DAI Assignment-1

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Contents

1 Mathemagic

1.1 Task A

For a Bernoulli random variable $X \sim Ber(p)$, $P[X=0] = 1-p$, $P[X=1] = p$ and $P[X=n] = 0 \forall n \geq 2$.

$$\begin{aligned} G_{Ber}(z) &= \sum_{n=0}^{\infty} P[X=n]z^n \\ &= P[X=0] + P[X=1]z \\ G_{Ber} &= (1-p) + pz \end{aligned}$$

1.2 Task B

When $X \sim Bin(n,p)$, $P[X=k] = \binom{n}{k} p^k (1-p)^{n-k}\ \forall k \in \mathbb{Z}, 0 \leq k \leq n$. $P[X=k] = 0\ \forall k > n$.

$$\begin{aligned} G_{Bin}(z) &= \sum_{k=0}^{\infty} P[X=n]z^k \\ &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} z^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1-p)^{n-k} \\ &= (pz + (1-p))^n \\ G_{Bin}(z) &= G_{Ber}(z)^n \end{aligned}$$

1.3 Task C

Let X_1 and X_2 be two random variables which take up non-negative integers and let $X = X_1 + X_2$.

$$
\begin{aligned}
G_X(z) &= \sum_{n=0}^{\infty} P[X = n]z^n \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} (P[X_1 = i]P[X_2 = n-i])z^n \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (P[X_1 = i]P[X_2 = n-i])z^n \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (P[X_1 = i]z^i)(P[X_2 = n-i]z^{n-i}) \\
&= \sum_{i=0}^{\infty} P[X_1 = i]z^i \sum_{n=i}^{\infty} P[X_2 = n-i]z^{n-i} \\
&= \sum_{i=0}^{\infty} P[X_1 = i]z^i \sum_{n=0}^{\infty} P[X_2 = n]z^n \\
G_X(z) &= G_{X_1}(z)G_{X_2}(z)
\end{aligned}
$$

For $k = 1$, $G_\Sigma(z) = G(z)^k$.
If for $k = n-1$, $G_\Sigma(z) = G(z)^k$, then for $k = n$,

$$
\begin{aligned}
G_\Sigma(z) &= G_{X_1+X_2+\cdots+X_{n-1}}(z)G_{X_n}(x) \\
&= \prod_{i=1}^{n} G_{X_i}(z)G_\Sigma(z) && = G(z)^k
\end{aligned}
$$

From the *Principle of Mathematical Induction*, for all $k \in \mathbb{N}$, $G_\Sigma(z) = G(z)^k$

1.4 Task D

When $X \sim Geo(p)$, $P[X = n] = (1-p)^{n-1}p$ and $P[X = 0] = 0$.

$$
\begin{aligned}
G_{Geo}(z) &= \sum_{n=0}^{\infty} P[X = n]z^n \\
&= \sum_{n=1}^{\infty} (1-p)^{n-1}pz^n \\
&= pz \sum_{n=1}^{\infty} ((1-p)z)^{n-1} \\
&= \frac{pz}{1-(1-p)z}
\end{aligned}
$$

1.5 Task E

$X \sim Bin(n,p)$ and $Y \sim NegBin(n,p)$. $Y = \sum_{k=1}^{n} Y_k$ where $Y_k \sim Geo(p)$.

$$\begin{aligned}
G(Y) &= G(Y_1 + Y_2 + \cdots + Y_n) \\
&= G(Y_1)G(Y_2)\cdots G(Y_n) \\
&= \left(\frac{pz}{1-(1-p)z}\right)^n \\
G_Y^{(n,p)}(z) &= \left(\frac{pz}{1-(1-p)z}\right)^n \\
&= \left(\frac{1}{\frac{1}{pz} + (1-\frac{1}{p})}\right)^n \\
G_X^{(n,p^{-1})}(z^{-1}) &= \left(\frac{1}{pz} + (1-\frac{1}{p})\right)^n \\
\implies G_Y^{(n,p)}(z) &= \left(G_X^{(n,p^{-1})}(z^{-1})\right)^{-1}
\end{aligned}$$

1.6 Task F

$P[Y=k] = \binom{k-1}{n-1} p^n (1-p)^{k-n}$.

$$\begin{aligned}
G_Y^{(n,p)}(z) &= \sum_{k=0}^{\infty} P[Y=k] z^k \\
&= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k \\
\left(\frac{1}{pz} + \left(1 - \frac{1}{p}\right)\right)^{-n} &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k
\end{aligned}$$

Substitute $x = z$, $p = 2$

$$
\begin{aligned}
\left(\frac{1}{2x}+\frac{1}{2}\right)^{-n} &= \sum_{k=n}^{\infty}\binom{k-1}{n-1}2^n(-1)^{k-n}x^k \\
x^n(1+x)^{-n} &= \sum_{k=n}^{\infty}\binom{k-1}{n-1}(-1)^{k-n}x^k \\
(1+x)^{-n} &= \sum_{k=n}^{\infty}\binom{k-1}{n-1}(-1)^{k-n}x^{k-n} \\
(1+x)^{-n} &= \sum_{r=0}^{\infty}\binom{n+r-1}{n-1}(-1)^r x^r \\
(1+x)^{-n} &= \sum_{r=0}^{\infty}\binom{n+r-1}{r}(-1)^r x^r
\end{aligned}
$$

1.7 Task G

$$
\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} P[X=n]z^n \\
G'(z) &= \sum_{n=0}^{\infty} nP[X=n]z^{n-1} \\
G'(1) &= \sum_{n=0}^{\infty} nP[X=n] \\
G'(1) &= E[X]
\end{aligned}
$$

For Bernoulli random variable $X \sim Ber(p)$,

$$
\begin{aligned}
G_{Ber}(z) &= (1-p)+pz \\
G'_{Ber}(z) &= p \\
G'_{Ber}(1) &= p \\
E[X] &= p
\end{aligned}
$$

For Binomial random variable $X \sim Bin(n,p)$,

$$
\begin{aligned}
G_{Bin}(z) &= (pz+(1-p))^n \\
G'_{Bin}(z) &= np(pz+(1-p))^{n-1} \\
G'_{Bin}(1) &= np \\
E[X] &= np
\end{aligned}
$$

For Geometric random variable $X \sim Geo(p)$,

$$G_{Geo}(z) = \frac{pz}{1-(1-p)z}$$
$$G'_{Geo}(z) = \frac{p}{(1-(1-p)z)^2}$$
$$G'_{Geo}(1) = \frac{p}{p^2}$$
$$E[X] = \frac{1}{p}$$

For Negative Binomial random variable $X \sim NegBin(n, p)$,

$$G_{NegBin}(z) = \left(\frac{pz}{1-(1-p)z}\right)^n$$
$$G'_{NegBin}(z) = n\left(\frac{pz}{1-(1-p)z}\right)^{n-1} \frac{p}{(1-(1-p)z)^2}$$
$$G'_{NegBin}(1) = n\left(\frac{p}{p}\right)^{n-1} \frac{p}{p^2}$$
$$E[X] = \frac{n}{p}$$