MATH 524 – Lecture 14 (10/05/2023)

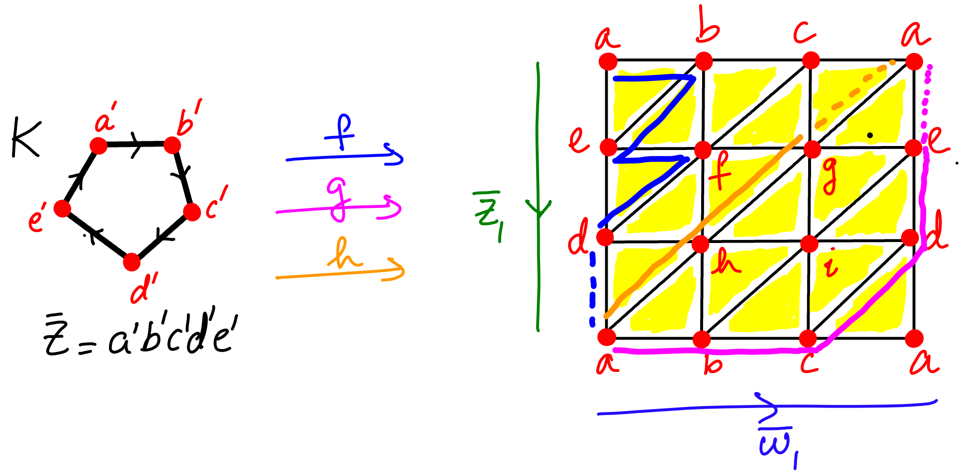
Today: * simplicial maps and induced homo'ms
* chain homotopy

Recall the example :

$f: a' \to a$, $b' \to b$, $c' \to e$, $d' \to f$, $e' \to d$

$h: a' \to a$, $b' \to h$, $c' \to h$, $d' \to g$, $e' \to g$

$g: a' \to a$, $b' \to b$, $c' \to c$, $d' \to d$, $e' \to e$



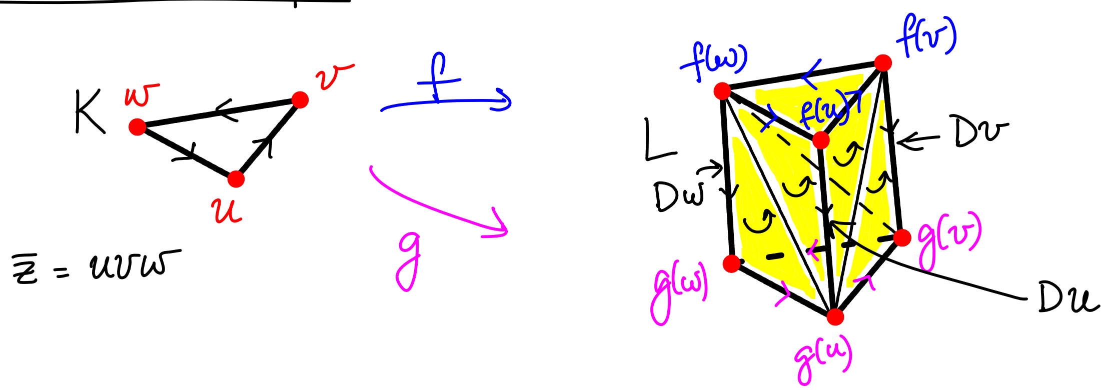
We can check that $f_\#(\bar{z}) \sim \bar{z}_1$, $g_\#(\bar{z}) \sim \bar{w}_1 - \bar{z}_1$, and $h_\#(\bar{z}) \sim \bar{w}_1 - \bar{z}_1$. Hence g_* and h_* are equal as homomorphisms of the first homology group.

It can be checked that g_* and h_* are equal as homomorphisms of the 0-dimensional homology groups as well.

When can this observation hold in general?

Given simplicial maps $f, g : K \to L$, we want to find conditions under which $f_\#(\bar{z}) \sim g_\#(\bar{z})$ $\forall \bar{z} \in Z_p(K)$. Thus we want to find a $(p+1)$-chain $D\bar{z}$ of L such that $f_\#(\bar{z}) - g_\#(\bar{z}) = \partial D\bar{z}$.

Here's an example where we can find Dz straightforwardly.



L consists of 6 triangles such that $|L|$ is the cylinder. K is made of 3 edges forming a cycle, which we term z. f and g are two simplicial maps which map z to the top and bottom cycles, respectively, in L. The triangles in L can be oriented consistently, e.g., ccw when looking from outside.

Here, Dz can be chosen to be the 2-chain made of the 6 triangles in the middle.

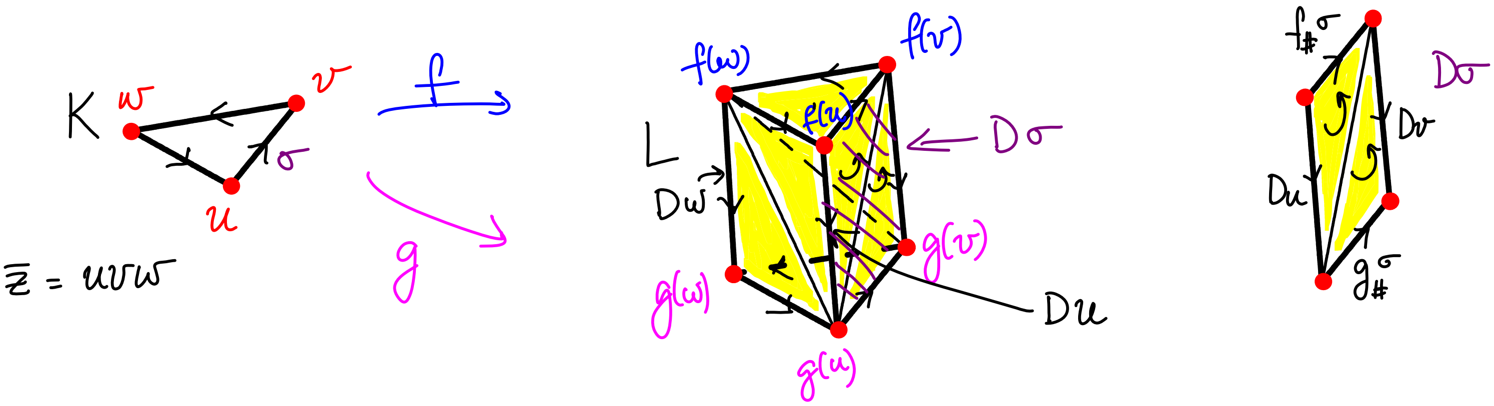
But for a different pair of maps f' and g', it might not be as straightforward to identify Dz in all cases.

When can we find Dz easily for all cycles z? Could we specify some sufficient conditions for the existence of Dz?

We develop some machinery toward answering these questions. Rather than specify the requirements for all $z \in Z_p(K)$, we specify them for $C_p(K)$, and in particular for elementary chains in K. Then we can apply the results directly to the case of $z \in Z_p(K)$ as well.

We continue with the example where we could identify $D\bar{z}$. But now we identify $D\sigma$ for elementary chains $\sigma \in K$, starting with vertices and proceeding to higher dimensional simplices. Our goal is to identify some sort of formula that $D\bar{c}$ should satisfy for a general p-chain $\bar{c} \in C_p(K)$.

For vertex $v \in K^{(0)}$, define Dv to be the edge in L connecting $f(v)$ and $g(v)$.



For edge uv, with $\sigma = uv$, $D\sigma$ is the sum of the two triangles between $(f(u), f(v))$ and $(g(u), g(v))$.

Notice that we get

$$\partial(D\sigma) = g_{\#}(\sigma) - Dv - f_{\#}(\sigma) + Du.$$

In other words, we have $\partial(D\sigma) = g_{\#}(\sigma) - f_{\#}(\sigma) - D(\underbrace{\partial\sigma}_{v-u})$.

This example in fact suggests the form that $D\sigma$ should satisfy in general. We want $D(\partial\sigma) + \partial(D\sigma) = g_{\#}(\sigma) - f_{\#}(\sigma)$.

We define the existence of such a $(p+1)$-chain for each p-simplex as the required sufficient condition in general for all p.

Def Let $f, g : K \to L$ be simplicial maps. Suppose that for all p, there is a homomorphism $D : C_p(K) \to C_{p+1}(L)$ which satisfies

$$\partial D + D \partial = g_\# - f_\# .$$

Then D is said to be a **chain homotopy** between $f_\#$ and $g_\#$.

Intuitively, the images of each p-simplex σ under f and g are "close" to each other if there is a chain homotopy. Notice that the requirement is specified for all dimensions.

We could be more precise in writing the equation by including subscripts of dimension $(p, p+1)$ and simplicial complexes (K and L). We express the maps in detail as follows.

$$\begin{tikzcd}
& C_{p+1}(L) \arrow[d, "(\partial_{p+1})_L"] \\
C_p(K) \arrow[ur, "D_p"] \arrow[r, shift left, "(f_\#)_p"] \arrow[r, shift right, "(g_\#)_p"'] \arrow[d, "(\partial_p)_K"'] & C_p(L) \\
C_{p-1}(K) \arrow[ur, "D_{p-1}"'] &
\end{tikzcd}$$

The detailed relation we want is the following:

$$(\partial_{p+1})_L D_p + D_{p-1} (\partial_p)_K = (g_\#)_p - (f_\#)_p .$$

But we usually will write $\partial D + D \partial = g_\# - f_\#$, for brevity.

The following theorem describes why we want to study chain homotopies.

Theorem 12.4 [M] If there is a chain homotopy between $f_{\#}$ and $g_{\#}$, then the induced homomorphisms f_* and g_*, for both reduced and absolute homology, are equal.

Proof If $\bar{z} \in Z_p(K)$, then

$$g_{\#}(\bar{z}) - f_{\#}(\bar{z}) = \partial D\bar{z} + D\partial\bar{z} = \partial D\bar{z} + 0.$$

So, $g_{\#}(\bar{z}) \sim f_{\#}(\bar{z})$, and hence $g_*(\{\bar{z}\}) = f_*(\{\bar{z}\})$.

We now give a sufficient condition for existence of a chain homotopy.

Def Two simplicial maps $f, g : K \to L$ are said to be **contiguous** if for every simplex $\sigma = (v_0 \cdots v_p)$ of K, the points $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$ span a simplex τ of L.

Note: 1. $0 \leq \dim(\tau) \leq 2p+1$.
2. $f(\sigma)$ and $g(\sigma)$ are both faces of a (possibly) larger simplex τ of L.
i.e, $f(\sigma)$ and $g(\sigma)$ are "close" to each other

Theorem 12.5 [M] If $f, g : K \to L$ are contiguous simplicial maps, then a chain homotopy exists between $f_{\#}$ and $g_{\#}$.

Proof (outline; see [M] for details)

For $\sigma = v_0, \dots, v_p$ of K, let $L(\sigma)$ be the subcomplex of L made of the simplex spanned by $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$, and all its faces. We should have the following results.

(1) $L(\sigma)$ is nonempty, $\tilde{H}_i(L(\sigma)) = 0 \ \forall i$.

(2) If τ is a face of σ, then $L(\tau) \subset L(\sigma)$.

(3) for every oriented simplex σ, $f_\#(\sigma)$ and $g_\#(\sigma)$ are both carried by $L(\sigma)$.

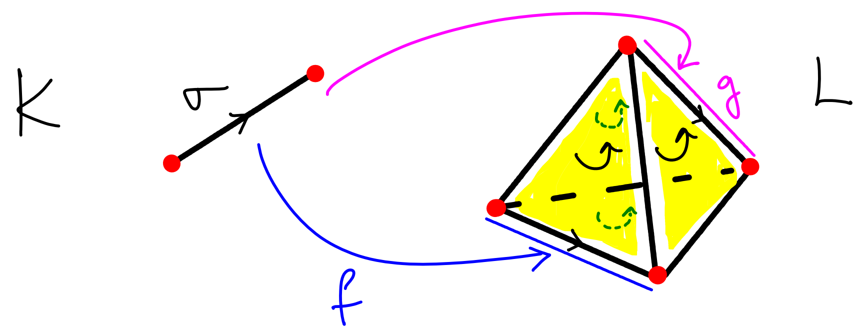
We now show that $D\sigma$ exists for each p-simplex σ using induction on p.

$p=0$ Notice that $\epsilon(g_\#(v) - f_\#(v)) = 1 - 1 = 0$. Hence $g_\#(v) - f_\#(v) \in \tilde{H}_0(L(v))$. But $\tilde{H}_0(L(v)) = 0$, so we can choose a 1-chain Dv of L carried by $L(v)$ such that

$$\partial(Dv) = g_\#(v) - f_\#(v).$$

(See [M] for the induction step going from $p-1$ to p). $\square$

Notice that the theorem guarantees the existence of some $D\sigma$ for each σ — the choice may not be unique. Indeed, consider the case where a 1-simplex σ gets mapped by f and g to two opposite edges of a tetrahedron. Then there are two choices for $D\sigma$ — the two triangles of the tetrahedron visible in front, or the other two tetrahedron lying behind.



Application to relative homology

Def Let $K_0 \subseteq K$ and $L_0 \subseteq L$ be subcomplexes. Let $f, g : (K, K_0) \to (L, L_0)$ be two simplicial maps. We say f and g are **contiguous as maps of pairs** if for every simplex $\sigma = v_0 \cdots v_p$ of K, the points $f(v_0), \ldots, f(v_p)$, $g(v_0), \ldots, g(v_p)$ span a simplex of L, and if σ is contained in K_0, then they span a simplex of L_0.

With maps that are contiguous as maps of pairs, we can extend the concept of chain homotopy to the case of relative homology, and how equal homomorphisms are induced on relative homology groups.

Theorem 12.6 [M] Let $f, g : (K, K_0) \to (L, L_0)$ be contiguous as maps of pairs. Then there exists a homomorphism $D : C_p(K, K_0) \to C_{p+1}(L, L_0)$ for all p such that

$$\partial D + D\partial = g_{\#} - f_{\#}.$$

Thus, f_* and g_* are equal as maps of the relative homology groups.

See [M] for proof details.

The main point is to notice that D maps $C_p(K_0)$ to $C_{p+1}(L_0)$.