$= \alpha(0) + \beta(0)$

$= 0.$

Linear combination

Let V be a vector space over F. and $v_1, v_2, v_3 \dots v_n \in V$. Any vector v of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, $\alpha_1, \alpha_2 \dots \alpha_n \in F$ is the linear combination of a vector $v_1, v_2 \dots v_n$

1. Let $V = R^2$, $S = \{(1,2)(2,1)\}$, $v = (2,2) \in$ and $\alpha, \beta \in F$. Check wether v is the linear combination of v_1 and v_2.

Solution

Let $v = \alpha v_1 + \beta v_2$

$(2,2) = \alpha(1,2) + \beta(2,1)$

$(2,2) = (\alpha, 2\alpha) + (2\beta, \beta)$

$(2,2) = (\alpha + 2\beta, 2\alpha + \beta)$ —(1)

Equaling

$2 = \alpha + 2\beta$ —(2)

$2 = 2\alpha + \beta$ —(3)

$(2) \times 2 \Rightarrow 4 = 2\alpha + 4$

$(3) \Rightarrow 2 = 2\alpha + \beta$

$2 = 3\beta$

$\beta = \frac{2}{3}$ —(3)

Substitute (3) in (2)

$2 = \alpha + 2\left(\frac{2}{3}\right)$

$6 = 3\alpha + 4$

$2 = 3\alpha$

$\alpha = \frac{2}{3}$ —— (4)

(1) => $(2) = \alpha + 2\beta$

$2 = \frac{2}{3} + 2\left(\frac{2}{3}\right)$

$2 = \frac{2}{3} + \frac{4}{3}$

$2 = \frac{6}{3}$

$2 = 2$

Hence 2 is the vector space.

2. Let $V = R^3$, $S = \{(1,2,0),(0,-5,-7)\}$ $v = (2,-5,7) \in V$ verify whether v is linear combination are not.

Solution

Let $v = \alpha v_1 + \beta v_2$

$(2,-5,7) = \alpha(1,2,0) + \beta(0,-5,-7)$

$(2,-5,7) = (\alpha, 2\alpha, 0) + (0,-5\beta,-7\beta)$

$(2,-5,7) = (\alpha+0, 2\alpha-5\beta, 0-7\beta)$

$(2,-5,7) = (\alpha, 2\alpha-5\beta, -7\beta)$

Equating

$\boxed{2 = \alpha}$

$(2,-5,7)$ $(2,9,-1)$

$+7 = -7\beta$

$\boxed{\beta = -1}$

$2(2)-5(-1) = 4+5 = 9$

$-5 \neq 9$

Hence 9 is not a linear combination

3 Determine wether ... expressed as a linear ... $(2,-1,1)$ in R.
of $u(1,-3,2)$ and $v,(2,$

Solution

Let $v : \alpha_1 u + \beta_2 v$

$(1,7,-4) = \alpha(1,-3,2) + \beta(2,-1,1)$

$(1,7,-4) : (\alpha,-3\alpha,2\alpha) + (2\beta,-\beta,+\beta)$

$(1,7,-4) : (\alpha+2\beta, -3\alpha-\beta, 2\alpha+\beta)$

Equating

$\alpha+2\beta : 1$ — (1)

$-3\alpha-\beta : 7$ — (2)

$2\alpha+\beta : -4$ — (3)

$\alpha : -3$

$\beta : 2$

$(1) \Rightarrow -3+2(2) : 1$

$-3+4 : 1$

$1 : 1$

Hence 1 is the linear combination

---

4 Write a vector $v=(1,-2,5)$ as the linear combination of vectors $e_1 : (1,1,1)$, $e_2 : (1,2,3)$, $e_3 = (2,-1,1)$ in R^3

Let $v : \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$(1,-2,5) : \alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(2,-1,1)$

$: (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$

$(1,-2,5) : (\alpha_1+\alpha_2+2\alpha_3) + (\alpha_1+2\alpha_2+3\alpha_2$
$+ (\alpha_1+3\alpha_2+\alpha_3)$ — (1)

3 Determine wether R^3 $(1,7,-4)$ is expressed as a linear combination of $u:(1,-3,2)$ and $v:(2,-1,1)$ in R^3.

Solution

Let $v : \alpha_1 u + \beta_2 v$

$(1,7,-4) = \alpha(1,-3,2) + \beta(2,-1,1)$

$(1,7,-4) : (\alpha, -3\alpha, 2\alpha) + (2\beta, -\beta, +\beta)$

$(1,7,-4) : (\alpha + 2\beta, -3\alpha - \beta, 2\alpha + \beta)$

Equating

$\alpha + 2\beta : 1$ — (1)

$-3\alpha - \beta : 7$ — (2)

$2\alpha + \beta : -4$ — (3)

$\alpha : -3$

$\beta : 2$

$(1) \Rightarrow -3 + 2(2) : 1$

$-3 + 4 : 1$

$1 : 1$

Hence 1 is the linear combination

---

4 Write a vector $v = (1,-2,5)$ as the linear combination of vectors $e_1 : (1,1,1)$, $e_2 : (1,2,3)$, $e_3 = (2,-1,1)$ in R

Let $v : \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$(1,-2,5) : \alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(2,-1,1)$

$: (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$

$(1,-2,5) : (\alpha_1 + \alpha_2 + 2\alpha_3) + (\alpha_1 + 2\alpha_2 + 3\alpha$

$+ (\alpha_1 + 3\alpha_2 + \alpha_3)$ — (1)

Equating

$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = -2$

$\alpha_1 + 3\alpha_2 + \alpha_3 = 5$

Solving these equation, we get

$\alpha_1 = -6$

$\alpha_2 = 3$

$\alpha_3 = 2$

$(2) \Rightarrow -6 + 3 + 2(2) = 1$

$-6 + 3 + 4 = 1$

$-6 + 7 = 1$

$1 = 1$

Hence v is the linear combination.

---

5. Consider the vectors (1, 2, 3) and v = (2, 3, 1) in R^3. For what values of k is the vector w = (1, k, 4) is the linear combination of u and v.

Solution

$v = \alpha_1 u + \alpha_2 v$

$(1, k, 4) = \alpha_1 (1, 2, 3) + \alpha_2 (2, 3, 1)$

$(1, k, 4) = (\alpha_1, 2\alpha_1, 3\alpha_1) + (2\alpha_2, 3\alpha_2, \alpha_2)$

$(1, k, 4) = (\alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2, 3\alpha_1 + \alpha_2)$ — (1)

$\alpha_1 + 2\alpha_2 = 1$ — (2)

$2\alpha_1 + 3\alpha_2 = k$ — (3)

$3\alpha_1 + \alpha_2 = 4$ — (4)

$\alpha_1 = \frac{7}{5}$

$\alpha_2 = -\frac{1}{5}$

in eqn (3)

$(3) \Rightarrow 2\left(\frac{7}{5}\right) + 3\left(-\frac{1}{5}\right) = k$

$k = \frac{11}{5}$

6. Consider the vector $u = (1, 2, -1)$ and $v = (6, 4, 2)$ in R^3. Show that $w = (9, 2, 7)$ is the linear combination of u and v.

$V = \alpha_1 u + \alpha_2 V$

$(9, 2, 7) = \alpha_1 (1, 2, -1) + \alpha_2 (6, 4, 2)$

$(9, 2, 7) = (\alpha_1, 2\alpha_1, -\alpha_1) + (6\alpha_2, 4\alpha_2, 2\alpha_2)$

$(9, 2, 7) = (\alpha_1 + 6\alpha_2, 2\alpha_1 + 4\alpha_2, -\alpha_1 + 2\alpha_2)$

$\alpha_1 + 6\alpha_2 = 9$ —— (1)

$2\alpha_1 + 4\alpha_2 = 2$ —— (2)

$-\alpha_1 + 2\alpha_2 = 7$ —— (3)

$\alpha_1 = -3 \quad \alpha_2 = 2$

$(1) \Rightarrow -3 + 6(2) = 9$

$9 = 9.$

Hence 9 is the vector space.

---

7. Check the polynomial $2x^3 - 2x^2 + 12x$ is the linear combination of $3x^3 - 5x^2 - 4x - 9$ and $x^3 - 2x^2 - 5x - 3$ in P_3

Solution

In $2x^3 - 2x^2 + 12x + 6 = \alpha_1(x^3 - 2x^2 - 5x - 3) + \alpha_2(3x^3 - 5x^2 - 4x - 9)$

$2x^3 - 2x^2 + 12x - 6 = (\alpha_1 x^3 - 2\alpha_1 x^2 - 5\alpha_1 x - 3\alpha_1) + (3\alpha_2 x^3 - 5\alpha_2 x^2 - 4\alpha_2 x - 9\alpha_2)$

$2x^3 - 2x^2 + 12x - 6 = x^3(\alpha_1 + 3\alpha_2) + x^2(-2\alpha_1 - 5\alpha_2) + x(-5\alpha_1 - 4\alpha_2) + (-3\alpha_1 - 9\alpha_2)$ —— (1)

Equating

$2 = \alpha_1 + 3\alpha_2$ —— (2)

$-2 = -2\alpha_1 - 5\alpha_2$ —— (3)

$12 = -5\alpha_1 - 4\alpha_2$ —— (4)

$-6 = -3\alpha_1 - 9\alpha_2$ —— (5)

$12 = -5(-4) - 4(2)$
$12 = +20 - 8$
$12 = +12$

$\alpha = -4$

$\alpha_2 = 2$

$-6 = -3(4) - 9(2)$
$-6 = +12 - 18$
$-6 = -6$

$(2) \Rightarrow -4 + 3(2) = 2$

$-4 + 6 = 2$

$2 = 2$

Hence the polynomial is a linear combination.

---

8. Check the polynomial $3x^3 - 2x^2 + 7x + 8$ is the linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$ in $P_3(\mathbb{R})$

Solution.

$3x^3 - 2x^2 + 7x + 8 = \alpha_1(x^3 - 2x^2 - 5x - 3) + \alpha_2(3x^3 - 5x^2 - 4x - 9)$

$3x^3 - 2x^2 + 7x + 8 = (\alpha_1 x^3 - 2\alpha_1 x^2 - 5\alpha_1 x - 3\alpha_1) + (3\alpha_2 x^3 - 5\alpha_2 x^2 - 4\alpha_2 x - 9\alpha_2)$

$3x^3 - 2x^2 + 7x + 8 = x^3(\alpha_1 + 3\alpha_2) + x^2(-2\alpha_1 - 5\alpha_2) + x(-5\alpha_1 - 4\alpha_2) + (3\alpha_1 - 9\alpha_2)$ —— (1)

Equating

$\alpha_1 + 3\alpha_2 = 3$ —— (2)

$-2\alpha_1 - 5\alpha_2 = -2$ —— (3)

$-5\alpha_1 - 4\alpha_2 = 7$ —— (4)

$3\alpha_1 - 9\alpha_2 = 8$ —— (5)

$-5(-9) - 4(4) = 7$
$45 - 16 = 7$
$29 \neq 7$

$\alpha_1 = -9$

$\alpha_2 = 4$

$3(-9) - 9(4) = 8$
$-27 - 36$
$-9 \neq 8$

$-9+12$
$3 = 3$

Hence the polynomial is ~~the~~ not linear combination.

9. Write the matrix $E = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as a linear combination of matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$.

Solution

Let $E = \alpha_1 A + \alpha_2 B + \alpha_3 C$ — (I)

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_2 & \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_3 \\ 0 & -\alpha_3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 + 2\alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_2 - \alpha_3 \end{bmatrix}$$

Equating

$\alpha_1 = 3$ — (1)

$\alpha_1 + 2\alpha_3 = 1$ — (2)

$\alpha_1 + \alpha_2 = 1$ — (3)

$\alpha_2 - \alpha_3 = -1$ — (4)

$(3) \Rightarrow \alpha_2 + 3 = 1$

$\alpha_2 = 1 - 3$

$\alpha_2 = -2$

$(4) \Rightarrow \alpha_2 - \alpha_3 = -1$

$-2 - \alpha_3 = -1$

$-\alpha_3 = -1 + 2$

$\alpha_3 = -1$

$(2) \Rightarrow 3+2(-1)=1$

$3-2=1$

$1=1$

$(I)\ E = 3A - 2B - C$

Linear Span

Definition.

Let V be a vector space over a field F and S be a non empty subset of V. Then the set of all linear combination of the finate subset S is called the linear span of S. And it is denoted by L(S).

(ie) $L(S) = \{\alpha_1 v_1 + \alpha_2 v_3 + \dots + \alpha_n v_n \,/\, \alpha_i \in F,\ v_i \in S\}$

Note.

* $L(S) \subseteq V$
* If $S = \phi$, then $L(S) = 0$.

---

1. Let $S = \{(1,2),(2,1)\}$, $V = R^2$. Prove that V is the linear span of S.

Solution

We know that $L(S) \subseteq V$ — (1)

Let us consider $(x,y) \in V$.

$(x,y) \in V$

$(x,y) = \alpha_1(1,2) + \alpha_2(2,1)$ — (2)

$= (\alpha_1, 2\alpha_1) + (2\alpha_2, \alpha_2)$

$(x,y) = (\alpha_1 + 2\alpha_2,\ 2\alpha_1 + \alpha_2)$

Equating on both sides

$\alpha_1 + 2\alpha_2 = x$ — (3)

$2\alpha_1 + \alpha_2 = y$ — (4)

$(2) \times 2 \Rightarrow 2\alpha_1 + 4\alpha_2 = 2x$

$(4) \Rightarrow 2\alpha_1 + \alpha_2 = y$

$3\alpha_2 = 2x - y$

$\alpha_2 = \frac{2x-y}{3}$

$(3) \Rightarrow \alpha_1 + 2\left[\frac{2x-y}{3}\right] = x$

$3\alpha_1 + 4x - 2y = 3x$

$3\alpha_1 = 3x - 4x + 2y$

$3\alpha_1 = -x + 2y$

$\alpha_1 = \frac{2y-x}{3}$

Sub α_1, α_2 in (2)

$(x,y) = \left(\frac{2y-x}{3}\right)(1,2) + \left(\frac{2x-y}{3}\right)(2,1)$

Hence x, y is the linear combination of S.

$\therefore (x,y) \in L(S)$

$(x,y) \in V \Rightarrow (x,y) \in L(S)$

$\therefore V \subseteq L(S)$ — (5)

From (1), (2)

$L(S) = V$.

The S generates V.

---

2. Prove that $V_2(R)$, $(3,7) \in$ liner span $\{(1,2),(0,1)\}$

Let $S = \{(1,2),(0,1)\}$

$v_1 = (1,2)$

$v_2 = (0,1)$

$\therefore V = \alpha_1 V_1 + \alpha_2 V_2$

$(x,y) = \alpha_1(1,2) + \alpha_2(0,1)$ — (1)

$= (\alpha_1, 2\alpha_1) + (0, \alpha_2)$

$(x,y) = (\alpha_1, 2\alpha_1 + \alpha_2)$

Equating

$\alpha_1 = x$

$2\alpha_1 + \alpha_2 = y$

$\alpha_2 = y - 2\alpha_1$

$(1) \Rightarrow (x,y) = x(1,2) + (y - 2\alpha_1)(0,1)$

Check $(3,7) \in L(S)$

Here $x = 3,\ y = 7$

$(3,7) = 3(1,2) + (7 - 2(3))(0,1)$

$= (3,6) + (7-6)(0,1)$

$= (3,6) + (0,1)$

$= (3,6) + (0,1)$

$(3,7) = (3,7)$

Which is true.

$\therefore (3,7) \in L(S)$

3. Determine wether the set of vectors $x_1 = (1,1,2),\ x_2 = (1,0,1),\ x_3 = (2,1,3)$ Span R^3.

Let $S = \{(1,1,2), (1,0,1), (2,1,3)\}$

WKT. $L(S) \subseteq R^3$ — (1)

Let $V \in R^3$, $V = (a,b,c)$

$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

$(a,b,c) = \alpha_1(1,1,2) + \alpha_2(1,0,1) + \alpha_3(2,1,3)$

$= (\alpha_1, \alpha_1, 2\alpha_1) + (\alpha_2, 0, \alpha_2) + (2\alpha_3, \alpha_3, 3\alpha_3)$

$(a,b,c) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 0\alpha_2 + \alpha_3,$

$2\alpha_1 + \alpha_2 + 3\alpha_3)$

$\therefore v = \alpha_1 v_1 + \alpha_2 v_2$

$(x,y) = \alpha_1(1,2) + \alpha_2(0,1)$ —— (1)

$= (\alpha_1, 2\alpha_1) + (0, \alpha_2)$

$(x,y) = (\alpha_1, 2\alpha_1 + \alpha_2)$

Equating

$\alpha_1 = x$

$2\alpha_1 + \alpha_2 = y$

$\alpha_2 = y - 2\alpha_1$

$(1) \Rightarrow (x,y) = x(1,2) + (y - 2\alpha_1)(0,1)$

Check $(3,7) \in L(S)$

Here $x = 3, y = 7$

$(3,7) = 3(1,2) + (7 - 2(3))(0,1)$

$= (3,6) + (7-6)(0,1)$

$= (3,6) + (0,1)$

$= (3,6) + (0,1)$

$(3,7) = (3,7)$

Which is true.

$\therefore (3,7) \in L(S)$

3. Determine wether the set of vectors $x_1 = (1,1,2)$, $x_2 = (1,0,1)$, $x_3 = (2,1,3)$ Span R^3.

Let $S = \{(1,1,2), (1,0,1), (2,1,3)\}$

WKT. $L(S) \subseteq R^3$ —— (1)

Let $v \in R^3$, $v = (a,b,c)$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$(a,b,c) = \alpha_1(1,1,2) + \alpha_2(1,0,1) + \alpha_3(2,1,3)$

$= (\alpha_1, \alpha_1, 2\alpha_1) + (\alpha_2, 0, \alpha_2) + (2\alpha_3, \alpha_3, 3\alpha_3)$

$(a,b,c) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 0\alpha_2 + \alpha_3,$
$2\alpha_1 + \alpha_2 + 3\alpha_3)$

Equating

$\alpha_1 + \alpha_2 + 2\alpha_3 = a$ — (2)

$\alpha_1 + 0\alpha_2 + \alpha_3 = b$ — (3)

$2\alpha_1 + \alpha_2 + 3\alpha_3 = c$ — (4)

Solve (1), (2), (3)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[A, B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$[A, B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_2 \end{array}$$

$\alpha_1 + \alpha_2 + 2\alpha_3 = a$

$0\alpha_1 - \alpha_2 - \alpha_3 = b - a$

$$[A, B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-a-b \end{array}\right] R_3 \to R_3 - R_2$$

$0\alpha_1 + 0\alpha_2 + 0\alpha_3 = c - a - b - b$

$0 = c - a - b.$

Which is not possible.

Given vector does not span R^3

Gen

4. Prove that the vectors $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ generate R^3.

Solution.

Generate: A subset S of a subs vector space generate (or) span V, if $L(S) = V$.

Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$

W.K.T $L(S) \subseteq R^3$ —— (1)

Let $v \in R^3$, $v = (a,b,c)$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$(a,b,c) = \alpha_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(0,1,1)$

$= (\alpha_1, \alpha_1, 0\alpha_1) + (\alpha_2, 0\alpha_2, \alpha_2) + (0\alpha_3, \alpha_3, \alpha_3)$

$(a,b,c) = (\alpha_1 + \alpha_2 + 0\alpha_3, \alpha_1 + 0\alpha_2 + \alpha_3, 0\alpha_1 + \alpha_2 + \alpha_3)$

$\alpha_1 + \alpha_2 + 0\alpha_3 = a$ —— (2)

$\alpha_1 + 0\alpha_2 + \alpha_3 = b$ —— (3)

$0\alpha_1 + \alpha_2 + \alpha_3 = c$ —— (4)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[A, B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 1 & 1 & c \end{array}\right] R_2 \to R_2 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 2 & c+b-a \end{array}\right] R_3 \to R_3 + R_2$$

$2\alpha_3 = c + b - a$ —— (5)

$-\alpha + \alpha_3 = b - a$ —— (6)

$\alpha_1 + \alpha_2 = a$ —— (7)

$(5) \Rightarrow \alpha_3 = \frac{1}{2}(c + b - a)$

$(6) \Rightarrow \alpha_2 = b - a - \alpha_3$

$\alpha_2 = -b + a + \alpha_3$

$\alpha_2 = -b + a + \frac{1}{2}(c + b - a)$

$= \frac{1}{2}(-2b + 2a + c + b - a)$

$\alpha_2 = \frac{1}{2}(-b + a + c)$

$(4) \Rightarrow \alpha_1 + \frac{1}{2}(-b+a+c) = a$

$\alpha_1 = a - \frac{1}{2}(-b+a+c)$

$\alpha_1 = \frac{1}{2}(2a+b-a-c)$

$\alpha_1 = \frac{1}{2}(a+b-c)$

substitute the values of $\alpha_1, \alpha_2, \alpha_3$ in (1)

$(I) \Rightarrow v = \frac{1}{2}(a+b-c)(1,1,0) + \frac{1}{2}(a+c-b)(1,$
$+ \frac{1}{2}(c+b-a)(0,1,1)$

$v \in L(S)$

$\therefore R^3 \subseteq L(S)$ — (8)

from (1) & (8)

we get, $L(S) = R^3$.

$\therefore S$ generates R^3.

Prove that the polynomials (x^2+3x-2), $(2x^2+5x-3)$ and $-x^2-4x+4$ generates $P_2(R)$.

Let $S = \{p(x), q(x), r(x)\}$

$L(S) \subseteq P_2(R)$ — (1)

$f(x) \in P_2(R)$

then $f(x) = ax^2+bx+c$.

Let $f(x) = \alpha_1 p(x) + \alpha_2 q(x) + \alpha_3 r(x)$

$ax^2+bx+c = \alpha_1(x^2+3x-2) + \alpha_2(2x^2+5x-3) + \alpha_3(-x^2-4x+4)$ — (I)

$= (\alpha_1 x^2 + 3\alpha_1 x - 2\alpha_1) + (2\alpha_2 x^2 + 5\alpha_2 x - 3\alpha_2) + (-\alpha_3 x^2 - \alpha_3 4x + 4\alpha_3)$

$ax^2+bx+c = x^2(\alpha_1 + 2\alpha_2 - \alpha_3)x^2 + (3\alpha_1 + 5\alpha_2 - 4\alpha_3)x$
$+(-2\alpha_1 - 3\alpha_2 + 4\alpha_3)$

$\alpha_1 + 2\alpha_2 - \alpha_3 = a$ —— (2)

$3\alpha_1 + 5\alpha_2 - 4\alpha_3 = b$ —— (3)

$-2\alpha_1 - 3\alpha_2 + 4\alpha_3 = c$ —— (4)

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -4 \\ -2 & -3 & 4 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[A, B] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 3 & 5 & 4 & b \\ -2 & -3 & 4 & c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & -1 & b-3a \\ 0 & 1 & 2 & c+2a \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & -1 & b-3a \\ 0 & 0 & 1 & c+b-a \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 + R_2 \end{array}$$

$\alpha_3 = c + b - a$ —— (5)

$-\alpha - \alpha_3 = b - 3a$ —— (6)

$\alpha_1 + 2\alpha_2 - \alpha_3 = a$ —— (7)

$(5) \Rightarrow \alpha_3 = c + b - a.$

$(6) \Rightarrow \alpha_2 = \alpha_3 + b - 3a$

$\alpha_2 = -\alpha_3 - b + 3a$

$\alpha_2 = -(c + b - a) - b + 3a$

$= -c - b + a - b + 3a$

$\alpha_2 = 4a - 2b - c$

$(7) \Rightarrow \alpha_1 + 2\alpha_2 - \alpha_3 = a$

$\alpha_1 = a - 2\alpha_2 + \alpha_3$

$= a - 2(4a - 2b - c) + (c + b - a)$

$= a - 8a + 4b + 2c + c + b - a$

$\alpha_1 = -8a + 5b + 3c$

substitute eqn $\alpha_1, \alpha_2, \alpha_3$ in I

$$(ax^2+bx+c) = (-8a+5b+3c)(x^2+3x-2)$$
$$(4a-2b-c)(2x^2+5x-3)$$
$$(c+b-a)(-x^2-4x+4)$$

$$P_2(R) \subseteq L(S) \text{ --- } (8)$$

From (1) & (8)

$$L(S) = P_2(R)$$

$\therefore S$ generate $P_2(R)$