DAI Assignment-1

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Contents

1 Mathemagic

1.1 Task A

For a Bernoulli random variable $X \sim Ber(p)$, $P[X = 0] = 1 - p$, $P[X = 1] = p$ and $P[X = n] = 0 \forall n \geq 2$.

$$\begin{aligned}
G_{Ber}(z) &= \sum_{n=0}^{\infty} P[X = n] z^n \\
&= P[X = 0] + P[X = 1]z \\
G_{Ber} &= (1 - p) + pz
\end{aligned}$$

1.2 Task B

When $X \sim Bin(n, p)$, $P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$ $\forall k \in \mathbb{Z}, 0 \leq k \leq n$. $P[X = k] = 0$ $\forall k > n$.

$$\begin{aligned}
G_{Bin}(z) &= \sum_{k=0}^{\infty} P[X = n] z^k \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} z^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1-p)^{n-k} \\
&= (pz + (1 - p))^n \\
G_{Bin}(z) &= G_{Ber}(z)^n
\end{aligned}$$

1.3 Task C

Let X_1 and X_2 be two random variables which take up non-negative integers and let $X = X_1 + X_2$.

$$
\begin{aligned}
G_X(z) &= \sum_{n=0}^{\infty} P[X = n] z^n \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} (P[X_1 = i] P[X_2 = n - i]) z^n \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (P[X_1 = i] P[X_2 = n - i]) z^n \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (P[X_1 = i] z^i)(P[X_2 = n - i] z^{n-i}) \\
&= \sum_{i=0}^{\infty} P[X_1 = i] z^i \sum_{n=i}^{\infty} P[X_2 = n - i] z^{n-i} \\
&= \sum_{i=0}^{\infty} P[X_1 = i] z^i \sum_{n=0}^{\infty} P[X_2 = n] z^n \\
G_X(z) &= G_{X_1}(z) G_{X_2}(z)
\end{aligned}
$$

For $k = 1$, $G_\Sigma(z) = G(z)^k$.
If for $k = n - 1$, $G_\Sigma(z) = G(z)^k$, then for $k = n$,

$$
\begin{aligned}
G_\Sigma(z) &= G_{X_1 + X_2 + \cdots + X_{n-1}}(z) G_{X_n}(x) \\
&= \prod_{i=1}^{n} G_{X_i}(z) G_\Sigma(z) && = G(z)^k
\end{aligned}
$$

From the *Principle of Mathematical Induction*, for all $k \in \mathbb{N}$, $G_\Sigma(z) = G(z)^k$