MATH 524 – Lecture 4 (08/31/2023)

Today:
* properties of $|K|$
* star, closed star, link
* simplicial maps
* ~~Abstract simplicial complexes~~ didn't get to it ☹!

Properties of $|K|$

Munkres – Elements of Algebraic Topology

Lemma 2.2 [M] If $L \subseteq K$ is a subcomplex, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.

$|\sigma|$ to be precise, but notice σ and $|\sigma|$ are identical!

Lemma 2.3 [M] A map $f: |K| \to X$ is continuous iff $f|_\sigma$ is continuous for each $\sigma \in K$.

f restricted to σ

Recall the barycentric coordinates of $\bar{x} \in \sigma$ ($t_{\bar{a}_i}(\bar{x})$ for vertices $\bar{a}_i$).
We can naturally extend the barycentric coordinates to $\bar{x} \notin \sigma$.

Def If $\bar{x} \in |K|$, then $\bar{x}$ is interior to precisely one simplex in K, whose vertices are, say, $\bar{a}_0, \dots, \bar{a}_n$. Then

$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i, \quad \text{where} \quad t_i > 0 \ \forall i, \ \sum_{i=0}^{n} t_i = 1.$$

If $\bar{v}$ is an arbitrary vertex of K, then the barycentric coordinate of $\bar{x}$ w.r.t. $\bar{v}$, $t_{\bar{v}}(\bar{x})$, is defined as $t_{\bar{v}}(\bar{x}) = 0$ if $\bar{v} \notin \{\bar{a}_0, \dots, \bar{a}_n\}$, and $t_{\bar{v}}(\bar{x}) = t_i$ if $\bar{v} = \bar{a}_i$.

Notice that $t_{\bar{v}}(\bar{x})$ is continuous on $|K|$, as $t_{\bar{a}_i}(\bar{x})$ are continuous, as we noted in the last lecture, and then by Lemma 2.3.

Lemma 2.4 [M] $|K|$ is Hausdorff.

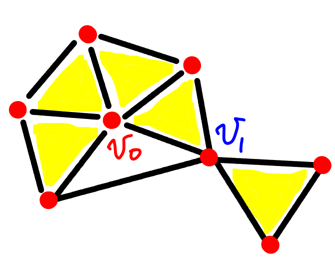
A space X is Hausdorff if every pair of distinct points $\bar{x}, \bar{y} \in X$ can be surrounded by open sets $\mathcal{U}, \mathcal{V} \subseteq X$ s.t. $\bar{x} \in \mathcal{U}$, $\bar{y} \in \mathcal{V}$, $\mathcal{U} \cap \mathcal{V} = \phi$.

Proof For $\bar{x}_i \neq \bar{x}_j$ in $|K|$, by definition, there exists at least one $\bar{v}$ (vertex) s.t. $t_{\bar{v}}(\bar{x}_i) \neq t_{\bar{v}}(\bar{x}_j)$. Choose r in between $t_{\bar{v}}(\bar{x}_i)$ and $t_{\bar{v}}(\bar{x}_j)$ and define $\mathcal{U} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) < r\}$ and $\mathcal{V} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) > r\}$ as the required open sets.

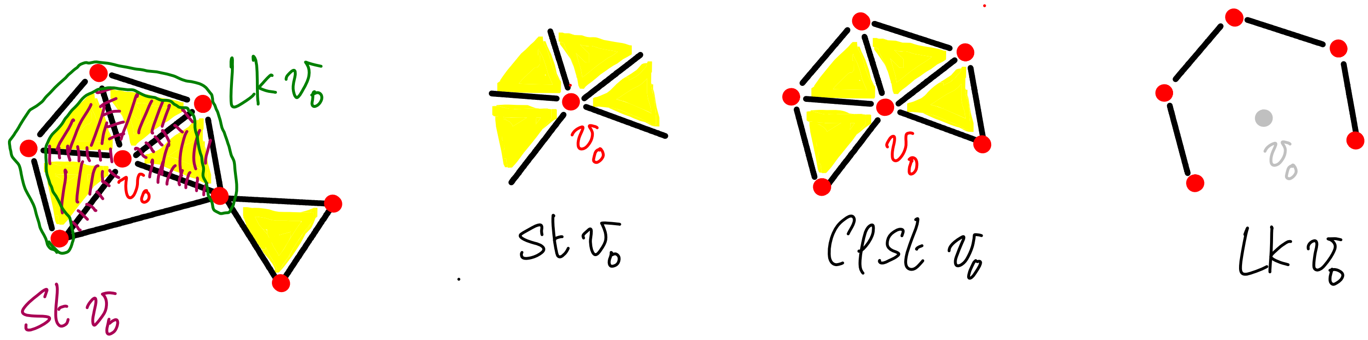
We now study some important subspaces of $|K|$.

Three Subspaces of $|K|$

Def If $\bar{v}$ is a vertex of K, then the **star** of $\bar{v}$ in K, denoted $St\,\bar{v}$ (or $St(\bar{v}, K)$) is the union of the interiors of all simplices in K that contain $\bar{v}$ as a vertex. The closure of $St\,\bar{v}$, denoted $\overline{St\,\bar{v}}$ or $Cl\,St\,\bar{v}$, is the **closed star** of $\bar{v}$. It is the union of all simplices of K which have $\bar{v}$ as a vertex. $Cl\,St\,\bar{v}$ is a polytope of a subcomplex of K. $Cl\,St\,\bar{v} - St\,\bar{v}$ is called the **link** of $\bar{v}$, denoted $Lk\,\bar{v}$.

K_6 

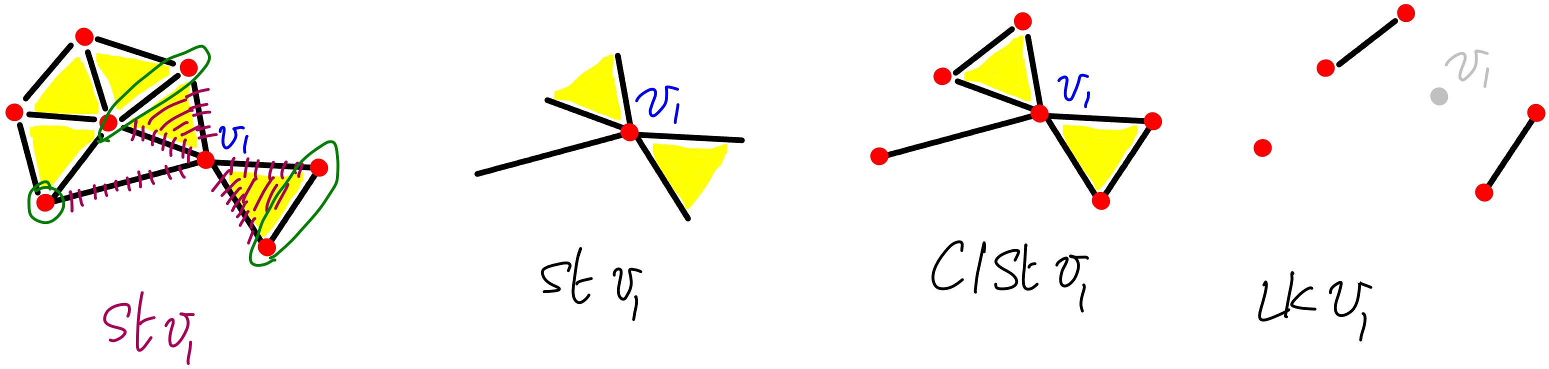
We illustrate these subcomplexes on K_6 for vertices v_0 and v_1. Note that the unshaded triangle below v_0 is not part of K_6.



add to get ClSt v_0

Note that $Lk\, v_0 = Cl\,St\, v_0 - St\, v_0$.

Also note that $v_0 \in St\, v_0$ (indeed, $Int\, v_0 = v_0$, and v_0 is a simplex that contains v_0 as a vertex, trivially).



Properties of star, closed star, link

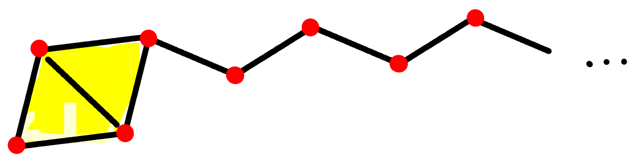
* St $\bar{v}$ is open in $|K|$. → We could use $t_{\bar{v}}(\cdot)$ to prove.
* The complement of St $\bar{v}$ is the union of all simplices that do not contain $\bar{v}$ as a vertex, and hence it is the polytope of a subcomplex of K.
* Lk $\bar{v}$ is the polytope of a subcomplex of K.
* $Lk\,\bar{v} = Cl\,St\,\bar{v} \cap (\text{Complement of } St\,\bar{v})$.
* St $\bar{v}$ and ClSt $\bar{v}$ are both path-connected.

  X is path-connected if $\forall \bar{u}, \bar{v} \in X, \bar{u} \neq \bar{v}$, $\exists$ a path connecting $\bar{u}$ and $\bar{v}$ in X.
* Lk $\bar{v}$ need not be connected.

Def A simplicial complex K is **locally finite** if each vertex of K belongs to only finitely many simplices of K. Equivalently, K is locally finite iff each closed star is the polytope of a finite subcomplex of K.

Note: A locally finite simplicial complex could be infinite, e.g., K_7.

K_7



(the edges continue forever)

Simplicial Maps

We study maps between simplicial complexes as a first step toward developing the tools to compare spaces modeled by the simplicial complexes.

Def Let K, L be simplicial complexes. A function $f: |K| \to |L|$ is a (linear) **simplicial map** if it takes simplices of K linearly onto simplices of L. In other words, if $\sigma \in K$, then $f(\sigma) \in L$.

linearly: If $\sigma = \text{conv}\{\bar{v}_0, \dots, \bar{v}_n\}$ and $\bar{x} = \sum_{i=0}^{n} t_i \bar{v}_i$, $t_i \geq 0$, $\sum_{i=0}^{n} t_i = 1$, then $f(\bar{x}) = \sum_{i=0}^{n} t_i f(\bar{v}_i)$.

Note that $\{f(\bar{v}_0), \dots, f(\bar{v}_n)\}$ span a simplex τ of L, which could be of a lower dimension than σ.

Munkres takes a slightly different approach in defining simplicial maps.

[M]: Starts with $f: K^{(0)} \to L^{(0)}$, then insist that when $\{\bar{v}_0, \dots, \bar{v}_n\}$ span $\sigma \in K$, $\{f(\bar{v}_0), \dots, f(\bar{v}_n)\}$ span $\tau \in L$.

f is a continuous map of σ onto τ, and hence as a map of σ onto $|L|$. Then by Lemma 2.3, it is a continuous map from $|K|$ to $|L|$.

If $g : |K| \to |L|$ and $h : |L| \to |M|$ are simplicial maps, then $f = h \circ g$ is a simplicial map from $|K|$ to $|M|$.

If we further insist that $f : K^{(0)} \to L^{(0)}$ is a **bijective** correspondence such that vertices $\bar{v}_0, \dots, \bar{v}_n$ of K span a simplex of K iff $f(\bar{v}_0), \dots, f(\bar{v}_n)$ span a simplex of L, then the induced simplicial map $g : |K| \to |L|$ is a homeomorphism. We call this map an **isomorphism** of K with L (or a simplicial homeomorphism).