How to find the boundary of a Metaball?

Metaball function: $M(x,y) = \sum_n \left(\frac{{r_n}^2}{(x-x_n)^2+(y-y_n)^2}\right)$

Where n: Control Ball Count

(x_n, y_n): Center of n^{th} Control Ball

r_n: Radius of n^{th} Control Ball

Definition:

- Given a threshold t, all points (x,y) that $M(x,y) \geq t$ belong to Metaball.
- All points (x,y) that $M(x,y) = t$ produce the boundary of the Metaball.

Thinking:

1. Initialize a boundary B that can obviously cover all Metaball boundaries.
2. For each control points $C_n = C(c_n, c_n)$ on B, find $M(c_n, c_n)$ and $\nabla M(c_n, c_n)$, where $\nabla M(c_n, c_n)$ is the Gradient of $M(x,y)$ at point (x,y).
3. If $M(c_n, c_n) > t$, move control point along the $-\nabla M(c_n, c_n)$ direction
   If $M(c_n, c_n) < t$, move control point along the $\nabla M(c_n, c_n)$ direction
4. Repeat step 2,3 until all control point match $M(c_n, c_n) = t$

Gradient of Metaball function:

$$\nabla M(x,y) = \left(\frac{\partial}{\partial x} M(x,y), \frac{\partial}{\partial y} M(x,y)\right)$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\begin{aligned}
\frac{\partial}{\partial x} M(x,y) &= \frac{\partial}{\partial x}\left[\sum_n \left(\frac{{r_n}^2}{(x-x_n)^2+(y-y_n)^2}\right)\right] \\
&= \sum_n \left[\frac{\partial}{\partial x}\left(\frac{{r_n}^2}{(x-x_n)^2+(y-y_n)^2}\right)\right] \\
&= \sum_n \left[\frac{\left(\frac{\partial}{\partial x}{r_n}^2\right)\left((x-x_n)^2+(y-y_n)^2\right) - \left({r_n}^2\right)\left(\frac{\partial}{\partial x}\left((x-x_n)^2+(y-y_n)^2\right)\right)}{\left((x-x_n)^2+(y-y_n)^2\right)^2}\right] \\
&= \sum_n \left[\frac{-\left({r_n}^2\right)\left(\frac{\partial}{\partial x}(x-x_n)^2 + \frac{\partial}{\partial x}(y-y_n)^2\right)}{\left((x-x_n)^2+(y-y_n)^2\right)^2}\right] \\
&= \sum_n \left[\frac{-\left({r_n}^2\right)(2x+2x_n)}{\left((x-x_n)^2+(y-y_n)^2\right)^2}\right] \\
&= \sum_n \left[\frac{-2X_n{r_n}^2}{\left((X_n)^2+(Y_n)^2\right)^2}\right], where \begin{cases} X_n = x - x_n \\ Y_n = y - y_n \end{cases}
\end{aligned}$$

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\begin{aligned}
\frac{\partial}{\partial y} M(x,y) &= \frac{\partial}{\partial y}\left[\sum_n \left(\frac{{r_n}^2}{(x-x_n)^2+(y-y_n)^2}\right)\right] \\
&= \cdots \\
&= \sum_n \left[\frac{-2Y_n {r_n}^2}{((X_n)^2+(Y_n)^2)^2}\right], where \begin{cases} X_n = x - x_n \\ Y_n = y - y_n \end{cases}
\end{aligned}
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\begin{aligned}
\nabla M(x,y) &= \left(\frac{\partial}{\partial x} M(x,y), \frac{\partial}{\partial y} M(x,y)\right) \\
&= \left(\sum_n \left[\frac{-2X_n {r_n}^2}{((X_n)^2+(Y_n)^2)^2}\right], \sum_n \left[\frac{-2Y_n {r_n}^2}{((X_n)^2+(Y_n)^2)^2}\right]\right)
\end{aligned}
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where \begin{cases} X_n = x - x_n \\ Y_n = y - y_n \end{cases}
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