MATH 524: Lecture 8 (09/11/2025)

Today: * cycles, boundaries, homology group
* Examples

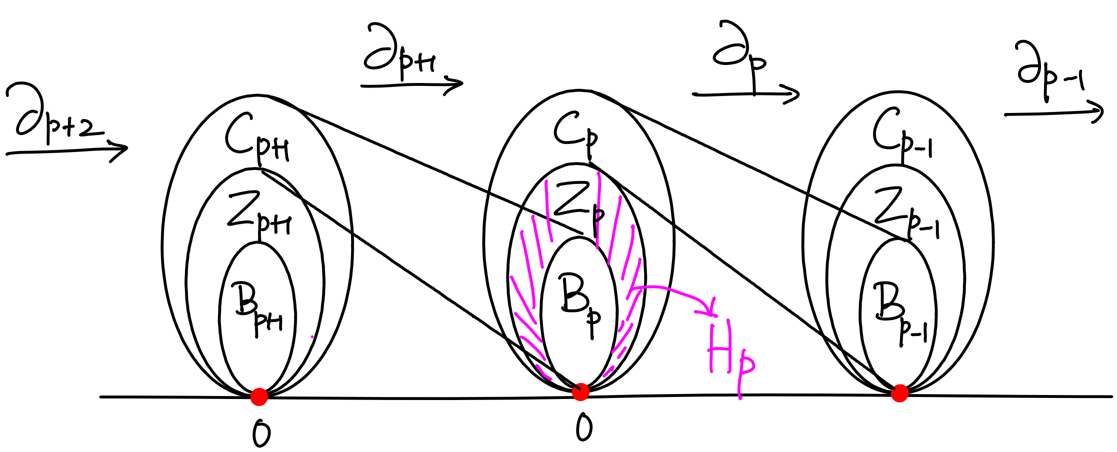
Recall: $\partial_p \circ \partial_{p+1} = 0$

Def The kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$ is the group of **p-cycles**, denoted $Z_p(K)$. The image of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is the group of **p-boundaries**, denoted $B_p(K)$.

Since $\partial_p \partial_{p+1} = 0$ by the above lemma, each boundary of a $(p+1)$-chain is automatically a p-cycle. Hence, $B_p(K) \subset Z_p(K) \subset C_p(K)$.

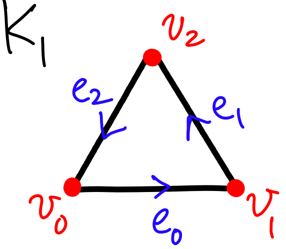
We now define $H_p(K) = Z_p(K)/B_p(K)$, and call it the **p-th homology group** of K.

The various groups and ∂ homomorphisms have the following structure:



Examples

1. K_1



$C_1(K_1)$ is free abelian, generated by, e.g., $\{e_0, e_1, e_2\}$.

We use vector notation for chains

Any 1-chain in K can be given as $\bar{c} = n_0 e_0 + n_1 e_1 + n_2 e_2$ for $n_i \in \mathbb{Z}$.

ideally, $\bar{c} = n_0\bar{e}_0 + n_1\bar{e}_1 + n_2\bar{e}_2$

When is $\bar{c}$ a cycle?

$$\partial_1 \bar{c} = n_0(v_1 - v_0) + n_1(v_2 - v_1) + n_2(v_0 - v_2)$$
$$= (n_2 - n_0)v_0 + (n_0 - n_1)v_1 + (n_1 - n_2)v_2.$$

So, $\partial_1 \bar{c} = 0$ iff $n_0 = n_1 = n_2$. Thus $\bar{c}$ is a 1-cycle iff $n_0 = n_1 = n_2$.

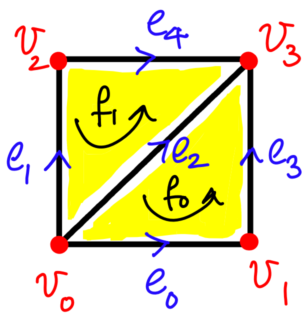
We can see that $Z_1(K_1)$ is infinite cycle, generated by $\bar{e}_0 + \bar{e}_1 + \bar{e}_2$. In other words, we can pick an integer, and that number tells us how many times we go around the cycle. If $n_0 = n_1 = n_2 = -3$, for instance, we go around in the opposite direction (ie., clockwise) 3 times.

There are no 2-simplices, so $B_1(K_1)$ is trivial. In other words, there are no 1-boundaries. Hence $H_1(K_1) = Z_1(K_1) \simeq \mathbb{Z}$.

Also, $\beta_1(K) = rk(H_1(K_1)) = 1$.

Example 2

K_2



This is the same example 2 in [M], but with different choices of orientations.

$|K_2|$ is a square.

A general 1-chain in K_2 is $\bar{c} = \sum_{i=0}^{4} n_i \bar{e}_i$. Then $\partial_1 \bar{c} = \sum_i n_i \partial_1(\bar{e}_i)$.

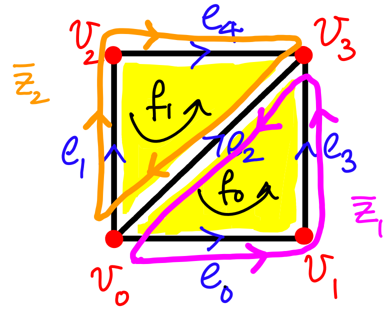
When is $\bar{c}$ as 1-cycle?

We need $n_0 = n_3$ (at v_1), and $n_1 = n_4$ (at v_2).

Similarly, $n_2 = -(n_0+n_1)$ at v_0, and $n_2 = -(n_3+n_4)$ at v_3.

Hence we can choose n_0, n_1 arbitrarily, and the other n_i's are fixed. So $Z_1(K_2)$ is free abelian with rank 2. One basis is $\{\underbrace{\bar{e}_0+\bar{e}_3-\bar{e}_2}_{n_0=1,\, n_1=0},\ \underbrace{\bar{e}_1+\bar{e}_4-\bar{e}_2}_{n_0=0,\, n_1=1}\}$.

Let's call these cycles as $\{\bar{z}_1, \bar{z}_2\}$.

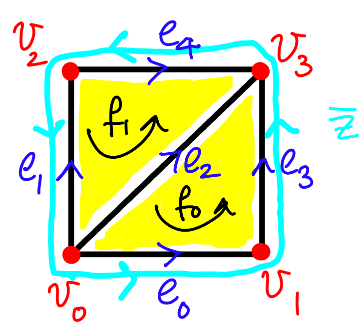


Indeed, any other cycle in K_1 can be written as a sum of $\bar{z}_1$ and $\bar{z}_2$. For instance, let $\bar{z}$ be the 1-cycle $e_0+e_3-e_4-e_1$

Indeed, $\bar{z}$ can be written as $\bar{z}_1 - \bar{z}_2$

$$= (\bar{e}_0+\bar{e}_3-\bar{e}_2) - (\bar{e}_1+\bar{e}_4-\bar{e}_2).$$

note that the e_2 portions from $\bar{z}_1$ and $\bar{z}_2$ cancel.



Now let's characterize $B_1(K_2)$. Notice that both 1-cycles $\bar{z}_1$ and $\bar{z}_2$ are also 1-boundaries. Indeed, we have

$\partial_2 \bar{f}_0 = \bar{e}_0 + \bar{e}_3 - \bar{e}_2 = \bar{z}_1$ and $\partial_2 \bar{f}_1 = \bar{e}_2 - \bar{e}_4 - \bar{e}_1 = -\bar{z}_2$.

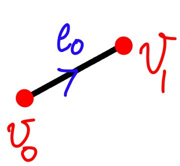
So $B_1(K_2) = Z_1(K_2)$. Hence $H_1(K_2) = Z_1(K_2)/B_1(K_2) = 0$.

(i.e., the first homology group is trivial; there are no 1-cycles that are not 1-boundaries).

Likewise, $H_2(K_2) = 0$. The general 2-chain is $\bar{d} = m_0\bar{f}_0 + m_1\bar{f}_1$. And $\partial_2 \bar{d} = 0$ iff $m_0 = m_1 = 0$. There are no 2-cycles. And $H_p(K_2) = 0$ for $p \geq 3$ trivially.

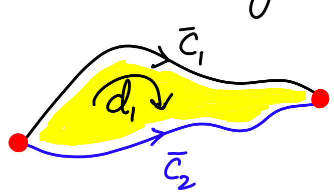
We now present some definitions we will use subsequently.

Def A chain $\bar{c}$ is **carried by a subcomplex** L if $\bar{c}(\sigma) = 0 \;\forall\, \sigma \notin L$. Two p-chains $\bar{c}_1, \bar{c}_2$ are **homologous** if $\bar{c}_1 - \bar{c}_2 = \partial_{p+1}\bar{d}$ for some $(p+1)$-chain $\bar{d}$. In particular, if $\bar{c} = \partial_{p+1}\bar{d}$, then $\bar{c}$ is homologous to zero, or we say that $\bar{c}$ **bounds**, i.e., $\bar{c}$ is a **boundary**. we write $\bar{c}_1 \sim \bar{c}_2$.

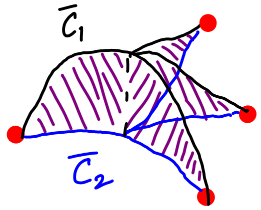


Here, the 2 0-chains v_0 and v_1 are homologous, since $v_1 - v_0 = \partial_1 e_0$.

Consider two 1-chains $\bar{c}_1, \bar{c}_2$ representing two 1D curves starting and ending at the same pair of vertices as shown.

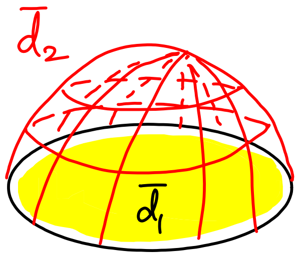


Then $\bar{c}_1$ and $\bar{c}_2$ are homologous here, as $\bar{c}_1 - \bar{c}_2 = \partial_2 \bar{d}_1$, where $\bar{d}_1$ is the 2-chain representing the 2D patch in between $\bar{c}_1$ & $\bar{c}_2$.



Notice that $\bar c_1$ and $\bar c_2$ need not be just simple open curves. Here, $\bar c_1$ and $\bar c_2$ both represent Y-shaped 1D curves. Again, $\bar c_1 - \bar c_2$ is the boundary of the 2D patch in between the two Y-shaped curves.

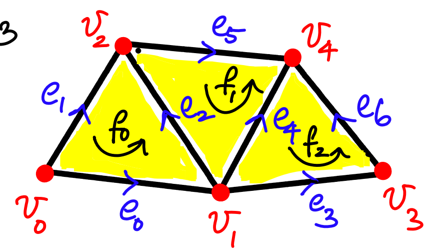
Consider two 2-chains $\bar d_1$ and $\bar d_2$, one representing a disc, and another representing the upper hemispherical surface that has the same boundary as the disc.



$\bar d_1$ and $\bar d_2$ are homologous, as $\bar d_1 - \bar d_2$ represents the boundary of the 3D solid hemisphere bounded by the two surfaces.

Example 3

K_3



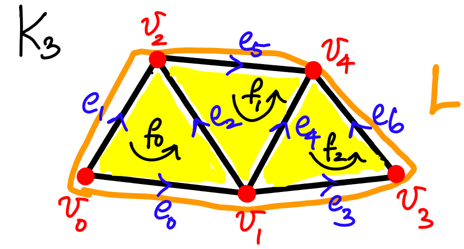
$$\bar c = \sum_{i=0}^{6} n_i e_i$$

Let $\bar c$ be a general 1-chain on K_3, and let its value on e_2 be c_2, i.e., $\bar c(e_2) = c_2$. Then the 1-chain $\bar c' = \bar c + \partial_2(c_2 \bar f_1)$ has $\bar c'(e_2) = 0$. Thus, we have "pushed $\bar c$ off e_2". Now, let $\bar c'(e_4) = c_4$. Then $= \bar c' + \partial_2(c_4 \bar f_2)$ has $\bar c''(e_4) = 0$. Notice that $\bar c''(\bar e_2) = 0$ still, as $\bar e_2 \notin \partial_2 \bar f_2$. So we have pushed $\bar c'$ off of e_4.

By combining the two steps, we have pushed $\bar c$ off of e_2 and e_4.

So, a 1-chain $\bar c$ on K_3 is homologous to $\bar c''$ carried by L_3, which is the subcomplex of K_3 made of $\{e_0, e_1, e_3, e_5, e_6\}$. Hence, $\bar c$ is a 1-cycle iff $\bar c''$ is.

But $\bar{c}''$ is a 1-cycle iff it is a multiple of $e_0+e_3+e_6-e_5-e_1$.



Hence $Z_1(K_3)$ has rank 1 (a basis is $\{e_0+e_3+e_6-e_5-e_1\}$).

But notice that this 1-cycle is also a 1-boundary. Precisely, it is $\partial_2(f_0+f_1+f_2)$. So $B_1(K_3) = Z_1(K_3)$.

Hence $H_1(K_3) = Z_1(K_3)/B_1(K_3) = 0$.

We also get that $H_p(K_3)=0$ for $p \geq 2$.

We used the two triangles to push off the general chain to the subcomplex which consists of the boundary of K_3. It's simpler to come up with the criterion for when a chain carried by this subcomplex is a cycle.

Notice that $H_p(K_2) = H_p(K_3)$ for $p \geq 1$, and that $|K_2| \approx |K_3|$.

This follows from the fundamental result that the homology groups depend only on the underlying space, and not on the particular simplicial complex chosen.

We could study homology in the "continuous" setting, i.e., without considering simplicial complexes. This homology, termed singular homology, can be shown to be equivalent to simplicial homology.

We can apply the techniques illustrated so far to compute the homology groups of more complicated simplicial complexes...

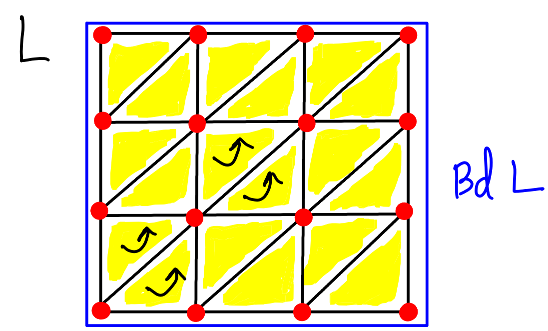
Homology Groups of Surfaces

If K if finite, $C_p(K)$ has finite rank, so does $Z_p(K)$ and $B_p(K)$. Also, $H_p(K)$ is finitely generated, and we can apply the fundamental theorem of finitely generated abelian groups to find the structure of $H_p(K)$. In particular, we want to compute the betti number and torsion coefficients of finite simplicial complexes representing surfaces, e.g., torus, Klein bottle, Möbius strip etc.

We first formalize the idea of "pushing chains off to the boundary of the simplicial complex".

Lemma 6.1 [M]

Let L be the simplicial complex such that $|L|$ is a rectangle.
Let $Bd\,L$ be the subcomplex of L representing the edges making up the boundary of the rectangle.



Orient each triangle counterclockwise (CCW), and the edges arbitrarily. Then the following results hold.

(1) Every 1-cycle of L is homologous to a 1-cycle carried by $Bd\,L$.

(2) If $\bar{d}$ is a 2-chain of L, and if $\partial\bar{d}$ is carried by $Bd\,L$, then $\bar{d}$ is a multiple of the chain $\sum \sigma_i$, where σ_i are all the triangles.