.

(a) $\bar{x}$ lies in each of its neighborhood.

(b) Intersection of two neighborhoods of $\bar{x}$ is itself a neighborhood of $\bar{x}$.

(c) If N is a neighborhood of $\bar{x}$, and $\mathcal{U} \subseteq X$ contains N, then $\mathcal{U}$ is a neighborhood of $\bar{x}$.

(d) If N is a neighborhood of $\bar{x}$, $\mathring{N}$, the **interior** of N is also a neighborhood of $\bar{x}$.

The interior of N is $\mathring{N} = \{\bar{y} \in N \mid N \text{ is a neighborhood of } \bar{y}\}$.

Intuitively, every point of N not on its boundary is in its interior.

We can extend the definition of continuous functions to functions defined between topological spaces.

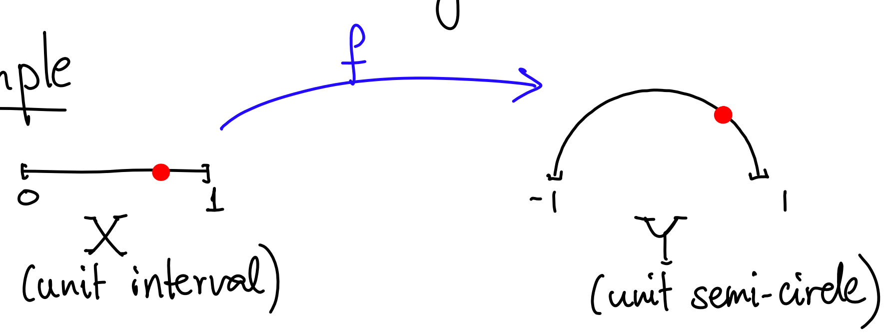
Def Let X, Y be topological spaces. $f: X \to Y$ is continuous if $\forall\, \bar{x} \in X$ and for every neighborhood N of $f(\bar{x})$ in Y, the set $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in X.

We are interested in studying when two topological spaces are similar. There are a few different notions of topological similarity, and the strongest notion is that of homeomorphism. For two spaces to be homeomorphic, we need a function between them that is "nicer" than just a continuous function.

Def A function $f: X \to Y$ is a **homeomorphism** if it is one-to-one, onto, continuous, and has a continuous inverse.

When such a function exists between two spaces X and Y, we say they are **homeomorphic**, or are topologically equivalent. We denote this fact by $X \approx Y$.

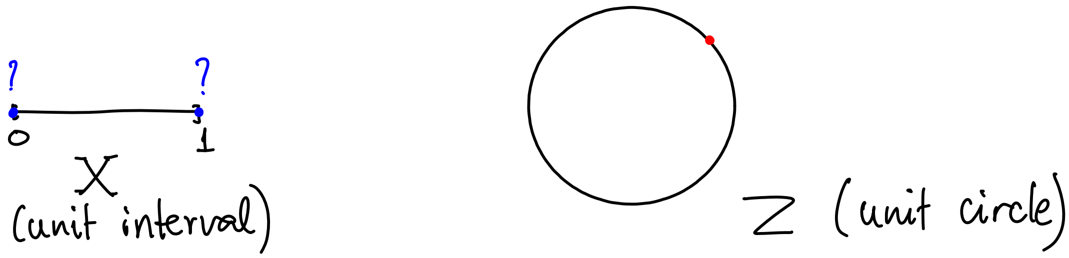
Example



$X \approx Y$. Can you define the function f?

Think of X & Y as subsets of $\mathbb{R}^2$, and write down the form of f, as well as f^{-1}. You can show f satisfies all requirements for being a homeomorphism.

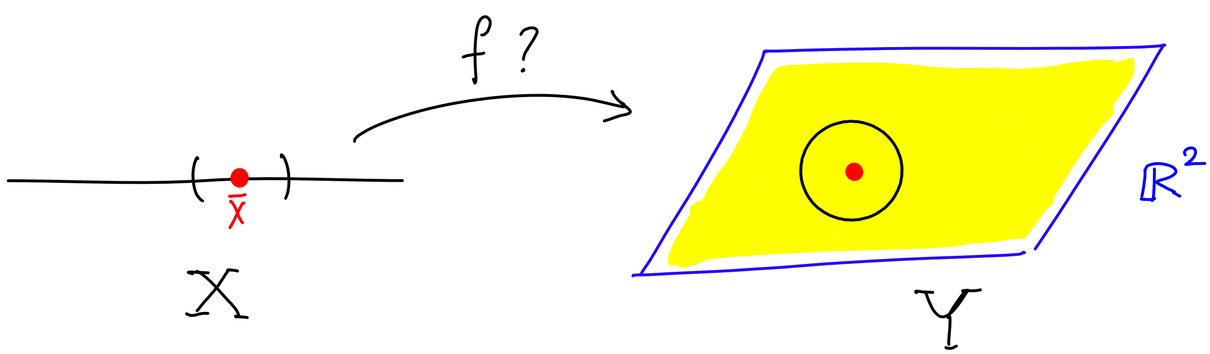
Showing two spaces are **not** homeomorphic could be harder — we need to show that no such function exists between X and Y.



Here, $X \not\cong Z$. Where do things breakdown?

Intuitively, one can notice the two end points of X behave distinctly from any point in Y.

Here is another example. Perhaps the simplest example of a topological space is $\mathbb{R}^d$ under the usual definition of neighborhoods, which specifies that any set $N \subseteq \mathbb{R}^d$ big enough to contain $B(\bar{x}, r)$ for some $r > 0$ is a neighborhood of $\bar{x} \in \mathbb{R}^d$. But notice that $\mathbb{R}^1 \not\cong \mathbb{R}^2$, for instance. It is not straightforward to prove this fact rigorously. But, how would one "argue" for it?



One method is to appeal to how the two spaces are connected. Recall that topologically similar spaces are "connected" the same way. Here, if we remove one point from both $X = \mathbb{R}^1$ and $Y = \mathbb{R}^2$, we can see that it affects the connectivity differently. Removing one point leaves X disconnected (into two pieces). But removing a point from Y still leaves it connected – its just like poking a hole in the "sheet" that is $\mathbb{R}^2$, which remains connected.

More formally, we could try to define a homeomorphism from X to Y. But we can observe that neighborhoods in X are 1-dimensional, while those in Y are 2D. Hence we cannot define a bijection between them.

We will talk about open sets in the next lecture, and define a topology using open sets. That definition is equivalent to the one introduced earlier today, i.e., Def I.

MATH 524 : Lecture 2 (08/21/2025)

Today: * open sets, topology using open sets
* simplices, properties of simplices

We now consider topology defined in terms of open sets. This is the default approach taken in most textbooks. We first define open sets using the concept of neighborhoods.

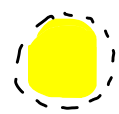
Def $O \subseteq X$ is **open** if it is a neighborhood of each of its points. By (c) of **Def I**, union of any collection of open sets is also open. Also, by (b) of **Def I**, the intersection of any finite number of open sets is open.

We mention unions and finite intersections of open sets as they are both required to be open in a topology. See below.

Notice, $\mathring{N}$ (interior of neighborhood N) is always open.

Alternatively, we can start by defining open sets directly.

Def A set $A \subseteq \mathbb{R}^n$ is **open** if each $\bar{x} \in A$ can be surrounded by a ball of positive radius that lies entirely inside the set.



We can also define open sets more generally, starting with collections of subsets of some set X.

We could define neighborhoods in terms of open sets.

Def A subset $N \subseteq X$ is a neighborhood of $\bar{x}$ if there exists an open set O s.t. $\bar{x} \in O \subseteq N$.

We now formally state the definition of topology in terms of open sets. This definition sees more use than the one using neighborhoods.

Def II A **topology** on a set X is a collection of open sets of X such that any union and finite intersection of open sets is open, and ϕ (empty set) and X are open.
The set X along with the topology is called a **topological space**.

We can define continuous functions also in terms of open sets.

Def $f: X \to Y$ is **continuous** iff (if and only if) the inverse image of each open set of Y is open in X.

We now start the discussion of homology, which is a less strict version of topological similarity than homeomorphism. We study in detail simplicial homology, where the spaces are made of "gluing" "nice" objects called simplices together, and are hence are very "regular".

As we will see, it is also much easier to algebraize questions about homology (than those about homeomorphism).

There is a "continuous" version of homology defined on spaces not composed to regular pieces (simplices), termed singular homology. It turns out singular homology is equivalent to simplicial homology.

We start by defining simplices, which are the building blocks.

Simplices

We define simplices in the usual geometric setting first, and then define them abstractly. We need some concepts from geometry first.

Def The set $\{\bar{a}_0, \ldots, \bar{a}_n\}$ of points in $\mathbb{R}^d$ is **geometrically independent** (**GI**) if for any scalars $t_i \in \mathbb{R}$, the equations

$$\sum_{i=0}^{n} t_i = 0, \quad \sum_{i=0}^{n} t_i \bar{a}_i = \bar{0} \quad \text{imply that } t_0 = t_1 = \cdots = t_n = 0.$$

Here are some observations about GI sets.

* $\{\bar{a}_i\}$ is GI $\forall i$. (singleton sets)

* $\{\bar{a}_0, \ldots, \bar{a}_n\}$ is GI $\iff$ ← if and only if
  $\{\bar{a}_1 - \bar{a}_0, \bar{a}_2 - \bar{a}_0, \ldots, \bar{a}_n - \bar{a}_0\}$ is linearly independent (LI).

$\bar{a}_0$ is chosen as the "origin", so to speak. But any $\bar{a}_i$ could play the role of $\bar{a}_0$ here.

IDEA: $\sum_{i=1}^{n} t_i(\bar{a}_i - \bar{a}_0) = \bar{0} \implies t_i = 0 \ \forall i$ (LI)

$$\sum_{i=1}^{n} t_i \bar{a}_i + \underbrace{\left(-\sum_{i=1}^{n} t_i\right)}_{t_0} \bar{a}_0 = \bar{0} \quad \left.\right\} \quad \sum_{i=0}^{n} t_i \bar{a}_i = 0 \ \& \ \sum_{i=0}^{n} t_i = 0 \implies t_i = 0 \ \forall i$$

* 2 distinct points in $\mathbb{R}^d$ are GI,
  3 non-collinear points are GI,
  4 non-coplanar points are GI, and so on.

Notice the relationship/correspondence to LI vectors. For instance, $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ is GI, but of course the set is not LI.

Def Given GI set $\{\bar{a}_0, \dots, \bar{a}_n\}$, the **n-plane** P spanned by these points consists of all $\bar{x}$ such that

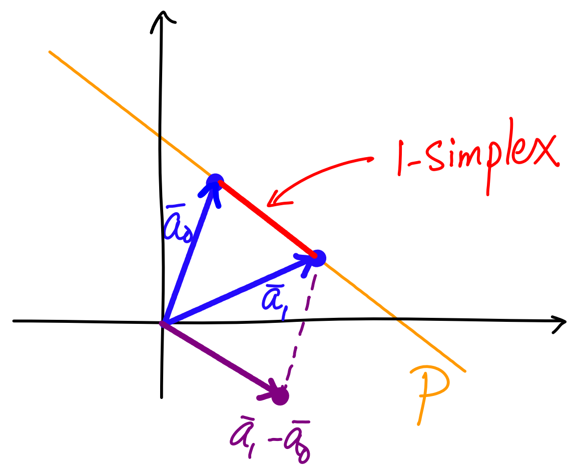
$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i \quad \text{for scalars } t_i \text{ with } \sum_{i=0}^{n} t_i = 1.$$

The scalars t_i are uniquely determined by $\bar{x}$.

Notice that t_i could be ≥ 0 or ≤ 0 here.

P can also be described as the set of $\bar{x}$ such that

$$\bar{x} = \bar{a}_0 + \sum_{i=1}^{n} t_i (\bar{a}_i - \bar{a}_0).$$



Hence P is the plane through $\bar{a}_0$ parallel to the vectors $\bar{a}_i - \bar{a}_0$.

Going back to the previous example with $\{[1], [2]\}$, the plane P is the line generated by one of the two vectors.

Q. What is the set described by $\bar{x} = \sum_{i=0}^{n} t_i a_i$, $\sum t_i = 0$?

e.g., consider $n=1$: $\bar{x} = t_0 \bar{a}_0 + t_1 \bar{a}_1$ with $t_0 + t_1 = 0 \Rightarrow t_0 = -t_1$.

$\Rightarrow \bar{x} = t_0(\bar{a}_0 - \bar{a}_1)$, i.e., it's the line generated by $\bar{a}_0 - \bar{a}_1$.

We now define a simplex as the set "spanned" by a set of GI points.

Def Let $\{\bar{a}_0, \dots, \bar{a}_n\}$ be a GI set in $\mathbb{R}^d$. The **n-simplex** σ spanned by $\bar{a}_0, \dots, \bar{a}_n$ is the set of points $\bar{x} \in \mathbb{R}^d$ s.t. $\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i$ with $\sum_{i=0}^{n} t_i = 1$, $t_i \geq 0 \; \forall i$.

The t_i are uniquely determined by $\bar{x}$, and are called the **barycentric coordinates** of $\bar{x}$ (in σ) w.r.t. $\bar{a}_0, \dots, \bar{a}_n$.

we will later extend definition of t_i to $\bar{x} \notin \sigma$. the

0-simplex : a point

1-simplex : line segment

<u>2-simplex</u>

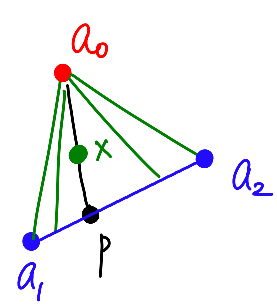
$\bar{x} = \bar{a}_0$ is trivial to consider.

Assume $\bar{x} \neq \bar{a}_0$, i.e., $t_0 \neq 1$. Now consider

$$\bar{x} = \sum_{i=0}^{2} t_i a_i = t_0 \bar{a}_0 + (1-t_0) \underbrace{\left[\frac{t_1}{1-t_0} \bar{a}_1 + \frac{t_2}{1-t_0} \bar{a}_2 \right]}_{\bar{p}}$$

Since $\sum_{i=0}^{2} t_i = 1$, $1 - t_0 = t_1 + t_2$. Hence $\left(\frac{t_1}{1-t_0}\right)\bar{a}_1 + \left(\frac{t_2}{1-t_0}\right)\bar{a}_2$ is a point $\bar{p}$ on the line segment $\overline{\bar{a}_1 \bar{a}_2}$, and $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{p}$ is a point on the line segment $\overline{\bar{a}_0 \bar{p}}$.

Hence the 2-simplex is the union of such line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\overline{a_1 a_2}$, i.e., the triangle $a_0 a_1 a_2$ ($\triangle a_0 a_1 a_2$).



This result extends to higher order simplices. For instance, a tetrahedron is the union of all line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\triangle a_1 a_2 a_3$.

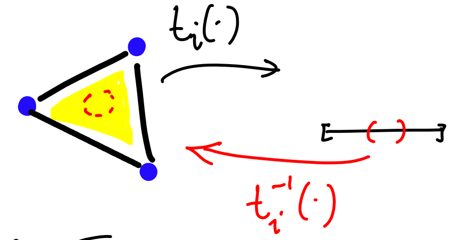
Properties of Simplices

(1) $t_i(\bar{x})$ are continuous functions of $\bar{x}$.

IDEA: $t_i : \mathbb{R}^d \to \mathbb{R}$

domain $\to$ $\mathrm{Dom}(t_i) = \mathrm{conv}(\{\bar{a}_0, \dots, \bar{a}_n\})$ $\to$ convex hull $\{\bar{x} \mid \bar{x} = \sum_{i=0}^{n} t_i a_i,\ t_i \geq 0,\ \sum_i t_i = 1\}$

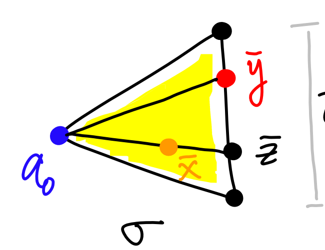
$\mathrm{Range}(t_i) = [0, 1]$



Prove that $t_i^{-1}(\text{open set in } [0,1])$ is open in σ.

(2) σ is the union of all line segments joining $\bar{a}_0$ to points of the simplex spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$.
Two such line segments intersect only at $\bar{a}_0$.
$\hookrightarrow$ proof?

Assume two such line segments from $\bar{a}_0$ to $\bar{y}, \bar{z} \in \tau$, the simplex spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$, meet at $\bar{x} \neq \bar{a}_0$. $\to$ $\bar{y} \neq \vec{z}$



Then $\bar{x} = t_0 \bar{a}_0 + (1 - t_0)\bar{y} = s_0 \bar{a}_0 + (1 - s_0)\bar{z}$, for $t_0, s_0 \in [0,1]$,
where $t_0 \neq s_0$ by assumption (else $\bar{y} = \bar{z}$!).

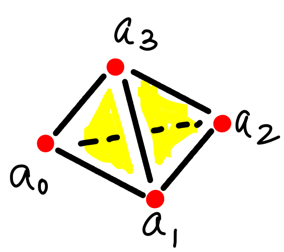
$\Rightarrow$ $\bar{a}_0 = u\bar{y} + v\bar{z}$, where $u, v \in \mathbb{R}$ with $u + v = 1$.

$\Rightarrow$ $\bar{a}_0 \in P(\{\bar{y}, \bar{z}\}) \in P(\tau)$ $\to$ $(n-1)$-plane spanned by $\{\bar{a}_1, \dots, \bar{a}_n\}$.

which contradicts the GI of $\{\bar{a}_0, \dots, \bar{a}_n\}$.

Def The points $\bar{a}_0, \dots, \bar{a}_n$ which span σ are called its **vertices**. The **dimension** of σ is n $(\dim(\sigma) = n)$. A simplex spanned by a nonempty subset of $\{\bar{a}_0, \dots, \bar{a}_n\}$ is a **face** of σ. The face spanned by $\{\bar{a}_0, \dots, \hat{\bar{a}}_i, \dots \bar{a}_n\}$ where $\hat{\bar{a}}_i$ means $\bar{a}_i$ is not included, is the **face opposite** $\bar{a}_i$. Faces of σ distinct from σ itself are its **proper faces**, their union is its **boundary**, $\mathrm{Bd}\,\sigma$ or $\partial\sigma$.

$\partial(\bar{a}_0) = \phi \longrightarrow$ there are no proper faces of a vertex.



tetrahedron $a_0a_1a_2a_3 = \sigma \longrightarrow$ a 3-simplex

proper faces : $\Delta a_0a_1a_2, \Delta a_0a_2a_3, \dots$ (4)

edges $\rightarrow$ $\overline{a_0a_1}, \overline{a_0a_2}, \dots$ (6)

vertices $\rightarrow$ $\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3$ (4)

$\partial\sigma = \bigcup(\text{proper faces})$ (triangles, edges, vertices)

$\longrightarrow$ the "hollow" tetrahedron

Def The **interior** of σ, $\mathrm{Int}(\sigma)$ or $\mathring{\sigma}$, is $\mathrm{Int}(\sigma) = \sigma - \mathrm{Bd}\,\sigma$.

$\mathrm{Int}(\sigma)$ is called an open simplex.

$\mathrm{Int}(\bar{a}_0) = \bar{a}_0 . \longrightarrow$ as $\partial a_0 = \phi$.

$\mathrm{Bd}\,\sigma$ consists of all $\bar{x} \in \sigma$ with at least one $t_i(\bar{x}) = 0$.

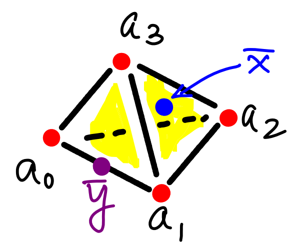
$\mathrm{Int}\,\sigma$ consists of all $\bar{x} \in \sigma$ with $t_i(\bar{x}) > 0 \ \forall i$.

could be σ itself!

Given $\bar{x} \in \sigma$, there is exactly one face τ s.t. $\bar{x} \in \operatorname{Int} \tau$.
τ is that face of σ spanned by those $\bar{a}_i$ for which $t_i(\bar{x}) > 0$.

$\bar{x}$ is interior to $\Delta a_1 a_2 a_3$

$\bar{y}$ is interior to $\overleftrightarrow{a_0 a_1}$



(3) σ is a compact, convex set in $\mathbb{R}^d$, and is the intersection of all convex sets in $\mathbb{R}^d$ containing $\bar{a}_0, \dots, \bar{a}_n$.

(4) There exists one and only one GI set of points $\{\bar{a}_0, \dots, \bar{a}_n\}$ spanning σ.

(5) $\operatorname{Int} \sigma$ is convex, and is open in P, and $Cl(\operatorname{Int} \sigma) = \sigma$. $\operatorname{Int} \sigma$ is the union of all "open line segments" joining $\bar{a}_0$ with points in $\operatorname{Int} \tau$, where τ is the face opposite $\bar{a}_0$.

closure