Steinhaus Filtration: Movie Recs using Topology!

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SoCG 2025

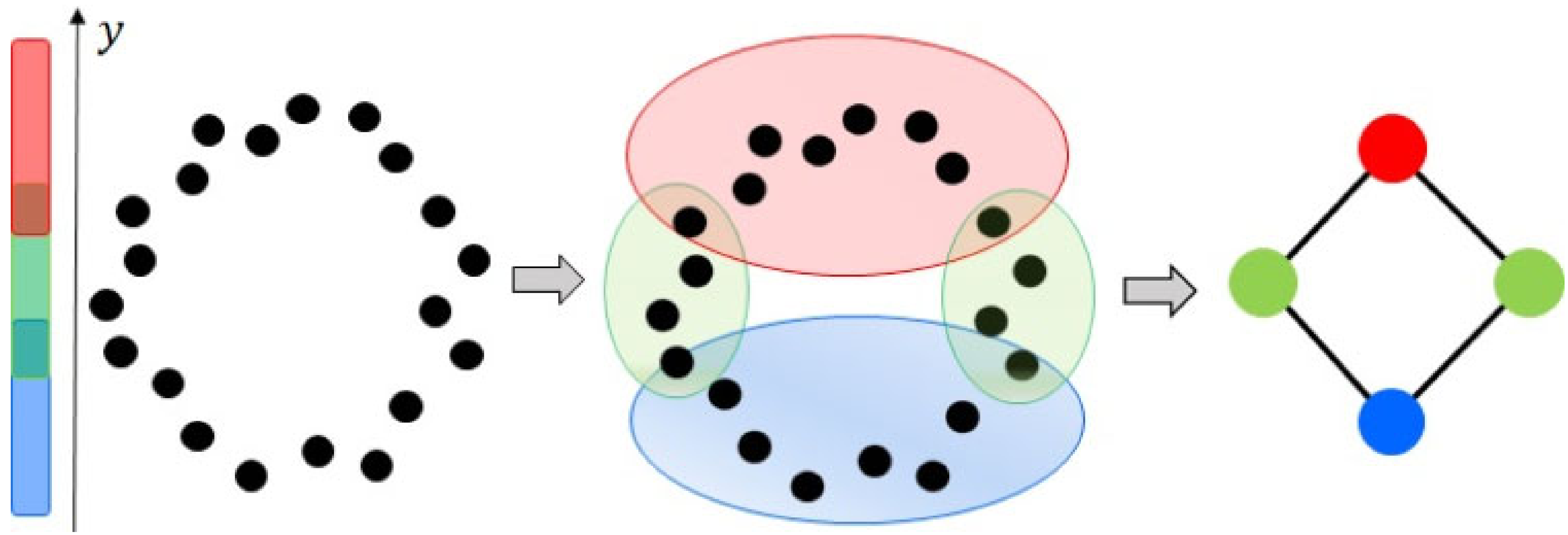
Movie Recommendations

?

(images: Wikipedia)

MAPPER

Singh, Mémoli, Carlsson (2007)



lens → cover → pullback → refine → nerve

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- → use 1-skeleton for interpretation
- → 1D features: paths, flares, loops, ...

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→ most applications still use a single mapper

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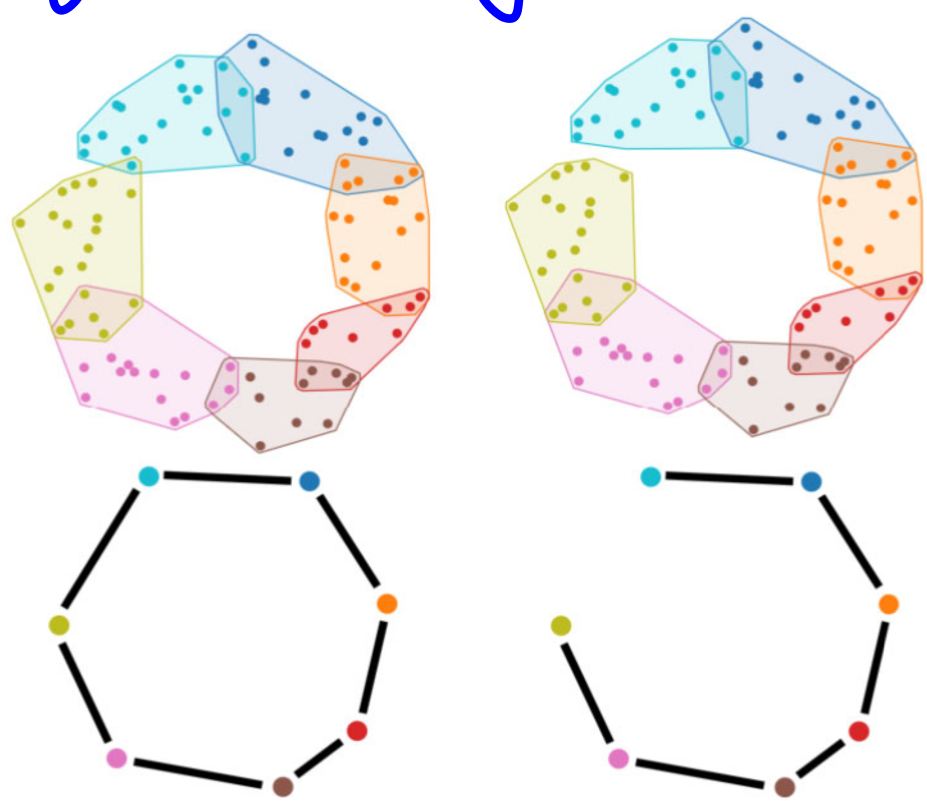

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✓ Applications
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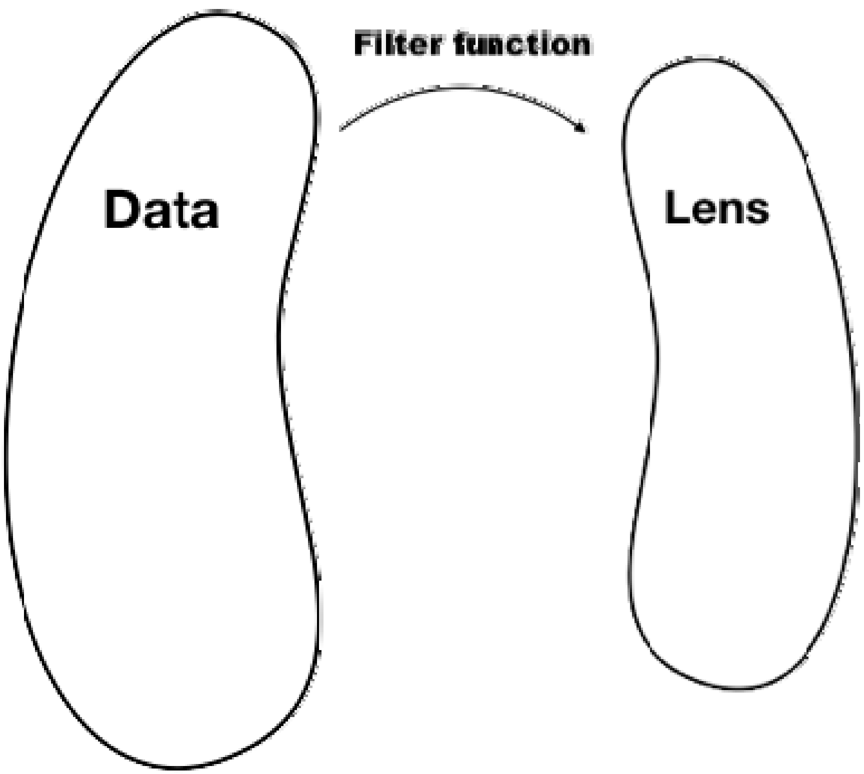

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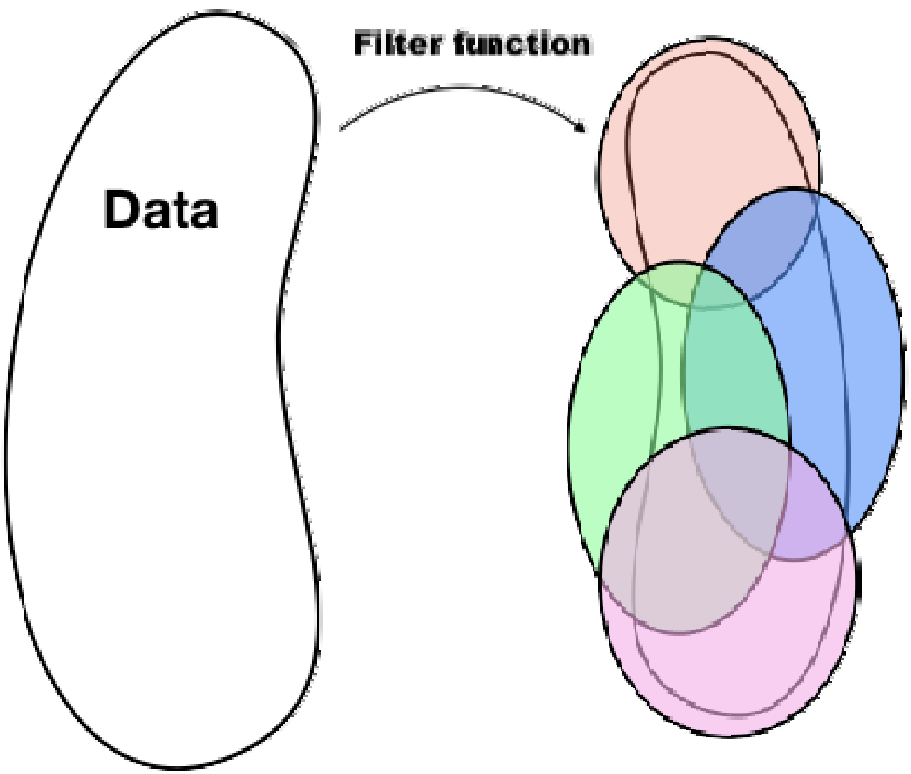

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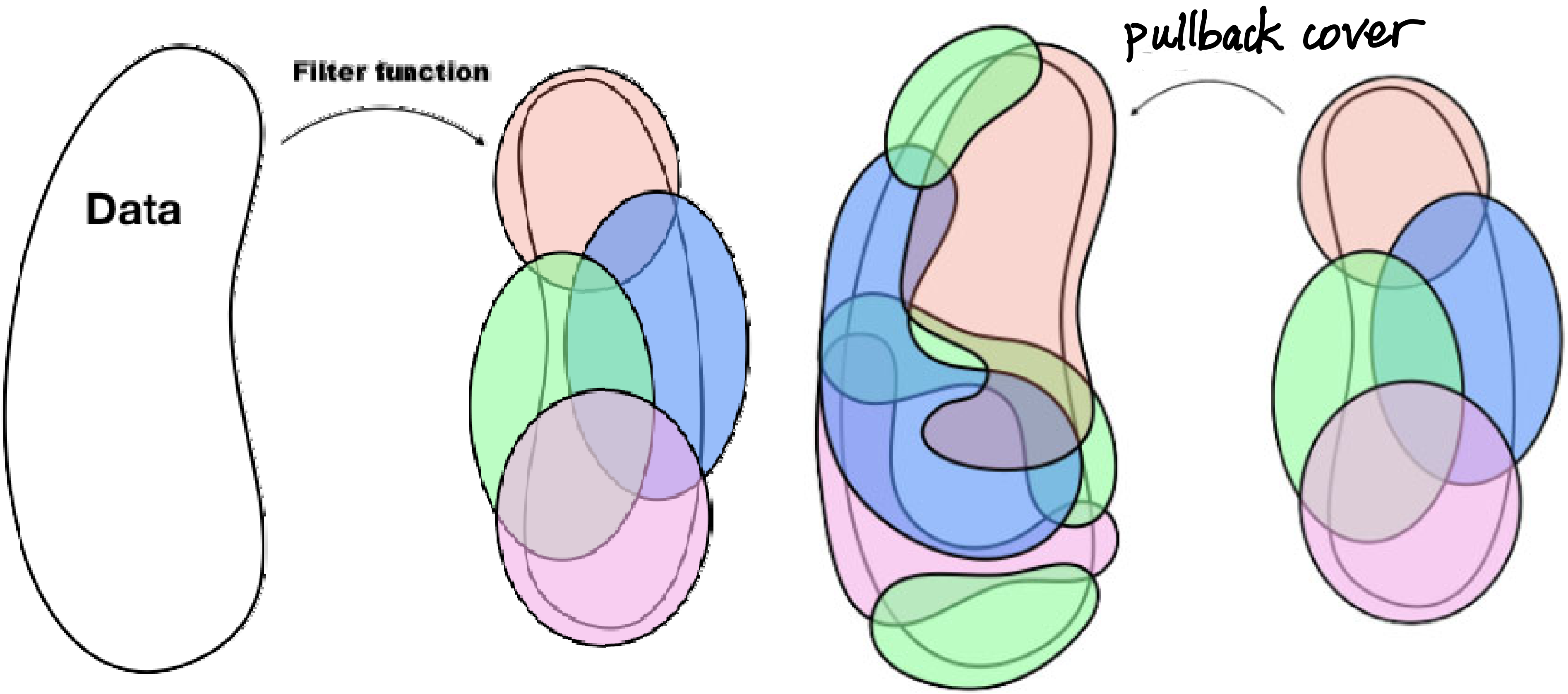

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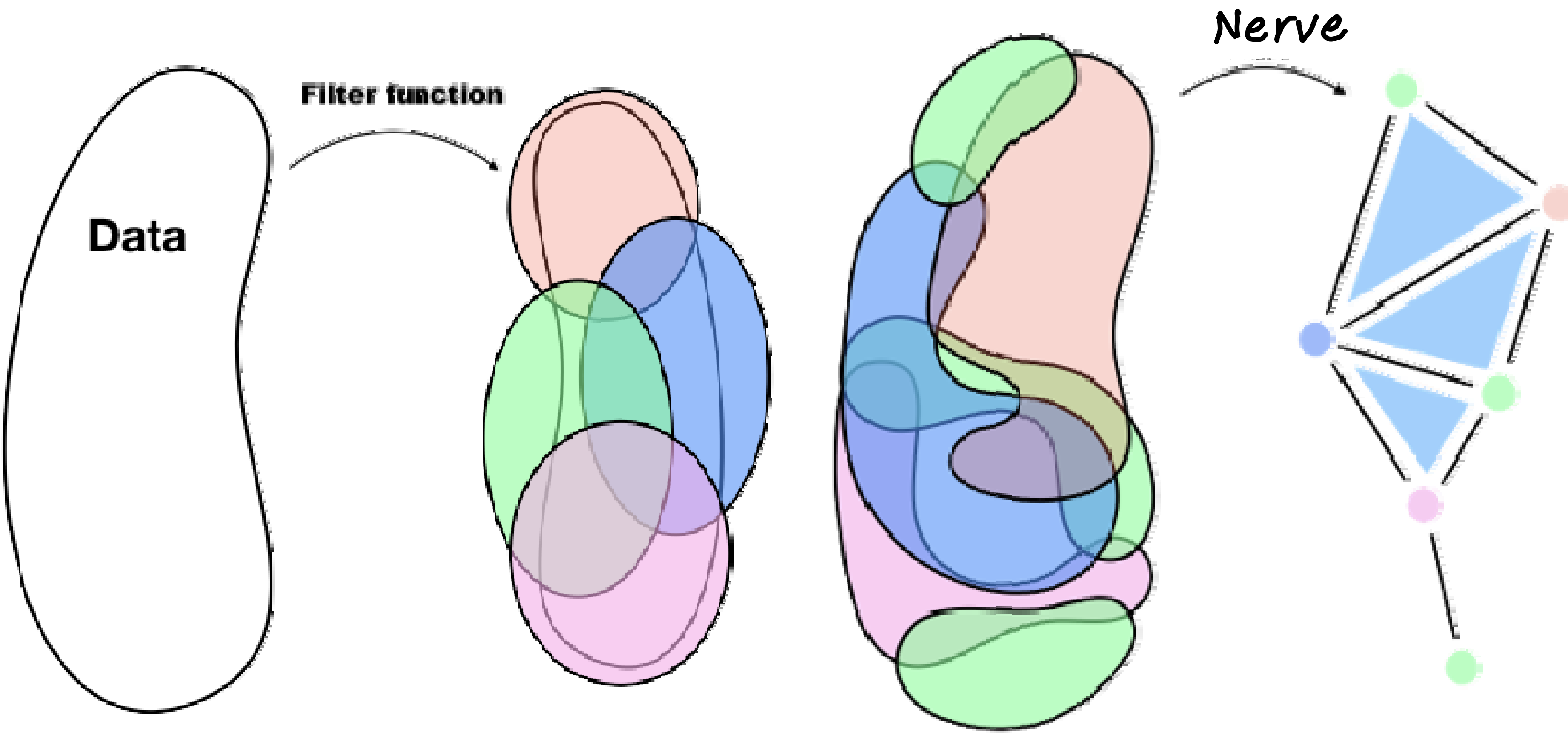

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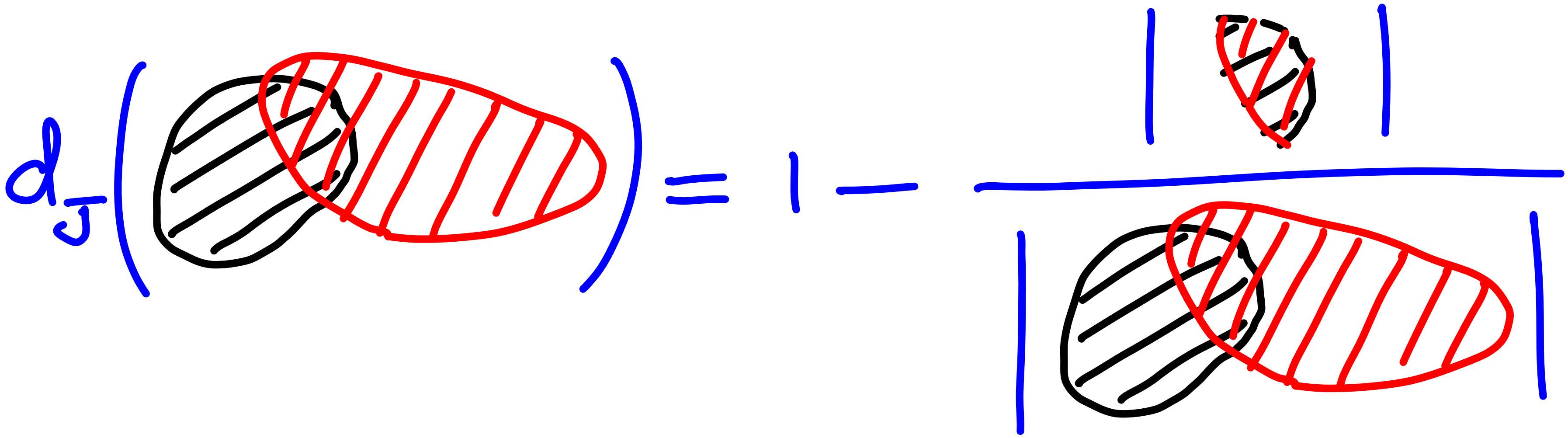

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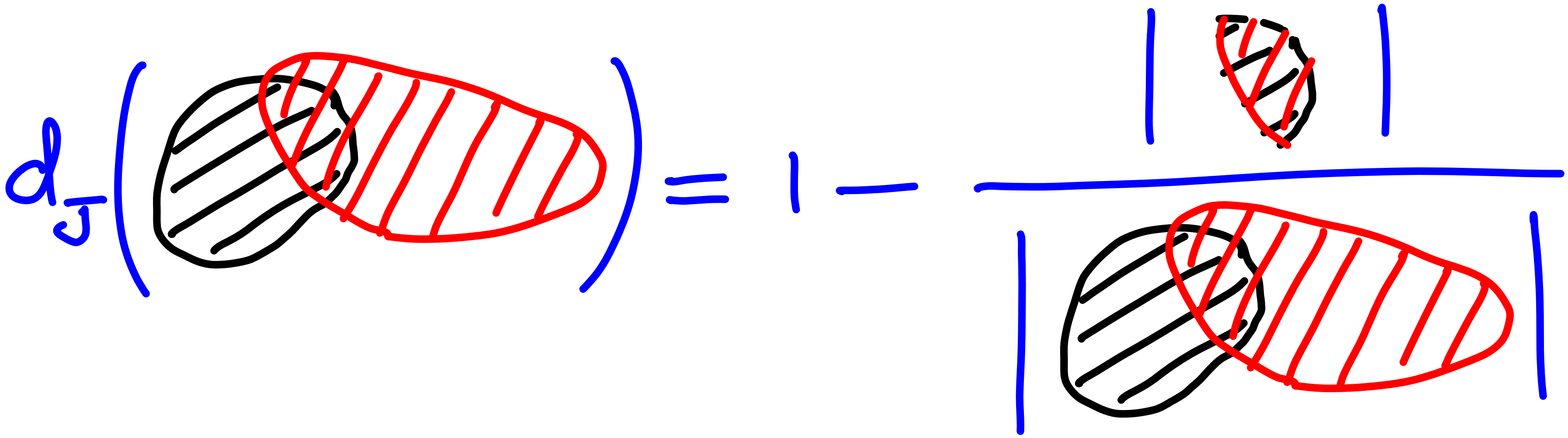
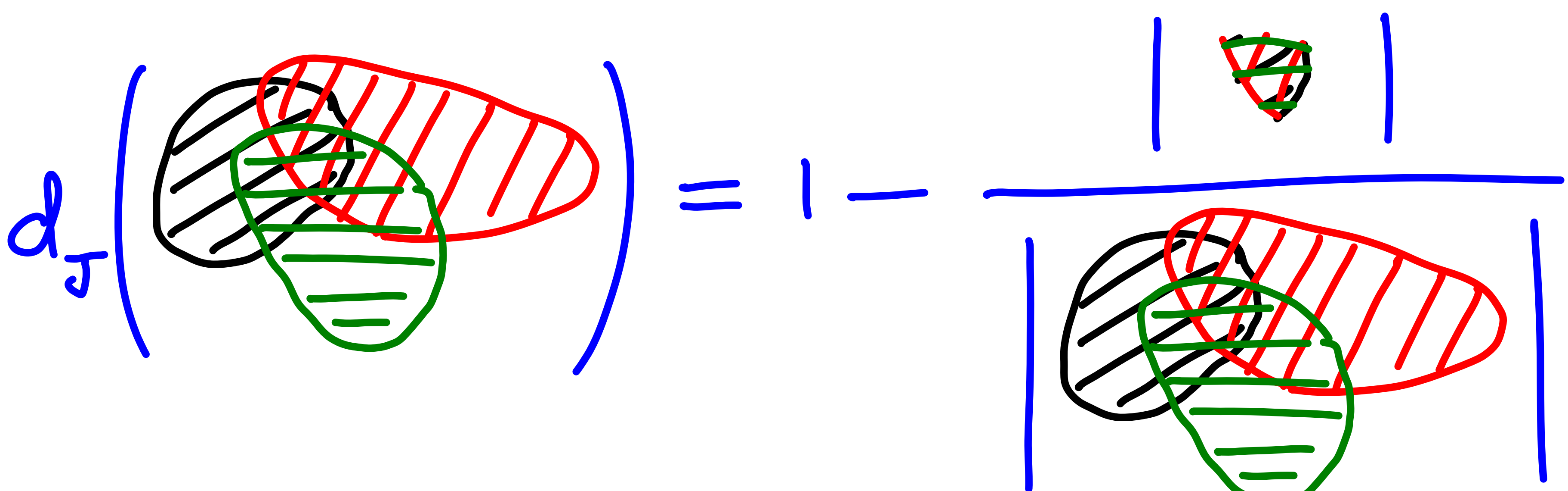
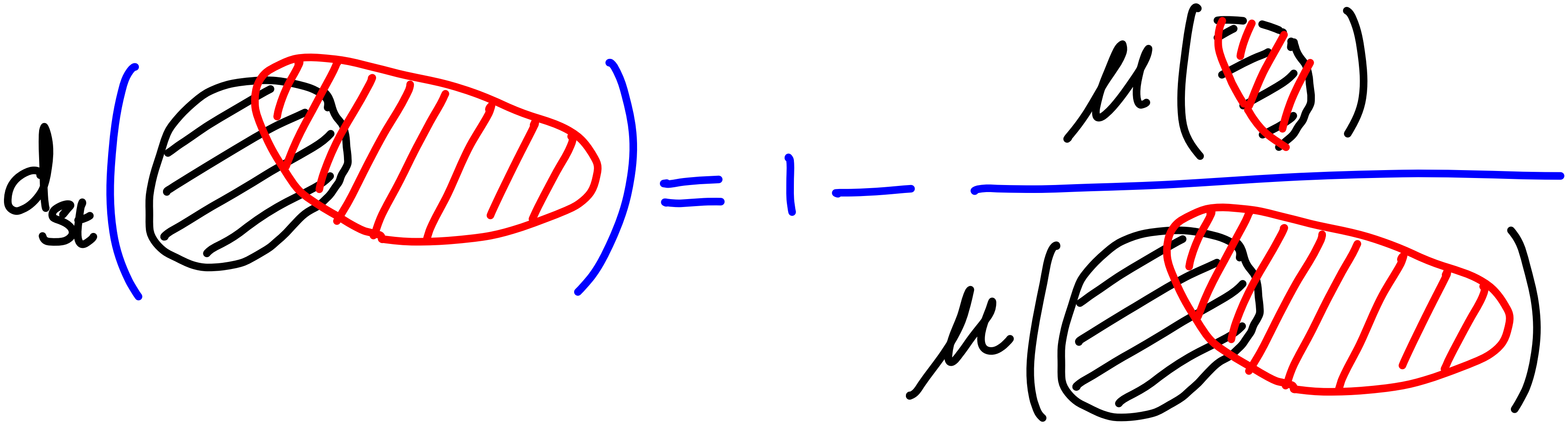


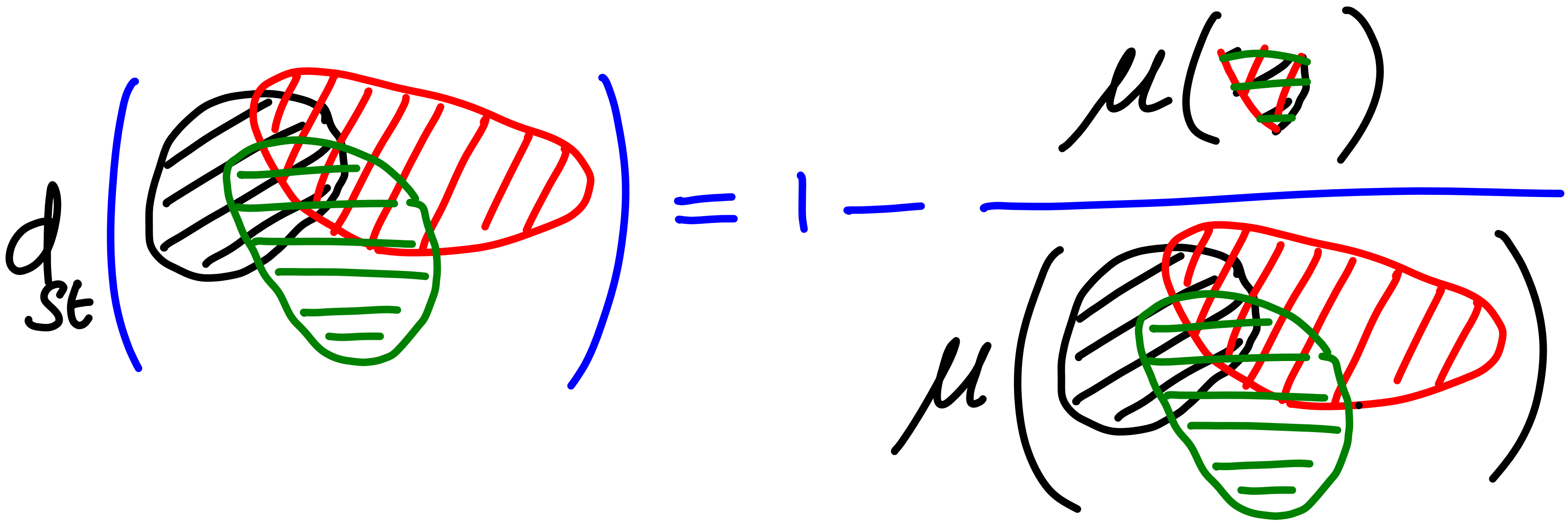
Generalized Jaccard distance:



STEINHAUS DISTANCE



Generalized Steinhaus distance: (μ: measure)



Steinhaus Nerve

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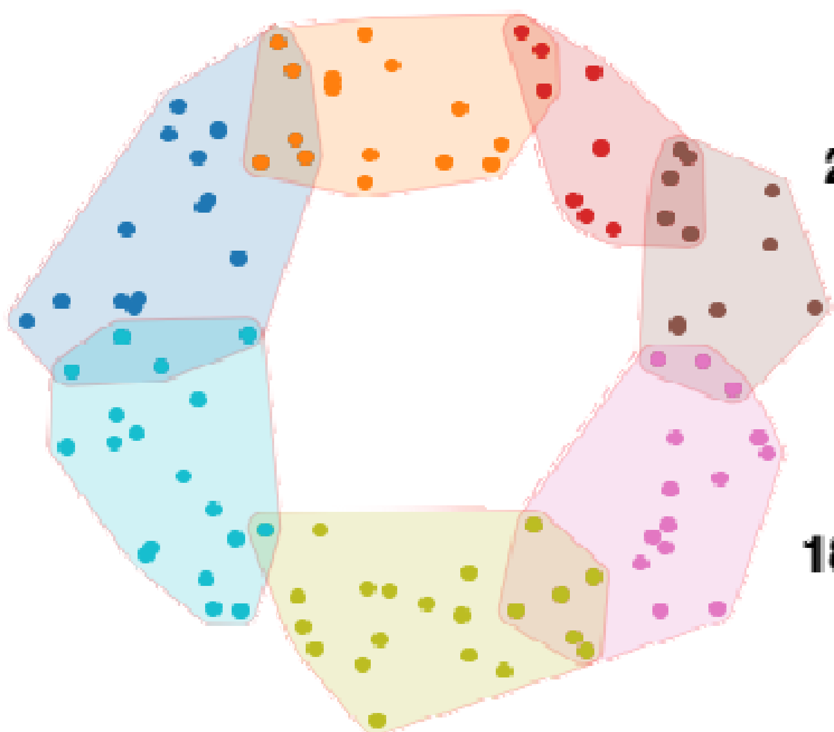
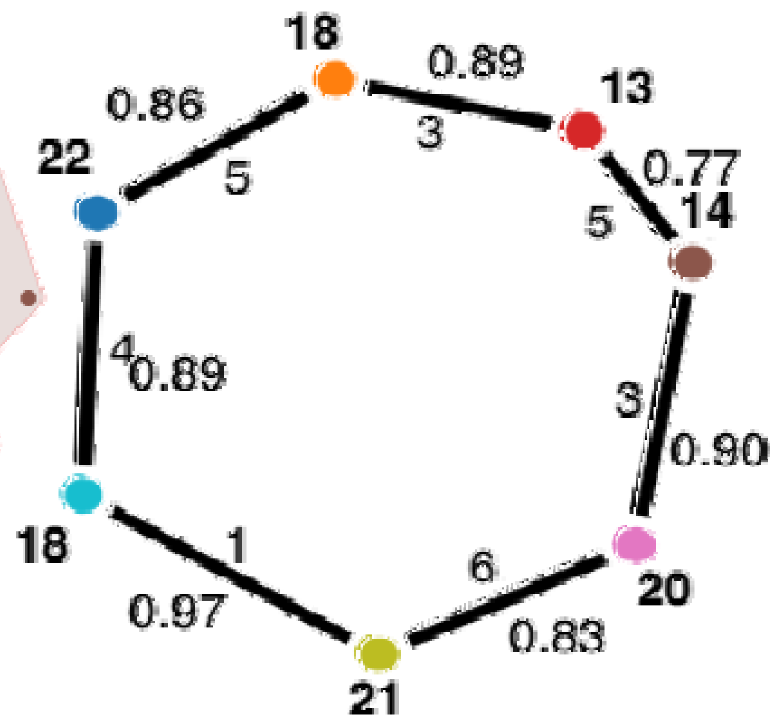
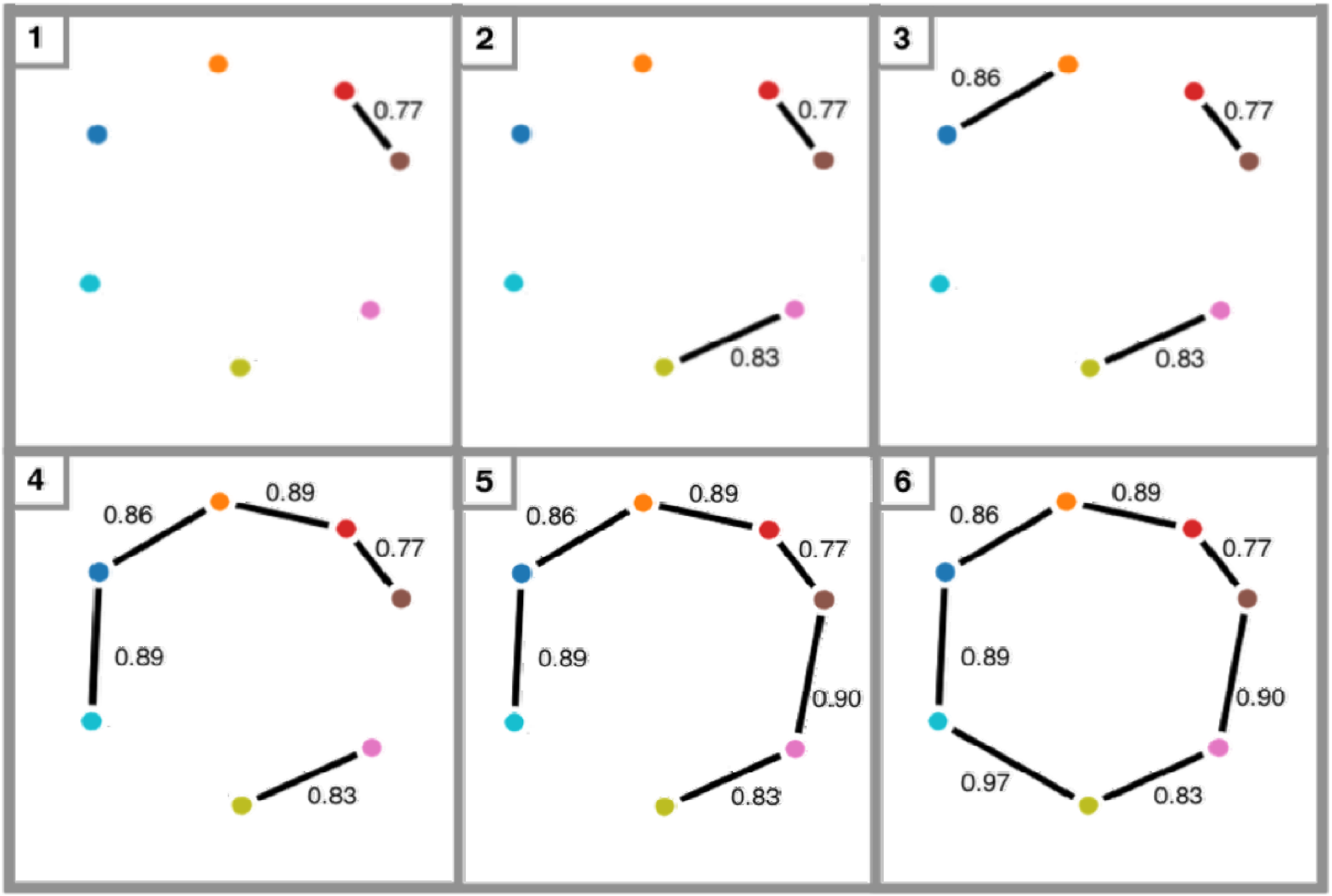
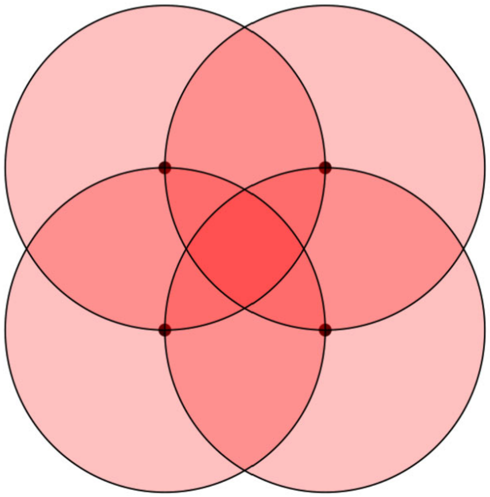




Steinhaus Filtration



STEINHAUS vs ČECH/VR



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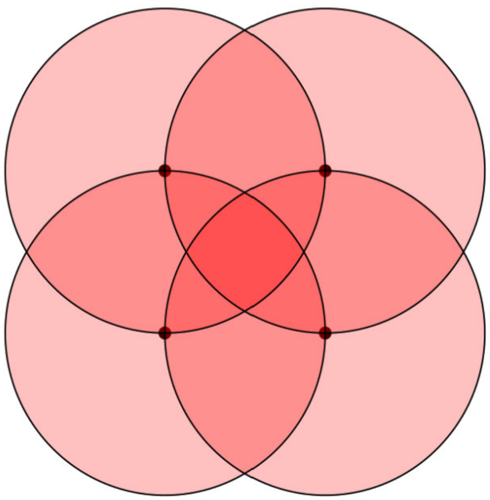

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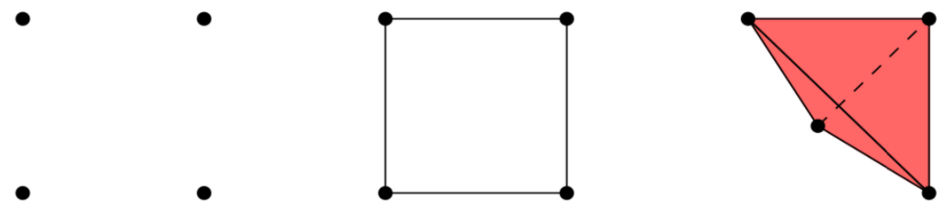

$r = 0$ $\qquad r = \frac{1}{2}$ $\qquad r = \frac{\sqrt{2}}{2}$

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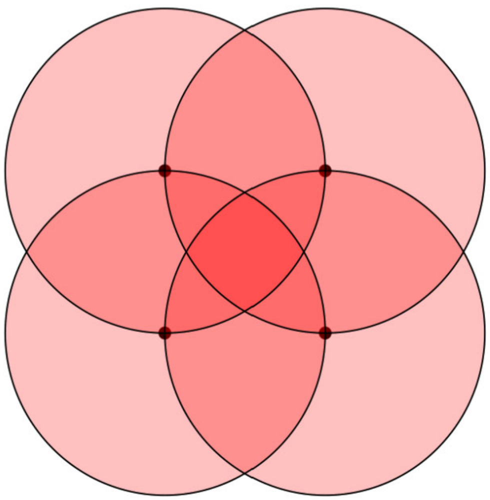

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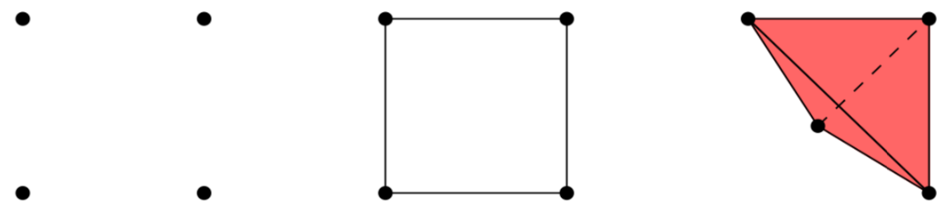

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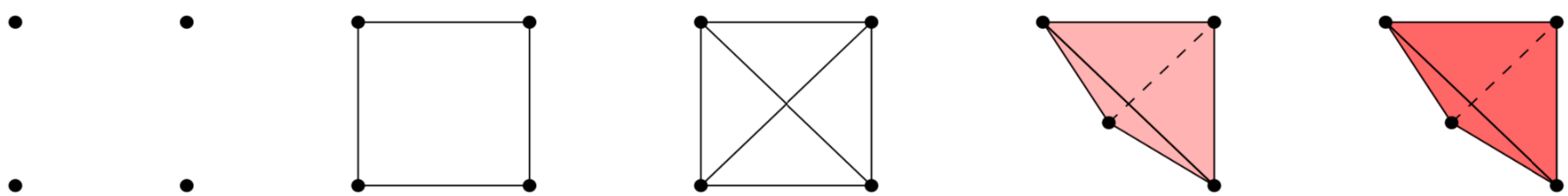
Steinhaus filtration w/ Lebesgue (vol.) measure:



$d_{St} = 0$ $\quad d_{St} \approx 0.756$ $\quad d_{St} \approx 0.9$ $\quad d_{St} \approx 0.935$ $\quad d_{St} \approx 0.959$

STEINHAUS vs ČECH/VR

Theorem 2. For finite $X \subset \mathbb{R}$ (in 1D), $\check{C}ech(X) \equiv$ Steinhaus filtration on X w/ {R-balls centered @ each pt in X} for $R >$ diam(X) w/ Lebesgue msr.

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Lemma 3. VR filtration determines the 1-skeleton of Steinhaus filtration in arbitrary dimensions.

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Theorem (Chazal et al., 2009). If $\mathbb{U}, \mathbb{V}$ are q-tame (i.e., finite rank) persistence modules that are ϵ-interleaved, then $d_B(\mathrm{dgm}(\mathbb{U}), \mathrm{dgm}(\mathbb{V})) \le \epsilon$

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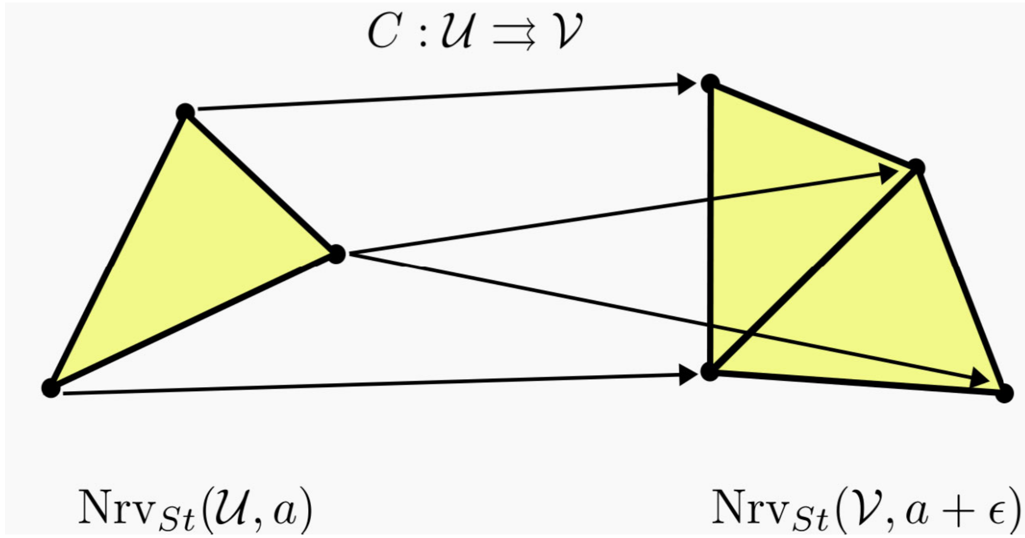
<u>Proposition</u> (Chazal, de Silva, Oudot, 2013).
If $C : X \rightrightarrows Y$ and $C^T : Y \rightrightarrows X$ are ϵ-simplicial multivalued maps, then they induce an ϵ-interleaving between $H_*(\mathbb{U})$ and $H_*(\mathbb{V})$.

Our Pseudometric



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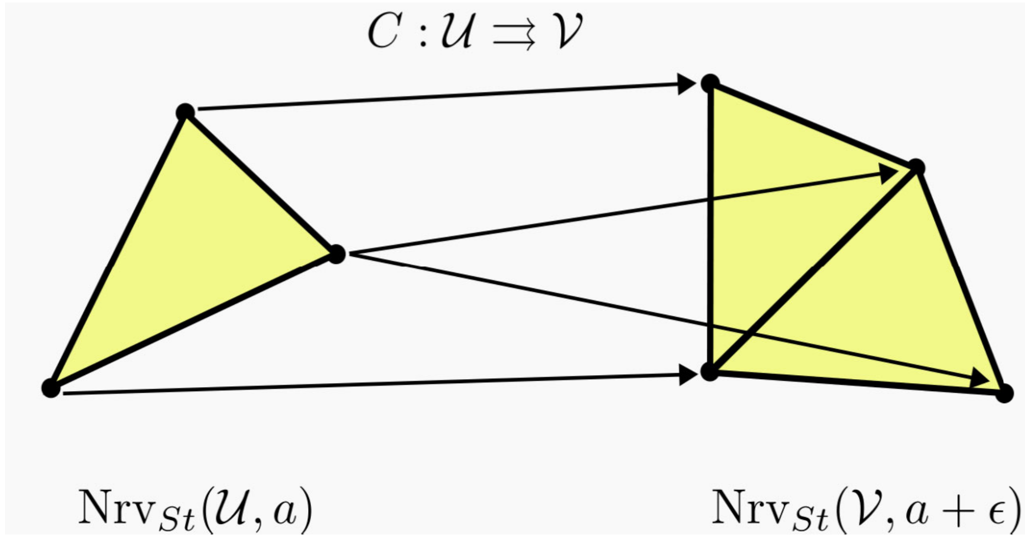

→ Find correspondence C w/ smallest distortion

$\mathrm{dis}(C) =$

$$\sup\{|d_{st}(\{U_i \mid i \in \sigma\}) - d_{st}(\{V_j \mid j \in \tau\})| : (\sigma, \tau) \in C\}$$

to define $d(\mathcal{U}, \mathcal{V})$

ϵ-INTERLEAVING

Proposition 4. Let $H^a(\mathcal{U}) := H_*(\mathrm{Nrv}_{St}(\mathcal{U}, a))$. We can find maps $\phi \in \mathrm{Hom}^{\epsilon}(H(\mathcal{U}), H(\mathcal{V}))$ & $\psi \in \mathrm{Hom}(H(\mathcal{V}), H(\mathcal{U}))$ s.t.

$$\begin{tikzcd}
H^a(\mathcal{U}) \arrow[rr] \arrow[dr, "\phi_a"] & & H^b(\mathcal{U}) \arrow[dr, "\phi_b"] & \\
& H^{a+\epsilon}(\mathcal{V}) \arrow[rr] & & H^{b+\epsilon}(\mathcal{V})
\end{tikzcd}
\qquad
\begin{tikzcd}
H^{a-\epsilon}(\mathcal{U}) \arrow[rr] \arrow[dr, "\phi_{a-\epsilon}"] & & H^{a+\epsilon}(\mathcal{U}) \\
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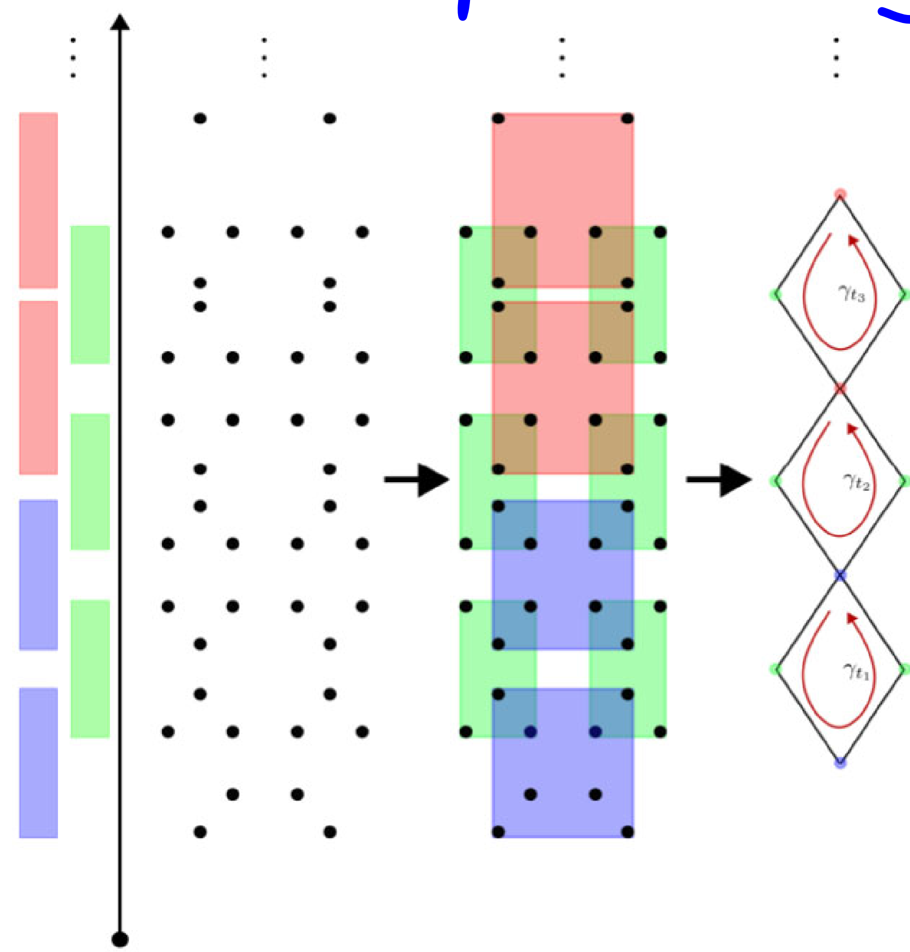
for all d_{St}'s $a \leq b$ when $\epsilon > d(\mathcal{U}, \mathcal{V})$.

NON q-TAME!

✓ non-q-tame Steinhaus complex in totally bounded space



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<u>Theorem 5.</u> If $H_*(\mathrm{Nrv}_{st}(\mathcal{U}))$ and $H_*(\mathrm{Nrv}_{st}(\mathcal{V}))$ are q-tame, then

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✓ q-tame: holds when $\mathcal{U}, \mathcal{V}$ finite

MOVIE RECOMMENDATIONS

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(images: Wikipedia)

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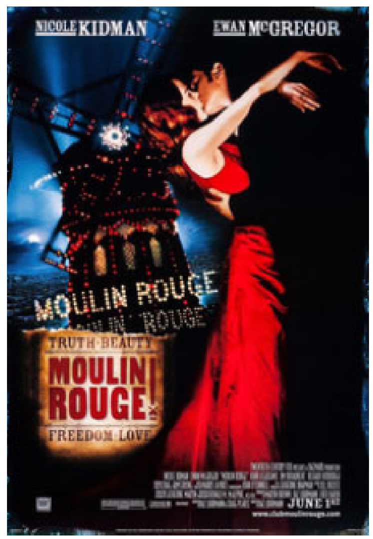

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→ Construct 1-skeleton of Steinhaus filtration w/ counts ($\mu(v) = |v|$)

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→ short v/s stable : pareto-frontier

Mulan → Moulin Rouge

(images: WWW)

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shortest path

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- → Rathore et al. (2021)

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