CEMC Summer Problem Solving Course 2024

Assignment #2

This assignment is out of 30 possible marks with each question worth 5 marks. Your solutions will be graded based on their quality as well as their correctness.

1. The integers a, b, and c satisfy $a^2b = 2700$, $b^2c = 1152$, and $c^2a = 960$. Determine the integers a, b, and c.

2. The sequence $a_1, a_2, a_3, \ldots$ is arithmetic with common difference $d > 0$. The numbers a_6, a_{14}, and a_{34} form a geometric sequence. If $a_1 = 1$, determine a_{2025}.

3. Jolene writes down a sequence $x_1, x_2, x_3, \ldots$ of real numbers starting with $x_1 = \frac{2}{3}$. For each integer $n \geq 2$, $x_n = \frac{7 - 5x_{n-1}}{4 - 3x_{n-1}}$. Determine the sum of the first 2025 numbers in the sequence.

4. A *permutation* of the ordered 8-tuple $\mathbf{a} = (1, 2, 3, 4, 5, 6, 7, 8)$ is an 8 tuple that contains each of the integers from 1 through 8 exactly once. For example, $(2, 1, 3, 4, 5, 6, 7, 8)$, $(8, 7, 6, 5, 4, 3, 2, 1)$, and $(5, 2, 3, 8, 1, 6, 7, 4)$ are all permutations of $\mathbf{a}$.

   For a permutation $\sigma = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$, define

   $$f(\sigma) = |a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8|$$

   For example, $f(3, 5, 1, 4, 2, 6, 8, 7) = |3 - 5| + |1 - 4| + |2 - 6| + |8 - 7| = 2 + 3 + 4 + 1 = 10$.

   Determine the average value of $f(\sigma)$ as σ ranges over all possible permutations of $\mathbf{a}$.

5. Determine all real x that satisfy $\log_{2x}(48\sqrt[3]{3}) = \log_{3x}(162\sqrt[3]{2})$.

   If you are unfamiliar with logarithms, please consult these lessons in our courseware.

6. Dikshant writes down $2k+1$ positive integers in a list where k is a positive integer. The integers are not necessarily all distinct, but there are at least three distinct integers in the list. The average (mean) of all $2k+1$ integers is itself an integer and appears at least once in the list. The average of the smallest $k+1$ integers in the list and the average of the largest $k+1$ integers in the list by less than $\frac{1}{2025}$. Determine the smallest possible value of k.

   Clarification: If m_1 is the average of the smallest $k+1$ integers and m_2 is the average of the largest $k+1$ integers, then the condition is $|m_1 - m_2| < \frac{1}{2025}$.