MATH401: Lecture 1 (08/19/2025)

This is Introduction to Analysis I

I'm Bala Krishnamoorthy (Call me Bala).

Today: * syllabus, logistics → see the course web page for details
* proof techniques
  - contrapositive proof
  - proof by contradiction
  - proof by induction

Book: Lindstrøm: Spaces–An Intro to Real Analysis (LSIRA)

LSIRA 1.1

Logical statements and notation.

If A then B (or $A \Rightarrow B$) "implies"

$A \Rightarrow B$ typically does not mean $B \Rightarrow A$.

e.g., A: p a natural number, is divisible by 6

B: p is divisible by 3.

$A \Rightarrow B$ holds, but $B \not\Rightarrow A$ (B does not imply A),

e.g., $p=9$.

But if $A \Rightarrow B$ and $B \Rightarrow A$ hold, we say A $\underbrace{\text{if and only if}}_{\text{iff}}$ B, or

$A \Leftrightarrow B$ (or A is equivalent to B).

To prove $A \Leftrightarrow B$, we often prove $A \Rightarrow B$ and $B \Rightarrow A$ ($A \Leftarrow B$) separately.

We start by reviewing certain standard techniques to construct proofs of mathematical statements.

1. Contrapositive Proof

To show $A \Rightarrow B$, equivalently show not $B \Rightarrow$ not A $(\neg B \Rightarrow \neg A)$.

$\neg$ "negation" or "not"

"If A happened then B happened" This statement is equivalent to "If B did not happen then A did not happen."

LSIRA 1.1 Prob 3. Prove the following Lemma.

Lemma 1 If n is a natural number such that n^2 is divisible by 3, then n is divisible by 3.

This is $A \Rightarrow B$ where A: $3 \mid n^2$ (n^2 is divisible by 3). → or 3 divides n^2

B: $3 \mid n$ (n is divisible by 3).

Let's try to prove $A \Rightarrow B$ directly: $n^2 = 3k$ (as $n^2|3$) $\Rightarrow$ $n = \sqrt{3}\sqrt{k}$ (taking square root on both sides)

Hard to conclude that $n|3$ ☹! → Would have to argue $k|3$, which is not obvious!

Let's try proving $\neg B \Rightarrow \neg A$.

$\neg B$: n is not divisible by 3.

$\Rightarrow$ $n = 3p+1$ or $n = 3q+2$, for $p, q \in \mathbb{N}$. (set of natural numbers)

Case 1. $n = 3p+1$

$$\Rightarrow n^2 = (3p+1)^2$$
$$= 9p^2 + 6p + 1$$
$$= 3(3p^2+2p)+1$$
$$= 3k+1 \text{ for } k = 3p^2+2p$$

$\Rightarrow n^2$ is not divisible by 3

Case 2. $n = 3q+2$

$$\Rightarrow n^2 = (3q+2)^2$$
$$= 9q^2 + 12q + 4$$
$$= 9q^2 + 12q + 3 + 1$$
$$= 3\underbrace{(3q^2+4q+1)}_{=k'}+1$$
$$= 3k'+1$$

$\Rightarrow n^2$ is not divisible by 3.

Hence we have proved that if n is not divisible by 3, then n^2 is not divisible by 3. Hence, by the contrapositive, we have $n^2|3 \Rightarrow n|3$. □

Should we always try to build a contrapositive proof?
Not necessarily! In cases where $A \Rightarrow B$ could be concluded directly, the contrapositive argument might make life harder! It is one of the different proof approaches that you should be aware of.

2. Proof by Contradiction

Assume opposite of what you want to prove, and end up with a contradiction (or an obviously wrong statement). Hence the original assumption must be wrong, i.e., you have proved the statement.

<u>LSIRA 1.1 Prob 3 (continued)</u> Prove the following Theorem.

<u>Theorem 2</u> $\sqrt{3}$ is irrational.

Assume $\underbrace{\sqrt{3} \text{ is rational}}$. → the opposite of what you want to prove

$\Rightarrow \left(\sqrt{3} = \frac{p}{q}\right)^2$, $p, q \in \mathbb{N}$ with no common factors. → by definition, any positive rational number can be written in the form p/q as specified.

Let's square both sides, and cross multiply.

$\Rightarrow 3q^2 = p^2 \Rightarrow 3 \mid p^2$ (p^2 is divisible by 3).

Hence by Lemma 1, $3 \mid p$. Let $p = 3k$. ($k \in \mathbb{N}$). Plug $p=3k$ back in:

$\Rightarrow 3q^2 = (3k)^2 = 9k^2$ (divide both sides by 3)

$\Rightarrow q^2 = 3k^2$, i.e., $3 \mid q^2$ (q^2 is divisible by 3).

Again by Lemma 1, $3 \mid q$.

Since we started with the assumption that p and q have no common factors

Thus p and q have a common factor of 3, which is a contradiction.

Hence $\sqrt{3}$ is irrational. $\square$

3. Proof by Induction

To show a statement $P(n)$ holds for all $n \in \mathbb{N}$,

1. show $P(1)$ holds;
2. Assume $P(k)$ holds for some $k \in \mathbb{N}$.
3. Show $P(k+1)$ holds under Assumption 2.

<u>Example</u>

"for all"

Show that $P(n) = 3+5+\cdots+2n+1 = n(n+2) \ \forall n \in \mathbb{N}$.

1. $P(1) = 3 = 1(1+2)$ (so $P(1)$ is true).
2. Assume $P(k) = k(k+2)$ for some $k \in \mathbb{N}$.
3. $P(k+1) = P(k) + 2(k+1)+1 = P(k) + 2k+3$

$$= k(k+2) + 2k+3 \quad \text{by induction assumption.}$$
$$= k(k+2) + k + k+3$$
$$= k(k+3) + k+3$$
$$= (k+1)(k+3) = n(n+2) \text{ for } n = k+1.$$

$$\Rightarrow P(n) = n(n+2) \ \forall n \in \mathbb{N}. \qquad \square$$

MATH 401 : Lecture 2 (08/21/2025)

Today : $*$ sets and operations

Sets and Operations (LS IRA 1.2)

Set: Collection of mathematical objects.

They can be finite, e.g., $\{2,5,9,1,6\}$, or infinite, e.g., $[0,1]$, the collection of all $x \in \mathbb{R}$ with $0 \leq x \leq 1$.

"element of" set of all real numbers

Given sets A, B we have

$A \subseteq B$: A is a subset of, or equal to, B.

$A \subset B$: A is a strict subset of B, i.e., there is at least one $x \in B$ such that $x \notin A$.

But $\forall x \in A$, $x \in B$ holds.

To prove $A = B$, we often prove $A \subseteq B$ and $A \supseteq B$ (or $B \subseteq A$).

Here are ~~some~~ standard sets we will use regularly.

$\emptyset$: empty set.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, set of all natural numbers

$\mathbb{R}$ = set of all real numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, set of all integers

$\mathbb{Q}$ = set of rational numbers, $\mathbb{C}$ = set of complex numbers.

$\mathbb{R}^n$: set of all real n-tuples, or n-vectors

Notation for sets : $[-2, 1] = \{x \in \mathbb{R} \mid -2 \leq x \leq 1\}$.

closed interval from -2 to 1

"such that"
could also use ":" instead of "|".

More generally, $A = \{a \in B \mid P(a)\}$.

bigger set than A property

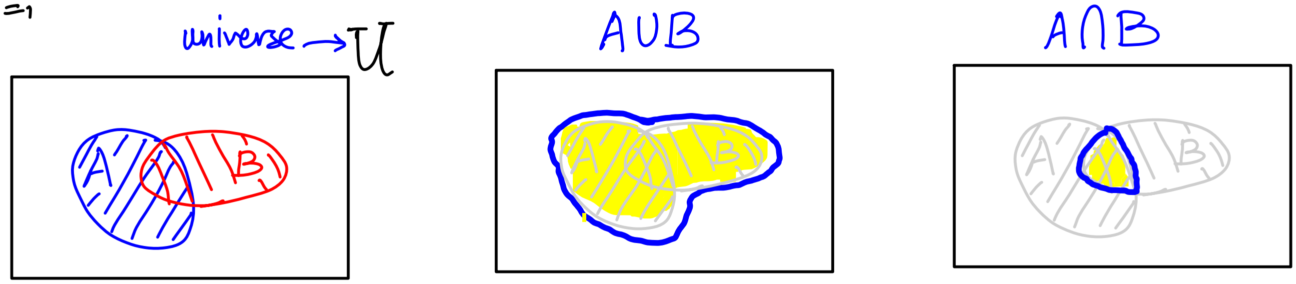
Union and Intersection

If A_i are sets for $i=1,\ldots,n$, i.e., $A_1, A_2, \ldots, A_n$ are sets, then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{a \mid a \in A_i \text{ for at least one } i\}$$ is their union,

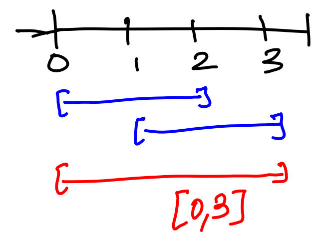
"for all"

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{a \mid a \in A_i \ \forall i\}$$ is their intersection.



LSIRA 1.2 Prob1

Show $[0,2] \cup [1,3] = [0,3]$.



We show $[0,2] \cup [1,3] \subseteq [0,3]$ and
$[0,2] \cup [1,3] \supseteq [0,3]$.

The result is an obvious one. But we go through the steps of a formal proof more for practice!

($\subseteq$) Let $x \in [0,2] \cup [1,3]$

$\Rightarrow x \in [0,2]$ or $x \in [1,3]$ (definition of $\cup$).

$x \in [0,2] \Rightarrow x \in [0,3]$ (as $[0,3]$ contains $[0,2]$)

$x \in [1,3] \Rightarrow x \in [0,3]$. In either case, $x \in [0,3]$.

Hence $[0,2] \cup [1,3] \subseteq [0,3]$.

($\supseteq$) Let $x \in [0,3]$. Hence $0 \le x \le 3$. Then we get that either $x \le 2$, and hence $x \in [0,2]$, or $x \in (2,3]$.

But if $x \in (2,3]$ then $x \in [1,3]$ (as $[1,3]$ includes $(2,3]$).

$\Rightarrow x \in [0,2] \cup [1,3]$.

Hence $[0,3] \subseteq [0,2] \cup [1,3]$. $\square$

Distributive Laws of Union and Intersection

For all sets $B, A_1, \dots, A_n$, we have

LSIRA (1.2.1) $$B \cap (A_1 \cup A_2 \cup \dots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n).$$

Using more compact notation, we can write

$$B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (B \cap A_i)$$

Proof

We will prove

$$B \cap (A_1 \cup \dots \cup A_n) \subseteq (B \cap A_1) \cup \dots \cup (B \cap A_n), \text{ and}$$

$$B \cap (A_1 \cup \dots \cup A_n) \supseteq (B \cap A_1) \cup \dots \cup (B \cap A_n).$$

('$\subseteq$') Let $x \in B \cap (A_1 \cup \dots \cup A_n)$.

$\Rightarrow$ $x \in B$ and $x \in (A_1 \cup \dots \cup A_n)$ (definition of $\cap$)

$\Rightarrow$ $x \in B$ and $x \in A_i$ for at least one A_i. (defn. of $\cup$)

$\Rightarrow$ $x \in B \cap A_i$ for at least one A_i.

$\Rightarrow$ $x \in (B \cap A_1) \cup \dots \cup (B \cap A_n)$.

('$\supseteq$') Let $x \in (B \cap A_1) \cup \dots \cup (B \cap A_n)$.

$\Rightarrow$ $x \in (B \cap A_i)$ for at least one A_i.

$\Rightarrow$ $x \in B$ and $\underbrace{x \in A_i \text{ for at least one } A_i}$

$\Rightarrow$ $x \in B$ and $x \in (A_1 \cup \dots \cup A_n)$

$\Rightarrow$ $x \in B \cap (A_1 \cup \dots \cup A_n)$. $\square$

LSIRA (1.2.2) is assigned in Homework 1.

Set Difference and Complement

We write $A \setminus B$ or $A - B$ "setminus"

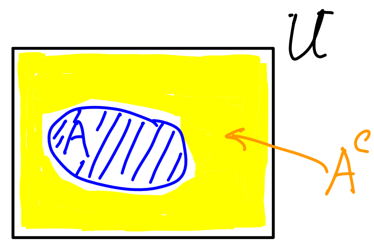
Caution! * $A \setminus B \neq B \setminus A$!

"A setminus B" is $A \setminus B = \{a \mid a \in A, a \notin B\}$.

If $\mathcal{U}$ is the universe, i.e., $A \subseteq \mathcal{U}$ for all sets A, then

$$A^c = \mathcal{U} \setminus A = \{a \in \mathcal{U} \mid a \notin A\}$$ is the

complement of A (or A-complement).



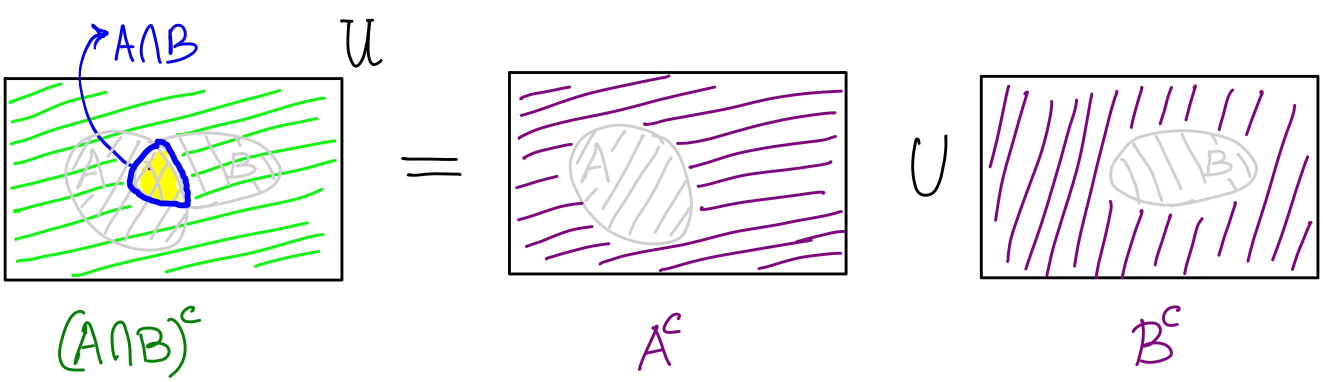
De Morgan's Laws

LSIRA (1.2.3) $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$. "complement of union = intersection of complements"

LSIRA (1.2.4) $(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c$. complement of intersection = union of complements.

See LSIRA for the proof.

Let's illustrate (1.2.4) for $n=2$, i.e., with A_1 and A_2 first.



We will prove subset inclusion in both directions.

($\subseteq$) Let $x \in (A_1 \cap \cdots \cap A_n)^c$.

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$ (definition of complement)

$\Rightarrow x \notin A_j$ for some j. (definition of $\cap$)

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \in A_1^c \cup \cdots \cup A_n^c$.

Hence $(A_1 \cap \cdots \cap A_n)^c \subseteq A_1^c \cup \cdots \cup A_n^c$.

($\supseteq$) Let $x \in A_1^c \cup \cdots \cup A_n^c$.

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \notin A_j$ for some j

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$. since $x \notin A_j$ for some j, it cannot be in the intersection of all A_i's.

$\Rightarrow x \in (A_1 \cap \cdots \cap A_n)^c$.

Hence $A_1^c \cup \cdots \cup A_n^c \subseteq (A_1 \cap \cdots \cap A_n)^c$. $\square$

Cartesian Products

For A, B: sets, we define

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$ cartesian product of A and B

given A_i, $i = 1, \ldots, n$ $(A_1, \ldots, A_n)$, we define

compact notation

$\prod$: product

$$A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \mid \underbrace{a_i \in A_i \ \forall i}_{a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n}\}.$$

e.g., $\mathbb{R}^n$: set of n-tuples of real numbers
(or set of real n-vectors)

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

We'll finish the proof in the next lecture...

MATH 401: Lecture 3 (08/26/2025)

Today: * families of sets, properties
* functions, images, pre images

We first do a problem on Cartesian products...

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A\cup B)\times C = (A\times C)\cup(B\times C)$.

'$\subseteq$' Let $(x,y)\in(A\cup B)\times C$.

$\Rightarrow x\in A\cup B,\ y\in C$ (Definition of cartesian product)

$\Rightarrow x\in A$ or $x\in B$, $y\in C$

If $x\in A$ then $(x,y)\in A\times C$, and

if $x\in B$ then $(x,y)\in B\times C$.

$\Rightarrow (x,y)\in A\times C$ or $(x,y)\in B\times C$

$\Rightarrow (x,y)\in(A\times C)\cup(B\times C)$.

'$\supseteq$' Let $(x,y)\in(A\times C)\cup(B\times C)$

$\Rightarrow (x,y)\in A\times C$ or $(x,y)\in B\times C$

$\Rightarrow x\in A, y\in C$ or $x\in B, y\in C \Rightarrow (x\in A$ or $x\in B), y\in C$.

$\Rightarrow x\in A\cup B, y\in C \Rightarrow (x,y)\in(A\cup B)\times C$.

$\square$

LSIRA 1.3 Families of Sets

Recall: $B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (B \cap A_i)$. compact notation for distributive law (from Lecture 2)

We could write, instead, $B \cap \left(\bigcup_{i \in \mathcal{I}} A_i \right) = \bigcup_{i \in \mathcal{I}} (B \cap A_i)$, where $\mathcal{I} = \{1, 2, \ldots, n\}$.

We now generalize $\mathcal{I}$ to be infinite in some cases, or indexing more general collections in general.

Def A collection of sets is a **family**.

e.g., $\mathcal{A} = \{[a,b] \mid a, b \in \mathbb{R}\}$ is the family of all closed intervals on $\mathbb{R}$.

Union and Intersection

We extend union, intersection, as well as their distribution to families.

$$\bigcup_{A \in \mathcal{A}} A = \{a \mid a \in A \text{ for some } A \in \mathcal{A}\}.$$

collection of elements that belong to at least one set in the family

$$\bigcap_{A \in \mathcal{A}} A = \{a \mid a \in A \text{ for all } A \in \mathcal{A}\}$$

collection of elements that belong to every set in the family.

We naturally extend distributive and De Morgan's laws to families...

$$B \cap \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (B \cap A), \quad \left(\bigcap_{A \in \mathcal{A}} A \right)^c = \bigcup_{A \in \mathcal{A}} A^c, \text{ etc.}$$

We now work on some problems involving families of sets.

LSIRA 1.3 Prob 1 (Pg 12)

Show that $\bigcup_{n\in\mathbb{N}} [-n,n] = \mathbb{R}$.

('$\subseteq$') $\mathbb{R}$ is the universe here, so $\bigcup_{n\in\mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

Or, observe that $[-n,n] \subseteq \mathbb{R}$ for each $n\in\mathbb{N}$, hence $\bigcup_{n\in\mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

('$\supseteq$') Let $x\in\mathbb{R}$. To be more careful, we could consider $x=0$ separately. Note that $x=0 \in [-n,n]$ $\forall n\in\mathbb{N}$.

Let $m = \lceil |x| \rceil$, ceiling of absolute value of x, i.e., the smallest natural number $\geq |x|$.

$\lceil x \rceil = \text{ceil}(x)$ = smallest integer $\geq x$.

Then $x \in [-m,m] = [-\lceil |x|\rceil, \lceil |x|\rceil]$, as

$x \leq |x| \leq \lceil |x| \rceil = m$, and $x \geq -|x| \geq -\lceil |x| \rceil$.

e.g., $x=-3.3 \Rightarrow x \geq -|-3.3| = -3.3 \geq -4$.

$\Rightarrow x \in \bigcup_{n\in\mathbb{N}} [-n,n]$.

$\square$

LSIRA 1.3 Prob 4

Show $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}] = \emptyset$ (empty set).

('$\supseteq$') $\emptyset \subseteq A$ for any set A (trivially).

('$\subseteq$') We show $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}] \subseteq \emptyset$.

$\emptyset^c = \mathbb{R}$. Hence we show $x\in\mathbb{R}$ is not in $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

For $x\in\mathbb{R}$, we show $x \notin \bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

If $x \leq 0$, then clearly, $x \notin (0,\frac{1}{n}]$ $\forall n\in\mathbb{N}$.

If $x \geq 1$, then $x \notin (0,\frac{1}{2}]$ for $n=2$, for instance.

Let $0<x<1$. Consider $m = \lceil \frac{1}{x} \rceil + 1$.

Then $x \notin (0, \frac{1}{m}]$ as $x > \frac{1}{m} = \frac{1}{\lceil \frac{1}{x} \rceil + 1}$. $\left(\lceil \frac{1}{x} \rceil + 1 > \frac{1}{x} \right)$

$$\Rightarrow \quad x \notin \bigcap_{n \in \mathbb{N}} (0, \tfrac{1}{n}].$$

Q. Why take $\lceil \frac{1}{x} \rceil + 1$?

Consider $x = \frac{1}{5}$, for instance.
Then $\lceil \frac{1}{x} \rceil = \lceil 5 \rceil = 5$.
Hence $x \in (0, \frac{1}{m}]$ here!

$\square$

<u>LSIRA 1.3 Prob 5 (Pg 12)</u>

Prove that $B \cup \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (B \cup A)$.

($\subseteq$) Let $x \in B \cup \left(\bigcap_{A \in \mathcal{A}} A \right)$

$\Rightarrow x \in B$ or $x \in \bigcap_{A \in \mathcal{A}} A$

$\Rightarrow x \in B$ or $x \in A$ for each $A \in \mathcal{A}$.

$\Rightarrow x \in B \cup A$ for each $A \in \mathcal{A}$.

$\Rightarrow x \in \bigcap_{A \in \mathcal{A}} (B \cup A)$.

('$\supseteq$') Let $x \in \bigcap_{A \in \mathcal{A}} (B \cup A)$

$\Rightarrow x \in B \cup A$ for every $A \in \mathcal{A}$.

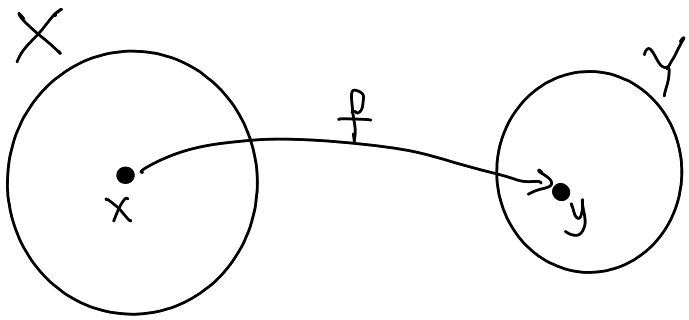
$\Rightarrow x \in B$ or $\underbrace{x \in A \text{ for every } A \in \mathcal{A}}$.

$\Rightarrow x \in B$ or $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in B \cup \left(\bigcap_{A \in \mathcal{A}} A \right)$.

$\square$

LSIRA 1.4 **Functions**

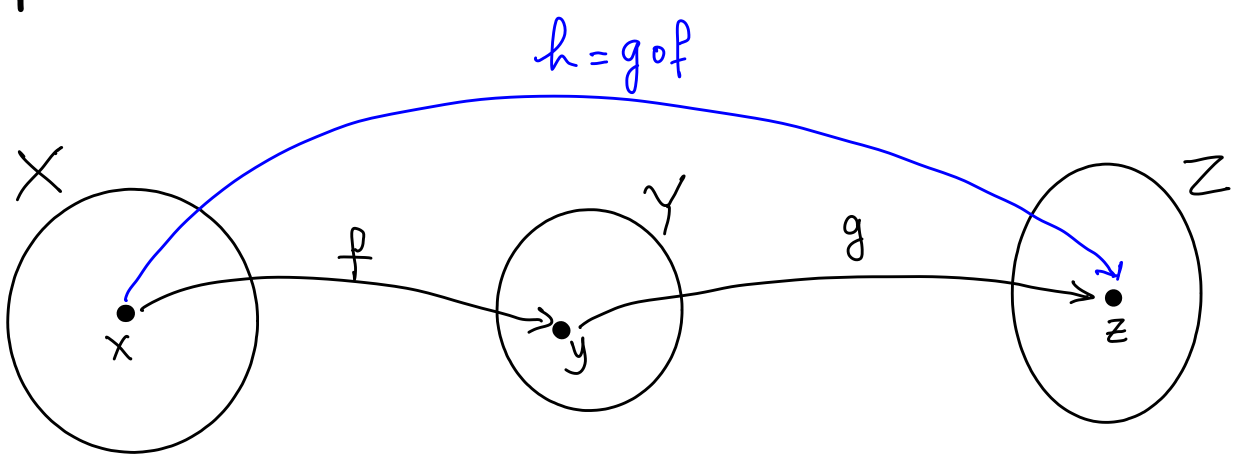
A function $f: X \to Y$ for sets X, Y is a rule that assigns for each $x \in X$ a **unique** $y \in Y$. We write $f(x) = y$, or $x \mapsto y$ "maps to".



Rather than the graphs of functions you may have seen previously, we think of such visualizations for functions now.

X is the domain and Y the codomain of f.

Compositions

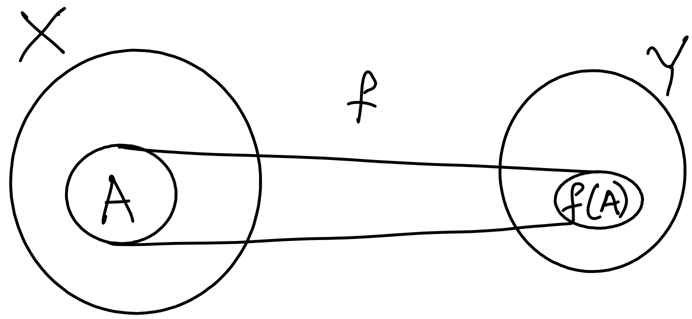


Let $f: X \to Y$ and $g: Y \to Z$ be functions. Then their composition is specified as $h: X \to Z$ defined as $h(x) = g(f(x))$. The composition is written as $\underline{g \circ f}$, with $g \circ f(x) = g(f(x))$.

"composition of f and g"

$f_1(f_2(\cdots f_n(x)))\cdots)$ → composition of n functions $f_1, f_2, \ldots, f_n$

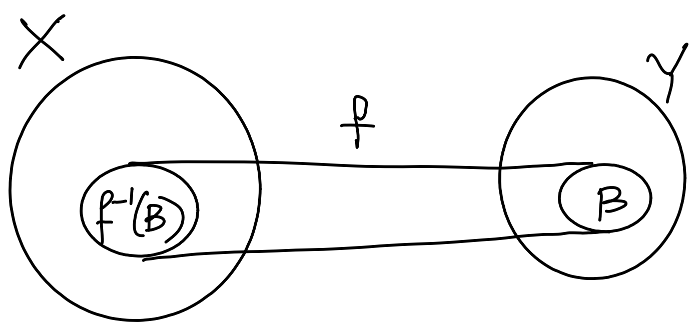
Function: $f: X \to Y$. We now define images and preimages under f.



For $A \subseteq X$, $f(A) \subseteq Y$ is defined as

$$f(A) = \{ f(a) \mid a \in A \},$$

and is called the **image** of A under f.



For $B \subseteq Y$, the set $f^{-1}(B) \subseteq X$ defined as

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}$$

is the **inverse image** or **preimage** of B under f.

In the next lecture, we consider how preimages and images commute with unions and intersections, or not...

MATH 401: Lecture 4 (08/28/2025)

Today:
- ✱ images/preimages and unions/intersections
- ✱ injective/surjective functions
- ✱ relations, equivalence relations, partitions

We now consider how images and inverse images commute (or not) with unions and intersections of families of sets.

Prop 1.4.1 Inverse images commute with arbitrary unions and intersections:

$$f^{-1}\Big(\bigcup_{B\in\mathcal{B}} B\Big) = \bigcup_{B\in\mathcal{B}} f^{-1}(B) \quad \text{and}$$

"inverse of union = union of inverses"

$$f^{-1}\Big(\bigcap_{B\in\mathcal{B}} B\Big) = \bigcap_{B\in\mathcal{B}} f^{-1}(B)$$

"inverse of intersection = intersection of inverses"

Proof (of the second statement) ⟶ See LSIRA for proof of first statement

($\subseteq$) Let $x \in f^{-1}\Big(\bigcap_{B\in\mathcal{B}} B\Big) \Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B$

$\Rightarrow f(x) \in B$ for every $B \in \mathcal{B}$.

$\Rightarrow x \in f^{-1}(B)$ for every $B\in\mathcal{B}$.

$\Rightarrow x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$.

($\supseteq$) Let $x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$

$\Rightarrow x \in f^{-1}(B)$ for every $B\in\mathcal{B}$.

$\Rightarrow f(x) \in B$ for every $B\in\mathcal{B}$.

$\Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B \Rightarrow x \in f^{-1}\Big(\bigcap_{B\in\mathcal{B}} B\Big)$.

$\square$

We saw that inverse images commute with unions and intersections. But forward images behave a bit differently.

<u>Prop 1.4.2</u> $f: X \to Y$ is a function, $\mathcal{A}$ is a family of subsets of X.

Then $f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f(A)$, $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

<u>Proof</u>

('$\subseteq$') Let $y \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$

"There exists"

$\Rightarrow \exists\, x \in \bigcup_{A\in\mathcal{A}} A$ such that $f(x) = y$.

$\Rightarrow x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x) = y$

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow y \in \bigcup_{A\in\mathcal{A}} f(A)$.

($\supseteq$) Let $y \in \bigcup_{A\in\mathcal{A}} f(A)$.

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow \exists\, x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x) = y$.

$\Rightarrow \exists x \in \bigcup_{A\in\mathcal{A}} A$ such that $f(x) = y$.

$\Rightarrow y \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$

LSIRA gives a slightly different proof for ($\supseteq$):

"for all"

$A \subseteq \bigcup_{A\in\mathcal{A}} A \quad \forall A \in \mathcal{A}$

$\Rightarrow f(A) \subseteq f\left(\bigcup_{A\in\mathcal{A}} A\right) \quad \forall A \in \mathcal{A}$

Since this result holds for every $A \in \mathcal{A}$, we can write

$\bigcup f(A) \subseteq f\left(\bigcup_{A\in\mathcal{A}} A\right)$.

$\square$

We consider intersections now: $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

Proof for ('$\subseteq$')

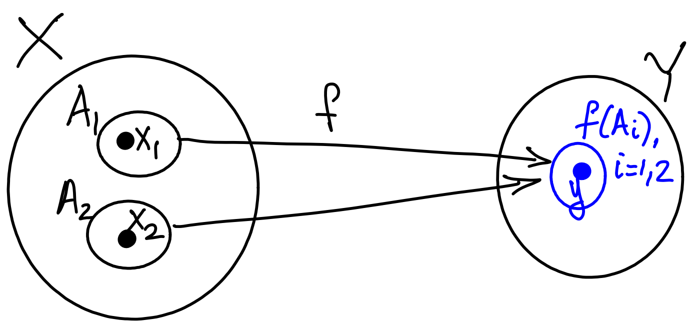
$$\bigcap_{A\in\mathcal{A}} A \subseteq A \quad \forall A\in\mathcal{A}$$

$$\Rightarrow f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq f(A) \;\forall A\in\mathcal{A}.$$

Since this inclusion holds for every $A\in\mathcal{A}$, we get

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A).$$

Counterexample for ('$\supseteq$') for $\cap$



For $x_1 \neq x_2$, $x_1, x_2 \in X$, let

$f(x_i) = y$, $i=1,2$.

Let $A_i = \{x_i\}$, $i=1,2$. $\Rightarrow \bigcap_{i=1,2} A_i = \phi$ (empty set).

But note that $f(A_i) = \{y\}$, $i=1,2$.

$\Rightarrow f\left(\bigcap_{i=1,2} A_i\right) = \phi$. But $\bigcap_{i=1,2} f(A_i) = \{y\} \neq \phi$.

$$\Rightarrow \bigcap_{i=1,2} f(A_i) \not\subseteq f\left(\bigcap_{i=1,2} A_i\right).$$

But we get this reverse inclusion if we specify that f is injective.

Def Let $f: X \to Y$ be a function.

f is **injective** (1-to-1) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Equivalent definition: For any $y \in Y$, there is **at most** one $x \in X$ s.t. $f(x) = y$.

→ there could be no $x \in X$

f is **surjective** (onto) if for every $y \in Y$, there is **at least** one $x \in X$ such that $f(x) = y$.

→ there could be more than one

f is **bijective** if it is both injective and surjective.

LSIRA 1.4 prob 4 (pg 17)

Let $f: \underset{X}{\mathbb{R}} \to \underset{Y}{\mathbb{R}}$ be a strictly increasing function, i.e.,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for } x_i \in \mathbb{R},\ i = 1, 2.$$

1. Show that f is injective.
2. Does it have to be surjective?

→ Either give a proof or a counterexample.

1. We show $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Without loss of generality (WLOG), let $x_1 < x_2$.

→ The same result holds when $x_2 < x_1$ as well.

Then $f(x_1) < f(x_2)$, as f is strictly increasing.

Hence $f(x_1) \neq f(x_2)$, and so f is injective.

2. No. $f = \arctan(x)$ is strictly increasing.

$f: \mathbb{R} \to \mathbb{R}$, but $\arctan(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}$.

So f need not be surjective.

Another example is $f = e^x$.

Relations (LSIRA 1.5)

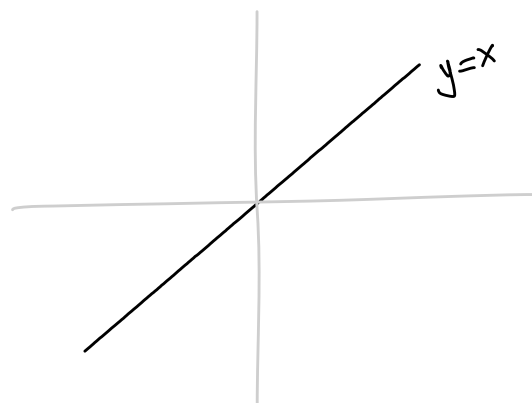
We had seen functions, where a unique $y \in Y$ is assigned for each $x \in X$ by $f: X \to Y$. But entities are related in other ways — numbers are $>$ or $<$ each other, lines are parallel, etc. We define relations formally to describe such dependencies.

Def A relation R on a set X is a subset of $\underbrace{X \times X}$.

We write xRy, $(x,y) \in R$, or $x \sim y$.

Cartesian product of X with itself

e.g., $R = \{(x,y) \in \mathbb{R}^2 \mid x = y\}$.



Recall, $y=x$ is the 45° line through $(0,0)$. All points are "related" by them belonging to this line.

Here is another relation (on integers):

$P = \{(x,y) \in \mathbb{Z}^2 \mid x, y \text{ have same parity}\}$.

So, all odd integers are related, and so are all even integers.

Some relations have more structure than default — as defined below.

Equivalence Relations

Def A relation $\sim$ on X is an **equivalence relation** if it is

(i) reflexive, i.e., $x \sim x \ \forall x \in X$;

(ii) symmetric, i.e., $x \sim y \Rightarrow y \sim x \ \forall x, y \in X$; and

(iii) transitive, i.e., $x \sim y, y \sim z \Rightarrow x \sim z \ \forall x, y, z \in X$.

Note that $<$ is not reflexive, or symmetric, e.g., $5 \not< 5$, and $3 < 5 \not\Rightarrow 5 < 3$.

Def Given an equivalence relation $\sim$ on X, we define the **equivalence class** $[x]$ of $x \in X$ as

$$[x] = \{y \in X \mid x \sim y\}.$$

→ the set of all "relatives" of x

The collection of equivalence classes forms a partition of X.

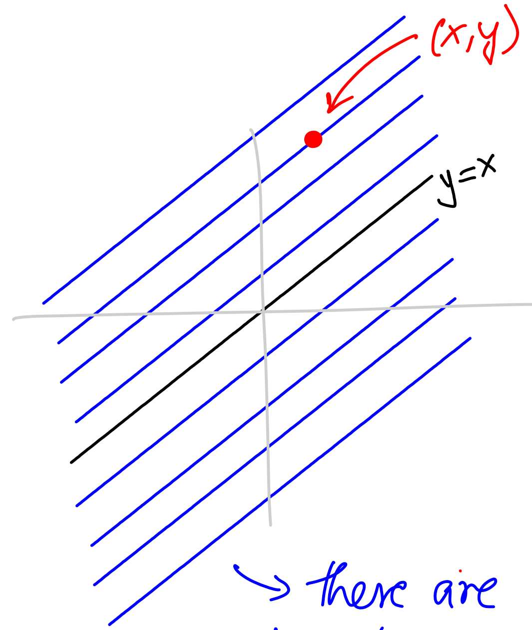
Def A **partition** $\mathcal{P}$ of X is a family of nonempty subsets of X such that $x \in X$ satisfies $x \in P \in \mathcal{P}$ for exactly one P in $\mathcal{P}$ (for every $x \in X$).

The elements P of $\mathcal{P}$ are called **partition classes** of $\mathcal{P}$.

e.g., $\mathcal{P} = \{ \underbrace{\{2k, k \in \mathbb{Z}\}}_{\text{even integers}}, \underbrace{\{2k+1, k \in \mathbb{Z}\}}_{\text{odd integers}} \}$ is a partition of $\mathbb{Z}$.

Here is a direct example of a partition of $\mathbb{R}^2$.

The collection of all lines with slope $= 1$ ($45°$) is a partition of $\mathbb{R}^2$.



→ there are infinitely many lines with slope $m=1$.

Any point in $\mathbb{R}^2$ belongs to exactly one line with a slope of $m=1$ (i.e., $45°$ degree slope).

We have not checked that the defining relation is an equivalence relation, but this can be done easily.

→ recall, the point-slope form of the equation of a line: $\frac{y-y_0}{x-x_0} = m$, given slope m and one point (x_0, y_0).

MATH 401: Lecture 5 (09/02/2025)

Today: * equivalence relations and partitions
* countability

Recall: * $\sim$ is an equivalence relation on X: $x \sim x$, $x \sim y \Rightarrow y \sim x$, $x \sim y, y \sim z \Rightarrow x \sim z$

* partition of X $\mathcal{P} = \{P\}$

We show that equivalence relations naturally define partitions.

Prop 1.5.3 If $\sim$ is an equivalence relation on X, then the collection of equivalence classes $\mathcal{P} = \{[x] \mid x \in X\}$ is a partition of X.

Proof We show each $x \in X$ belongs to exactly one equivalence class.

$x \sim x$ $\quad$ $\sim$ is equivalence relation, so is reflexive ((i))

$\Rightarrow x \in [x] \longrightarrow$ So, each $x \in X$ belongs to at least its own class.

We now show if $x \in [y]$ for $y \in X$, $y \neq x$, then $[x] = [y]$.

We show $[x] \subseteq [y]$ and $[x] \supseteq [y]$.

($\subseteq$) Let $z \in [x]$

$\Rightarrow x \sim z$ $\quad$ Definition of $[x]$

We assumed $x \in [y] \Rightarrow y \sim x$

$\sim$ is an equivalence relation, so $\underbrace{y \sim x, x \sim z \Rightarrow y \sim z}_{\sim \text{ is transitive ((iii))}}$.

$\Rightarrow z \in [y]$.

($\supseteq$) let $z \in [y]$ $\Rightarrow$ $y \sim z$

Also, $x \in [y] \Rightarrow y \sim x$

$\sim$ is equivalence relation $\Rightarrow x \sim y$ $\quad$ ($\sim$ is symmetric (ii))

$\Rightarrow x \sim y, y \sim z \Rightarrow x \sim z$ $\quad$ ($\sim$ is transitive (iii))

$\Rightarrow z \in [x]$. $\square$

LSIRA 1.5 Prob 5 (Pg 20) Let $\sim$ be a relation on $\mathbb{R}^3$ defined as

$$(x,y,z) \sim (x',y',z') \iff 3x-y+2z = 3x'-y'+2z'.$$

Show that $\sim$ is an equivalence relation. Describe its equivalence classes.

We check that $\sim$ is reflexive, symmetric, and transitive.

Reflexive: $(x,y,z) \sim (x,y,z)$, as $3x-y+2z = 3x-y+2z$. ✓

Symmetric: $(x,y,z) \sim (x',y',z') \Rightarrow (x',y',z') \sim (x,y,z)$ holds as

$3x-y+2z = 3x'-y'+2z' \Rightarrow$ $\quad$ $a=b \Rightarrow b=a$ for $a,b \in \mathbb{R}$.

$3x'-y'+2z' = 3x-y+2z$. ✓

Transitive: $(x,y,z) \sim (x',y',z')$ and $(x',y',z') \sim (x'',y'',z'')$

$\Rightarrow (x,y,z) \sim (x'',y'',z'')$ also holds, as

$3x-y+2z = 3x'-y'+2z'$ and $3x'-y'+2z' = 3x''-y''+2z''$

$\Rightarrow 3x-y+2z = 3x''-y''+2z''$. ✓

$$[(x,y,z)] = \{(x',y',z') \in \mathbb{R}^3 \mid 3x-y+2z = 3x'-y'+2z'\}$$

If we set $3x-y+2z = d \in \mathbb{R}$, then

$$[(x,y,z)] = \{\underbrace{(x',y',z') \in \mathbb{R}^3 \mid 3x'-y'+2z' = d}\}$$

plane with normal vector $(3,-1,2)$ (or $\begin{bmatrix}3\\-1\\2\end{bmatrix}$) through (x,y,z).

We can describe the equivalence classes as follows.
The equivalence class of a point in $\mathbb{R}^3$ is the plane with normal $(3,-1,2)$ passing through that point.

We write $\mathbb{R}^3/\sim$ for the set of all equivalence classes of $\sim$.

Def If $\sim$ is an equivalence relation on X, then $X/\sim$ is the set of all equivalence classes under $\sim$. "X quotient $\sim$"

$\mathbb{R}^3/\sim$ here is the set of all planes with normal $(3,-1,2)$.

Note that any point $(x,y,z) \in \mathbb{R}^3$ belongs to exactly one plane with normal $(3,-1,2)$. Also, all such parallel planes together cover all of $\mathbb{R}^3$, i.e., $\mathbb{R}^3/\sim$ is indeed a partition of $\mathbb{R}^3$. Note the similarity to previous example of 45° lines in $\mathbb{R}^2$.

Another example on equivalence classes and partitions

Let X be the set of all fruits in a grocery store. We can group them into fruit types (classes), e.g., apples, citrus, grapes, tomatoes, plums, etc. Note that apples could include honeycrisp, red delicious, etc. (varieties of apples)

$\mathcal{P}$: A partition of X into fruit classes may look like this $\longrightarrow$

$$\mathcal{P} = \{\underset{P_1}{\text{apples}}, \underset{P_2}{\text{grapes}}, \underset{P_3}{\text{citrus}}, \underset{P_4}{\text{plums}}, \dots\}$$

Note that any individual fruit belongs to exactly one class. $\mathcal{P}$ is indeed a partition of X.

Equivalence relation $\sim$ on X associated with $\mathcal{P}$

For fruits x, y, $x \sim y$ if x and y are the same fruit type.
$\sim$ is indeed an equivalence relation (can check it's reflexive, symmetric, transitive).

What is the equivalence class $[x]$ of a fruit x?

$[x]$ is the set of all fruits of its type in the store.
e.g., x = Valencia orange, $[x] = \{$set of all citrus fruits$\}$.

What is the quotient space $X/\sim$? $X/\sim$ is the set of all fruit types.

So $X/\sim = \{$apples, citrus, $\dots\}$

Check all problems on equivalence relations from LSIRA.

LSIRA 1.6 Countability

We typically count a set of objects as 1,2,3,..., i.e., by numbering or indexing the first element, then the second one, etc. We can talk about sets being countable (or not) in general.

Def A set A is **countable** if it is possible to list all elements of A as $a_1, a_2, \ldots, a_n, \ldots$

e.g., $\mathbb{N}$ is countable — just list the elements as $1, 2, 3, \ldots$.

set of natural numbers

We could use a little more formal definition of a countable set, than the one given above (as listed in LSIRA).

Def A set A is countable if there exists an injective function $f: A \to \mathbb{N}$.

The function f is the "indexing" or "numbering" function that assigns a separate natural number to each element of A.

Note that finite sets are always countable — we can always list the elements in a sequence. Things are more interesting for infinite sets.

Def If f is also surjective, i.e., it is bijective, then A is **countably infinite**, i.e., it is countable and is infinite.

e.g., $\mathbb{Z}$ is countable.

We can list all integers as

This is just one way to list all integers. Other ways could be devised as well.

$$\begin{array}{lccccccccc} & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \ldots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \text{index} & 1 & & 3 & & 5 & & 7 & \ldots \\ & & 2 & & 4 & & 6 & \ldots & \end{array}$$

Note how the indices are listed. The positive integers are the even entries in the list, and the negative integers (& 0) are the odd entries in the list

Or, we can define $f: \mathbb{Z} \to \mathbb{N}$ as

$$f(z) = \begin{cases} 2z, & z > 0 \\ 1-2z, & z \leq 0 \end{cases}$$

We can specify $f^{-1}(\cdot)$ as follows:

$$f^{-1}(n) = \begin{cases} n/2, & n \text{ even} \\ \frac{-n+1}{2}, & n \text{ odd.} \end{cases}$$

f is bijective, and hence $\mathbb{Z}$ in countably infinite.

Proposition 1.6.1 If A, B are countable, then so is $\underline{A \times B}$. $\hookrightarrow$ Cartesian product

A, B are countable $\Rightarrow \exists$ lists $\{a_n\}, \{b_n\}$ containing all elements of A and B, respectively.

$$\Rightarrow \{\underbrace{(a_1,b_1)}_{1+1}, \underbrace{(a_1,b_2)}_{\substack{1+2\\=3}}, \underbrace{(a_2,b_1)}_{\substack{2+1\\=3}}, \underbrace{(a_1,b_3)}_{\substack{1+3\\=4}}, \underbrace{(a_2,b_2)}_{\substack{2+2\\=4}}, \underbrace{(a_3,b_1)}_{\substack{3+1\\=4}}, \dots\}$$

index

is a list containing all elements of $A \times B$.

Note the index trick: we list pairs of elements (a_i, b_j) with $a_i \in \{a_n\}$ and $b_j \in \{b_n\}$ such that the sum of their indices increase as natural numbers. Thus, $i+j=2$, and then all options for $i+j=3$, followed by all options for $i+j=4$, and so on.

This index trick could be used to show other sets are countable, e.g., the cartesian product of k countable sets is countable. ($A_1 \times A_2 \times \dots \times A_k$, where A_i is countable for $1 \leq i \leq k$).

LSIRA 1.6 Prob 1 (Pg 22) Show that the subset of a countable set is countable.

Let $B \subset A$, where A is countable.

As A is countable, there is a list $a_1, a_2, \ldots, a_n, \ldots$ such that every $a_i \in A$ is included in the list.

from the list

Let $n_1 \in \mathbb{N}$ be the smallest natural number such that $a_{n_1} \in B$.

And let $n_2 \in \mathbb{N}$, $n_2 > n_1$, be the smallest number such that $a_{n_2} \in B$, and let $n_3 > n_2$, $n_3 \in \mathbb{N}$, be the smallest number such that $a_{n_3} \in B$, and so on.

We form a new list with $b_i = a_{n_i}$, $i = 1, 2, 3, \ldots$

$\Rightarrow$ $b_1, b_2, b_3, \ldots$ is a listing of all elements in B, ensuring that B is countable. $\square$

indeed, we will miss no elements of B in this process, and all of them are included in the new list.

Check Prop 1.6.2 (in LSIRA): $\bigcup_{n \in \mathbb{N}} A_n$ is countable when A_n is countable $\forall n$.

We can use a similar indexing trick as in Prop. 1.6.1.

Countability is one way to compare two infinite sets. We know $\mathbb{R} \supseteq \mathbb{Q}$, but both have infinitely many elements. Intuitively, we know $\mathbb{R}$ is bigger as it contains irrational numbers in addition to rationals.

We'll first show that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is, in fact, uncountable. More in the next lecture...