

# DAI Assignment-2

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# 1 Mathemagic

## 1.1 Task A

For a Bernoulli random variable  $X \sim Ber(p)$ ,  $P[X = 0] = 1 - p$ ,  $P[X = 1] = p$  and  $P[X = n] = 0 \forall n \geq 2$ .

$$\begin{aligned} G_{Ber}(z) &= \sum_{n=0}^{\infty} P[X = n] z^n \\ &= P[X = 0] + P[X = 1] z \\ G_{Ber} &= (1 - p) + pz \end{aligned}$$

## 1.2 Task B

When  $X \sim Bin(n, p)$ ,  $P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k} \forall k \in \mathbb{Z}, 0 \leq k \leq n$ .  $P[X = k] = 0 \forall k > n$ .

$$\begin{aligned} G_{Bin}(z) &= \sum_{k=0}^{\infty} P[X = k] z^k \\ &= \sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k} z^k \\ &= \sum_{k=0}^n \binom{n}{k} (pz)^k (1 - p)^{n-k} \\ &= (pz + (1 - p))^n \\ G_{Bin}(z) &= G_{Ber}(z)^n \end{aligned}$$

## 1.3 Task C

Let  $X_1$  and  $X_2$  be two random variables which take up non-negative integers and let  $X = X_1 + X_2$ .

$$\begin{aligned}
G_X(z) &= \sum_{n=0}^{\infty} P[X = n]z^n \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^n (P[X_1 = i]P[X_2 = n - i])z^n \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (P[X_1 = i]P[X_2 = n - i])z^n \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (P[X_1 = i]z^i)(P[X_2 = n - i]z^{n-i}) \\
&= \sum_{i=0}^{\infty} P[X_1 = i]z^i \sum_{n=i}^{\infty} P[X_2 = n - i]z^{n-i} \\
&= \sum_{i=0}^{\infty} P[X_1 = i]z^i \sum_{n=0}^{\infty} P[X_2 = n]z^n \\
G_X(z) &= G_{X_1}(z)G_{X_2}(z)
\end{aligned}$$

For  $k = 1$ ,  $G_{\Sigma}(z) = G(z)^k$ .

If for  $k = n - 1$ ,  $G_{\Sigma}(z) = G(z)^k$ , then for  $k = n$ ,

$$\begin{aligned}
G_{\Sigma}(z) &= G_{X_1+X_2+\dots+X_{n-1}}(z)G_{X_n}(z) \\
&= \prod_{i=1}^n G_{X_i}(z)G_{\Sigma}(z) = G(z)^k
\end{aligned}$$

From the *Principle of Mathematical Induction*, for all  $k \in \mathbb{N}$ ,  $G_{\Sigma}(z) = G(z)^k$

## 1.4 Task D

When  $X \sim \text{Geo}(p)$ ,  $P[X = n] = (1 - p)^{n-1}p$  and  $P[X = 0] = 0$ .

$$\begin{aligned}
G_{\text{Geo}}(z) &= \sum_{n=0}^{\infty} P[X = n]z^n \\
&= \sum_{n=1}^{\infty} (1 - p)^{n-1}pz^n \\
&= pz \sum_{n=1}^{\infty} ((1 - p)z)^{n-1} \\
&= \frac{pz}{1 - (1 - p)z}
\end{aligned}$$

### 1.5 Task E

$X \sim \text{Bin}(n, p)$  and  $Y \sim \text{NegBin}(n, p)$ .  $Y = \sum_{k=1}^n Y_k$  where  $Y_k \sim \text{Geo}(p)$ .

$$\begin{aligned}
 G(Y) &= G(Y_1 + Y_2 + \dots + Y_n) \\
 &= G(Y_1)G(Y_2) \dots G(Y_n) \\
 &= \left( \frac{pz}{1 - (1-p)z} \right)^n \\
 G_Y^{(n,p)}(z) &= \left( \frac{pz}{1 - (1-p)z} \right)^n \\
 &= \left( \frac{1}{\frac{1}{pz} + (1 - \frac{1}{p})} \right)^n \\
 G_X^{(n,p^{-1})}(z^{-1}) &= \left( \frac{1}{pz} + (1 - \frac{1}{p}) \right)^n \\
 \implies G_Y^{(n,p)}(z) &= \left( G_X^{(n,p^{-1})}(z^{-1}) \right)^{-1}
 \end{aligned}$$

### 1.6 Task F

$$P[Y = k] = \binom{k-1}{n-1} p^n (1-p)^{k-n}.$$

$$\begin{aligned}
 G_Y^{(n,p)}(z) &= \sum_{k=0}^{\infty} P[Y = k] z^k \\
 &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k \\
 \left( \frac{1}{pz} + \left( 1 - \frac{1}{p} \right) \right)^{-n} &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k
 \end{aligned}$$

Substitute  $x = z$ ,  $p = 2$

$$\begin{aligned}
\left(\frac{1}{2x} + \frac{1}{2}\right)^{-n} &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} 2^n (-1)^{k-n} x^k \\
x^n (1+x)^{-n} &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} (-1)^{k-n} x^k \\
(1+x)^{-n} &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} (-1)^{k-n} x^{k-n} \\
(1+x)^{-n} &= \sum_{r=0}^{\infty} \binom{n+r-1}{n-1} (-1)^r x^r \\
(1+x)^{-n} &= \sum_{r=0}^{\infty} \binom{n+r-1}{r} (-1)^r x^r
\end{aligned}$$

## 1.7 Task G

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} P[X = n] z^n \\
G'(z) &= \sum_{n=0}^{\infty} n P[X = n] z^{n-1} \\
G'(1) &= \sum_{n=0}^{\infty} n P[X = n] \\
G'(1) &= E[X]
\end{aligned}$$

For Bernoulli random variable  $X \sim \text{Ber}(p)$ ,

$$\begin{aligned}
G_{\text{Ber}}(z) &= (1-p) + pz \\
G'_{\text{Ber}}(z) &= p \\
G'_{\text{Ber}}(1) &= p \\
E[X] &= p
\end{aligned}$$

For Binomial random variable  $X \sim \text{Bin}(n, p)$ ,

$$\begin{aligned}
G_{\text{Bin}}(z) &= (pz + (1-p))^n \\
G'_{\text{Bin}}(z) &= np(pz + (1-p))^{n-1} \\
G'_{\text{Bin}}(1) &= np \\
E[X] &= np
\end{aligned}$$

For Geometric random variable  $X \sim Geo(p)$ ,

$$\begin{aligned} G_{Geo}(z) &= \frac{pz}{1 - (1-p)z} \\ G'_{Geo}(z) &= \frac{p}{(1 - (1-p)z)^2} \\ G'_{Geo}(1) &= \frac{p}{p^2} \\ E[X] &= \frac{1}{p} \end{aligned}$$

For Negative Binomial random variable  $X \sim NegBin(n, p)$ ,

$$\begin{aligned} G_{NegBin}(z) &= \left( \frac{pz}{1 - (1-p)z} \right)^n \\ G'_{NegBin}(z) &= n \left( \frac{pz}{1 - (1-p)z} \right)^{n-1} \frac{p}{(1 - (1-p)z)^2} \\ G'_{NegBin}(1) &= n \left( \frac{p}{p} \right)^{n-1} \frac{p}{p^2} \\ E[X] &= \frac{n}{p} \end{aligned}$$