Eigenfunctions of LSI Systems

$$x(n_1, n_2) = e^{j(\omega_1' n_1 + \omega_2' n_2)} \rightarrow \boxed{h(n_1, n_2)} \rightarrow y(n_1, n_2) = ?$$

(ω_1', ω_2')

$$y(n_1,n_2) = x(n_1,n_2) ** h(n_1,n_2)$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{j\omega_1'(n_1-k_1)} e^{j\omega_2'(n_2-k_2)} h(k_1,k_2)$$

$$= \underbrace{e^{j\omega_1' n_1} e^{j\omega_2' n_2}} \underbrace{\sum_{k_1} \sum_{k_2} h(k_1,k_2) e^{-j\omega_1' k_1} e^{-j\omega_2' k_2}}_{H(\omega_1', \omega_2') \triangleq \text{frequency response}}$$

2D Fourier Transform

$$\rightarrow \quad X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$\begin{cases} |X(\omega_1,\omega_2)| \\ \arg X(\omega_1,\omega_2) \end{cases}$

$$\rightarrow \quad x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$

Properties

$$X(\omega_1, \omega_2) = X(\omega_1 + 2\pi, \omega_2 + 2\pi)$$

$$x(n_1 - m_1, n_2 - m_2) \leftrightarrow e^{-j\omega_1 m_1 - j\omega_2 m_2} X(\omega_1, \omega_2)$$

$$x(n_1, n_2) e^{j\theta_1 n_1 + j\theta_2 n_2} \leftrightarrow X(\omega_1 - \theta_1, \omega_2 - \theta_2)$$ modulation

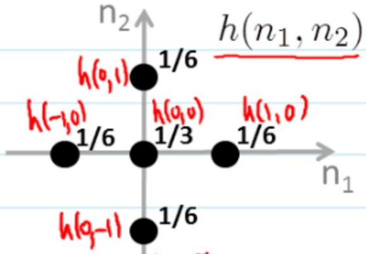
for $x(n_1, n_2)$ real : $|X(\omega_1, \omega_2)| = |X(-\omega_1, -\omega_2)|$ and $\arg X(\omega_1, \omega_2) = -\arg X(-\omega_1, -\omega_2)$

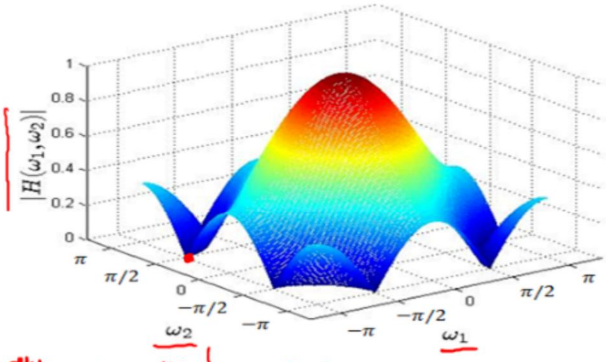
$$\underbrace{\sum_{n_1} \sum_{n_2} |x(n_1, n_2)|^2} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |X(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2$$ Parseval's theorem

↑ energy-density spectrum

Example

$h(n_1, n_2)$





$$H(\omega_1,\omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h(n_1,n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$= h(0,0) + h(-1,0) e^{j\omega_1} + h(1,0) e^{-j\omega_1} + h(0,-1) e^{j\omega_2} + h(0,1) e^{-j\omega_2}$$

$$= \frac{1}{3} + \underbrace{\frac{1}{6} e^{j\omega_1} + \frac{1}{6} e^{-j\omega_1}} + \frac{1}{6} e^{j\omega_2} + \frac{1}{6} e^{-j\omega_2}$$

$$= \frac{1}{3} + \frac{1}{6} 2\cos\omega_1 + \frac{1}{6} \cdot 2\cos\omega_2$$

$$= \frac{1}{3}\left(1 + \cos\omega_1 + \cos\omega_2\right) \qquad -1 = 1 \cdot e^{\pm j\pi}$$

$H(0,0) = 1$

$H(-\pi, \pi/2) = 0$

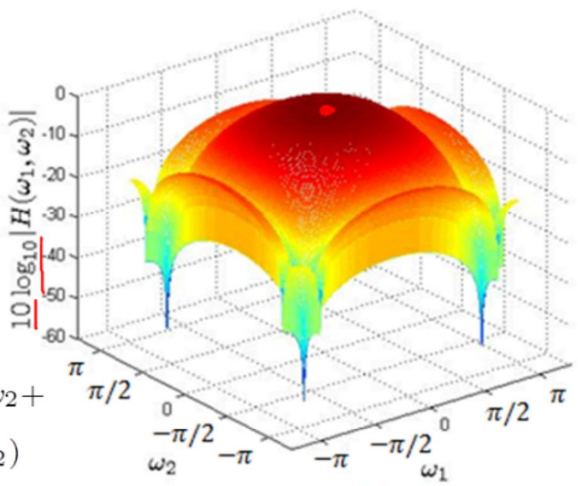
Example

$0.124 \cdot 2 \cdot \cos\omega_1$

h(-1,1) h(1,1) h(0,0)

$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

$0.075 \cdot 2 \cdot \cos(\omega_1 + \omega_2)$



$$H(\omega_1, \omega_2) = 0.204 + 0.124 \cdot 2 \cdot cos\omega_1 + 0.124 \cdot 2 \cdot cos\omega_2 + 0.075 \cdot 2 \cdot cos(\omega_1 + \omega_2) + 0.075 \cdot 2 \cdot cos(\omega_1 - \omega_2)$$

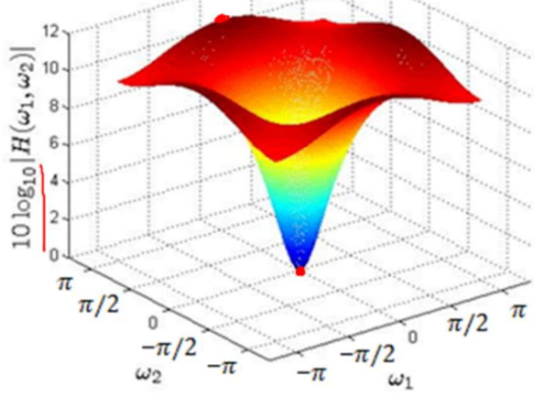
$H(0,0) = 1 \rightarrow \log_{10} H(0,0) = 0$

LPF

Example

h(0,0)

$$
h(n_1,n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}
$$



$$
H(\omega_1,\omega_2) = 9 - 2\cdot cos\omega_1 - 2\cdot cos\omega_2 - 2\cdot cos(\omega_1+\omega_2) - 2\cdot cos(\omega_1-\omega_2)
$$

$H(0,0)=1 \rightarrow \log H(0,0)=0$

$H(0,\pi)=13,\quad H(\pi,\pi)=9$

The Convolution Theorem

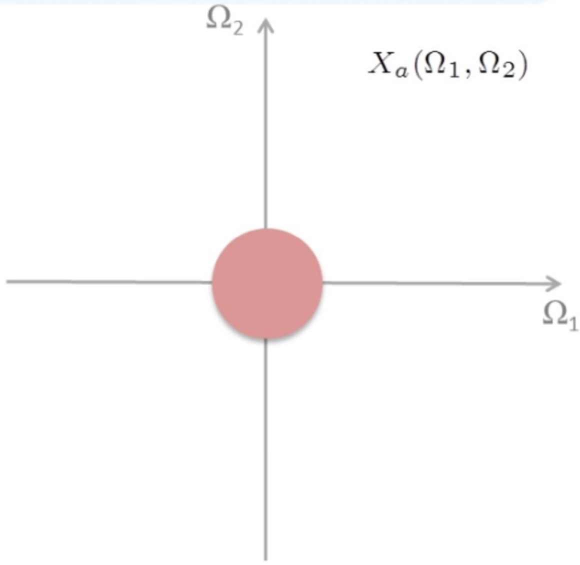
$$
\overset{X(\omega_1,\omega_2)}{x(n_1,n_2)} \longrightarrow \boxed{\overset{H(\omega_1,\omega_2)}{h(n_1,n_2)}} \longrightarrow \overset{Y(\omega_1,\omega_2)}{y(n_1,n_2)} = x(n_1,n_2) ** h(n_1,n_2)
$$

$$
Y(\omega_1,\omega_2) = X(\omega_1,\omega_2)\cdot H(\omega_1,\omega_2)
$$

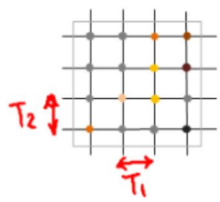
$$
\begin{aligned}
y(n_1,n_2) &= T[x(n_1,n_2)] \\
&= T\left[\frac{1}{4\pi^2}\int\!\!\int_{-\pi}^{\pi} X(\omega_1,\omega_2)\, e^{j\omega_1 n_1} e^{j\omega_2 n_2}\, d\omega_1 d\omega_2\right] \\
&= \frac{1}{4\pi^2}\int\!\!\int_{-\pi}^{\pi} X(\omega_1,\omega_2)\, \underbrace{T\left[e^{j\omega_1 n_1} e^{j\omega_2 n_2}\right]}\, d\omega_1 d\omega_2 \\
&= \frac{1}{4\pi^2}\int\!\!\int_{-\pi}^{\pi} \underbrace{X(\omega_1,\omega_2)\, H(\omega_1,\omega_2)}_{Y(\omega_1,\omega_2)}\, e^{j\omega_1 n_1} e^{j\omega_2 n_2}\, d\omega_1 d\omega_2
\end{aligned}
$$

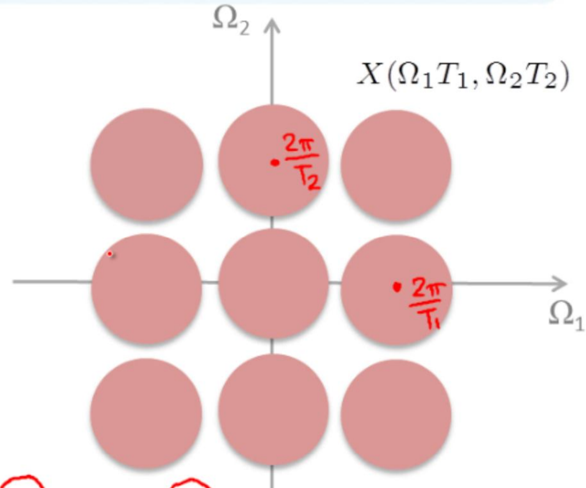
$$
\Rightarrow Y(\omega_1,\omega_2) = X(\omega_1,\omega_2)\cdot H(\omega_1,\omega_2)
$$

2D Sampling



2D Sampling



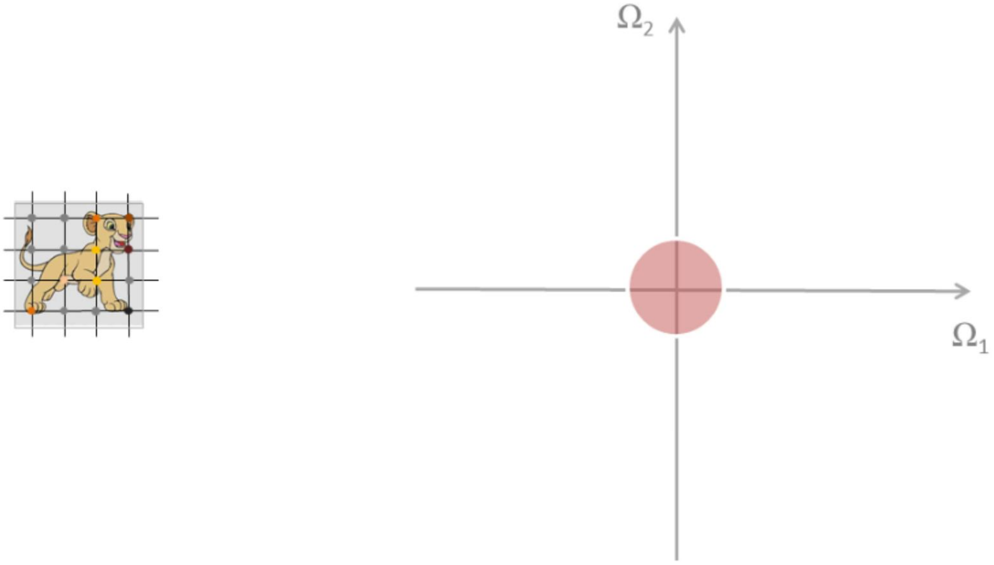


$X(\omega_1, \omega_2)$ $\quad \omega_1 = \Omega_1 T_1 \quad \omega_2 = \Omega_2 T_2$

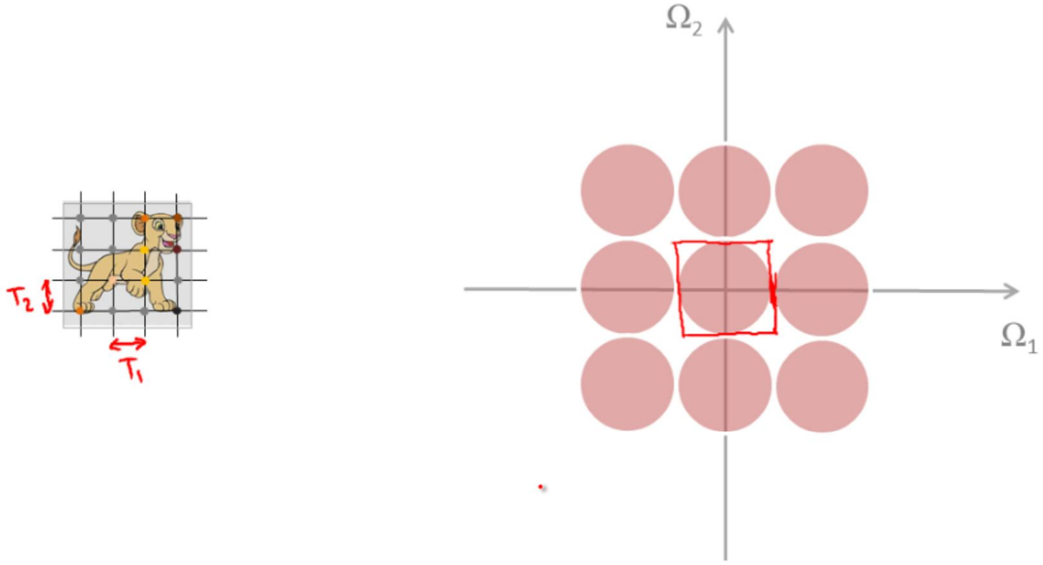
rads $\quad \frac{\text{rads}}{\text{mm}} \cdot \text{mm}$

$$X(\underline{\Omega_1 T_1}, \underline{\Omega_2 T_2}) = \frac{1}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} X_a\left(\Omega_1 - \frac{2\pi}{T_1}k_1, \Omega_2 - \frac{2\pi}{T_2}k_2\right)$$

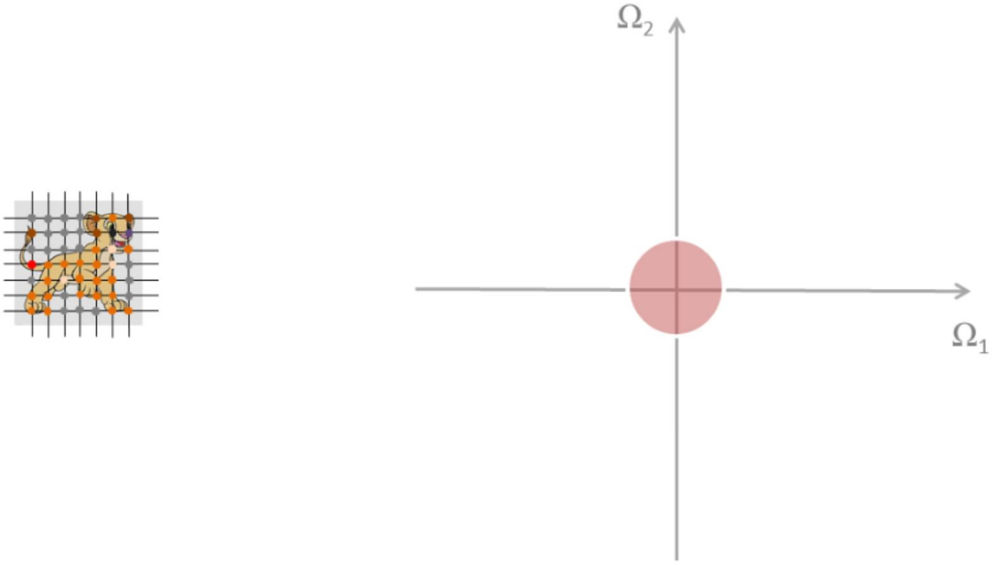
Critically Sampled



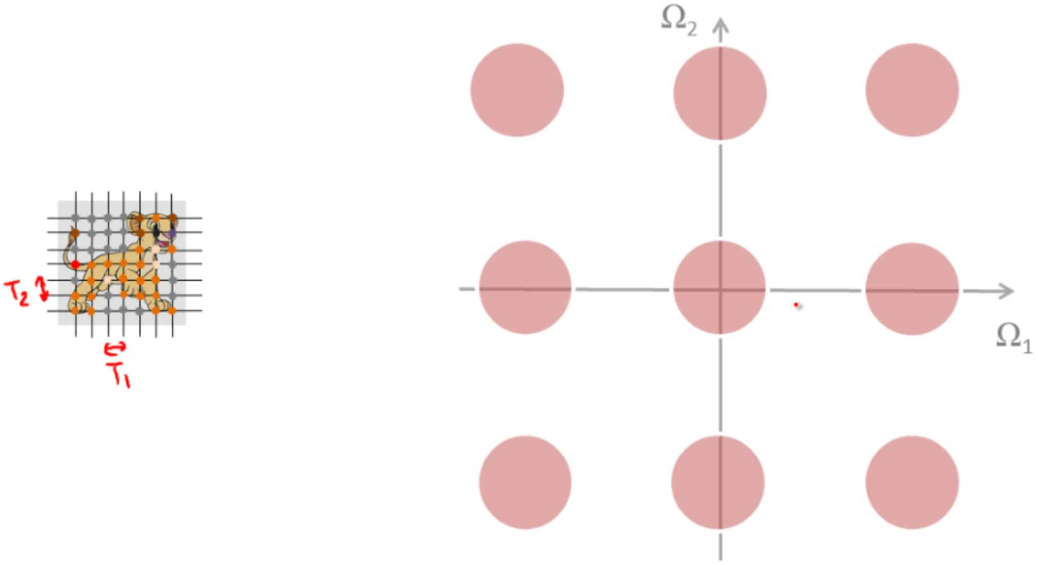
Critically Sampled



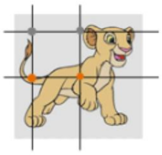
Over-Sampled

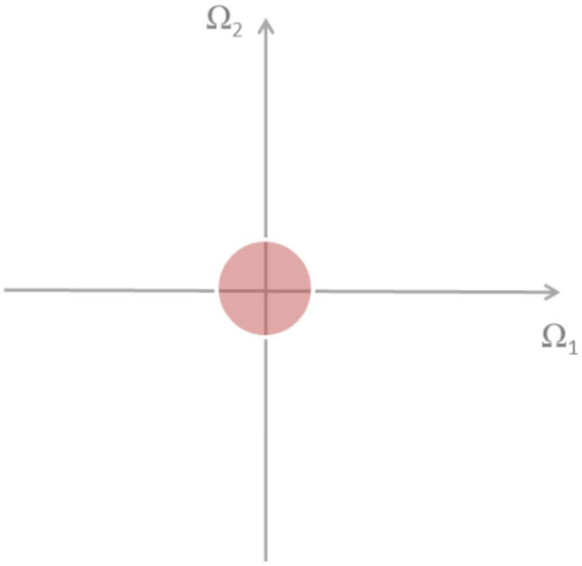


Over-Sampled

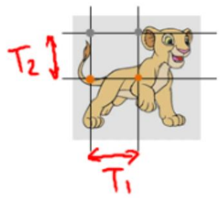


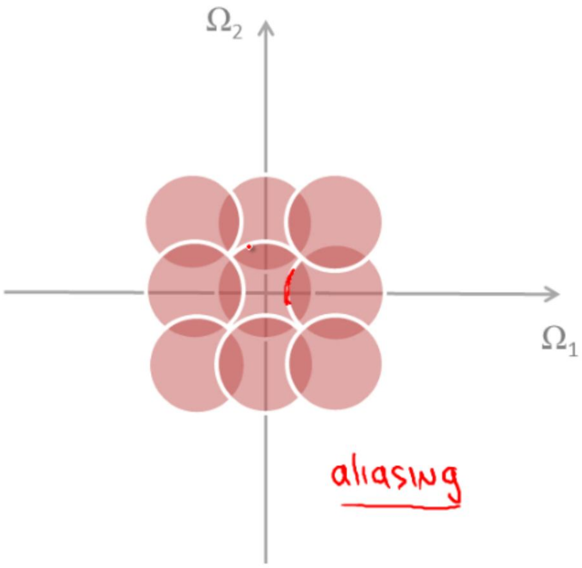
Under-Sampled



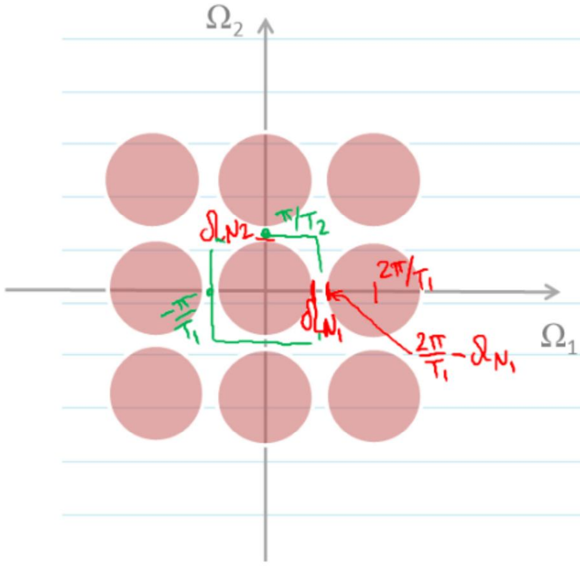


Under-Sampled





2D Nyquist Theorem

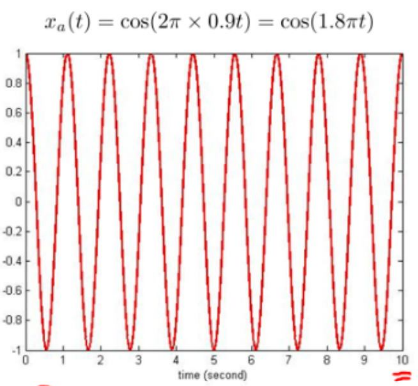


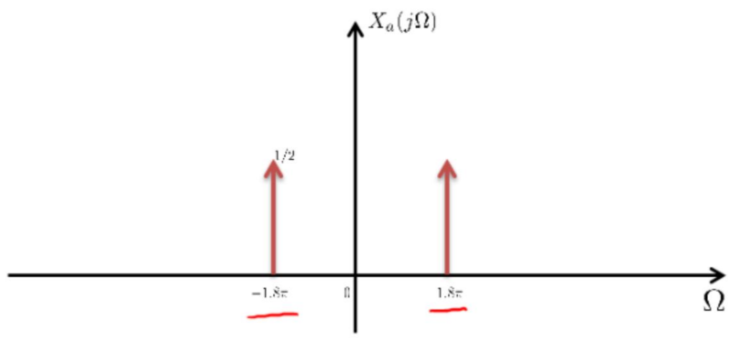
$$\frac{2\pi}{T_1} - \Omega_{N_1} \geq \Omega_{N_1} \Rightarrow \frac{2\pi}{T_1} \geq 2\Omega_{N_1}$$

$$\frac{2\pi}{T_2} \geq 2\Omega_{N_2}$$

$$F(\Omega_1, \Omega_2) = \begin{cases} T_1 T_2 & |\Omega_1| < \pi/T_1, \ |\Omega_2| < \pi/T_2 \\ 0, & \text{otherwise} \end{cases}$$

Analog Signal





$f = 0.9\text{Hz}$

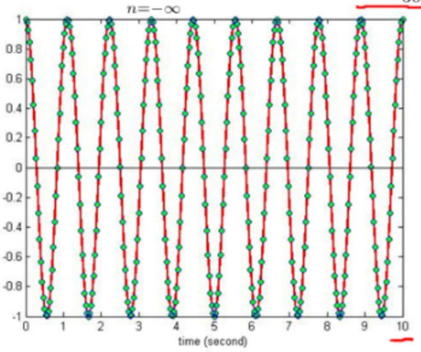
$\Omega = 2\pi f = 1.8\pi$ rads/sec

Nyquist freq: 1.8Hz / $2\pi \cdot 1.8 \cdot \frac{\text{rads}}{\text{sec}}$

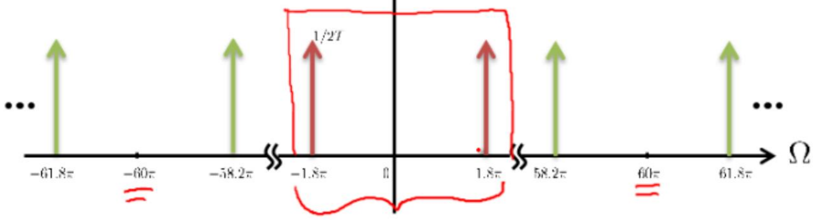
Over-Sampling

$\Omega_N = 1.8 \times 2\pi = 3.6\pi$ rads/sec

$$x_s = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t-nT) \quad T = \frac{1}{30}\text{s}, \Omega_s = \frac{2\pi}{T} = 60\pi \text{ rads/s}$$

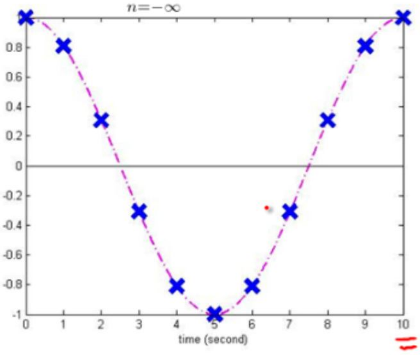


$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_s))$$



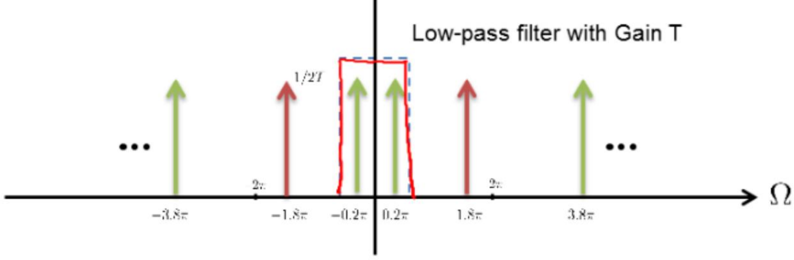
Reconstructed Aliased Signal

$$x_s = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t-nT) \quad T = 1\text{s}, \Omega_s = \frac{2\pi}{T} = 2\pi \text{ rads/s}$$

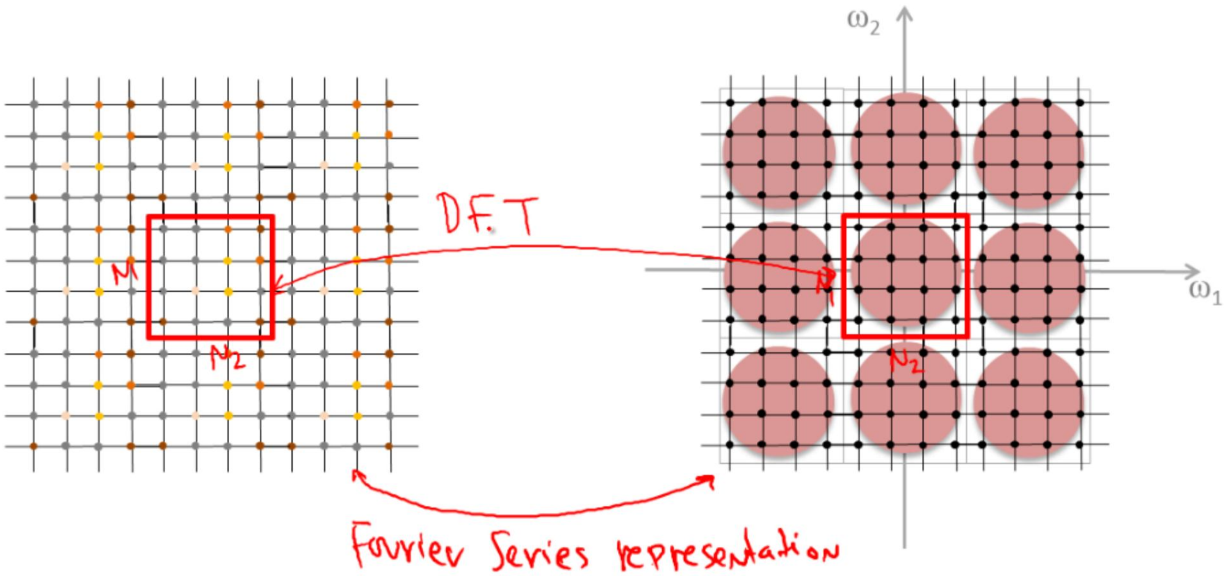


0.1 Hz
1 cycle

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_s))$$



Sampling in the Frequency Domain



Discrete Fourier Transform (DFT)

$$X(\omega_1,\omega_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}$$

$$X(k_1,k_2)=X(\omega_1,\omega_2)\big|_{\omega_1=\frac{2\pi}{N_1}k_1,\,\omega_2=\frac{2\pi}{N_2}k_2} \qquad \begin{cases}k_1=0,\dots,N_1-1\\ k_2=0,\dots,N_2-1\end{cases}$$

$$X(k_1,k_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,n_2)e^{-j\frac{2\pi}{N_1}n_1k_1}e^{-j\frac{2\pi}{N_2}n_2k_2} \qquad \begin{cases}k_1=0,\dots,N_1-1\\ k_2=0,\dots,N_2-1\end{cases}$$

$$x(n_1,n_2)=\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}X(k_1,k_2)e^{j\frac{2\pi}{N_1}n_1k_1}e^{j\frac{2\pi}{N_2}n_2k_2} \qquad \begin{cases}n_1=0,\dots,N_1-1\\ n_2=0,\dots,N_2-1\end{cases}$$

FT linear shifts ⟷ DFT circular shifts

Fast Fourier Transforms (FFTs)

$$X(k_1,k_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,n_2)\underline{e^{-j\frac{2\pi}{N_1}n_1k_1}}\,\underline{e^{-j\frac{2\pi}{N_2}n_2k_2}}\qquad \begin{matrix}k_1=0,\dots,N_1-1\\ k_2=0,\dots,N_2-1\end{matrix}$$

A. Direct Computations

For each (k_1,k_2) : N_1N_2 mults; Total $N_1^2N_2^2$ mults $(N_1=N_2=N:\ N^4)$

B. Row-Column Decomposition

$$X(k_1,k_2)=\sum_{n_1=0}^{N_1-1}\left[\underbrace{\sum_{n_2=0}^{N_2-1}x(n_1,n_2)e^{-j\frac{2\pi}{N_2}n_2k_2}}_{G(n_1,k_2)}\right]e^{-j\frac{2\pi}{N_1}n_1k_1}=\sum_{n_1=0}^{N_1-1}G(n_1,k_2)e^{-j\frac{2\pi}{N_1}n_1k_1}$$

$x(n_1,n_2)$ n_2 n_1

$G(n_1,k_2)$ k_2 n_1

r/c directly: $N_1\cdot N_2^2+N_2\cdot N_1^2=N_1N_2(N_1+N_2)$ $(N_1=N_2=N:\ 2N^3)$

r/c FFT: $N_1\cdot\frac{N_2}{2}\log_2 N_2+N_2\cdot\frac{N_1}{2}\log_2 N_1=\frac{N_1N_2}{2}\log(N_1N_2)$

$(N_1=N_2=N:\ N^2\log_2 N)$

1965 Cooley+Tukey

radix (2×2) FFT: $\frac{3}{4}N^2\log_2 N$

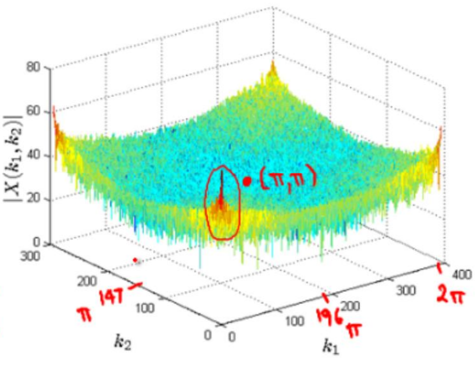
DFT

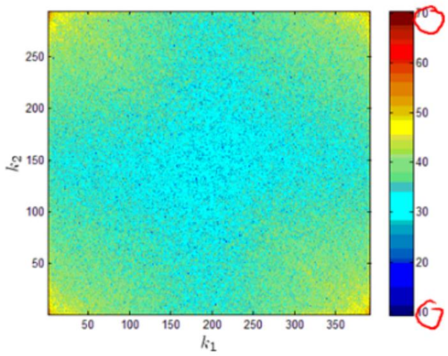
n_2

n_1

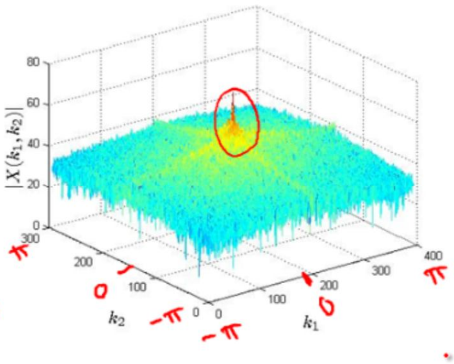
392x294

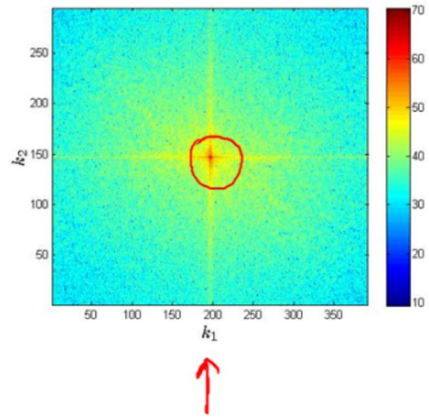
8bpp





DFT (centered)





$$(-1)^{n_1+n_2} = e^{-j\pi(n_1+n_2)}$$

2D Circular Convolution

$$x(n_1,n_2) \longrightarrow \boxed{h(n_1,n_2)} \longrightarrow y_L(n_1,n_2) = x(n_1,n_2) ** h(n_1,n_2)$$

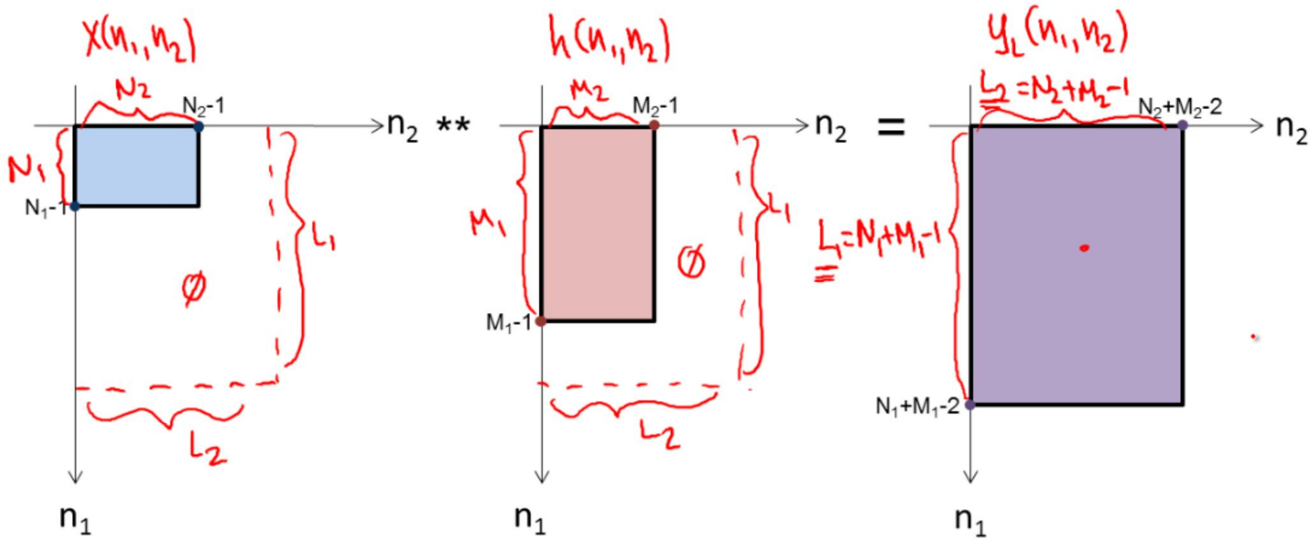
$$X(k_1,k_2) \longrightarrow \boxed{H(k_1,k_2)} \longrightarrow Y(k_1,k_2) = X(k_1,k_2) \cdot H(k_1,k_2)$$

$$y(n_1,n_2) = x(n_1,n_2) \circledast \circledast h(n_1 n_2)$$

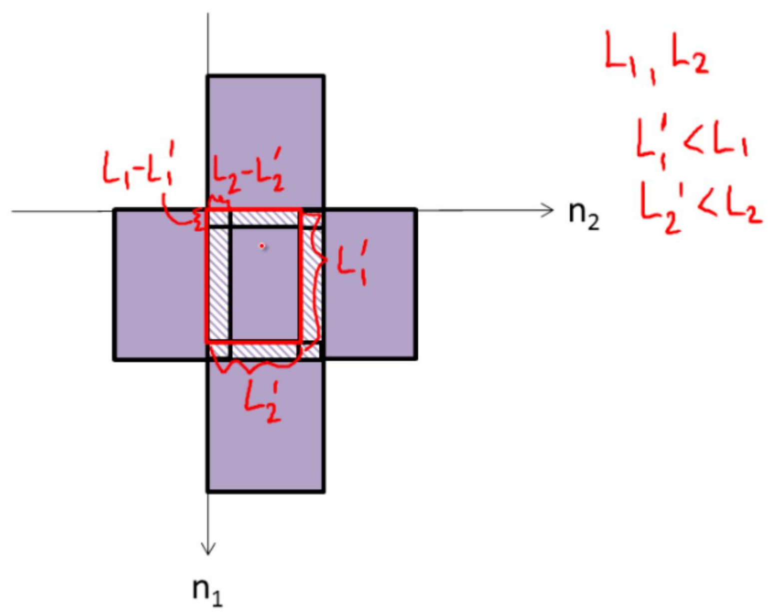
Circular Convolution

$$y(n_1,n_2) = \sum_{r_1} \sum_{r_2} y_L(n_1 - r_1 N_1, n_2 - r_2 N_2) \qquad \begin{cases} n_1 = 0,\ldots,N_1-1 \\ n_2 = 0,\ldots,N_2-1 \end{cases}$$

ROS of Linear Convolution

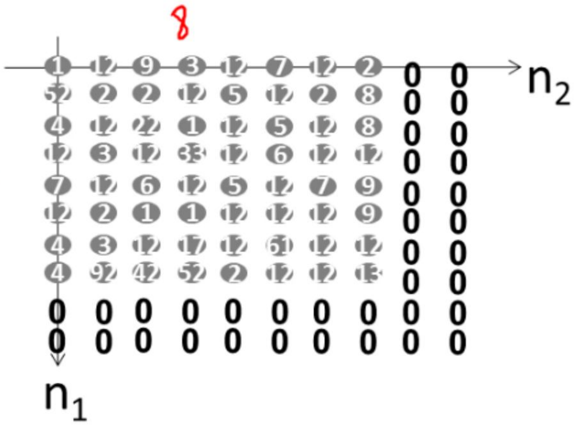


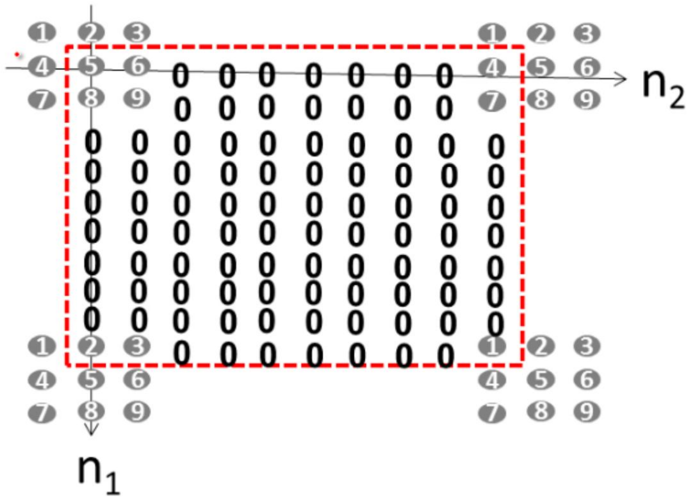
Spatial Aliasing



Use of DFT for Filtering

$8+3-1=10$

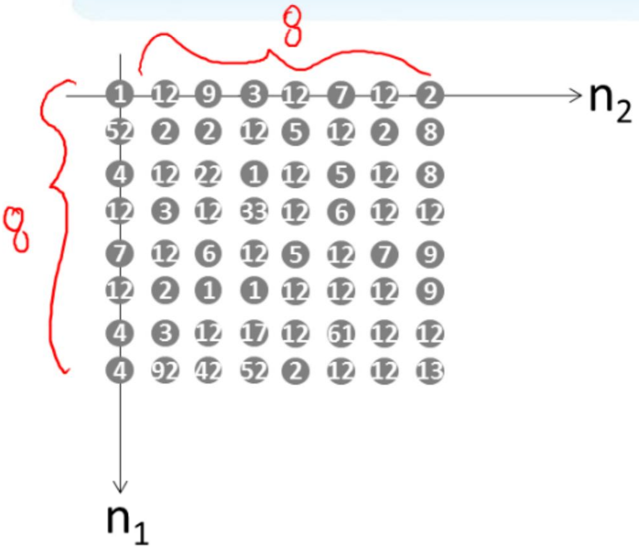


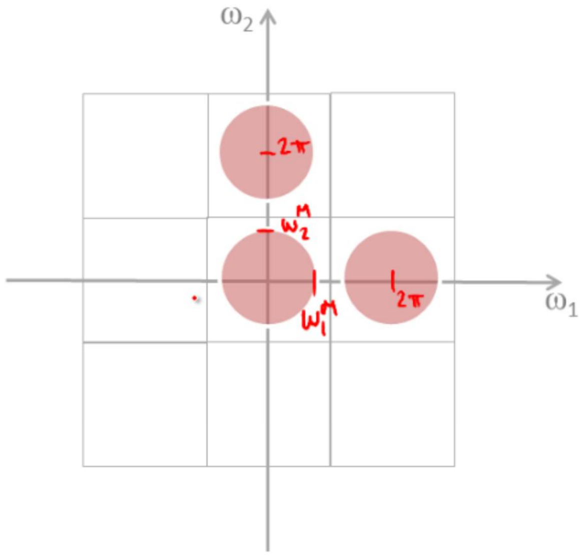


Frequency Filtering

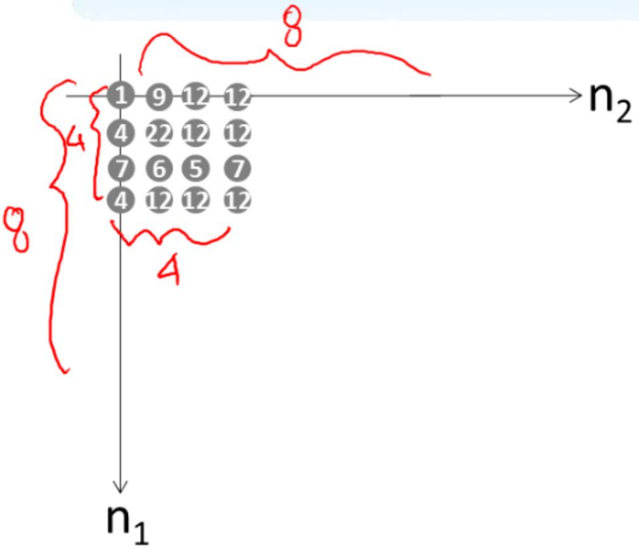
$y(n_1,n_2) = x(n_1,n_2) + \cos(0.1\pi n_2)$

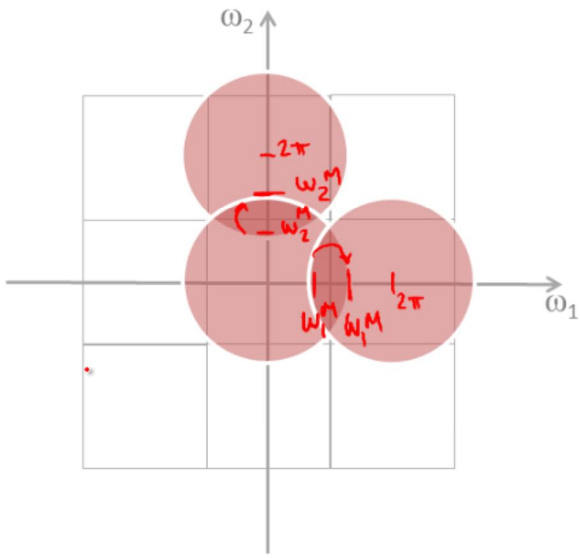
Down-Sampling





Down-Sampling

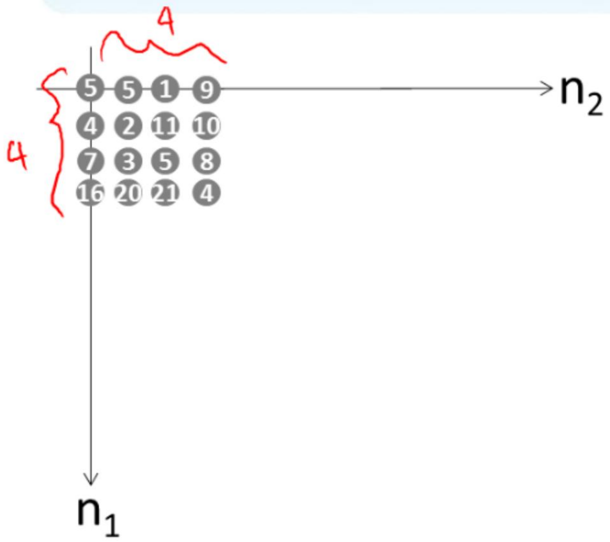


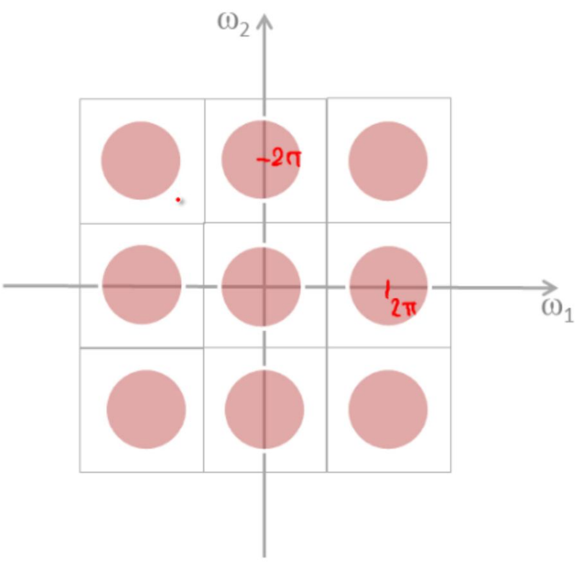


Down-Sampling Example

Resolution of original image: 4200 x 3000 pixels (not shown due to space limitations).
Left image: Down-sampled by a factor of 10, direct pixel removal.
Right image: LPF by a Gaussian 11x11 filter, then down-sampled by a factor of 10.

Up-Sampling



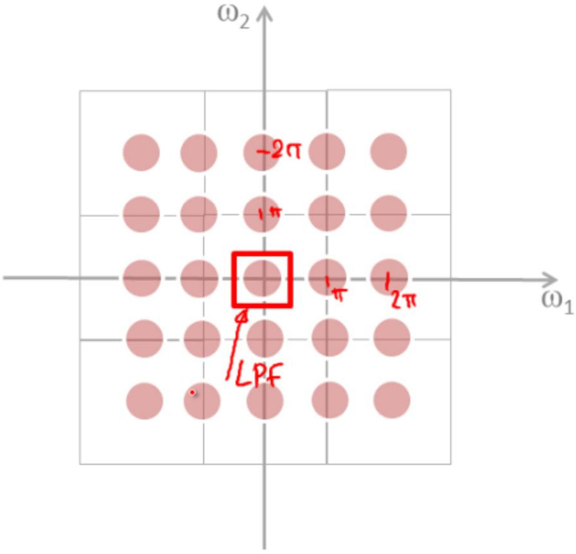


Up-Sampling

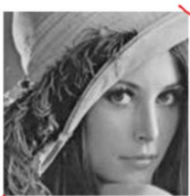
n_2 (horizontal axis), n_1 (vertical axis)

5	0	5	0	1	0	9	0
0	0	0	0	0	0	0	0
4	0	2	0	11	0	10	0
0	0	0	0	0	0	0	0
7	0	3	0	5	0	8	0
0	0	0	0	0	0	0	0
16	0	20	0	21	0	4	0
0	0	0	0	0	0	0	0

4   4   8   8

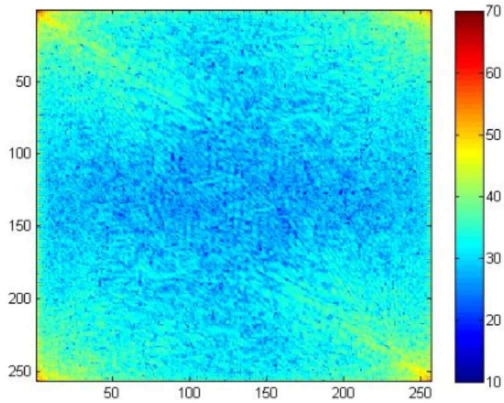


Up-Sampling Example



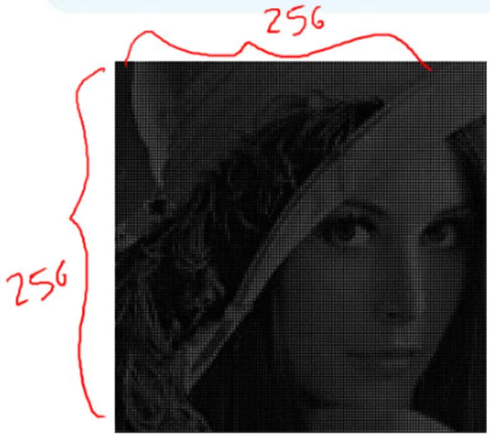
128x128 input image

Ø

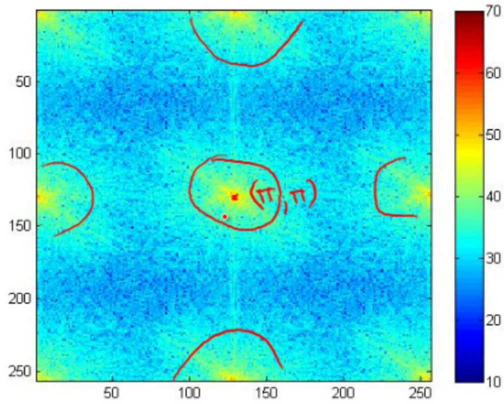


Magnitude of the 256x 256-point DFT, obtained by zero padding the input image to size 256x256 and then taking the DFT

Up-Sampling Example

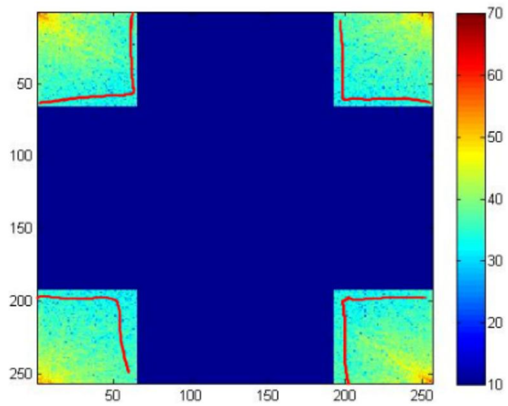


256x256 image obtained by inserting zero-columns and zero-rows to the original 128x128 image



Magnitude of 256x256-point DFT

Up-Sampling Example



LPF spectrum (blue area corresponds to zero values)

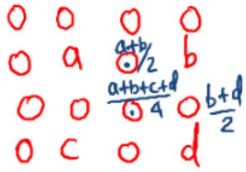
Inverse DFT

Up-Sampling Example

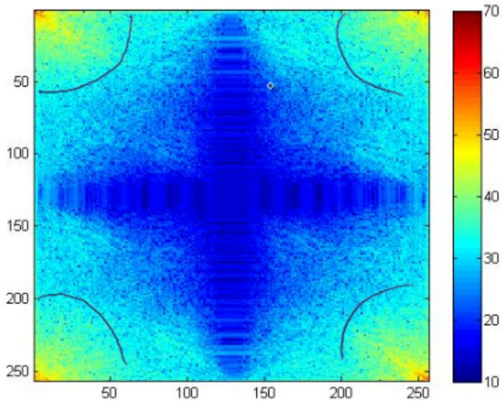
Up-sampled 256x256 image, with filter

$$\begin{vmatrix} 0.25 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{vmatrix}$$

$h(0,0)$



$$H(\omega_1,\omega_2) = 1 + \cos\omega_1 + \cos\omega_2 + \frac{1}{2}\cos(\omega_1+\omega_2) + \frac{1}{2}\cos(\omega_1-\omega_2)$$



Spectrum