MATH 524 – Lecture 29 (12/05/2023)

Today: * 0-dimensional cohomology
* relative cohomology groups

Zero-dimensional Cohomology

Theorem 42.1 [M] $H^0(K;G)$ is the group of all 0-cochains ϕ^0 such that $\langle \phi^0, v\rangle = \langle \phi^0, w\rangle$ whenever v, w belong to the same component of $|K|$. In particular, if $|K|$ is connected, then $H^0(K) \simeq \mathbb{Z}$, and is generated by the cochain whose value is 1 on each vertex of K.

Proof $H^0(K;G)$ equals the group of 0-cocycles trivially, as there are no (−1)-dimensional simplices. If v, w are vertices that belong to the same component of $|K|$, there exists a 1-chain $\bar{c}$ of K such that $\partial \bar{c} = v - w$. Then, for any 0-cocycle ϕ^0, we have

$$0 = \langle \delta\phi^0, \bar{c}\rangle = \langle \phi^0, \partial\bar{c}\rangle = \langle \phi^0, v\rangle - \langle \phi^0, w\rangle.$$

Conversely, let ϕ^0 be a 0-cochain such that $\langle \phi^0, v\rangle - \langle \phi^0, w\rangle = 0$ whenever v, w lie in the same component of $|K|$. Then for each oriented 1-simplex σ of K,

$$\langle \delta\phi^0, \sigma\rangle = \langle \phi^0, \partial\sigma\rangle = 0.$$

So we conclude that $\delta\phi^0 = 0$. $\square$

In general, $H^0(K) \simeq$ direct product of infinite cyclic groups, one for each component of $|K|$. On the other hand, $H_0(K) \simeq$ direct sum of this collection of groups.

Relative Cohomology Groups

Def Let $K_0 \subseteq K$ be a subcomplex. The group of relative cochains in dimension p is defined as

$$C^p(K, K_0; G) = \operatorname{Hom}(C_p(K, K_0), G).$$

The relative coboundary, also denoted δ, is defined as the dual of the relative boundary operator:

$$\delta^p : C^p(K, K_0; G) \longrightarrow C^{p+1}(K, K_0; G).$$

We let $Z^p(K, K_0; G) = \ker \delta^p$,

$B^p(K, K_0; G) = \operatorname{im} \delta^{p-1}$, and

$$H^p(K, K_0; G) = Z^p(K, K_0; G) / B^p(K, K_0; G).$$

These are the groups of relative cocycles, relative coboundaries, and the relative cohomology group in dimension p of K modulo K_0.

While the definition is presented in a straightforward manner, the correspondence to the structure of relative homology groups is specified in a dual manner.

For chains, we have the exact sequence

$$0 \longrightarrow C_p(K_0) \xrightarrow{i} C_p(K) \xrightarrow{j} C_p(K, K_0) \longrightarrow 0$$

which splits, because $C_p(K, K_0)$ is free.

For cochains, we get a similar sequence

$$0 \longleftarrow C^p(K_0; G) \xleftarrow{\tilde{i}} C^p(K; G) \xleftarrow{\tilde{j}} C^p(K, K_0; G) \longleftarrow 0$$

which is exact, and also splits.

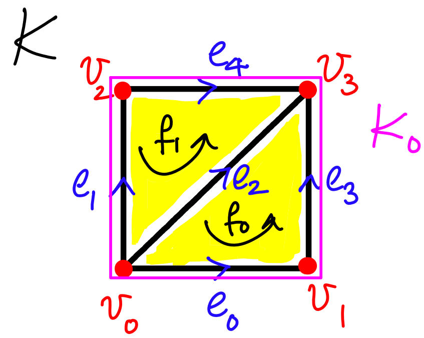
$C^p(K, K_0; G)$ is a subgroup of $C^p(K; G)$ — these are the cochains that vanish on simplices carried by K_0.
Equivalently, $C^p(K, K_0; G)$ is the group of cochains "carried by" $K - K_0$. Hence $\tilde{j}$ is an inclusion map.

$\tilde{i}$ is a restriction (or projection) — it is the restriction of cochain ϕ^p of $C^p(K; G)$ to simplices in K_0.

So, dual of inclusion i is projection $\tilde{i}$, and

dual of projection j is inclusion $\tilde{j}$.

Examples of Relative Cohomology



1. Let K_0 consist of $\{e_0, e_1, e_3, e_4\}$ and all vertices. Let's evaluate the relative cochains.

Notice that $H_2(K, K_0) \simeq \mathbb{Z}$, $\{f_0 + f_1\}$ being a generator.

f_0^*, f_1^* are relative 2-cochains; and each of them is a relative 2-cocycle (trivially, as there are no 3-simplices).

Is either of them a coboundary? No!

$\delta e_1^* = -f_1^*$, $\delta e_4^* = -f_1^*$
but $e_1, e_4 \in K_0$.

e_2^* is the only relative 1-cochain. And

$\delta e_2^* = f_1^* - f_0^*$. So f_1^* and f_0^* are cohomologous.

$\Rightarrow H^2(K, K_0) \simeq \mathbb{Z}$, and is generated by $\{f_0^*\}$ or $\{f_1^*\}$.

$H^1(K, K_0) = 0$, as there are no relative 1-cocycles.

$\delta e_2^* \neq 0$, $e_i^*, i = 0,1,3,4$ are trivial as those $e_i \in K_0$.

$H^0(K, K_0) = 0$, as all 0-cochains are carried by K_0.