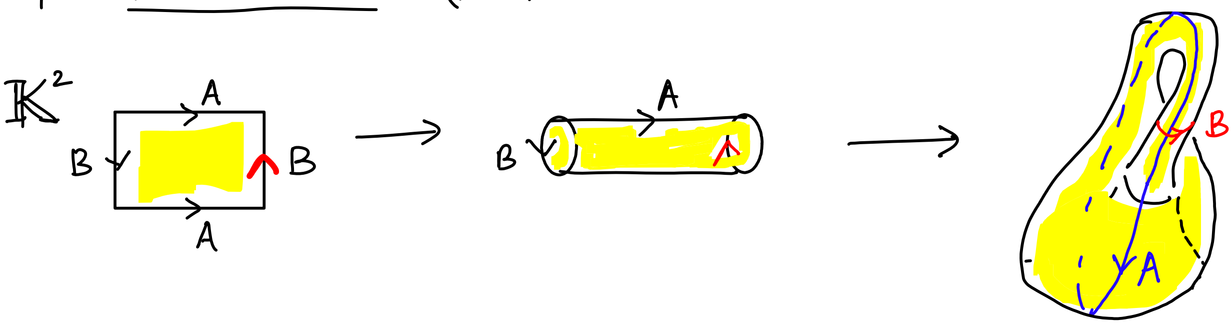
MATH 524 – Lecture 6 (09/07/2023)

Today: * ASC for Klein bottle
* Review of Abelian groups

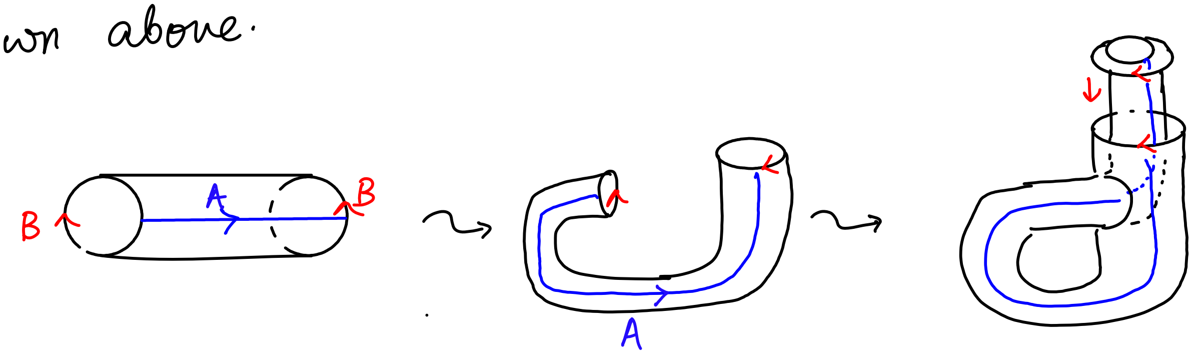
Examples for ASCs (continued...)

4. Klein bottle ($\mathbb{K}^2$) → \mathbb{K}^2 in LaTeX!



Here, we identify the opposite pairs of edges – one pair with a twist as in the Möbius strip (B here) and the other without (A here; similar to torus or cylinder). The Klein bottle does not have an embedding in $\mathbb{R}^3$, but has in $\mathbb{R}^4$. We must go to the higher dimension to avoid self-intersections.

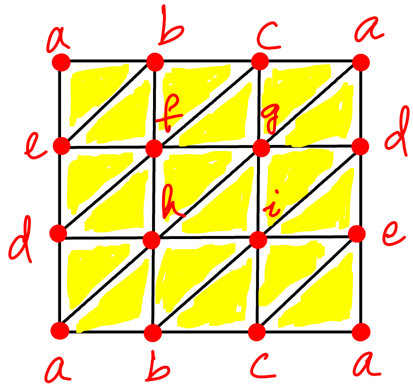
We do get an **immersion** in $\mathbb{R}^3$, which allows self intersection. Here is a schematic of how one arrives at the immersion shown above.



This instance illustrates the difficulty faced when working with geometric embeddings. We could instead work with the abstract space along with the quotient map!

We now construct an ASC for $\mathbb{K}^2$.

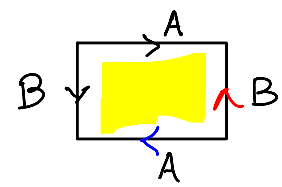
K_3



$|K_3| \approx \mathbb{K}^2 \rightsquigarrow$ one can check to make sure we are not gluing more than two edges anywhere.

Of course, $|K_3| \not\approx |K_2|$, and in general, $\mathbb{K}^2 \not\approx \mathbb{T}^2$.

Notice that we could start with the rectangular space (for L) and identify pairs of edges in several ways. For instance, when we glue both pairs of opposite edges with twists, we get the real projective plane ($\mathbb{RP}^2$).



$\mathbb{RP}^2$

The related question now is how to identify homeomorphic simplicial complexes K for any such quotient space. In particular, when do we get "nice" labelings (or gluings)?

See Lemma 3.2 in [M] for a condition given in terms of closed stars of vertices in K. This result is left as a **candidate for video tutorial**.

Review of Abelian Groups

We now review several properties and results from groups and homomorphisms between groups. The idea is to cast questions about similarity of topological spaces as corresponding questions on homomorphisms between groups defined on simplicial complexes that are homeomorphic to the spaces in question.

A good book – Fraleigh (First course in Abstract Algebra).

closure is assumed, i.e., $a+b \in G \; \forall a,b \in G$.

Group: set G with an operation $+$ "addition", such that

(1) there exists an **identity**, $0 \in G$, s.t.

$$a+0 = 0+a = a \quad \forall a \in G;$$

(2) $\forall a \in G$, there is an **inverse**, i.e., $-a \in G$ s.t.

$$a+(-a) = (-a)+a = 0; \text{ and}$$

(3) $a+(b+c) = (a+b)+c \quad \forall a,b,c \in G$; i.e., $+$ is **associative**.

(4) Further, if $a+b = b+a \; \forall a,b \in G$, then G is an **abelian group**.

In general, we will work with abelian groups in this class.

Notation: $\quad ng = \underbrace{g+g+\cdots+g}_{n \text{ times}}$ for $g \in G$.

<u>Homomorphisms</u> $f: G \to H$, G, H are groups is a homomorphism if $f(g_1 +_G g_2) = f(g_1) +_H f(g_2)$.

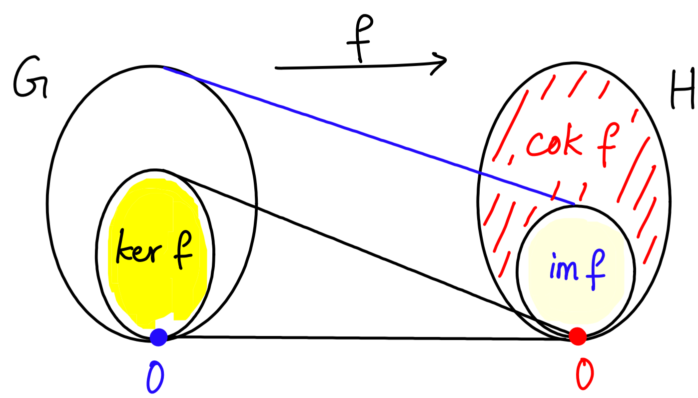
Intuitively, homomorphisms "preserve the structure" of groups.

We study subgroups specified by homomorphism f:

kernel of f: $f^{-1}(0)$, is a subgroup of G, denoted **ker f**.

image of f: $f(G)$, is a subgroup of H, denoted **im f**.

cokernel of f: quotient group of H given as $H/f(G)$, denoted **cok f**.



f is a monomorphism (injection) iff $\ker f = 0$.

f is an epimorphism (surjection) iff $\operatorname{cok} f = 0$, and in this case, f defines an isomorphism $G/_{\ker f} \simeq H$.

An abelian group G is **free** if it has a basis $\{g_\alpha\}$ of elements in G such that $\forall g \in G$, $g = \sum n_\alpha g_\alpha$ is a unique finite sum, for $n_\alpha \in \mathbb{Z}$.

This uniqueness (for the free abelian group) implies that each basis element g_α generates an infinite cyclic group $H = \{ n g_\alpha \mid n \in \mathbb{Z} \}$.

notation used in [M]

Note: $\mathbb{Z}/n$ (or $\mathbb{Z}_n$) has elements $\{0, 1, \dots, n-1\}$ with addition mod n.

More generally, if each $g \in G$ can be written as $\sum_i n_\alpha g_\alpha$, but not necessarily uniquely, then $\{g_\alpha\}$ **generates** G. If $\{g_\alpha\}$ is finite, we say that G is **finitely generated**.

We will work mostly with finitely generated abelian groups

Def If G is free, and has a basis of n elements, then every basis of G has n elements. The number of elements in a basis of G is its **rank**, denoted $rk(G)$.
or $rank(G)$

The **order** of G is the # elements in G, denoted $|G|$.

A crucial property: If $\{g_\alpha\}$ is a basis of G, any function f from $\{g_\alpha\}$ to abelian group H extends uniquely to a homomorphism from G to H.

Somewhat similar in flavor to a vertex map extending to the corresponding simplicial map

Let G be an abelian group. $g \in G$ has **finite order** if $ng = 0$ for some $n \in \mathbb{Z}_{>0}$. The set of all elements of finite order in G is a subgroup T of G, called the **torsion** subgroup. If T vanishes, we say G is **torsion-free**.

Notice that $0 \in G$ is a trivial case in this context, as $n0 = 0$ for any $n \in \mathbb{Z}$.

We now consider how to "combine" (abelian) groups to form bigger (abelian) groups. The intuition is similar to combining multiple individual dimensions to form a higher dimensional space.

[M] defines internal direct sums, direct products, and external direct sums. We discuss them all for the sake of completeness.

Internal direct sums

Let G be an abelian group, and let $\{G_\alpha\}_{\alpha \in J}$ be a collection of subgroups of G indexed bijectively by the index set J. If each $g \in G$ can be written uniquely as finite sum $g = \sum_\alpha^{\prime} g_\alpha$, where $g_\alpha \in G_\alpha$ for each $\alpha \in J$, then G is the **internal direct sum** of the groups G_α,

and is written $\quad G = \bigoplus_{\alpha \in J} G_\alpha .$

If $J = \{1, 2, \ldots, n\}$ for finite n, say, we also write

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \quad \text{or} \quad G = \bigoplus_{\alpha=1}^{n} G_\alpha$$

There is a similar distinction here to a basis vs generating set of a group.

If each $g \in G$ can be written as a finite sum $g = \sum_\alpha g_\alpha$, but not necessarily uniquely, then G is simply the **sum** of groups $\{G_\alpha\}$.
We write $G = \sum_\alpha G_\alpha$, or $G = G_1 + \cdots + G_n$ (if finite).

internal sum, to be precise

Here, we say $\{G_\alpha\}$ **generates** G.

Notice that if G is free abelian with basis $\{g_\alpha\}$, then G is the direct sum of subgroups $\{G_\alpha\}$, where G_α is the infinite cyclic group generated by g_α.

The converse is also true here, i.e., if G is the direct sum of $\{G_\alpha\}$ where G_α is the infinite cyclic group generated by g_α, then G is free abelian with basis $\{g_\alpha\}$.

Direct Products and External direct sums

Def Let $\{G_\alpha\}_{\alpha \in J}$ be an indexed family of abelian groups. The **direct product** $\prod_{\alpha \in J} G_\alpha$ is the group whose set is the cartesian product of sets G_α, and the operation is component-wise addition.

J can be infinite here; you could assume it is finite, though, to get the intuition. There is technical work required to extend the results and definitions to the infinite case — but it's not critical for us.

The **external direct sum** G is the subgroup of the direct product $\prod_{\alpha \in J} G_\alpha$ consisting of all tuples

$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{\alpha_i} \\ \vdots \end{pmatrix}$ such that $g_{\alpha_i} = 0_{\alpha_i}$ for all but finitely many values of α_i.

Examples

1. $G_1 = \mathbb{Z} \times \mathbb{Z}$ $\quad G_1$ has rank 2; basis is $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$. $rk(G_1) = 2$.

   operation is componentwise addition.

2. $G_2 = \mathbb{Z}/2 \times \mathbb{Z}/3$ $\quad$ (or $\mathbb{Z}_2 \times \mathbb{Z}_3$)

   componentwise addition mod 2 and mod 3.

   $1 +_2 1 = 0$ $\qquad$ $1 +_3 1 = 2$, $\; 1 +_3 2 = 0$

   G_2 is a cyclic group, $|G_2| = 6$, order

   $G_2 = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$.

   $rk(G_2) = 1$, as $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a basis.

   $1 \times \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, $\quad 2 \times \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$, $\quad 3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

   $4\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, $\quad 5\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, $\quad 6\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

Theorem The group $\prod_{i=1} \mathbb{Z}/t_i$ for $t_i \in \mathbb{Z}_{>0}$ is cyclic and is isomorphic to $\mathbb{Z}_{t_1 t_2 \cdots t_n}$ iff $\gcd(t_i, t_j) = 1 \ \forall i, j$.

t_i and t_j are relatively prime

Back to example 2: $\mathbb{Z}/_2 \times \mathbb{Z}/_3 \simeq \mathbb{Z}_6$.

If $n = (p_1)^{n_1}(p_2)^{n_2} \cdots (p_r)^{n_r}$ for primes $p_1, \dots, p_r$, then

$$\mathbb{Z}_n \simeq \mathbb{Z}/_{(p_1)^{n_1}} \times \mathbb{Z}/_{(p_2)^{n_2}} \times \cdots \times \mathbb{Z}/_{(p_r)^{n_r}}.$$