Quantum error correction IV:

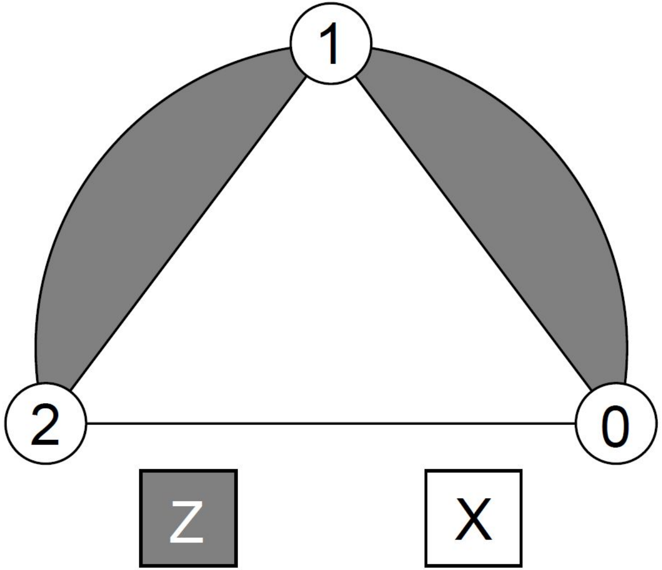
The surface code

Austin Fowler

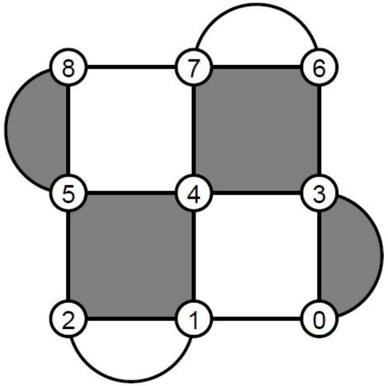
Last time: three ways of representing the same state

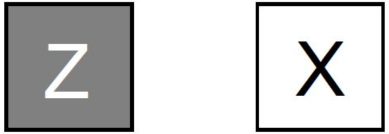
$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

```
+XXX
+ZZI
+IZZ
```



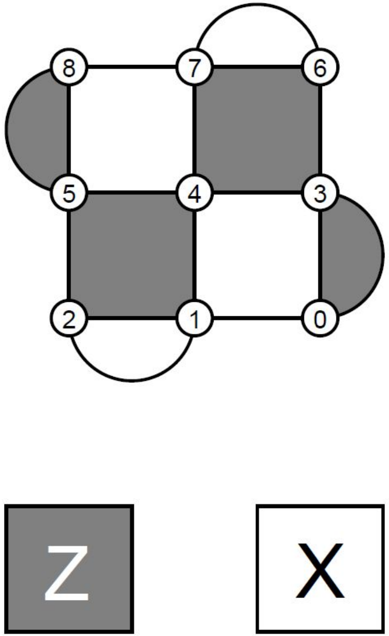
Introducing the surface code





This is the preferred representation

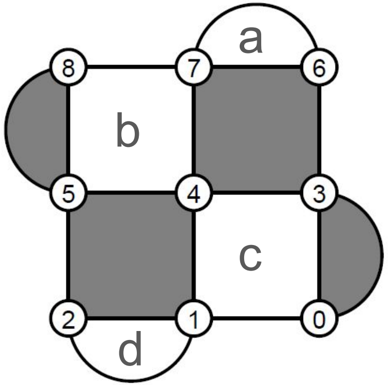
Introducing the surface code

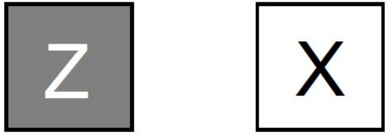


+IXXIIIIII
+ZIIZIIIII
+XXIXXIIII
+IZZIZZIII
+IIIZZIZZI
+IIIIXXIXX
+IIIIIZIIZ
+IIIIIIXXI

… as this is not easy to work with …

Introducing the surface code: logical zero

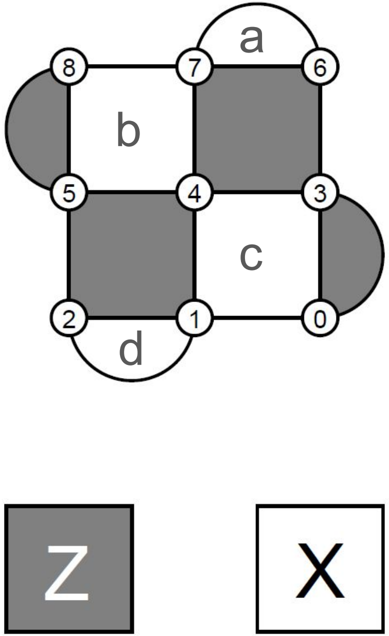




$$
\begin{aligned}
|0_L\rangle = \quad & |000000000\rangle & & +(-1)^{d}|000000110\rangle \\
& +(-1)^{a}|011000000\rangle & & +(-1)^{a+d}|011000110\rangle \\
& +(-1)^{b}|110110000\rangle & & +(-1)^{b+d}|110110110\rangle \\
& +(-1)^{a+b}|101110000\rangle & & +(-1)^{a+b+d}|101110110\rangle \\
& +(-1)^{c}|000011011\rangle & & +(-1)^{c+d}|000011101\rangle \\
& +(-1)^{a+c}|011011011\rangle & & +(-1)^{a+c+d}|011011101\rangle \\
& +(-1)^{b+c}|110101011\rangle & & +(-1)^{b+c+d}|110101101\rangle \\
& +(-1)^{a+b+c}|101101011\rangle & & +(-1)^{a+b+c+d}|101101101\rangle
\end{aligned}
$$

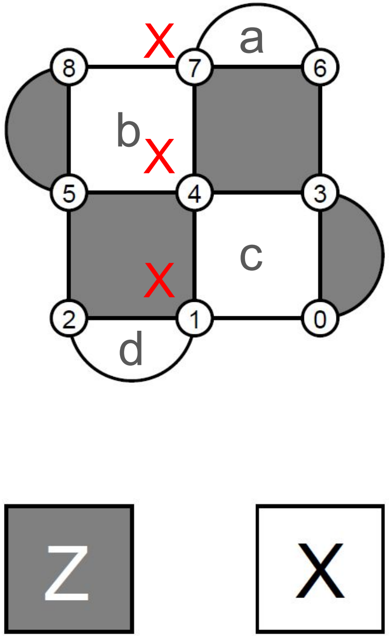
… and this is even more useless …

Introducing the surface code: logical zero



$$
\begin{aligned}
|0_L\rangle = \; & |000000000> && +(-1)^{d}|000000110> \\
& +(-1)^{a}|011000000> && +(-1)^{a+d}|011000110> \\
& +(-1)^{b}|110110000> && +(-1)^{b+d}|110110110> \\
& +(-1)^{a+b}|101110000> && +(-1)^{a+b+d}|101110110> \\
& +(-1)^{c}|000011011> && +(-1)^{c+d}|000011101> \\
& +(-1)^{a+c}|011011011> && +(-1)^{a+c+d}|011011101> \\
& +(-1)^{b+c}|110101011> && +(-1)^{b+c+d}|110101101> \\
& +(-1)^{a+b+c}|101101011> && +(-1)^{a+b+c+d}|101101101>
\end{aligned}
$$

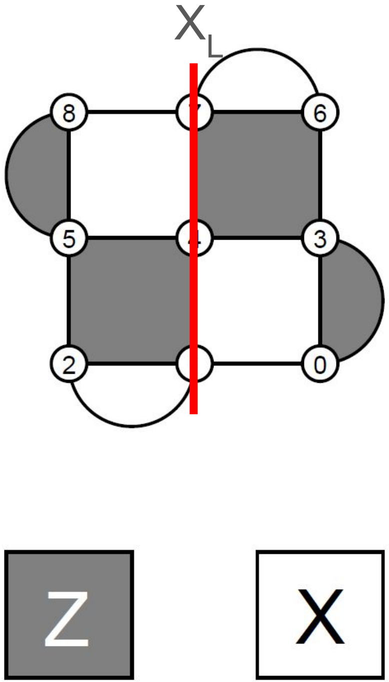
Introducing the surface code: logical one



$$
\begin{aligned}
|1_L\rangle = \; & |010010010\rangle \\
& +(-1)^{a}|001010010\rangle \\
& +(-1)^{b}|100100010\rangle \\
& +(-1)^{a+b}|111100010\rangle \\
& +(-1)^{c}|010001001\rangle \\
& +(-1)^{a+c}|001001001\rangle \\
& +(-1)^{b+c}|100111001\rangle \\
& +(-1)^{a+b+c}|111111001\rangle \\
& +(-1)^{d}|010010100\rangle \\
& +(-1)^{a+d}|001010100\rangle \\
& +(-1)^{b+d}|100100100\rangle \\
& +(-1)^{a+b+d}|111100100\rangle \\
& +(-1)^{c+d}|010001111\rangle \\
& +(-1)^{a+c+d}|001001111\rangle \\
& +(-1)^{b+c+d}|100111111\rangle \\
& +(-1)^{a+b+c+d}|111111111\rangle
\end{aligned}
$$

… though perhaps nice to see an explicit logical operation in action. Qubits 7, 4, and 1 have been flipped.

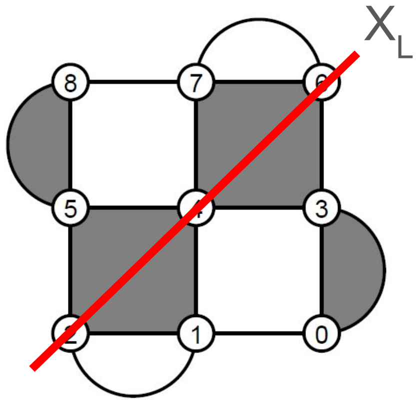
Introducing the surface code

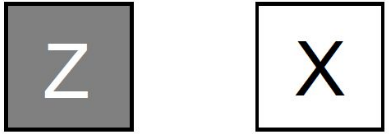


Let's write this more compactly:

$$
\begin{aligned}
|1_L\rangle &= X_L|0_L\rangle \\
&= X_7 X_4 X_1 |0_L\rangle
\end{aligned}
$$

Introducing the surface code



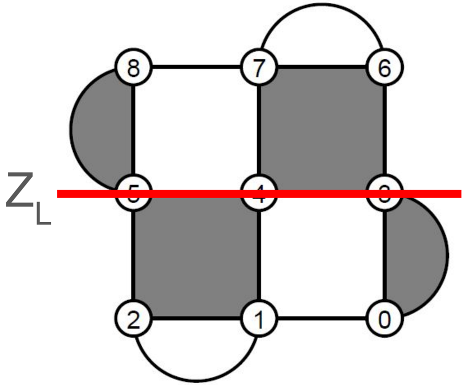


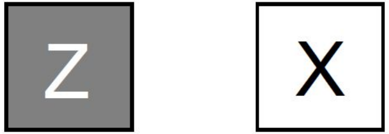
Any chain of X operators from top boundary to bottom boundary that commutes with all stabilizers transforms the state identically:

$$|1_L\rangle = X_6 X_4 X_2 |0_L\rangle$$

This is why the surface code is called a topological code. Many manipulations only depend on the global form rather than the low-level details.

Introducing the surface code

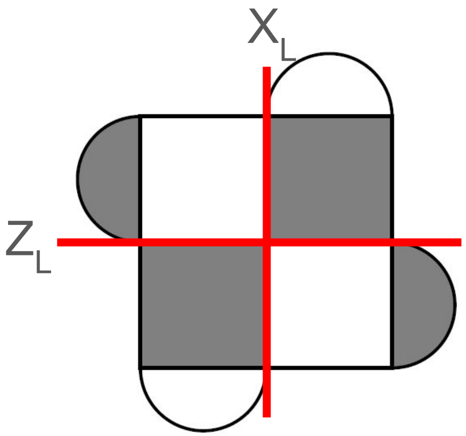


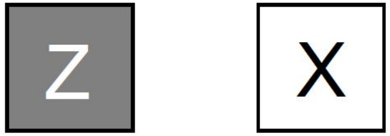


Can do the same for logical Z:

$$
\begin{aligned}
-|1_L\rangle &= Z_L|1_L\rangle \\
&= Z_5 Z_4 Z_3 |1_L\rangle
\end{aligned}
$$

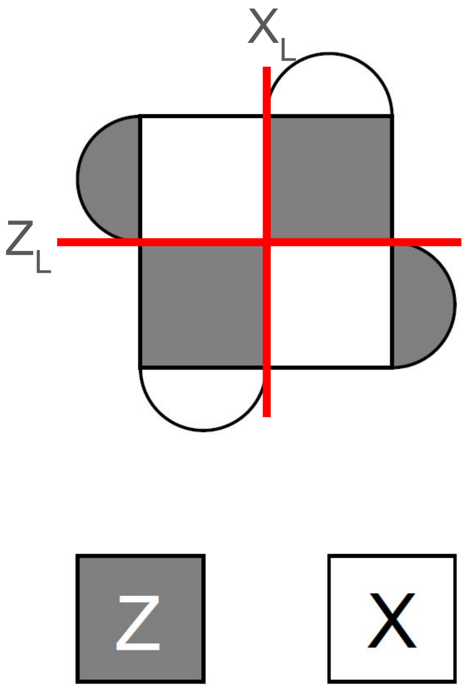
Introducing the surface code

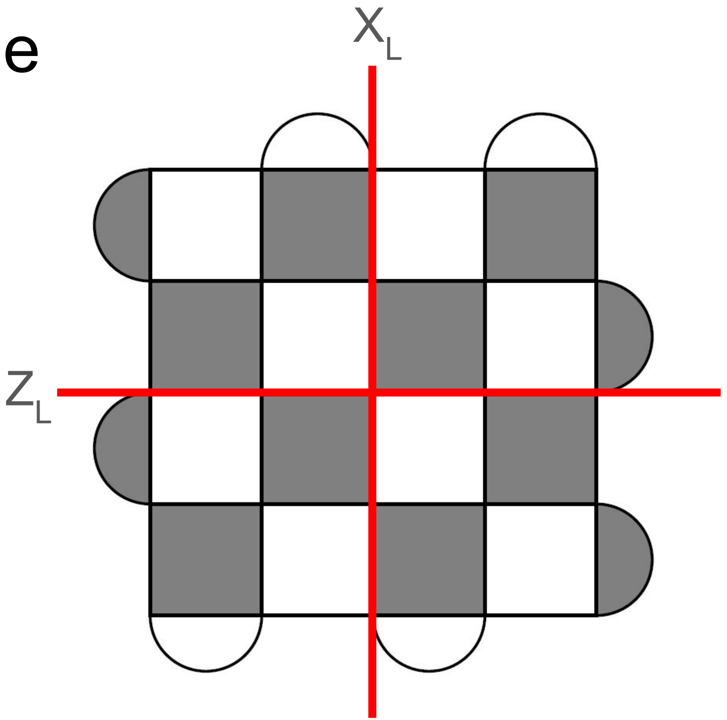




Don’t really need qubit numberings.

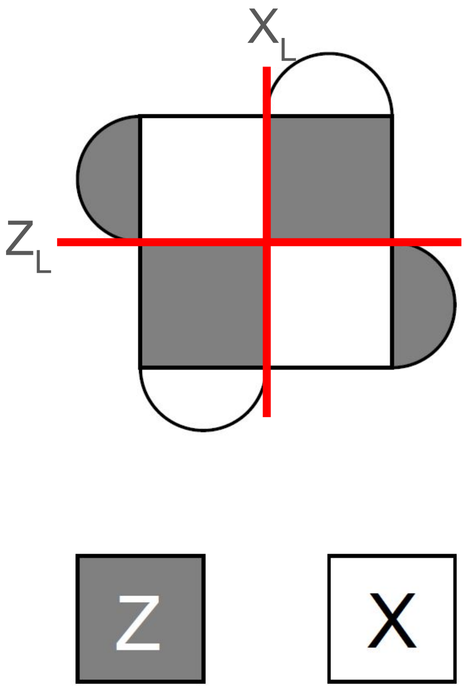
Introducing the surface code

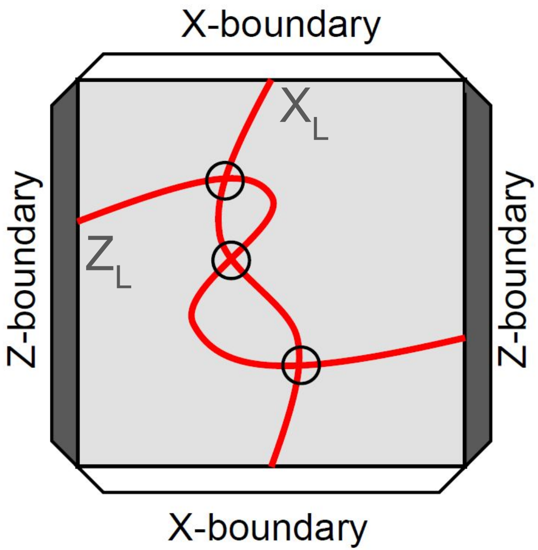




Here is a distance 5 surface code, would have been difficult to even write without all of the simplifications.

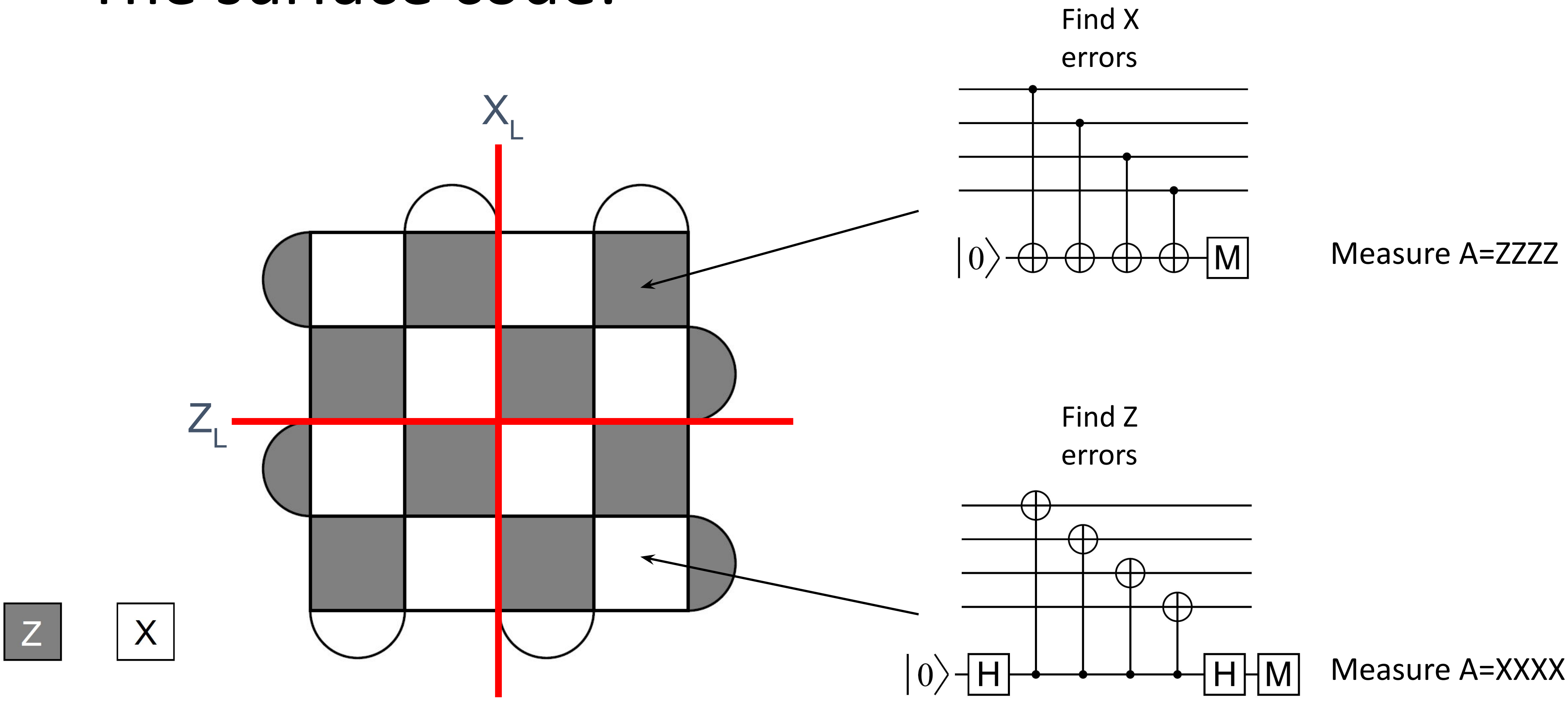
Introducing the surface code





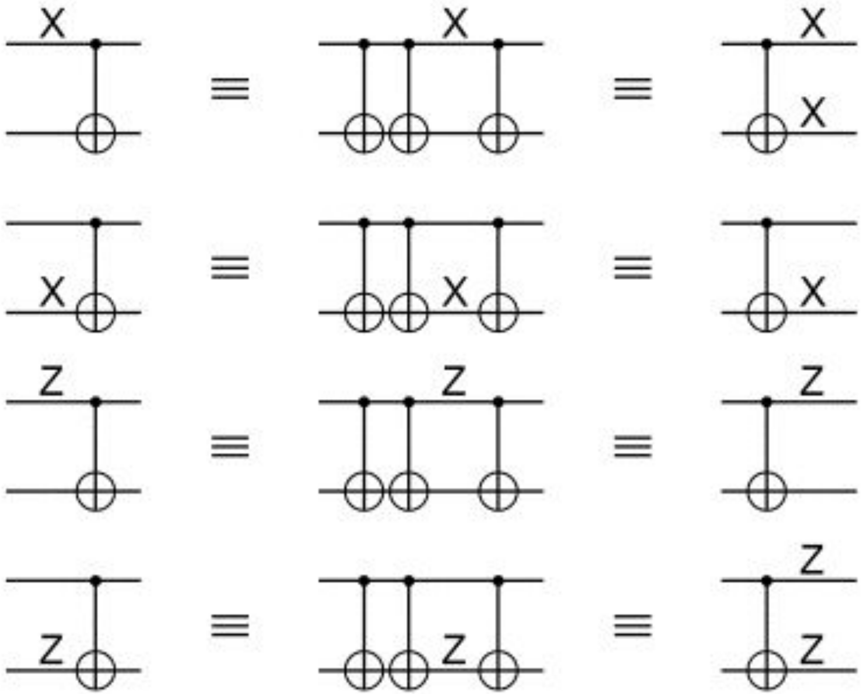
Here is an arbitrary distance surface code. Note that X_L and Z_L must always cross an odd number of time, ensuring anticommutation just like physical X and Z.

The surface code:

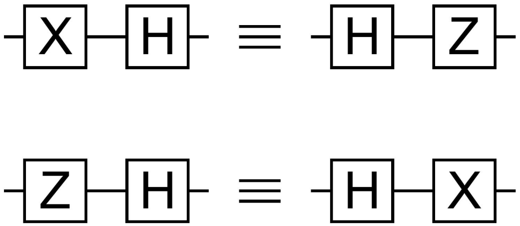


Question: in what order should a measure qubit touch its surrounding data qubits?

Error propagation

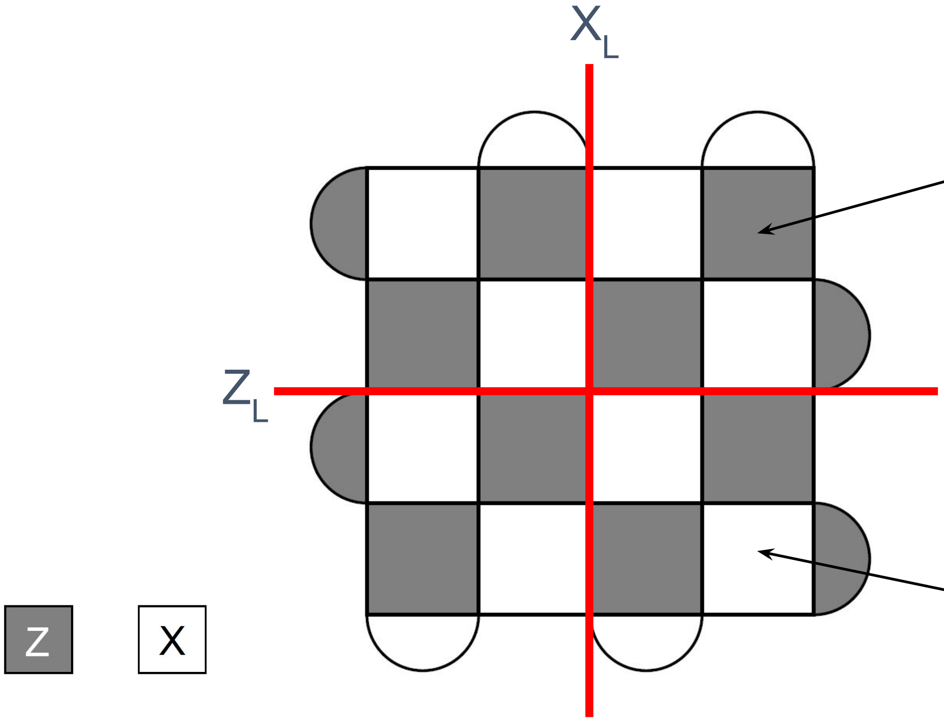


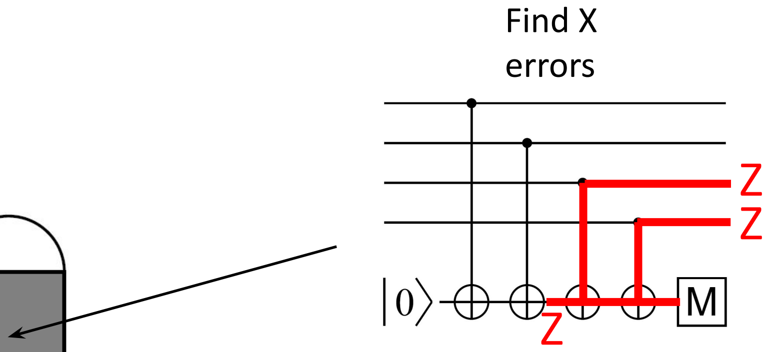
These examples can also be described as commuting errors through gates.

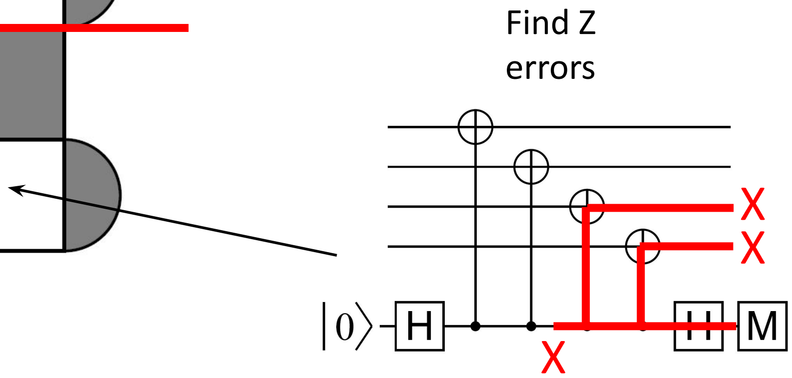


CNOT gates copy X errors on control, and Z errors on target.

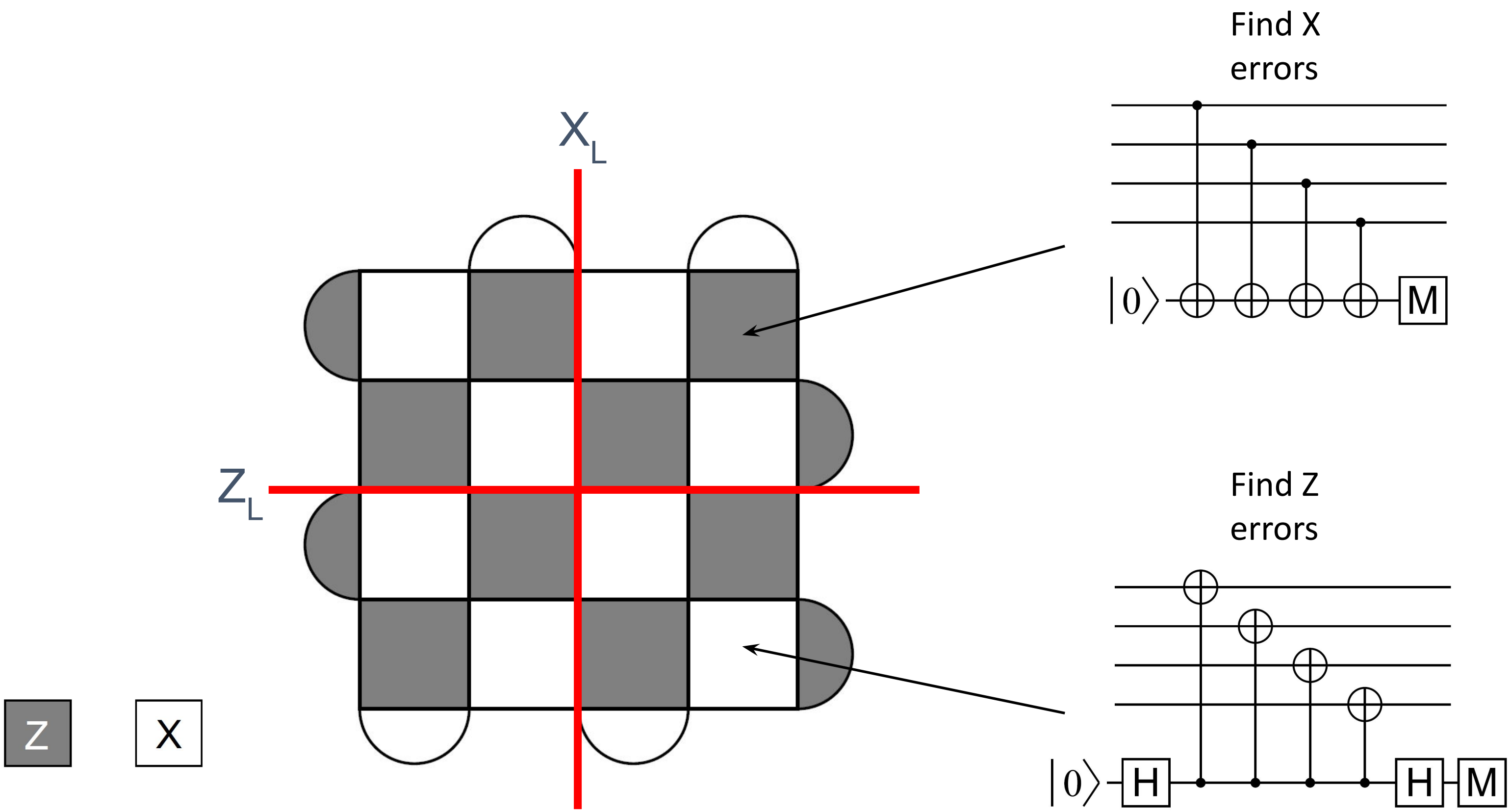
The surface code:





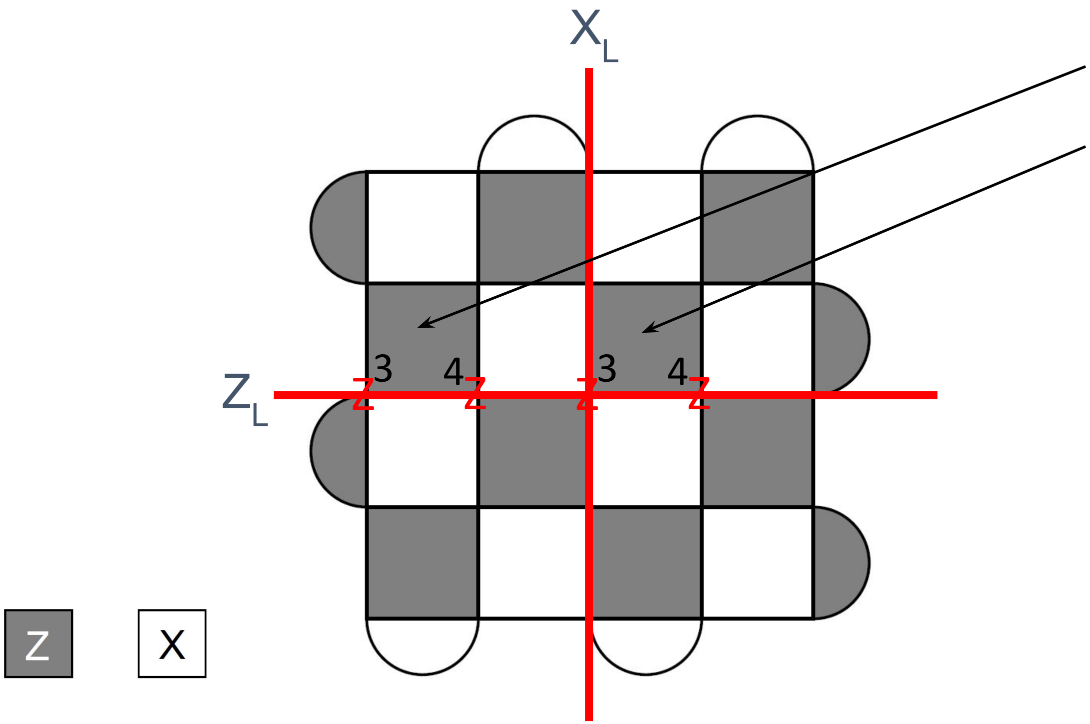


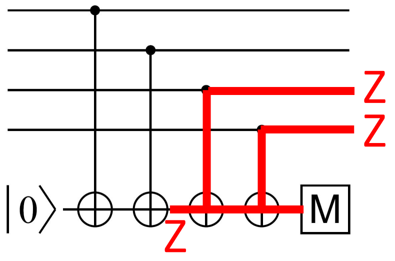
Errors on the measure qubits can propagate to two data qubits



Find X errors
Find Z errors
X_L
Z_L
$|0\rangle$
M
H
Z
X

The surface code:

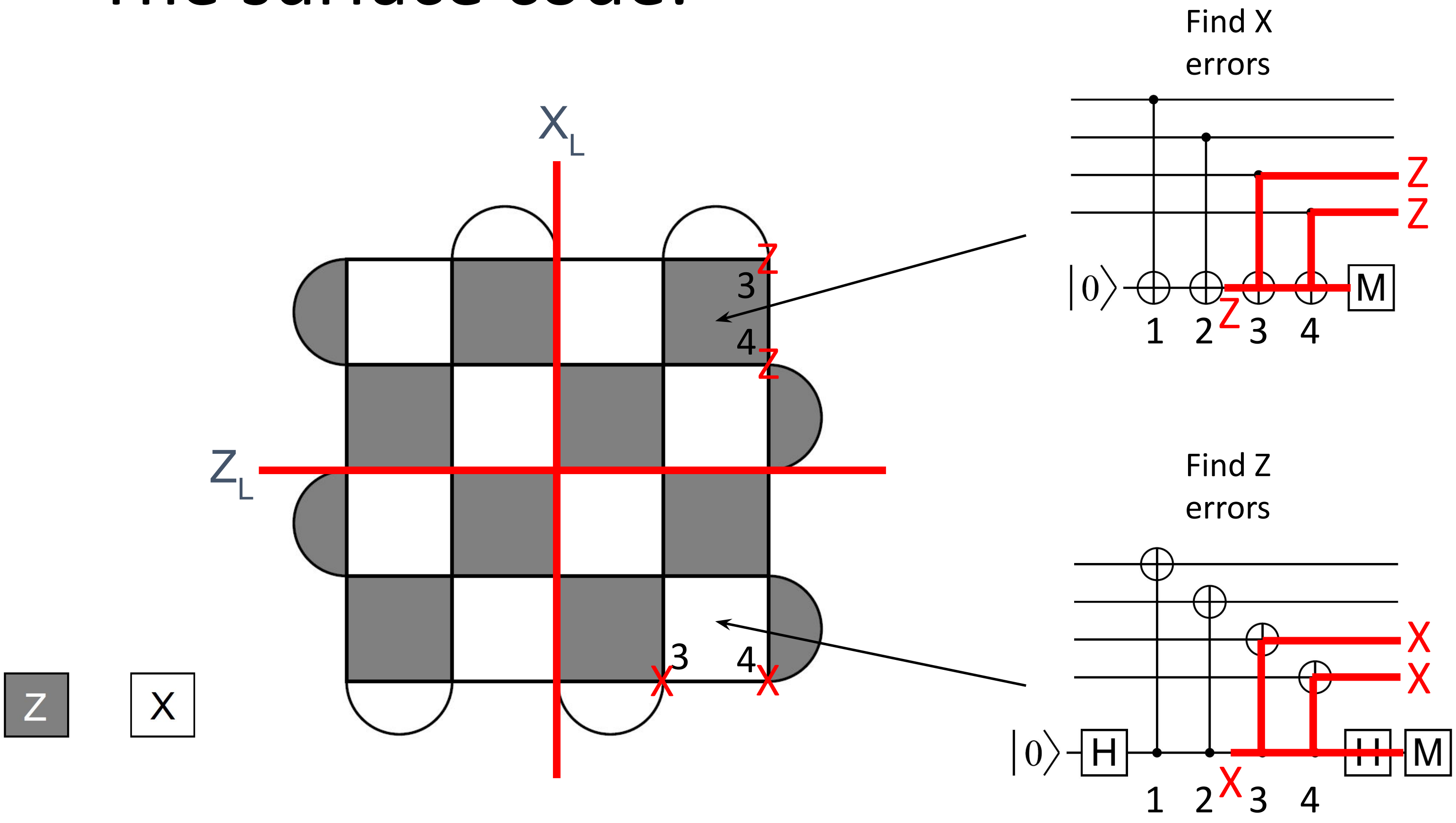




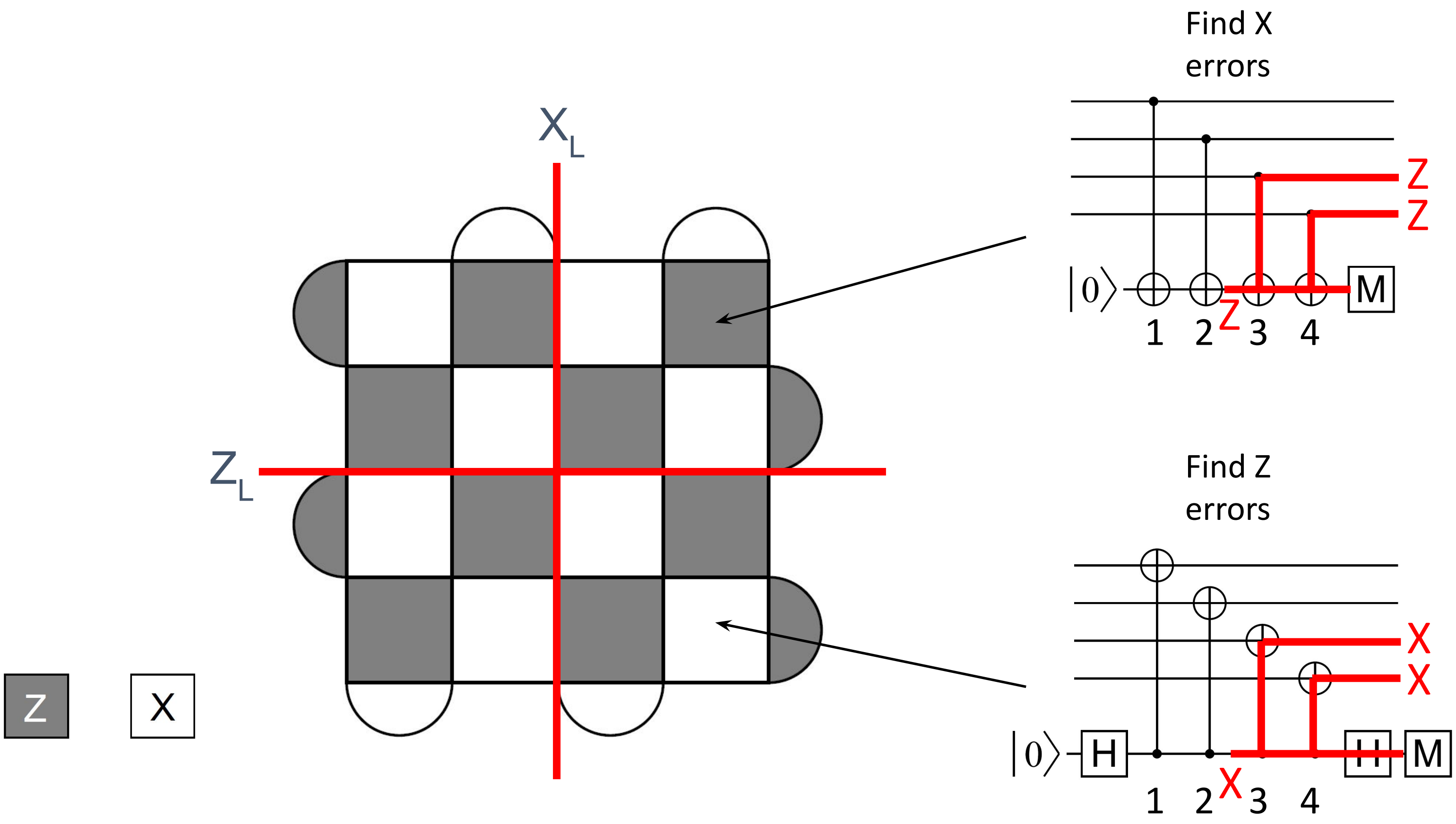
Two errors of probability p have led to four data qubit errors that will lead to a logical error.

d=5 codes are supposed to take 3 independent errors to fail.

The surface code:

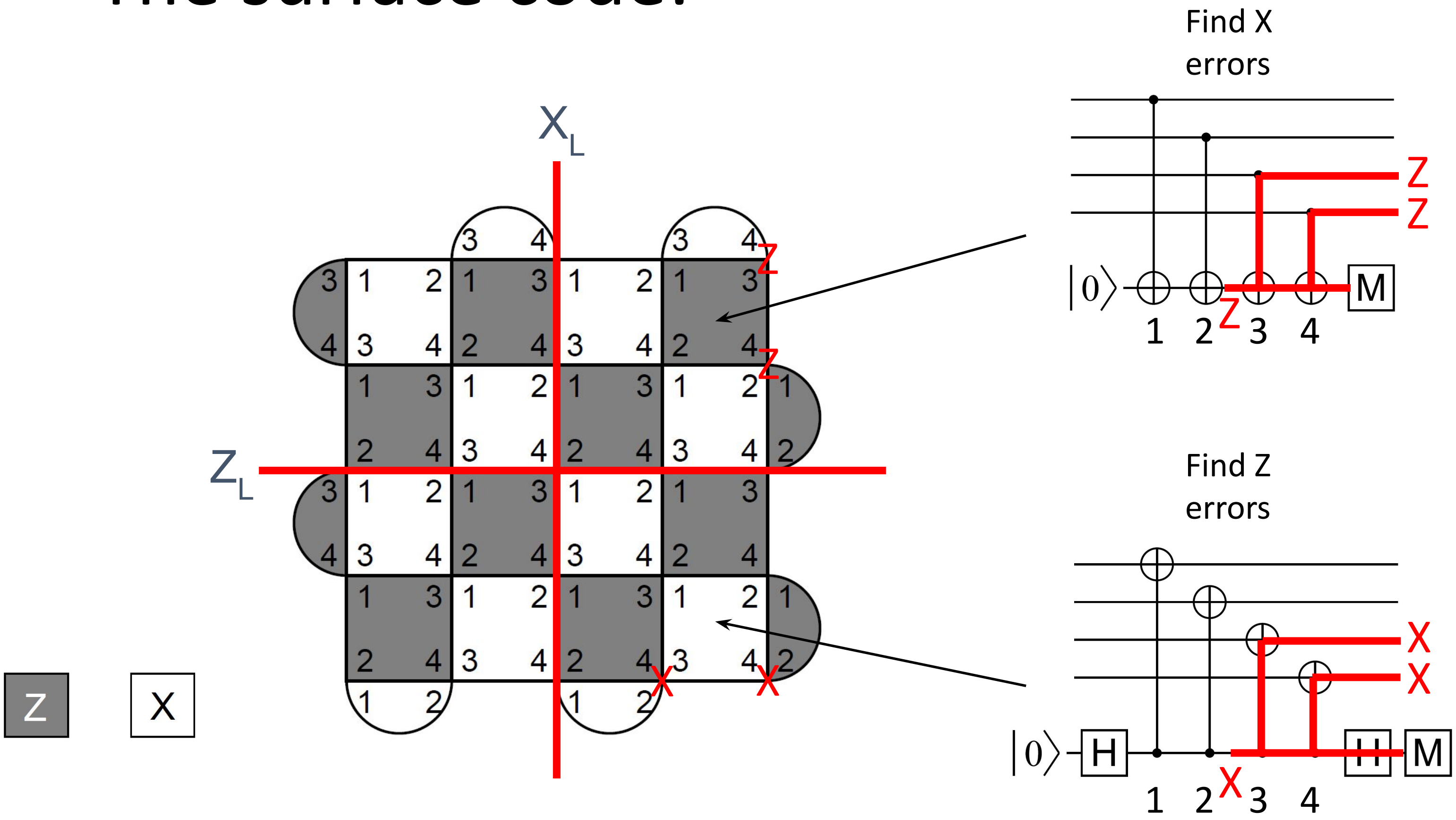


The last two qubits touched must propagate errors perpendicular to the logical operators.



X_L
Z_L
Z
X
Find X errors
|0⟩
M
1 2 3 4
Z
Z
Z
Find Z errors
|0⟩
H
H
M
1 2 3 4
X
X
X

The surface code:



Need global gate order that preserves the last two qubits touched rules and is perfectly parallel.

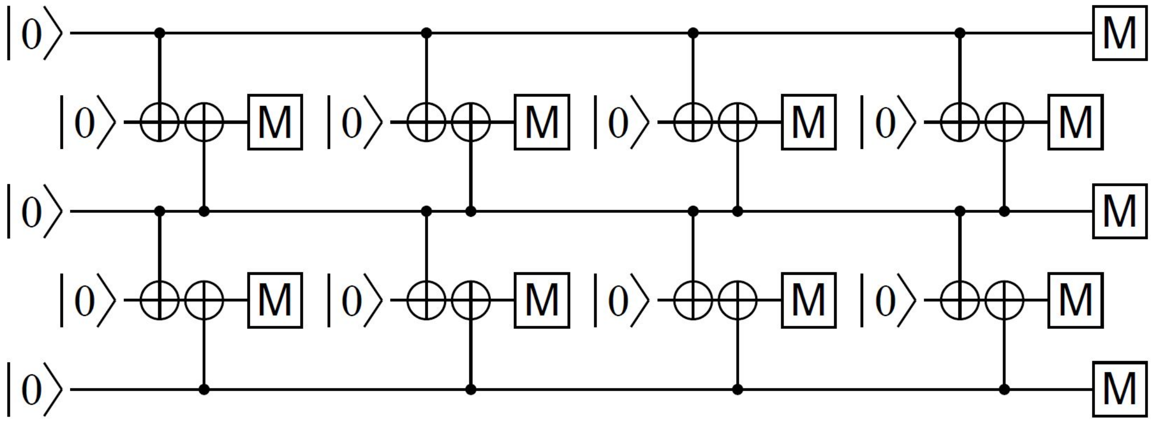
Initializing and measuring the surface code

- Initialize to $|0_L>$ by initializing all data qubits to $|0>$
- Initialize to $|+_L>$ by initializing all data qubits to $|+>$

- Measure in the logical Z basis (result $|0_L>$ or $|1_L>$) by measuring (M) each data qubit
- Measure in the logical X basis (result $|+_L>$ or $|-_L>$) by measuring (M_X) each data qubit

A surface code memory experiment

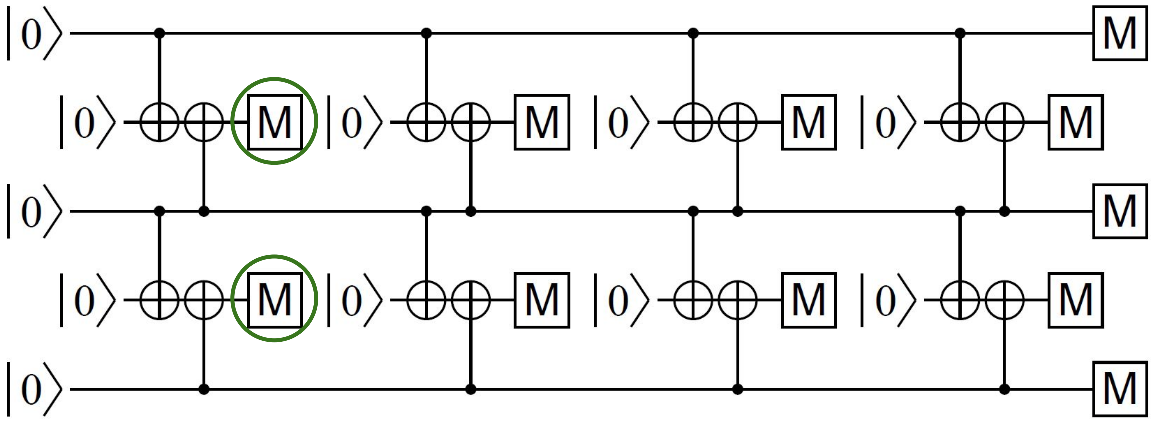
- Initialize to $|0_L\rangle$ by initializing all data qubits to $|0\rangle$
- Measure one or more rounds of surface code stabilizers
- Measure in the logical Z basis (result $|0_L\rangle$ or $|1_L\rangle$) by measuring (M) each data qubit



(the above is the repetition code, but illustrates the idea)

A surface code memory experiment

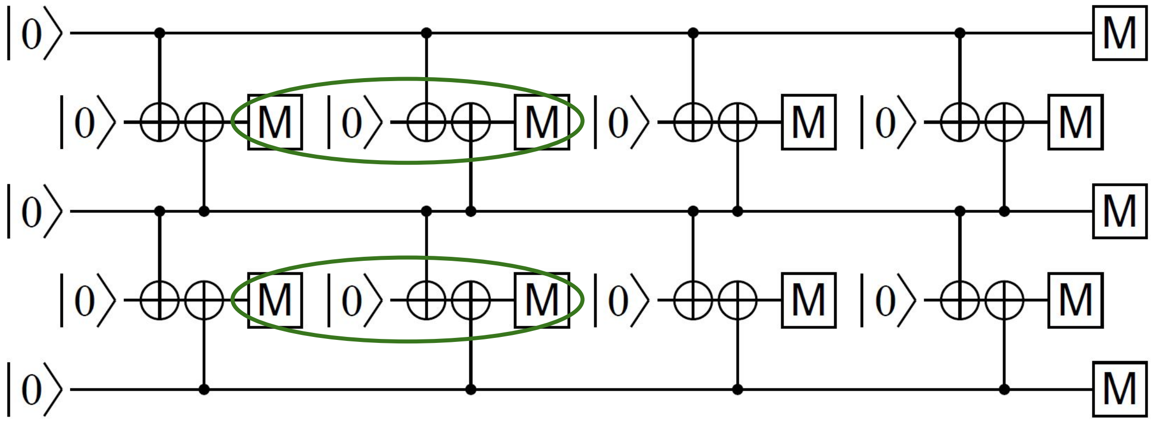
- Circuits that detect bit flips (Z stabilizers) have single measurement detectors at the beginning as data qubits initialized to |0> should lead to the first measurement being 0



(the above is the repetition code, but illustrates the idea)

A surface code memory experiment

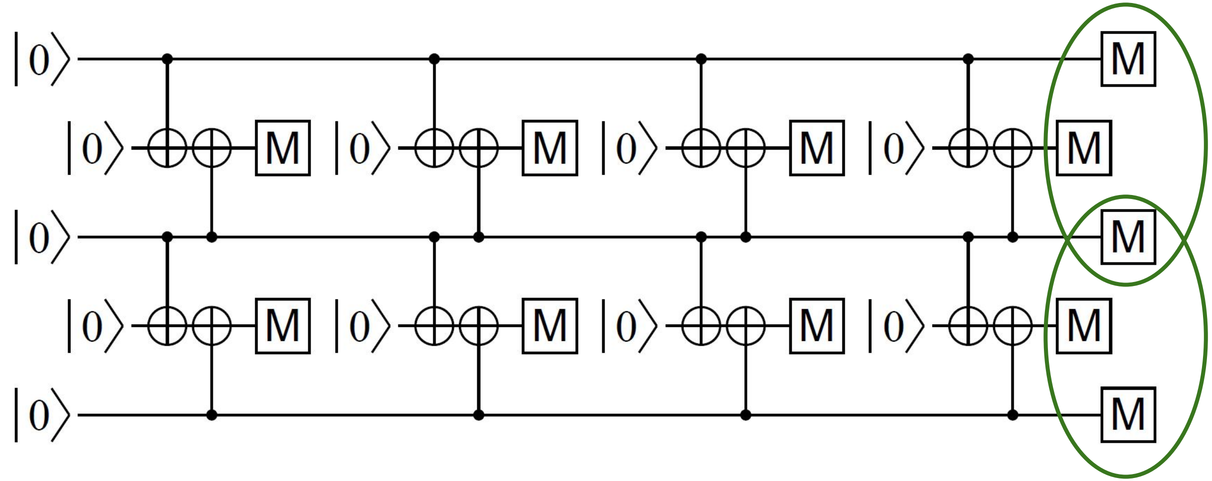
- In subsequent rounds detectors are made up of sequential pairs of measurements



(the above is the repetition code, but illustrates the idea)

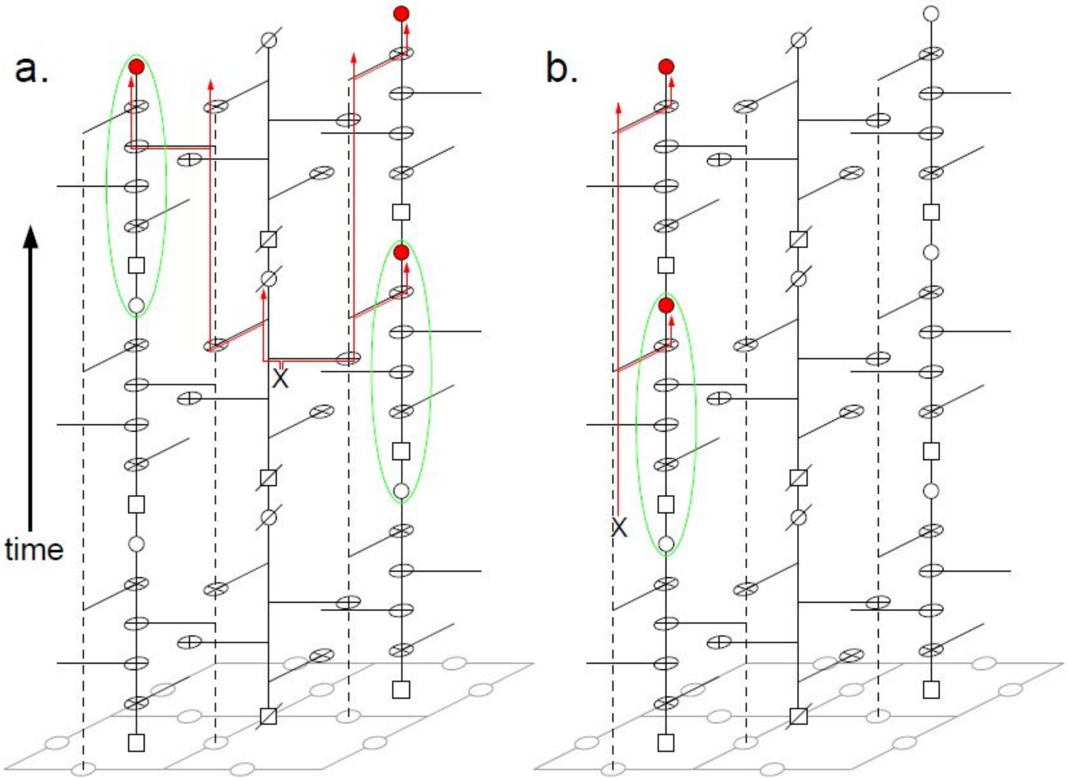
A surface code memory experiment

- The parity of neighboring final data qubit measurements measures the Z stabilizers one last time, and can be compared with last measure qubit measurement to form a detector



(the above is the repetition code, but illustrates the idea)

Surface code error detection: 3D graphs



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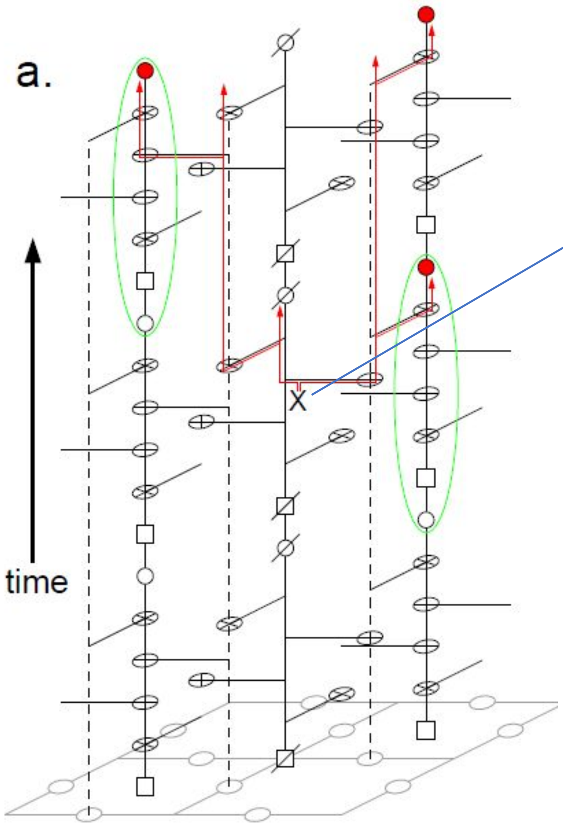
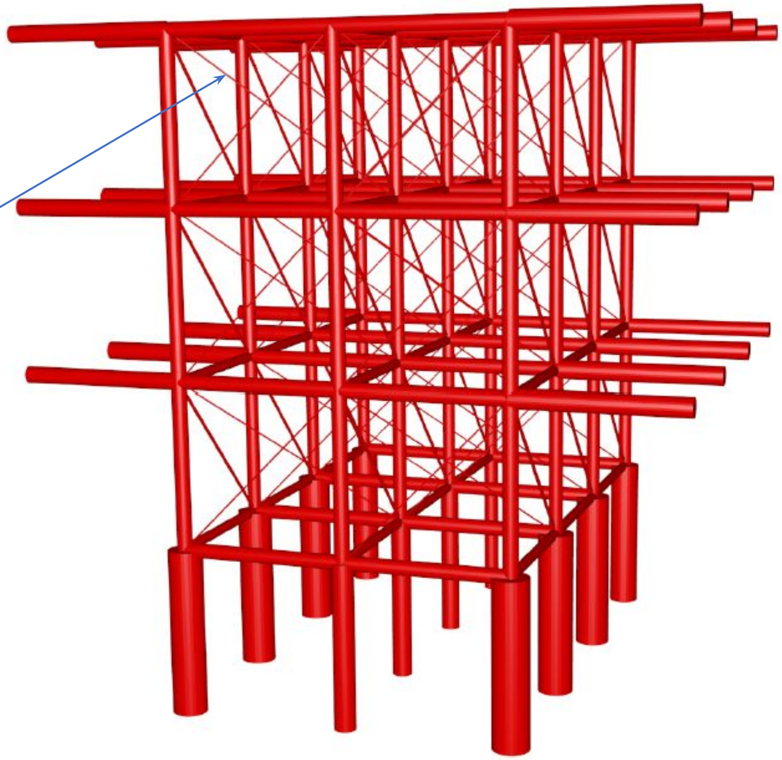
Let's draw our qubit chip as a horizontal plane, lay out the time sequence of surface code gates vertically

As before can study all possible errors and build a graph

Surface code error detection: 3D graphs





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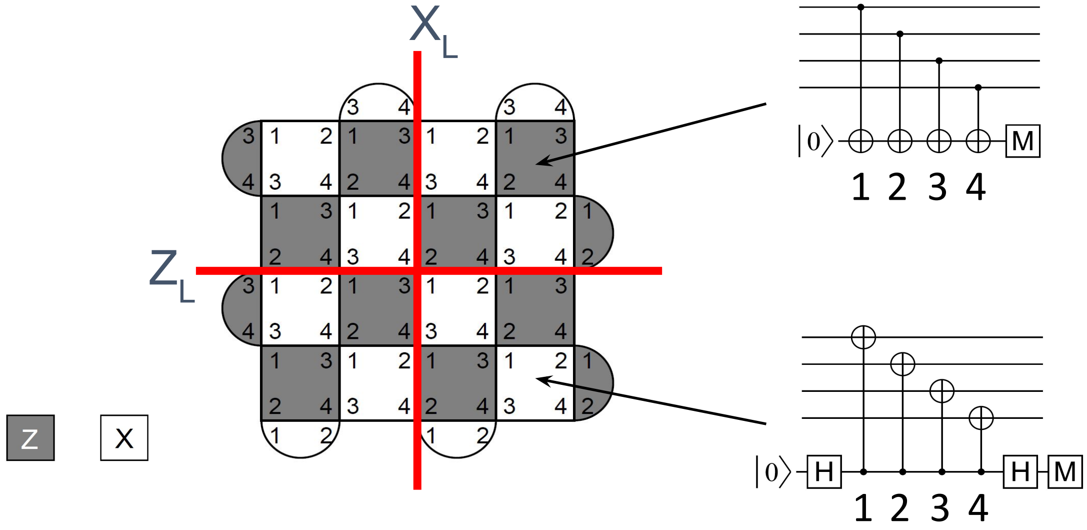
The thickness of each edge is proportional to the total probability of all errors leading to it.

Circuit on the left is just the front left corner of the circuit required to generate the graph on the right.

The surface code:

Even for distance 5, the surface code is a complex 49 qubit circuit requiring complex error analysis and decoding.



Further reading:
[arXiv:1808.06709](https://arxiv.org/abs/1808.06709)
[arXiv:1905.08916](https://arxiv.org/abs/1905.08916)

Next time: How to analyze circuits of this nature using Stim