Efficient Monte Carlo Technique for Locating Critical Slip Surface

By Venanzio R. Greco[1]

Abstract: The search for the critical slip surface in slope-stability analysis is performed by means of a minimization of the safety factor. The procedures most widely used are deterministic methods of nonlinear programming, and random search methods have been neglected, since they are considered to be generally less efficient. In this paper, an efficient Monte-Carlo method for locating the critical slip surface is presented. The procedure is articulated in a sequence of stages, where each new slip surface is randomly generated by an appropriate technique. From a comparative analysis, the proposed method provides solutions of the same quality as the best nonlinear programming methods. However, the structure of the presented method is very simple, and it can be more easily programmed, integrated, and modified for particular exigencies.

INTRODUCTION

More recently, nonlinear programming has been frequently used for locating the critical slip surface in a slope-stability analysis by means of a safety-factor minimization. Following this approach, Baker (1980) proposed the use of dynamic programming, while Celestino and Duncan (1981) and Li and White (1987) utilized the alternating variable method. Successively, Arai and Tagyo (1985) used the conjugate-gradient method; Nguyen (1985) and De Natale (1991) used the simplex method; Greco and Gullà (1985), and Basudhar et al. (1988) employed the sequential unconstrained minimization technique (SUMT). Other minimization methods have been utilized in comparative studies by Yamagami and Ueta (1988), Greco (1988), and Giam and Donald (1989).

Until the early 1960s, when the first deterministic methods of nonlinear programing began to emerge, the grid method or Monte Carlo methods were the only means of searching for the minimum of a function of several variables. The Monte Carlo methods are techniques of random search that are very simple in structure. In fact, they are based on a random generation of trial solutions that are compared with the current best solution to improve it. According to the way a trial slip surface is made, the Monte Carlo methods can be divided into two classes: random jumping methods and random walking methods.

Random jumping methods are based on the random generation of a large number of trial slip surfaces and assume the critical slip surface as that with the lowest safety factor (Siegel 1975; Boutrup and Lovell 1980; and Siegel et al. 1981). As these trial solutions are generated without taking the current best solution into consideration, the technique appears very crude and lacking in an effective search strategy. As a result, when the number of variables is not small, the possibility of effectively finding the minimum is only theoretical.

The random walking methods, on the contrary, generate random slip surfaces close to the current best solution by slightly modifying it. A sequence of improved approximations of the minimum is more easily obtained. A method of this type was presented by Cherubini and Greco (1987), but its results are clearly less reliable than those obtained by deterministic methods. As a result, the technique was not successful and was dropped.

The degree of efficiency of the two classes of methods is significantly different and depends on the particular search technique, but generally it is not high. In the past, when only random search methods and the grid method were available, several minimization problems remained unresolved, and this inadequacy discouraged the formulation of problems in terms of minimization of a function.

The advent of deterministic methods of nonlinear programming has given impetus to the use of optimization techniques. The power of these methods has also limited the use of random search ones, although random jumping methods are still used in problems of global minimization, where a deterministic minimization procedure starts from randomly generated, trial slip surfaces, to localize a local minimum (Chen 1992). This use, however, is not due to a particular merit of the random technique, but rather to the ineffectiveness of current methods of global minimization. Although the Monte Carlo methods have been neglected, they are useful for a large variety of problems. Indeed, they are even able to examine functions that are undifferentiable or affected by error, that have complicated shapes, shallow regions, and discontinuity points. In these cases, random search methods, if adequately structured, can be as effective as deterministic techniques, and for complicated functions they can be even more efficient. Random search methods, when efficient, are preferable to deterministic methods because they are easily explained and programmed. The relative computer programs are more easily understood and can be easily modified for particular exigencies. The main shortcoming of such methods, the larger number of slip surfaces to be examined, is becoming of secondary importance owing to the speed of modern computers. It is evident that if a random search method is capable of providing solutions of a quality comparable with deterministic methods of nonlinear programming, it can be accepted and used for solving slope-stability problems. Furthermore, no one method of nonlinear programming has yet been affirmed as the best, rather it is widely held that the behavior of the various minimization procedures varies significantly, according to the type of problem analyzed.

The present paper proposes a new Monte Carlo method, of the random walking type, for locating critical slip surfaces of general shape in a slope-stability analysis. It can be classified among direct search techniques. Only safety-factor computations, without derivatives, are required. The efficiency of this method is compared with some of the customary deterministic methods used to search for the critical slip surface.

SEARCH FOR CRITICAL SLIP SURFACE

This paper is concerned only with two-dimensional problems of slope stability. Assume a Cartesian system of reference, Oxy, and let $y = t(x)$ be a mathematical function that describes the topographic profile of the soil (Fig. 1). Analo-

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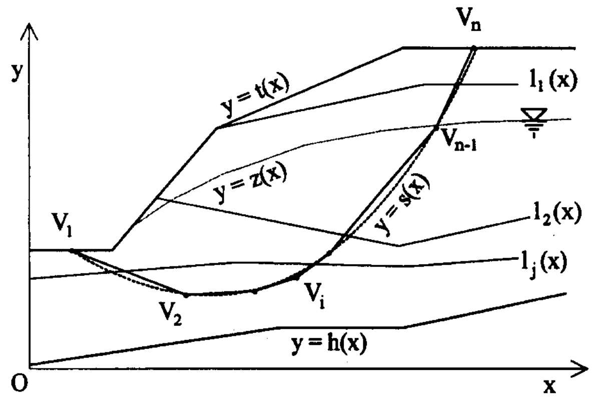

FIG. 1. Cross Section of Slope (Slip Surface Approximated by Broken Line with *n* Vertices)

gously, the slip surface is represented by another mathematical function, $y = s(x)$, and if a water table is present, it is represented by a further mathematical function, $y = z(x)$. Other functions, $l_j(x)$, can be introduced to describe discontinuity surfaces in layered soils.

For practical reasons, the region of the plane xy where the search is made has to be specified. A simple way is to assume that the abscissas of the slip surface are located within two boundaries

$$x_{\min} < x < x_{\max} \tag{1a}$$

The ordinates between a lower boundary, $h(x)$, and the topographic profile are represented by the function $t(x)$ where

$$h(x) \le s(x) \le t(x); \quad \forall\ x{:}x_{\min} \le x \le x_{\max} \tag{1b}$$

Position of Problem

A potential slip surface is approximated by a broken line with n vertices: $V_1, V_2, \ldots, V_n$, whose coordinates (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) are the unknowns of the problem for locating the critical slip surface. These coordinates can be considered as the components of a $2n$-dimensional array

$$\mathbf{S} = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}^T \tag{2}$$

Each distinct potential slip surface is represented by one point **S** of a $2n$-dimensional space. We are interested in locating, among all the potential slip surfaces, that surface with the minimum safety factor, called the critical slip surface. In this way, searching for the critical slip surface is mathematically expressed by the problem of minimizing the objective function, safety factor, F, with respect to array **S**.

$$\min F(\mathbf{S}) \tag{3}$$

For the slip surface to be geometrically feasible, some constraints have to be placed on the variables

$$x_i < x_{i+1} \text{ for } i = 1 \text{ to } n - 1 \tag{4a}$$

$$y_i = t(x_i) \text{ for } i = 1 \text{ and } i = n \tag{4b}$$

$$h(x_i) < y_i < t(x_i) \text{ for } i = 2 \text{ to } n - 1 \tag{4c}$$

Constraint (4a) guarantees that the vertices remain ordered, during the development of the automatic procedure leading to the minimum. Constraints (4b) and (4c) assure, respectively, that the external vertices of the slip surface stay on the topographic surface while the others stay below it. All the constraints in (4) are aimed at guaranteeing the geometrical feasibility of the slip surfaces tested during the search for the minimum.

Other constraints can be necessary to fix a shape type for the slip surface. For example, Basudhar et al. (1988) have considered further constraints to obtain a slip surface that is concave upward. In the opinion of the writer, such a shape is not always a priori justified. For homogeneous soils, the constraints in (4) are generally sufficient to give a critical slip surface of this shape without the necessity of introducing further conditions. In layered soils, the condition of concavity upward can lead to error, since, due to stratigraphic conditions, part of the slip surface could be convex.

Finally, the particular method used for stability analysis may necessitate further constraints on the variables to give inclinations of the slice bases that produce admissible solutions (Whitman and Bailey 1967; and Baker 1980). This question is examined in the following section.

Solution of Problem

Nonlinear programming procedures for searching for the minimum of a function of several variables start from a trial slip surface, $\mathbf{S}^0$, and proceed towards the minimum in an iterative way, generating a sequence of feasible slip surfaces, $\mathbf{S}^1, \mathbf{S}^2, \ldots, \mathbf{S}^k, \mathbf{S}^{k+1}, \ldots$, so that the sequence of the associated safety factors decreases

$$F(\mathbf{S}^0) > F(\mathbf{S}^1) > \cdots > F(\mathbf{S}^k) > F(\mathbf{S}^{k+1}) > \cdots \tag{5}$$

where

$$\mathbf{S}^k = \{x_1^k, y_1^k, x_2^k, y_2^k, \ldots, x_n^k, y_n^k\} \tag{6a}$$

$$\mathbf{S}^{k+1} = \{x_1^{k+1}, y_1^{k+1}, x_2^{k+1}, y_2^{k+1}, \ldots, x_n^{k+1}, y_n^{k+1}\} \tag{6b}$$

where (x_i^k, y_i^k) = coordinates of vertex i at stage k of the minimization procedure; and (x_i^{k+1}, y_i^{k+1}) = same vertex at the next stage.

The various procedures differ from one another essentially in the method of generating a new slip surface $\mathbf{S}^{k+1}$ (from $\mathbf{S}^k$) and in testing the optimality of $\mathbf{S}^{k+1}$, i.e., in stating if this slip surface can be reasonably assumed as the critical one. In the proposed method, each search stage is articulated in two phases: exploration and extrapolation.

During the exploration phase, each slip-surface vertex is shifted to a new position randomly obtained. If the safety factor of the modified slip surface is smaller than the previous one, then the vertex is fixed in this new position; otherwise, it returns to the previous position.

In the extrapolation phase, the total displacement obtained in the exploration is repeated, and the slip surface is updated if the corresponding safety factor is smaller than that obtained at the end of the exploration phase.

Exploration Phase

Each vertex of the current slip surface is randomly moved to obtain a reduction of the safety factor. Thus, vertex i is moved from point (x_i^k, y_i^k) to point (x_i^{k+1}, y_i^{k+1}), where

$$x_i^{k+1} = x_i^k + \xi_i^k \tag{7a}$$

$$y_i^{k+1} = t(x_i^{k+1}) \text{ for } i = 1 \text{ and } i = n$$

$$\text{and } y_i^{k+1} = y_i^k + \eta_i^k \text{ for } i = 2 \text{ to } n - 1 \tag{7b}$$

where ξ_i^k and η_i^k = random displacements of vertex i in directions x and y, respectively. These displacements are given by

$$\xi_i^k = N_x R_x Dx_i^k \tag{8a}$$

$$\eta_i^k = N_y R_y Dy_i^k \tag{8b}$$

where R_x and R_y = numbers randomly extracted from a population uniformly distributed in the range $[-0.5, +0.5]$; Dx_i^k and Dy_i^k = widths of the search steps in directions x and y for vertex i at stage k; and N_x and N_y = defined numbers, whose

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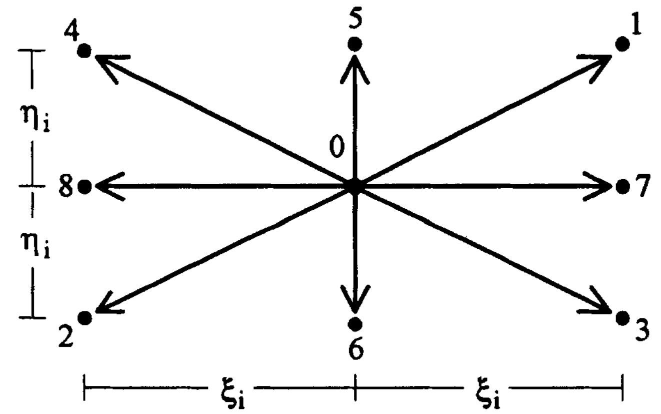

FIG. 2. Vertex *I* Moved from Position 0 to One of Positions 1–8 (Effective Positions of Points 1–8 Depend on Signs of Random Numbers R_x and R_y)

combination provides various displacements of vertex i for the same pair of random numbers R_x and R_y. The following eight combinations are given for parameters N_x and N_y:

$$N_x = 1 \quad N_y = 1 \tag{9a}$$

$$N_x = -1 \quad N_y = 1 \tag{9b}$$

$$N_x = 1 \quad N_y = -1 \tag{9c}$$

$$N_x = -1 \quad N_y = -1 \tag{9d}$$

$$N_x = 0 \quad N_y = 1 \tag{9e}$$

$$N_x = 0 \quad N_y = -1 \tag{9f}$$

$$N_x = 1 \quad N_y = 0 \tag{9g}$$

$$N_x = -1 \quad N_y = 0 \tag{9h}$$

In this way, eight random displacements are tried for every internal vertex of the slip surface. Fig. 2 gives a representation of the possible eight displacements. However, effective directions of displacements depend on the signs of R_x and R_y.

If one of these trials is successful, no further trials are made for this vertex, and the width of the search step is increased as follows:

$$Dx_i^{k+1} = Dx_i^k + |x_i^{k+1} - x_i^k| \tag{10a}$$

$$Dy_i^{k+1} + Dy_i^k + |y_i^{k+1} - y_i^k| \tag{10b}$$

If no trial is successful for vertex i, then the width of the search step for the successive step, $k + 1$, is reduced as follows:

$$Dx_i^{k+1} = Dx_i^k(1 - \varepsilon) \tag{11a}$$

$$Dy_i^{k+1} = Dy_i^k(1 - \varepsilon) \tag{11b}$$

where ε = a number between 0 and 1. The value of ε must be chosen opportunely, since a small value leads to long computational time while too high a value may lead to a premature arrest of the iterative procedure. On the basis of the acquired experience, a value of 0.5 is recommended.

Extrapolation Phase

When the exploration phase is terminated, the extrapolation phase begins. In this phase, the movements of the vertices of the slip surface that occurred during the exploration are repeated. A new slip surface is then generated with vertices given as

$$x_i^e = 2x_i^{k+1} - x_i^k \quad \text{for } i = 1 \text{ to } n \tag{12a}$$

$$y_i^e = 2y_i^{k+1} - y_i^k \quad \text{for } i = 2, \ldots, n - 1$$

$$\text{and } y_i^e = t(x_i^e) \quad \text{for } i = 1 \text{ or } i = n \tag{12b}$$

The slip surface is checked with respect to the boundaries in (4) on the coordinates, and they are modified if it is necessary. If the safety factor associated with the slip surface is less than the previous minimum value, then the best current slip surface is updated by the extrapolated slip surface as

$$x_i^{k+1} = x_i^e,\ y_i^{k+1} = y_i^e \quad \forall\ i{:}i = 1 \text{ to } n \tag{12c}$$

Otherwise, the new exploration phase begins starting with the slip surface obtained at the end of the previous one.

Criterion of Arrest

In numerical techniques of minimization, the minimum is searched by means of an iterative procedure that is repeated until a point, which can be assimilated to the point of minimum, is attained. These techniques are not able to find the true minimum, but only an approximation that should differ only slightly from it. Therefore, at the end of each search step, (including exploration and extrapolation) it is necessary to check if the current slip surface can be assumed as the critical one and if the procedure can be arrested.

In the proposed method, the iterative procedure is stopped, and the current point $\mathbf{S}^{k+1}$ is assumed as minimum when the following two criteria are simultaneously verified:

$$Dx_i^{k+1} < \Delta \text{ and } Dy_i^{k+1} < \Delta \quad \forall\ i{:}i = 1 \text{ to } n \tag{13a}$$

$$|F(\mathbf{S}^k) - F(\mathbf{S}^{k+1})| < \delta \tag{13b}$$

where Δ = lowest admissible width for the search range; and δ = tolerable difference between the values of safety factor in the subsequent iterations.

PRACTICAL QUESTIONS CONCERNING CODE IMPLEMENTATION

Writing a computer code, based on the technique exposed in the previous section, is very simple. However, the implementation of a minimization technique applied to a particular problem requires further assumptions typical of the problem type. These, although not always necessarily related to the minimization method used, aim at improving the effectiveness of the minimization procedure and take into account the physical and geometrical constraints. This section discusses some practical questions concerning the automatic search for the critical slip surface using the presented minimization method.

Although the use of unconstrained nonlinear programming methods is common in the search for the critical slip surface, this problem really requires constrained methods, such as the SUMT techniques or the cutting plane method. According to the rule, the use of unconstrained methods is possible only if the constraints do not have any influence on the search for the minimum. Generally, the constraints have little influence on the search when the vertices are well distanced, so that mutual interferences are avoided.

Li and White (1987) have indicated that the alternating variable method is very robust when the trial slip surface has a small number of vertices. In effect, this observation regards all the methods of searching for the critical slip surface. A large number of vertices, in fact, leads to interferences among them and difficulties in the descent towards the minimum. As a result, the search is slow, and a premature arrest of the iterative procedure is very probable.

On the other hand, it is evident that the degree of approximation of a slip surface increases with the number of vertices of the broken line. To overcome these opposing exigencies, Li and White proposed performing a first search with a few ver-

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tices; successively, new vertices are introduced at the midpoint of the straight line joining adjacent vertices, and a new search is started assuming the critical slip surface of the previous search as the trial slip surface of the next one. This procedure has the merit of allowing large displacements of the vertices in the initial search without the problem of interferences among vertices and resulting in an accurate solution owing to the high number of vertices. Following this suggestion, the writer has used trial slip surfaces with 4 vertices in the first search, 7 in the second, and 13 in the third. This seems to combine the speed and robustness of the search with few variables with the degree of accuracy obtainable with a large number of vertices. With more than 13 vertices, no practical improvement is generally attainable. For practical use, even slip surfaces with seven vertices are sufficiently good.

In the numerical applications, searches have been made starting from slip surfaces randomly generated with four vertices. The technique proposed by Boutrup and Lovell (1980) is able to randomly generate starting slip surfaces well. However, the following procedure has been used, where R_1, R_2, . . . , R_i are random numbers extracted from a population uniformly distributed in the range [0,1]:

1. The abscissas of vertices 1 and 4 are assumed or, alternatively, randomly generated in the range $[x_{\min}, x_{\max}]$

$$x_1 = x_{\min} + R_1(x_{\max} - x_{\min})/4 \tag{14a}$$

$$x_4 = x_{\max} - R_4(x_{\max} - x_{\min})/4 \tag{14b}$$

2. The ordinates of vertices 1 and 4 are given by

$$y_1 = t(x_1) \quad y_4 = t(x_4) \tag{14c}$$

3. The inclinations of the first and last segments of the slip surfaces are randomly generated

$$\alpha_1 = 40° - 15°R_5 \quad \alpha_3 = 45° + 15°R_6 \tag{14d}$$

4. Point P of coordinates (x_p, y_p) is determined as an intersection between the line passing through vertex 1 and inclined at α_1, and the line through vertex 4 and inclined at α_3.
5. Vertices 2 and 3 are then generated by

$$x_2 = x_1 + R_2(x_p - x_1) \quad y_2 = y_1 + (x_2 - x_1)\tan \alpha_1 \tag{14e}$$

$$x_3 = x_4 - R_3(x_4 - x_p) \quad y_3 = y_4 + (x_3 - x_4)\tan \alpha_3 \tag{14f}$$

In some cases, during the execution of the search procedure, two vertices of the slip surface may approach one another [see Fig. 3(a)]. In such an instance, the quality of the solution is inferior, because it seems like the two vertices are only one. Moreover, the introduction of an additional vertex between the two closest ones is ineffective for a better approximation of the slip surface. In applications at the end of a search, when two vertices are too close, one of them is taken from its position and introduced in the mean point of the longest segment of the slip surface. On the basis of experience, this modification is applied when the difference in abscissas between two successive vertices of the slip surface is less than $d_{\min}$, where

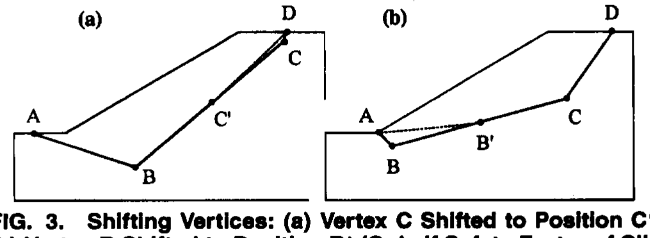

FIG. 3. Shifting Vertices: (a) Vertex C Shifted to Position C′; (b) Vertex B Shifted to Position B′ (Only If Safety Factor of Slip Surface AB′CD Is Not Higher than Slip Surface ABCD)

$$d_{\min} = (x_n - x_1)/4(n - 1) \tag{15a}$$

In some cases, such as that illustrated in Fig. 3(b), this correction may greatly affect the current solution, giving a new slip surface with a higher safety factor. Therefore, the modification is allowed only when it does not increase the safety factor higher than 0.05.

The values of the parameters for the criterion of arrest are selected differently for the various searches, so that the degree of accuracy increases from the initial search, with 4 vertices, to the final, with 13 vertices. The following values are advisable:

- The parameters Δ in (13a) is fixed in terms of the size of slide mass and the number of vertices by the following equation:

$$\Delta = (x_n - x_1)/200(n - 1) \tag{15b}$$

- The parameter δ in (13b) varies from 0.001, in the first search, to 0.00001, in the final one, being reduced by a factor 10 at each subsequent search.
- The initial widths of the search steps, Dx_i^o and Dy_i^o, are fixed as

$$Dx_i^o = (x_n - x_1)/2(n - 1) \tag{16a}$$

$$Dy_i^o = (y_{\max} - y_{\min})/n \tag{16b}$$

- where $y_{\max}$ and $y_{\min}$ = the maximum and the minimum value of the vertex ordinates of the slip surface.

When an impenetrable stratum is present, vertex displacements must be checked. The control technique used in Fig. 4 is where the current slip surface is ABCD. Point B should be moved to position B′, falling in the impenetrable stratum. Because this position is unfeasible, the vertex is moved vertically to position B″, a little above the top of the hard stratum. The new slip surface is then AB″ECD where point E is temporarily introduced to avoid constraint violation in (4c).

Methods that modify one variable in turn, such as the alternating variable and pattern-search methods, are better able to avoid interference among the vertices. However, in layered soils when a thin-inclined, weak layer is present, the vertices of the slip surface meet difficulty in moving into the layer. In Fig. 5(a), the displacements must be contained in the layer, and if it is thin, they must be appropriately short. Because the

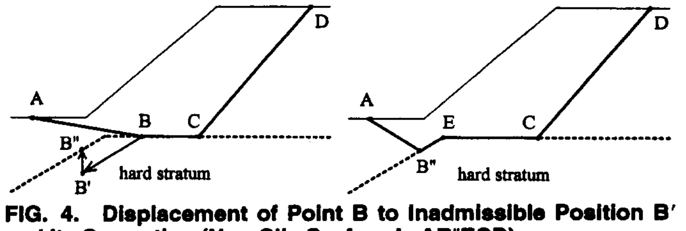

FIG. 4. Displacement of Point B to Inadmissible Position B′ and Its Correction (New Slip Surface Is AB″ECD)

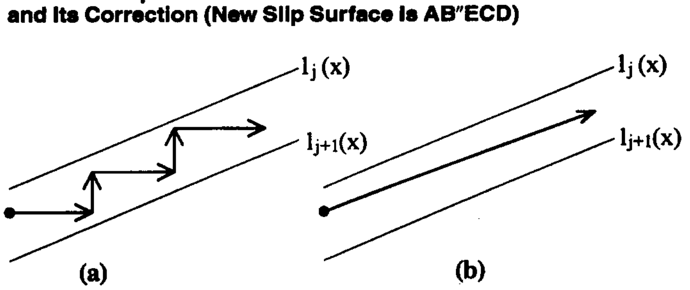

FIG. 5. Displacement of Vertex into Weak, Inclined Thin Layer: (a) Methods that Modify Coordinate in Turn; (b) Proposed Method

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differences in the safety factors associated with the closest slip surfaces are very small, the iterative-search procedure may have a premature arrest.

When all the vertex displacements in (7) lead the vertex out of the weak layer, and increase the safety factor, two further displacements are tried, moving vertex i into the layer by

$$x_i^{k+1} = x_i^k + \xi_i^k = x_i^k + N_x R_x D x_i^k \quad (17a)$$

$$y_i^{k+1} = 1/2[l_j(x_i^{k+1}) + l_{j+1}(x_i^{k+1})] \quad (17b)$$

where $N_x = 1$, in the first attempt, and $N_x = -1$, in the second; and $l_j(x)$ and $l_{j+1}(x)$ = mathematical functions describing the boundary surfaces of the layer where vertex i is placed. Contrary to the technique used by Li and White (1987), this assumption permits us to work with homogeneous variables: These are all vertex coordinates in the assumed Cartesian reference system.

Problems of convergence and admissibility of the solutions can affect the limit equilibrium methods in a slope-stablity analysis. For example, according to Whitman and Bailey (1967), results provided by Bishop's method are reliable only if for all the slices, dividing the slide mass, we have

$$m_{\alpha_i} = \cos \alpha_i + \sin \alpha_i \tan \phi_i/F \geq 0.2 \quad (18)$$

where ϕ_i = shear strength angle along the base of slice i; and α_i = inclination of the base of slice i, given by

$$\alpha_i = \arctan \left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right) \quad (19)$$

Constraint (18) also regards Janbu's method (1954). In the same way, according to Baker (1980), Spencer's method (1973) converges and yields reasonable results only when, for the various slices, coefficient p_{α_i} is

$$p_{\alpha_i} = \cos(\alpha_i - \theta) \left[1 + \frac{\tan \phi_i}{F} \tan(\alpha_i - \theta)\right] > 0.3 \div 0.4 \quad (20)$$

where θ = inclination of the interslice forces with respect to the horizontal.

Eqs. (18) to (20) introduce nonlinearities in the constraints on the variables that render the minimization to be more complicated. The simplest way to avoid such complications is to reject all the potential slip surfaces that do not verify (18) or (20). This technique has been tested and does not reduce the effectiveness of the minimization technique presented here.

However, Chowdhury and Zhang (1990, 1991) believe that no slip surface needs to be rejected for computational reasons, if adequate initial safety factor F_0 is used in the iterative procedure to compute the safety factor. The writer's experience of this proposed technique in problems where slip surfaces are automatically modified is not positive. Slip surfaces tend to assume shapes with peaks that render them clearly kinematically inadmissible. Thus, in applications of the present paper, the slip surfaces that do not verify (18) and (20) have been rejected.

NUMERICAL APPLICATIONS

This section exposes the results of the application of the proposed method to some slope-stability problems reported in the literature. For each problem, the search has been made by starting from 100 different randomly generated, trial slip surfaces using the presented method in comparison with the pattern-search method (Hooke and Jeeves 1961), the last being one of the most efficient methods for slope stability (Greco 1988; and Giam and Donald 1989). The most unfavorable trial slip surfaces are reported in the respective figures.

Example 1

Fig. 6 shows a slope used by Yamagami and Ueta (1988) in a comparative study on nonlinear programming methods applied to search for the critical slip surface. The soil is homogeneous, without pore pressures, with friction angle $\phi' = 10°$, cohesion c' = 9.8 kPa, and soil unit weight γ = 17.64 kN/m^3.

The authors used the method of Morgenstern and Price (1965) to calculate the safety factors associated with the various slip surfaces, considering the inclinations of the interslice forces as constant [corresponding to assumption $f(x) = 1$ in the original paper of Morgenstern and Price]. The minimization was made by employing two variable metric methods: the DFP method (Davidon 1959; and Fletcher and Powell 1963) and the BFGS method (Broyden 1970; Fletcher 1970; Goldfarb 1970; and Shanno 1970); and two methods of direct search (which do not require derivatives): the method of conjugate directions of Powell (1964) and the simplex method of Nelder and Mead (1965). Table 1 reports the values of the safety factors given by these minimization methods starting from three different slip surfaces.

In the present study, Monte-Carlo and pattern-search methods have been used to search for the critical slip surface, while safety factors have been calculated by Spencer's method (1973), as programmed by Baker (1980). Table 1 reports the range of values obtained by these methods in comparison with

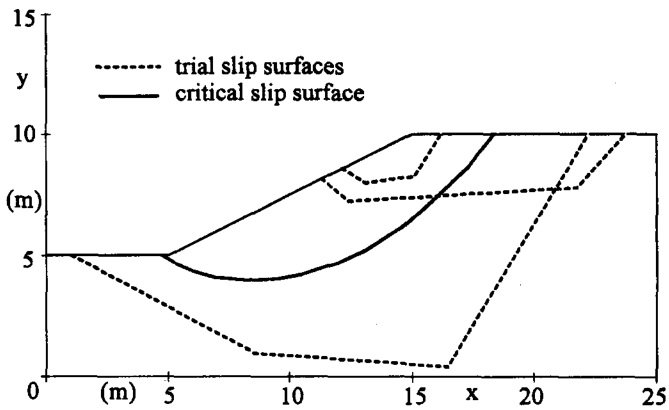

FIG. 6. Cross Section of Slope of Example 1

TABLE 1. Minimum Safety Factors Given by Minimization Procedures (Example 1)

Method (1)	Range of F (2)
(*a*) Yamagami and Ueta (1988)	
BFGS	1.338
DFD	1.338
Powell	1.338
Simplex	1.339–1.348
(*b*) Present study	
Pattern search	1.326–1.330
Monte Carlo	1.327–1.333

TABLE 2. Minimum Safety Factors versus Number of Vertices (Example 1)

Method (1)	Range of Values of Safety Factors for Slip Surfaces		
	4 vertices (2)	7 vertices (3)	13 vertices (4)
Pattern search	1.385–1.397	1.337–1.343	1.326–1.330
Monte Carlo	1.385–1.393	1.338–1.345	1.327–1.333

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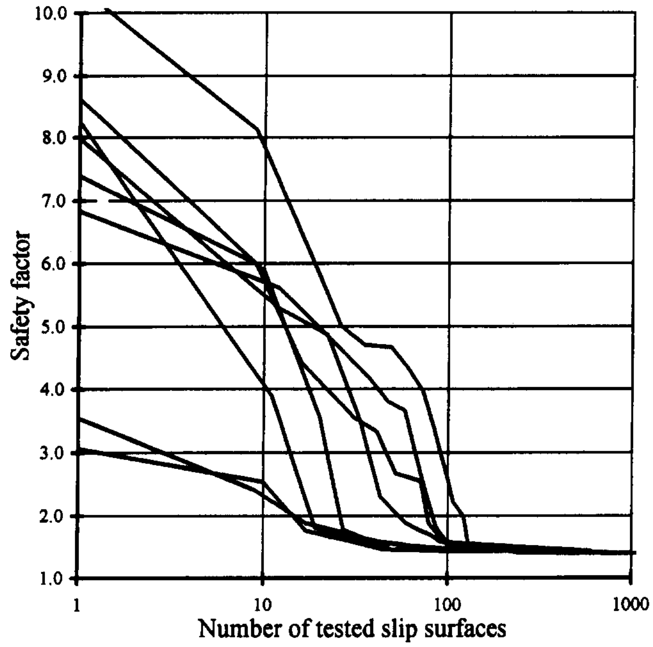

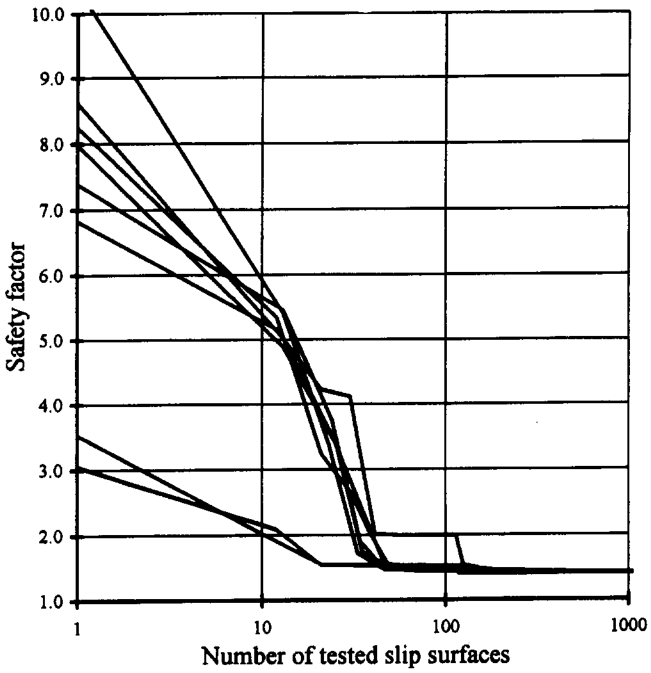

FIG. 7. Convergence Rates: (a) Monte Carlo Method for Example 1; (b) Convergence Rate of Pattern-Search Method for Example 1

those used by Yamagami and Ueta (1988), and Table 2 shows the range of values in terms of the number of vertices. In this case, the quality of the solution given by the proposed minimization method is similar to that of other nonlinear programming methods, although the initial slip surfaces used here are less favorable than those of Yamagami and Ueta (1988). The slightly higher accuracy is due to the larger number of vertices used in the present study. However it is not significant in practical terms.

Figs. 7(a) and 7(b) display variations in the safety factor values during the development of the two minimization procedures for the cases with the highest initial safety factor values. Apparently, no noticeable difference exists between the pattern-search and Monte Carlo methods in the search rate.

Example 2

In proposing the use of dynamic programming to search for critical slip surfaces, Baker (1980) showed a number of applications by computing safety factors by Spencer's method. Fig. 8 shows one of the slopes used by the author, taken here to examine the effectiveness of the Monte Carlo method. The soil is homogeneous with $\phi' = 20°$, $c' = 2.9$ t/m^2 and $\gamma = 1.92$ t/m^3.

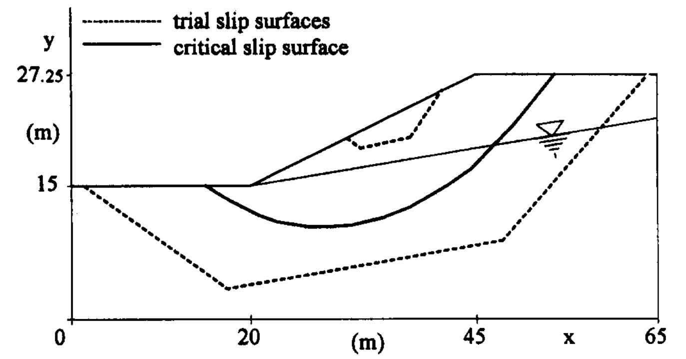

FIG. 8. Cross Section of Slope of Example 2

TABLE 3. Minimum Safety Factors for Example 2

Method (1)	Number of Vertices: 4 vertices (2)	7 vertices (3)	13 vertices (4)
Pattern search	1.828–1.830	1.758–1.760	1.744–1.745
Monte Carlo	1.828–1.833	1.758–1.766	1.744–1.751
Dynamic programming	1.77		

Table 3 shows the safety factor given by dynamic programming in comparison with those obtained in the present study by pattern-search and Monte Carlo methods, in terms of the number of vertices of the slip surface. In cases similar to this, broken lines with seven vertices are sufficient to approximate slip surfaces well while no difference exists in the values of the safety factors given by the minimization methods.

Figs. 9(a) and 9(b) show reduction in the safety factor obtained by Monte Carlo and pattern-search methods in terms of the number of tested slip surfaces in cases where the safety factors of the initial slip surfaces are among the largest. A comparison of these figures leads us to believe that the convergence rates of these two minimization methods are similar.

Example 3

This example is taken from Arai and Tagyo (1985) and concerns a layered slope where a layer of low resistance is interposed between two layers of higher strength. Geometrical features of slope and values of shear-strength parameters of various layers are reported in Fig. 10 and Table 4, respectively. Using the simplified method of Janbu (1954) to calculate the safety factor and the conjugate gradient method (Fletcher and Reeves 1964), the authors obtained a safety factor of 0.405 for the critical slip surface (Fig. 10) by a dotted line. The same problem has also been examined by Sridevi and Deep (1992), using the random-search technique RST-2 (Shanker and Mohan 1987). The critical slip surface, obtained by the last method, drawn in Fig. 10 with a dashed line, is slightly different from that of Arai and Tagyo, and the associated safety factor is a little smaller.

The critical slip surface obtained, using the Monte Carlo method (drawn by a bold line) is clearly different from those given by RST-2 and conjugate gradient methods (see Table 5). The condition of a concave, upward shaped slip surface would lead to error. Since the values of safety factors are small, their differences are also reduced.

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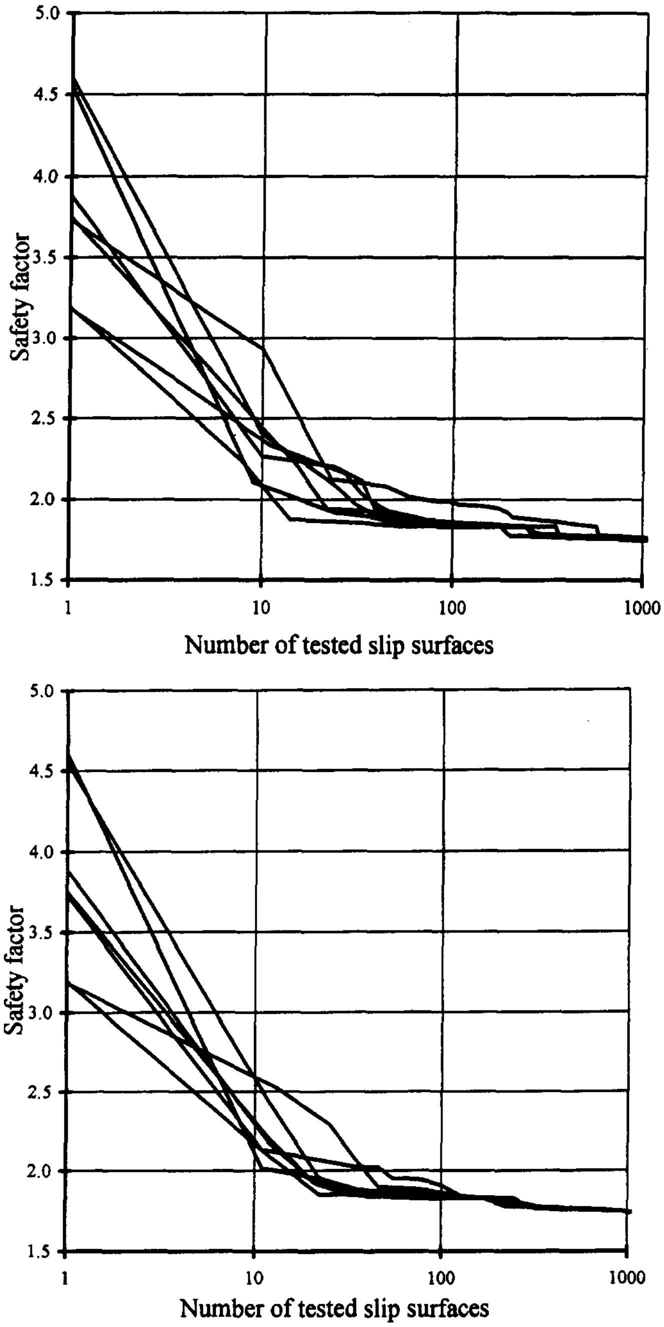

FIG. 9. Convergence Rates: (a) Monte Carlo Method for Example 2; (b) Pattern-Search Method for Example 2

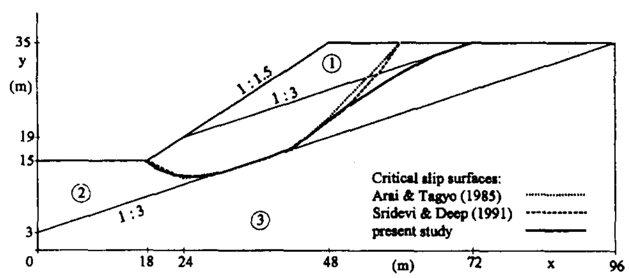

FIG. 10. Cross Section of Slope in Example 3

Example 4

This example is also referred to in Yamagami and Ueta's paper (1988), where a slope in layered soil is analyzed using the previous method of minimization. The safety factors were calculated by Morgenstern and Price assuming $f(x) = 1$. Fig. 11 shows the geometrical features of the analyzed slope, while Table 6 shows the geotechnical properties in layers 1 to 4.

Table 7*a*,*b* reports the safety factors associated with the critical slip surfaces obtained by the various minimization procedures, starting from the two trial slip surfaces drawn with dashed lines in Fig. 11. The solutions are seriously affected by the trial slip surfaces. This is not surprising, since nonlinear programming methods are only able to find a local minimum. However, it is more probable that a premature arrest has stopped the iterative procedure. In fact, BFGS and DFP methods require the calculation of the derivatives of the safety factors, which have been approximated by finite differences. The precision of these methods is seriously affected by truncation errors of the iterative procedure that calculates F and errors connected with the number of slices dividing the slip mass. Therefore the derivative calculation is unreliable, and the procedure is not able to find the minimum. The ineffectiveness of the simplex method (which does not require derivatives) is, on the contrary, typical of this technique in problems when the number of variables is not small.

In the present study, a stability analysis was performed using Spencer's method. Table 7*c* indicates a slightly higher effectiveness of Monte Carlo and pattern-search methods and

TABLE 4. Geotechnical Parameters for Example 3

Layer (1)	ϕ' (degrees) (2)	c' (kPa) (3)	γ (kN/m^3) (4)
1	12	29.4	18.82
2	5	9.8	18.82
3	40	294.0	18.82

TABLE 5. Safety Factors for Example 3

Method (1)	F (2)
Conjugate gradient	0.405
RST-2	0.401
(*a*) Present study	
Pattern search	0.388
Monte Carlo	0.388

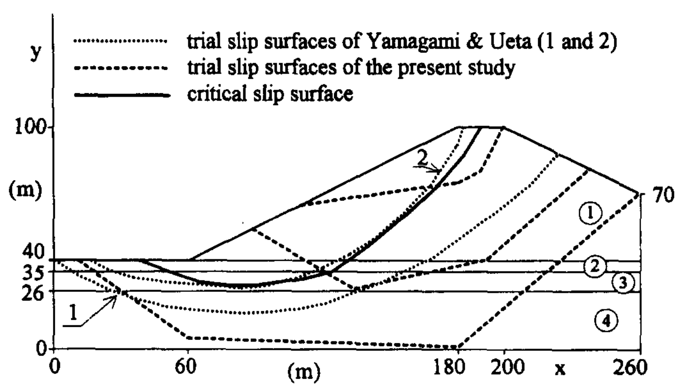

FIG. 11. Cross Section of Slope in Example 4

TABLE 6. Geotechnical Properties of Layers in Example 4

Layer (1)	c' (kPa) (2)	ϕ' (degrees) (3)	γ (kN/m^3) (4)
1	49.00	29	20.38
2	0.00	30	17.64
3	7.84	20	20.38
4	0.00	30	17.64

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TABLE 7. Minimum Safety Factors Given by Minimization Procedures (Example 4)

Method (1)	Range of F (2)
(*a*) Yamagami and Ueta (1988)—Slip Surface 1	
BFGS	1.626
DFP	1.942
Powell	1.481
Simplex	1.809
(*b*) Yamagami and Ueta (1988)—Slip Surface 2	
BFGS	1.423
DFP	1.453
Powell	1.402
Simplex	1.405
(*c*) Present study—average of 100 cases	
Pattern search	1.400
Monte Carlo	1.401

TABLE 8. Minimum Safety Factors versus Number of Vertices (Example 4)

Method (1)	Range of Values of Safety Factors for Slip Surfaces		
	4 vertices (2)	7 vertices (3)	13 vertices (4)
Pattern search	1.438–1.775	1.406–1.421	1.400–1.406
Monte Carlo	1.437–1.657	1.407–1.431	1.400–1.413

their greater independence of the trial slip surfaces. From Table 8, which shows the ranges of values of F in terms of the number of vertices, broken lines with seven vertices are sufficient to approximate slip surfaces for practical applications.

Example 5

Fig. 12 depicts a layered slope, analyzed by Chen and Shao (1988). The geotechnical properties of its layers are listed in Table 9. The authors searched for the critical slip surface using the simplex method, the steepest descent method, the DFP method, and an appropriately modified version of the last. The safety factors were calculated by the generalized method of slices (Chen and Morgenstern 1983) with two different assumptions about the inclination of the interslice forces. The values obtained with one of these assumptions that correspond to Spencer's method fall in the range 1.01–1.03, with little difference between the original and the modified version of the DFP method.

Table 10 shows the safety factors obtained by the Monte Carlo and pattern-search methods in terms of the number of vertices, in comparison with those obtained by Chen and Shao. In this case, pattern-search and Monte-Carlo methods give solutions slightly smaller than other methods, although the trial slip surfaces, being randomly generated, in some cases were unfavorable.

In the present problem, a local minimum is also present with $F = 1.08$. The presence of more minima is frequent in layered soils; therefore, slope-stability analysis in these cases must be carried out with care and caution, since there are still no reliable techniques for localizing global minima. Although, sometimes it is sufficient to make a number of searches with initial slip surfaces located in suspected regions, the probability of finding the global minimum is also dependent on the experience and judgment of the engineer. In the present case, starting from 100 randomly generated, trial slip surfaces the pattern-search method has localized the global minimum in 16% of the cases, and the Monte Carlo method in 57%.

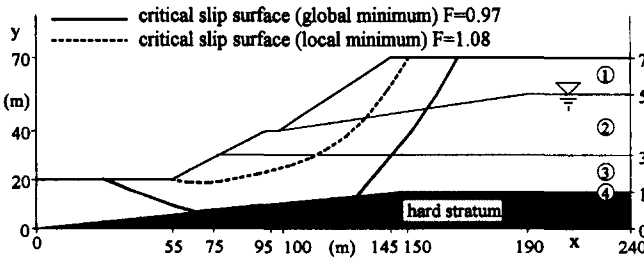

FIG. 12. Cross Section of Slope in Example 5

TABLE 9. Geotechnical Parameters for Example 5

Layer (1)	ϕ' (degrees) (2)	c' (kPa) (3)	γ (g/cm³) (4)
1	35°	9.8	2.00
2	25°	58.8	1.90
3	30°	19.8	2.15
4	16°	9.8	2.15

TABLE 10. Minimum Safety Factors for Example 5

Method (1)	No. of vertices (2)	Range of F (3)
(*a*) Chen and Shao (1988)		
Original DFP	—	1.011–1.035
Modified DFP	—	1.009–1.025
Simplex	—	1.025
Steepest descent	—	1.025
(*b*) Present study		
Pattern search[a]	4	1.127–1.386
Pattern search[a]	7	0.973–1.286
Pattern search[a]	13	0.973–1.033
Monte Carlo[b]	4	0.979–1.098
Monte Carlo[b]	7	0.975–0.976
Monte Carlo[b]	13	0.973–0.974

[a]Global minimum obtained in 16% of cases.
[b]Global minimum obtained in 57% of cases.

CONCLUSIONS

The present paper has proposed a Monte Carlo technique for locating the critical slip surface in slope-stability analysis by means of an iterative procedure, based on the generation of random numbers. The method does not require derivatives of the safety factor and is classifiable among direct search methods. In contrast to other methods of random search, the proposed Monte Carlo method is clearly efficient in the search for the critical slip surface in problems of slope-stability analysis. The quality of the solutions of the present method is certainly comparable with those of the nonlinear programming methods. Moreover, with respect to many nonlinear programming methods, the proposed technique is simple in structure and easily programmable. As a result, the relative computer program is intelligible and integrable, offering the possibility of being incorporated into more general procedures for slope-stability analyses and studies aimed at slope stabilization and back analysis.

In reference to the convergence rate, no appreciable difference has emerged in the comparison between pattern-search and Monte Carlo methods. Taking into account that some studies (Greco 1988; and Giam and Donald 1989) suggest the convergence rate of the pattern-search method as one of the highest, the rate of the Monte Carlo methods seems good.

The presented method also seems to be sufficiently robust for layered soils with weak, thin-inclined layers, which is a

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crucial problem in the techniques of slip-surface search. In this case, it seems more efficient than the pattern-search method.

In spite of its robustness and effectiveness, the theoretical background of the proposed method is very poor, since the minimization procedure is not based on a rational technique of descent towards the minimum, but simply articulated in a sequence of random displacements of the vertices of the broken line approximating the slip surface. Therefore, it can only be considered as an "ad hoc method" for slope-stability analysis.

The proposed technique does not ensure that the minimum found is the global one, but provides a local minimum only. The probability of finding a global minimum increases by restarting the iterative procedure from different trial slip surfaces and checking that the critical ones obtained are very similar. This is especially advisable for layered soils.

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