How to find the boundary of a Metaball?

Metaball strength function: $M(x,y) = \sum_n \left(\frac{{r_n}^2}{(x-x_n^c)^2+(y-y_n^c)^2} \right)$

Where n: Control Ball Count

(x_n^c, y_n^c): Center of n^{th} Control Ball

r_n^c: Radius of n^{th} Control Ball

Definition:

- Given a threshold t, all points (x,y) that $M(x,y) \geq t$ belong to Metaball.
- All points (x,y) that $M(x,y) = t$ produce the boundary of the Metaball.

Overall Algorithm:

1. For any one of Uncovered Control Ball C_n, repeat steps 2~4.
2. Find the first boundary point $B_1(x_1^b, y_1^b)$
   a. Set $B_1(x_1^b, y_1^b) = (x_n^c + r_n^c, y_n^c)$
   b. Calculate $M(x_1^b, y_1^b)$
   c. Move $B_1(x_1^b, y_1^b)$ according to ∇_1^b to make $M(x_1^b, y_1^b) = t$,
   where $\nabla_1^b = \nabla M(x_1^b, y_1^b)$ is the gradient of $M(x,y)$ at point B_1.
   *PS. Refer to **Algorithm A** for detail.*
3. Find next boundary point $B_2(x_2^b, y_2^b)$
   a. At the direction vertical to ∇_1^b, find the boundary point $B_2(x_2^b, y_2^b)$ that:
   $\|B_1B_2\| = E$, where $\|B_1B_2\|$ is the distance between these two points.
   b. Move $B_2(x_2^b, y_2^b)$ according to ∇_2^b to make $M(x_2^b, y_2^b) = t$,
   where $\nabla_2^b = \nabla M(x_2^b, y_2^b)$ is the gradient of $M(x,y)$ at point B_2.
   *PS. Refer to **Algorithm A** for detail.*
4. Repeat step 3 to find the next boundary points B_3, B_4, … B_m, until:
   $\|B_mB_1\| \leq E$
   then $\{B_1, B_2, B_3, \ldots, B_m, B_1\}$ is a close Metaball boundary
5. If there are any Control Balls not covered in this boundary, repeat steps 1~4

Algorithm A: of moving a point to find the best boundary point

1. For a point $P(x_p, y_p)$ on *B*, find $M(x_p, y_p)$ and $\nabla M(x_p, y_p)$, where $\nabla M(x_p, y_p)$ is the Gradient of $M(x, y)$ at point $P(x_p, y_p)$.
2. Set movement direction $\rho = \begin{cases} -\nabla M(x_p, y_p) \\ \nabla M(x_p, y_p) \end{cases}, \ when \begin{matrix} M(x_p, y_p) > t \\ M(x_p, y_p) < t \end{matrix}$
3. Calculate $P'(x_p', y_p')$ according to ρ, where $\overrightarrow{PP'} = \rho$ and $\|PP'\| = D$

   where D is the movement coefficient

   *PS. Refer to **Movement of Boundary Point** section for detail*
4. Recalculate $M(x_p', y_p')$ and $\nabla M(x_p', y_p')$, then set $P = P'$
5. If $[M(x_p, y_p) - t] \times [M(x_p', y_p') - t] < 0$, then set $D = D/2$
6. Repeat step 2~5 until $M(x_p, y_p) = t$

Gradient of Metaball function:

$$\nabla M(x,y) = \left(\frac{\partial}{\partial x}M(x,y), \frac{\partial}{\partial y}M(x,y)\right)$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\begin{aligned}
\frac{\partial}{\partial x}M(x,y) &= \frac{\partial}{\partial x}\left[\sum_n \left(\frac{{r_n}^2}{(x-x_n)^2 + (y-y_n)^2}\right)\right] \\
&= \sum_n \left[\frac{\partial}{\partial x}\left(\frac{{r_n}^2}{(x-x_n)^2 + (y-y_n)^2}\right)\right] \\
&= \sum_n \left[\frac{\left(\frac{\partial}{\partial x}{r_n}^2\right)\left((x-x_n)^2 + (y-y_n)^2\right) - ({r_n}^2)\left(\frac{\partial}{\partial x}\left((x-x_n)^2 + (y-y_n)^2\right)\right)}{\left((x-x_n)^2 + (y-y_n)^2\right)^2}\right] \\
&= \sum_n \left[\frac{-({r_n}^2)\left(\frac{\partial}{\partial x}(x-x_n)^2 + \frac{\partial}{\partial x}(y-y_n)^2\right)}{\left((x-x_n)^2 + (y-y_n)^2\right)^2}\right] \\
&= \sum_n \left[\frac{-({r_n}^2)(2x + 2x_n)}{\left((x-x_n)^2 + (y-y_n)^2\right)^2}\right] \\
&= \sum_n \left[\frac{-2X_n {r_n}^2}{\left((X_n)^2 + (Y_n)^2\right)^2}\right], where \begin{cases} X_n = x - x_n \\ Y_n = y - y_n \end{cases} \\
\frac{\partial}{\partial y}M(x,y) &= \frac{\partial}{\partial y}\left[\sum_n \left(\frac{{r_n}^2}{(x-x_n)^2 + (y-y_n)^2}\right)\right] \\
&= \cdots \\
&= \sum_n \left[\frac{-2Y_n {r_n}^2}{\left((X_n)^2 + (Y_n)^2\right)^2}\right], where \begin{cases} X_n = x - x_n \\ Y_n = y - y_n \end{cases}
\end{aligned}$$

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$$\begin{aligned}
\nabla M(x,y) &= \left(\frac{\partial}{\partial x}M(x,y), \frac{\partial}{\partial y}M(x,y)\right) \\
&= \left(\sum_n \left[\frac{-2X_n {r_n}^2}{\left((X_n)^2 + (Y_n)^2\right)^2}\right], \sum_n \left[\frac{-2Y_n {r_n}^2}{\left((X_n)^2 + (Y_n)^2\right)^2}\right]\right)
\end{aligned}$$

$$where \begin{cases} X_n = x - x_n \\ Y_n = y - y_n \end{cases}$$

Movement of Boundary Point:

In step 3 and 4: Move control point C_n to $C_n{}'$ along the direction,

where $\overline{\overline{C_n C_n{}'}} = \overline{\overline{(c_x^n, c_y^n)(c_x^{n\prime}, c_y^{n\prime})}} = \sqrt{(c_x^n - c_x^{n\prime})^2 + \left(c_y^n - c_y^{n\prime}\right)^2} = D$

Assume that moving direction is $\left(\nabla_x^c, \nabla_y^c\right)$, then $C_n' = \left(c_x^{n\prime}, c_y^{n\prime}\right) = \left(c_x^n + \mu\nabla_x^c, c_y^n + \mu\nabla_x^c\right)$

$\Rightarrow \sqrt{(c_x^n - c_x^{n\prime})^2 + \left(c_y^n - c_y^{n\prime}\right)^2} = \sqrt{(\mu\nabla_x^c)^2 + (\mu\nabla_x^c)^2} = D$

$\Rightarrow \mu = \sqrt{\dfrac{D^2}{(\nabla_x^c)^2 + (\nabla_x^c)^2}} = \dfrac{D}{\sqrt{(\nabla_x^c)^2 + (\nabla_x^c)^2}}$