DAI Assignment-1

Aditya Neeraje, Balaji Karedla, Moulik Jindal

September 3, 2024

Contents

1 Mathemagic

1.1 Task A

For a Bernoulli random variable $X \sim Ber(p)$, $P[X = 0] = 1 - p$, $P[X = 1] = p$ and $P[X = n] = 0 \forall n \geq 2$.

$$\begin{aligned}
G_{Ber}(z) &= \sum_{n=0}^{\infty} P[X = n] z^n \\
&= P[X = 0] + P[X = 1] z \\
G_{Ber} &= (1 - p) + pz
\end{aligned}$$

1.2 Task B

When $X \sim Bin(n, p)$, $P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}\ \forall k \in \mathbb{Z}, 0 \leq k \leq n$. $P[X = k] = 0\ \forall k > n$.

$$\begin{aligned}
G_{Bin}(z) &= \sum_{k=0}^{\infty} P[X = n] z^k \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} z^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1 - p)^{n-k} \\
&= (pz + (1 - p))^n \\
G_{Bin}(z) &= G_{Ber}(z)^n
\end{aligned}$$

1.3 Task C

Let X_1 and X_2 be two random variables which take up non-negative integers and let $X = X_1 + X_2$.

$$
\begin{aligned}
G_X(z) &= \sum_{n=0}^{\infty} P[X=n]z^n \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}(P[X_1=i]P[X_2=n-i])z^n \\
&= \sum_{i=0}^{\infty}\sum_{n=i}^{\infty}(P[X_1=i]P[X_2=n-i])z^n \\
&= \sum_{i=0}^{\infty}\sum_{n=i}^{\infty}(P[X_1=i]z^i)(P[X_2=n-i]z^{n-i}) \\
&= \sum_{i=0}^{\infty}P[X_1=i]z^i\sum_{n=i}^{\infty}P[X_2=n-i]z^{n-i} \\
&= \sum_{i=0}^{\infty}P[X_1=i]z^i\sum_{n=0}^{\infty}P[X_2=n]z^n \\
G_X(z) &= G_{X_1}(z)G_{X_2}(z)
\end{aligned}
$$

For $k=1$, $G_\Sigma(z)=G(z)^k$.
If for $k=n-1$, $G_\Sigma(z)=G(z)^k$, then for $k=n$,

$$
\begin{aligned}
G_\Sigma(z) &= G_{X_1+X_2+\cdots+X_{n-1}}(z)G_{X_n}(x) \\
&= \prod_{i=1}^{n}G_{X_i}(z)G_\Sigma(z) & &= G(z)^k
\end{aligned}
$$

From the *Principle of Mathematical Induction*, for all $k\in\mathbb{N}$, $G_\Sigma(z)=G(z)^k$

1.4 Task D

When $X\sim Geo(p)$, $P[X=n]=(1-p)^{n-1}p$ and $P[X=0]=0$.

$$
\begin{aligned}
G_{Geo}(z) &= \sum_{n=0}^{\infty}P[X=n]z^n \\
&= \sum_{n=1}^{\infty}(1-p)^{n-1}pz^n \\
&= pz\sum_{n=1}^{\infty}((1-p)z)^{n-1} \\
&= \frac{pz}{1-(1-p)z}
\end{aligned}
$$

1.5 Task E

$X \sim Bin(n,p)$ and $Y \sim NegBin(n,p)$. $Y = \sum_{k=1}^{n} Y_k$ where $Y_k \sim Geo(p)$.

$$
\begin{aligned}
G(Y) &= G(Y_1 + Y_2 + \cdots + Y_n) \\
&= G(Y_1)G(Y_2)\cdots G(Y_n) \\
&= \left(\frac{pz}{1-(1-p)z}\right)^n \\
G_Y^{(n,p)}(z) &= \left(\frac{pz}{1-(1-p)z}\right)^n \\
&= \left(\frac{1}{\frac{1}{pz} + (1-\frac{1}{p})}\right)^n \\
G_X^{(n,p^{-1})}(z^{-1}) &= \left(\frac{1}{pz} + (1-\frac{1}{p})\right)^n \\
\implies G_Y^{(n,p)}(z) &= \left(G_X^{(n,p^{-1})}(z^{-1})\right)^{-1}
\end{aligned}
$$