Quantum error correction III:

Introduction to stabilizers

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What is a stabilizer?

Let's start with some notation, we need signed tensor product of the matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

eg: $-X \otimes I \otimes Z \otimes Z \otimes Y$

or more compactly: $-XIZZY$

A signed tensor product will also simply be called an operator

What is a stabilizer?

The simplest stabilizers are just operators that leave some useful state unchanged, eg:

$$
\begin{aligned}
Z|0\rangle &= |0\rangle \\
-Z|1\rangle &= |1\rangle \\
X|+\rangle &= |+\rangle \\
-X|-\rangle &= |-\rangle
\end{aligned}
$$

These states are called eigenstates of their associated stabilizer, and are unique up to global phase given the stabilizer

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\end{aligned}
$$

$$
\begin{aligned}
|0\rangle &\rightarrow +Z \\
|1\rangle &\rightarrow -Z \\
|+\rangle &\rightarrow +X \\
|-\rangle &\rightarrow -X
\end{aligned}
$$

Since the eigenstates are uniquely specified by the stabilizers, we can write them instead of the states.

What is a stabilizer?

Consider $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

This state is stabilized by 3 independent stabilizers

```
+XXX
+ZZI
+IZZ
```

For complex states, the list of independent stabilizers can be far more compact than the state itself

Some algebra

Let’s work through one example of showing a stabilizer stabilizes a state:

$$
\begin{aligned}
ZZI|\Psi\rangle &= \frac{1}{\sqrt{2}}ZZI(|000\rangle + |111\rangle) \\
&= \frac{1}{\sqrt{2}}(ZZI|000\rangle + ZZI|111\rangle) \\
&= \frac{1}{\sqrt{2}}\Big(|000\rangle + (-1)^2|111\rangle\Big) \\
&= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
&= |\Psi\rangle
\end{aligned}
$$

What about errors?

Consider the effect of errors: $X_2|\Psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle)$

What does this do to our stabilizers?

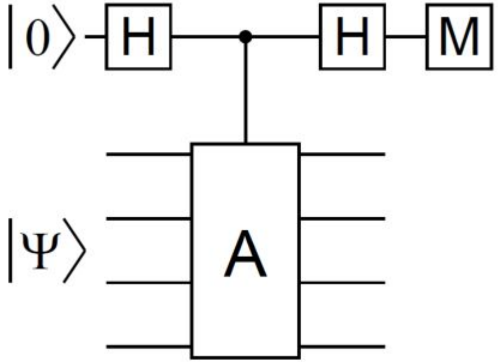
$$\begin{aligned} Z_2Z_1X_2|\Psi\rangle &= Z_2X_2Z_1|\Psi\rangle \\ &= -X_2Z_2Z_1|\Psi\rangle \\ &= -X_2|\Psi\rangle \end{aligned}$$

To say the same thing another way: $(-Z_2Z_1)X_2|\Psi\rangle = X_2|\Psi\rangle$ or:

```
+XXX
-ZZI
+IZZ
```

If we can work out how to measure the sign of a stabilizer, we can detect errors.

General operator measurement

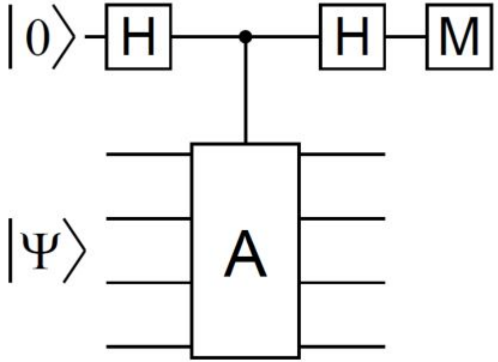


$A^2=I$

$|0\rangle|\Psi\rangle$

Try to work out the output of this circuit

General operator measurement



$A^2=I$

$$|0\rangle|\Psi\rangle$$

$$\text{H}\rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\Psi\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle|\Psi\rangle + |1\rangle|\Psi\rangle)$$

$$\text{controlled-A}\rightarrow \frac{1}{\sqrt{2}}(|0\rangle|\Psi\rangle + |1\rangle A|\Psi\rangle)$$

$$\text{H}\rightarrow \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\Psi\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)A|\Psi\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\frac{1}{\sqrt{2}}(|\Psi\rangle + A|\Psi\rangle) + |1\rangle\frac{1}{\sqrt{2}}(|\Psi\rangle - A|\Psi\rangle)\right)$$

A zero measurement means the output is the +1 eigenstate of A, one means -1 eigenstate

Locating an error with stabilizers

$$X_2|\Psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle) \longrightarrow \begin{array}{l} \texttt{+XXX} \\ \texttt{-ZZI} \\ \texttt{+IZZ} \end{array}$$

$$X_1|\Psi\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle)$$

What will the signs of the stabilizers be after these errors?

$$X_0|\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)$$

Locating an error with stabilizers

$$X_2|\Psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle) \longrightarrow \begin{array}{l}\texttt{+XXX}\\ \texttt{-ZZI}\\ \texttt{+IZZ}\end{array}$$

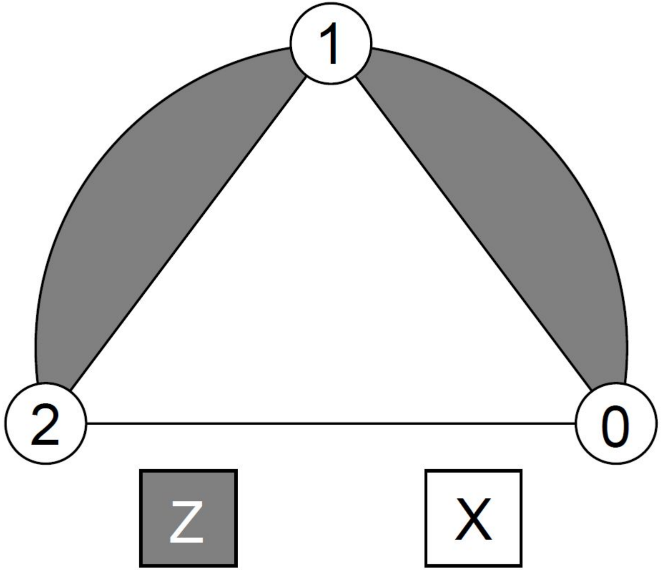
$$X_1|\Psi\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle) \longrightarrow \begin{array}{l}\texttt{+XXX}\\ \texttt{-ZZI}\\ \texttt{-IZZ}\end{array}$$

$$X_0|\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle) \longrightarrow \begin{array}{l}\texttt{+XXX}\\ \texttt{+ZZI}\\ \texttt{-IZZ}\end{array}$$

Representing stabilizers as a picture

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

+XXX
+ZZI
+IZZ



These are all the same state.
What happens after a Z error?

Representing stabilizers as a picture

$$Z|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$$

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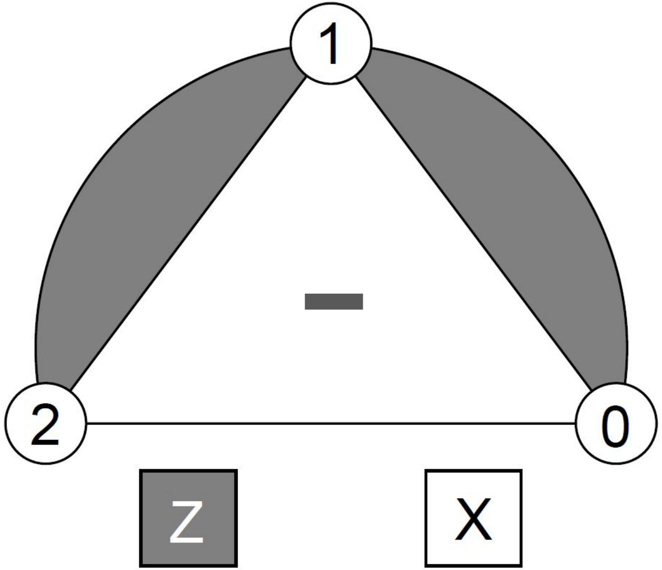
-XXX
+ZZI
+IZZ
```



X stabilizer is flipped.
Next time: the surface code