Leaf Energy Balance

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Quadratic Solutions for Leaf Temperature and Latent Heat Exchange

Isothermal Energy Balance

One Sided Leaf

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Taylor expansion Series

$$E[f(x)] = f(\bar{x}) + \frac{df(\bar{x})}{dx}(x-\bar{x}) + \frac{1}{2}\frac{d^2 f(\bar{x})}{dx^2}(x-\bar{x})^2$$

Linearization of Clausis Claperyon, saturation vapor pressure-temperature function, $e_s(T)$

$$e_s(T_s) = e_s(T_a) + \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2$$

$$(T_s - T_a)^2 = T_s^2 - T_a T_s + T_a^2$$

Linearization of Stefan Bolzmann, T^4

$$\varepsilon\sigma T_s^4 = \varepsilon\sigma T_a^4 + 4\varepsilon\sigma T_a^3(T_s - T_a) + 6\varepsilon\sigma T_a^2(T_s - T_a)^2$$

Solve for set of coupled energy balance equations, one Sided case

Sensible heat flux

$$H = \rho C_p g_h (T_s - T_a)$$

Latent heat flux

$$\lambda E = \frac{\rho m_v}{P m_a} \lambda g_w (e_s(T_s) - e_a)$$

$$g_w = 1/(r_b + r_s) = \frac{1}{1/g_b + 1/g_s} = \frac{g_s g_b}{g_s + g_b}$$

$$\lambda E = \frac{\rho m_v}{P m_a} \lambda g_w [e_s(T_a) + \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - e_a]$$

Isothermal radiation budget, single sided leaf

$$Q = R\downarrow - R\uparrow + L\downarrow - L\uparrow$$

$$L\downarrow - L\uparrow = L\downarrow - (1-\varepsilon)L\downarrow = \varepsilon L\downarrow$$

$$Q = R\downarrow(1-\rho) + \varepsilon L\downarrow$$

$$R_n = (1-\alpha)R_g + \varepsilon L\downarrow - \varepsilon\sigma T_s^4 = \lambda E + H$$

$$R_n = Q - \varepsilon\sigma T_s^4 = \lambda E + H$$

$$T_s = \frac{H}{\rho C_p g_h} + T_a$$

$$H = Q - \lambda E - \varepsilon\sigma T_s^4$$

$$H = \rho C_p g_h (T_s - T_a) = Q - \lambda E - \varepsilon\sigma T_s^4$$

$$H = Q - \lambda E - \varepsilon\sigma T_a^4 - 4\varepsilon\sigma T_a^3 (T_s - T_a) - 6\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

$$T_s - T_a = \frac{Q - \lambda E - \varepsilon\sigma T_a^4 - 4\varepsilon\sigma T_a^3 (T_s - T_a) - 6\varepsilon\sigma T_a^2 (T_s - T_a)^2}{\rho C_p g_h}$$

$$T_s - T_a = \frac{Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) + \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - e_a] - \varepsilon\sigma T_a^{\,4} - 4\varepsilon\sigma T_a^{\,3}(T_s - T_a) - 6\varepsilon\sigma T_a^2 (T_s - T_a)^2}{\rho C_p g_h}$$

$$T_s - T_a = \frac{Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) - e_a] - \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) - \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - \varepsilon\sigma T_a^{\,4} - 4\varepsilon\sigma T_a^{\,3}(T_s - T_a) - 6\varepsilon\sigma T_a^2 (T_s - T_a)^2}{\rho C_p g_h}$$

$$\begin{aligned}(T_s - T_a)\rho C_p g_h = Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) - e_a] - \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) - \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - \varepsilon\sigma T_a^{\,4} \\ - 4\varepsilon\sigma T_a^{\,3}(T_s - T_a) - 6\varepsilon\sigma T_a^2 (T_s - T_a)^2\end{aligned}$$

$$\begin{aligned}0 = Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) - e_a] - \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) - \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - \varepsilon\sigma T_a^{\,4} \\ - 4\varepsilon\sigma T_a^{\,3}(T_s - T_a) - 6\varepsilon\sigma T_a^2 (T_s - T_a)^2 - (T_s - T_a)\rho C_p g_h\end{aligned}$$

$$\begin{aligned}(T_s - T_a)^2[-\frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2} - 6\varepsilon\sigma T_a^2] + (T_s - T_a)[-\frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT} - \rho C_p g_h - 4\varepsilon\sigma T_a^{\,3}] \\ + Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) - e_a] - \varepsilon\sigma T_a^{\,4} = 0\end{aligned}$$

$$\begin{aligned}(T_s - T_a)^2[\frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2} + 6\varepsilon\sigma T_a^2] + (T_s - T_a)[\frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT} + \rho C_p g_h + 4\varepsilon\sigma T_a^{\,3}] \\ - Q + \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) - e_a] + \varepsilon\sigma T_a^{\,4} = 0\end{aligned}$$

$$a(T_s - T_a)^2 + b(T_s - T_a) + c = 0$$

$$a = [6\varepsilon\sigma T_a^2 + \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}]$$

$$b = [\rho C_p g_h + \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT} + 4\varepsilon\sigma T_a^{\,3}]$$

$$c = \frac{\rho m_v}{P m_a} \lambda g_w [e_s(T_a) - e_a] + \varepsilon\sigma T_a^{\ 4} - Q$$

Substitute g_w with g_s and g_b:

$$a = [6\varepsilon\sigma T_a^2 + \frac{\rho m_v}{P m_a} \lambda \frac{g_s g_b}{g_s + g_b} \frac{1}{2} \frac{d^2 e_s(T_a)}{dT^2}]$$

$$b = [\rho C_p g_h + \frac{\rho m_v}{P m_a} \lambda \frac{g_s g_b}{g_s + g_b} \frac{de_s(T_a)}{dT} + 4\varepsilon\sigma T_a^{\ 3}]$$

$$c = \frac{\rho m_v}{P m_a} \lambda \frac{g_s g_b}{g_s + g_b} [e_s(T_a) - e_a] + \varepsilon\sigma T_a^{\ 4} - Q$$

The leaf energy balance can also be used to derive a quadratic equation for latent heat exchange (λE; W m^{-2}):

$$a\, LE^2 + b\, LE + c = 0$$

We can solve directly for latent heat flux density

$$\lambda E = \frac{0.622 \lambda \rho_a (e_s(T_l) - e_a)}{P} \frac{g_s g_a}{g_a + g_s} = \frac{0.622 \lambda \rho_a (e_s(T_l) - e_a)}{P(r_s + r_a)}$$

Linearize $(e_s(T_l) - e_a)$ with 2[nd] order Taylor Expansion

$$\lambda E = \frac{0.622 \lambda \rho_a g_w}{P} [(e_s(T_a) + s(T_a)(T_l - T_a) + \frac{e_s(T_a)''}{2}(T_l - T_a)^2 - e_a]$$

I have second order terms of $(T_l - T_a)$

Substituting

Use 1st order expansion of Ts for LE

$$\varepsilon\sigma T_s^{\ 4} = \varepsilon\sigma T_a^{\ 4} + 4\varepsilon\sigma T_a^{\ 3}(T_s - T_a)$$

$$T_s - T_a = \frac{Q - \lambda E - \sigma\varepsilon T_a^4}{(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3)}$$

$$\lambda E = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a) + s(T_a)(\frac{Q - \lambda E - \sigma\varepsilon T_a^4}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3}) + \frac{e_s(T_a)''}{2}(\frac{Q - \lambda E - \sigma\varepsilon T_a^4}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})^2]$$

Multiply LE by $\rho_a C_p g_h + 4\varepsilon\sigma T_a^3$

$$\lambda E(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) + s(T_a)(Q - \lambda E - \sigma\varepsilon T_a^4)$$
$$+ \frac{e_s(T_a)''}{2}(\frac{(Q - \lambda E - \sigma\varepsilon T_a^4)^2}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$

The squared term is:

$$(Q - \lambda E - \sigma\varepsilon T_a^4)(Q - \lambda E - \sigma\varepsilon T_a^4) =$$
$$QQ - Q\lambda E - Q\sigma\varepsilon T_a^4 - \lambda EQ + \lambda E\lambda E + \lambda E\sigma\varepsilon T_a^4 - \sigma\varepsilon T_a^4 Q + \sigma\varepsilon T_a^4\lambda E + \sigma\varepsilon T_a^4\sigma\varepsilon T_a^4 =$$
$$QQ - 2Q\lambda E - 2Q\sigma\varepsilon T_a^4 + 2\lambda E\sigma\varepsilon T_a^4 + \sigma\varepsilon T_a^4\sigma\varepsilon T_a^4 + \lambda E\lambda E$$

Substitute squared term and Simplifying:

$$\lambda E(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) + s(T_a)(Q - \lambda E - \sigma\varepsilon T_a^4)$$
$$+ \frac{e_s(T_a)''}{2}(\frac{QQ - 2Q\lambda E - 2Q\sigma\varepsilon T_a^4 + 2\lambda E\sigma\varepsilon T_a^4 + \sigma\varepsilon T_a^4\sigma\varepsilon T_a^4 + \lambda E\lambda E}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$

Organize in terms of LE^2 LE and intercept

$$0 = \frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{\lambda E\lambda E}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$
$$-\lambda E(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) - \frac{0.622\lambda\rho_a g_w}{P}[s(T_a)\lambda E + \frac{e_s(T_a)''}{2}(\frac{2\lambda E\sigma\varepsilon T_a^4 - 2Q\lambda E}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$
$$\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) + s(T_a)(Q - \sigma\varepsilon T_a^4)$$
$$+ \frac{e_s(T_a)''}{2}(\frac{QQ - 2Q\sigma\varepsilon T_a^4 + \sigma\varepsilon T_a^4\sigma\varepsilon T_a^4}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$

$$0 = +\frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{\lambda E\lambda E}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{2\lambda E\sigma\varepsilon T_a^4 - 2Q\lambda E}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})] - \lambda E\rho_a C_p g_h - \lambda E 4\varepsilon\sigma T_a^3 - \frac{0.622\lambda\rho_a g_w}{P}s(T_a)\lambda E$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{QQ - 2Q\sigma\varepsilon T_a^4 + \sigma\varepsilon T_a^4\sigma\varepsilon T_a^4}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) + s(T_a)(Q - \sigma\varepsilon T_a^4)]$$

Separate out a, b and c for quadratic solution

$$a = \frac{0.622\lambda\rho_a g_w}{P(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3)}\frac{e_s(T_a)''}{2}$$

$$b = \frac{0.622\lambda\rho_a g_w}{P}[e_s(T_a)''(\frac{\sigma\varepsilon T_a^4 - Q}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})] - \rho_a C_p g_h - 4\varepsilon\sigma T_a^3 - \frac{0.622\lambda\rho_a g_w}{P}s(T_a)$$

$$c = \frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{QQ - 2Q\sigma\varepsilon T_a^4 + \sigma\varepsilon T_a^4\sigma\varepsilon T_a^4}{\rho_a C_p g_h + 4\varepsilon\sigma T_a^3})]$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(\rho_a C_p g_h + 4\varepsilon\sigma T_a^3) + s(T_a)(Q - \sigma\varepsilon T_a^4)]$$

Compute Latent Heat Exchange

1: **Hypostomatous** leaves have stomata on one side. Their two-sided net radiation balance is:

$$R_n = 2H + \lambda E = (R\downarrow - R\uparrow + L\downarrow - L\uparrow)_{top} + (R\uparrow - R\downarrow + L\uparrow - L\downarrow)_{bottom}$$

$$R_n = 2H + \lambda E = Q_{in} - 2\varepsilon\sigma T_{sfc}^4$$

$$H = \rho C_p g_h (T_s - T_a)$$

Latent heat flux

$$\lambda E = \frac{\rho m_v}{P m_a} \lambda g_w (e_s(T_s) - e_a)$$

$$g_w = 1/(r_b + r_s) = \frac{1}{1/g_b + 1/g_s} = \frac{g_s g_b}{g_s + g_b}$$

$$\lambda E = \frac{\rho m_v}{P m_a} \lambda g_w [e_s(T_a) + \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - e_a]$$

$$2H = Q_{in} - \lambda E - 2\varepsilon\sigma T_s^{\,4}$$

$$2H = 2\rho C_p g_h (T_s - T_a) = Q_{in} - \lambda E - 2\varepsilon\sigma T_s^{\,4}$$

1st order expansion

$$2H = 2\rho C_p g_h (T_s - T_a) = Q_{in} - \lambda E - 2\varepsilon\sigma T_a^{\,4} - 8\varepsilon\sigma T_a^{\,3}(T_s - T_a)$$

$$T_s - T_a = \frac{Q_{in} - \lambda E - 2\sigma\varepsilon T_a^4}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$

The leaf energy balance can also be used to derive a quadratic equation for latent heat exchange (λE; W m^{-2}):

$$a\,LE^2 + b\,LE + c = 0$$

We can solve directly for latent heat flux density

$$\lambda E = \frac{0.622\lambda\rho_a(e_s(T_l)-e_a)}{P}\frac{g_s g_a}{g_a+g_s} = \frac{0.622\lambda\rho_a(e_s(T_l)-e_a)}{P(r_s+r_a)}$$

Linearize $(e_s(T_l)-e_a)$ with 2nd order Taylor Expansion

$$\lambda E = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a)+s(T_a)(T_l-T_a)+\frac{e_s(T_a)''}{2}(T_l-T_a)^2-e_a]$$

Substituting

Use 1st order expansion of Ts for LE

$$\varepsilon\sigma T_s^{\,4} = \varepsilon\sigma T_a^{\,4} + 4\varepsilon\sigma T_a^{\,3}(T_s-T_a)$$

$$T_s - T_a = \frac{Q_{in}-\lambda E - 2\sigma\varepsilon T_a^4}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$

$$\lambda E = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a)-e_a)+s(T_a)(\frac{Q_{in}-\lambda E-2\sigma\varepsilon T_a^4}{2\rho_a C_p g_h+8\varepsilon\sigma T_a^3})+\frac{e_s(T_a)''}{2}(\frac{Q_{in}-\lambda E-2\sigma\varepsilon T_a^4}{2\rho_a C_p g_h+8\varepsilon\sigma T_a^3})^2]$$

Multiply LE by $2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3$

$$\lambda E(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3) = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a)-e_a)(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)+s(T_a)(Q_{in}-\lambda E-2\sigma\varepsilon T_a^4)$$
$$+\frac{e_s(T_a)''}{2}(\frac{(Q_{in}-\lambda E-2\sigma\varepsilon T_a^4)^2}{2\rho_a C_p g_h+8\varepsilon\sigma T_a^3})]$$

The squared term is:

$$(Q-\lambda E-2\sigma\varepsilon T_a^4)(Q-\lambda E-2\sigma\varepsilon T_a^4) =$$
$$QQ - Q\lambda E - Q2\sigma\varepsilon T_a^4 - \lambda E Q + \lambda E\lambda E + \lambda E 2\sigma\varepsilon T_a^4 - 2\sigma\varepsilon T_a^4 Q + 2\sigma\varepsilon T_a^4\lambda E + 2\sigma\varepsilon T_a^4 2\sigma\varepsilon T_a^4 =$$
$$QQ - 2Q\lambda E - 4Q\sigma\varepsilon T_a^4 + 4\lambda E\sigma\varepsilon T_a^4 + 4\sigma\varepsilon T_a^4\sigma\varepsilon T_a^4 + \lambda E\lambda E$$

Substitute squared term and Simplifying:

$$\lambda E(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3) = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3) + s(T_a)(Q - \lambda E - 2\sigma\varepsilon T_a^4)$$
$$+\frac{e_s(T_a)''}{2}(\frac{QQ - 2Q\lambda E - 4Q\sigma\varepsilon T_a^4 + 4\lambda E\sigma\varepsilon T_a^4 + 4\sigma\varepsilon T_a^4\sigma\varepsilon T_a^4 + \lambda E\lambda E}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})]$$

Organize in terms of LE[2] LE and intercept

$$0 = \frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{\lambda E\lambda E}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})]$$
$$-\lambda E(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3) - \frac{0.622\lambda\rho_a g_w}{P}[s(T_a)\lambda E + \frac{e_s(T_a)''}{2}(\frac{4\lambda E\sigma\varepsilon T_a^4 - 2Q\lambda E}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})]$$
$$\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3) + s(T_a)(Q - 2\sigma\varepsilon T_a^4)$$
$$+\frac{e_s(T_a)''}{2}(\frac{QQ - 4Q\sigma\varepsilon T_a^4 + 4\sigma\varepsilon T_a^4\sigma\varepsilon T_a^4}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})]$$

$$0 = +\frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{\lambda E\lambda E}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})]$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{4\lambda E\sigma\varepsilon T_a^4 - 2Q\lambda E}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})] - \lambda E 2\rho_a C_p g_h - \lambda E 8\varepsilon\sigma T_a^3 - \frac{0.622\lambda\rho_a g_w}{P}s(T_a)\lambda E$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{QQ - 4Q\sigma\varepsilon T_a^4 + 4\sigma\varepsilon T_a^4\sigma\varepsilon T_a^4}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})]$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3) + s(T_a)(Q - 2\sigma\varepsilon T_a^4)]$$

Separate out a, b and c for quadratic solution

$$a = \frac{0.622\lambda\rho_a g_w}{P(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}\frac{e_s(T_a)''}{2}$$

$$b = \frac{0.622\lambda\rho_a g_w}{P}[e_s(T_a)''(\frac{4\sigma\varepsilon T_a^4 - 2Q}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})] - 2\rho_a C_p g_h - 8\varepsilon\sigma T_a^3 - \frac{0.622\lambda\rho_a g_w}{P}s(T_a)$$

$$c = \frac{0.622\lambda\rho_a g_w}{P}[\frac{e_s(T_a)''}{2}(\frac{QQ - 4Q\sigma\varepsilon T_a^4 + 4\sigma\varepsilon T_a^4\sigma\varepsilon T_a^4}{2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3})]$$
$$+\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a)(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3) + s(T_a)(Q - 2\sigma\varepsilon T_a^4)]$$

Leaf Temperature

2nd order expansion

$$2H = 2\rho C_p g_h (T_s - T_a) = Q_{in} - \lambda E - 2\varepsilon\sigma T_a^{\;4} - 8\varepsilon\sigma T_a^{\;3}(T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

$$T_s - T_a = \frac{Q_{in} - \lambda E - 2\varepsilon\sigma T_a^{\;4} - 8\varepsilon\sigma T_a^{\;3}(T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2}{2\rho C_p g_h}$$

$$T_s - T_a = \frac{Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) + \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - e_a] - 2\varepsilon\sigma T_a^{\;4} - 8\varepsilon\sigma T_a^{\;3}(T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2}{2\rho C_p g_h}$$

$$T_s - T_a = \frac{Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) - e_a] - \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) - \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - 2\varepsilon\sigma T_a^{\;4} - 8\varepsilon\sigma T_a^{\;3}(T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2}{2\rho C_p g_h}$$

$$(T_s - T_a) 2\rho C_p g_h = Q - \frac{\rho m_v}{P m_a}\lambda g_w [e_s(T_a) - e_a] - \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) - \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - 2\varepsilon\sigma T_a^{\;4}$$
$$- 8\varepsilon\sigma T_a^{\;3}(T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

$$0 = Q - \frac{\rho m_v}{P m_a}\lambda g_w[e_s(T_a) - e_a] - \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) - \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - 2\varepsilon\sigma T_a{}^4$$
$$- 8\varepsilon\sigma T_a{}^3 (T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2 - (T_s - T_a) 2\rho C_p g_h$$

$$(T_s - T_a)^2[-\frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2} - 12\varepsilon\sigma T_a^2] + (T_s - T_a)[-\frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT} - 2\rho C_p g_h - 8\varepsilon\sigma T_a{}^3]$$
$$+ Q - \frac{\rho m_v}{P m_a}\lambda g_w[e_s(T_a) - e_a] - 2\varepsilon\sigma T_a{}^4 = 0$$

$$(T_s - T_a)^2[\frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2} + 12\varepsilon\sigma T_a^2] + (T_s - T_a)[\frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT} + 2\rho C_p g_h + 8\varepsilon\sigma T_a^3]$$
$$- Q + \frac{\rho m_v}{P m_a}\lambda g_w[e_s(T_a) - e_a] + 2\varepsilon\sigma T_a{}^4 = 0$$

$$a(T_s - T_a)^2 + b(T_s - T_a) + c = 0$$

$$a = [12\varepsilon\sigma T_a^2 + \frac{\rho m_v}{P m_a}\lambda g_w \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}]$$

$$b = [2\rho C_p g_h + \frac{\rho m_v}{P m_a}\lambda g_w \frac{de_s(T_a)}{dT} + 8\varepsilon\sigma T_a{}^3]$$

$$c = \frac{\rho m_v}{P m_a}\lambda g_w[e_s(T_a) - e_a] + 2\varepsilon\sigma T_a{}^4 - Q$$

Substitute g_w with g_s and g_b:

$$a = [12\varepsilon\sigma T_a^2 + \frac{\rho m_v}{P m_a}\lambda \frac{g_s g_b}{g_s + g_b}\frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}]$$

$$b = [2\rho C_p g_h + \frac{\rho m_v}{P m_a}\lambda \frac{g_s g_b}{g_s + g_b}\frac{de_s(T_a)}{dT} + 8\varepsilon\sigma T_a{}^3]$$

$$c = \frac{\rho m_v}{P m_a} \lambda \frac{g_s g_b}{g_s + g_b} [e_s(T_a) - e_a] + 2\varepsilon\sigma T_a^{\ 4} - Q$$

1: **Amphistomatous** leaves have stomata on both sides. Their two-sided net radiation balance is:

Basic Equations

$$R_n = H + \lambda E = (R \downarrow - R \uparrow + L \downarrow - L \uparrow)_{top} + (R \uparrow - R \downarrow + L \uparrow - L \downarrow)_{bottom}$$

$$Q = (R \downarrow - R \uparrow + L \downarrow)_{top} + (R \uparrow - R \downarrow + L \uparrow)_{bottom}$$

$$(L \uparrow)_{top} + (L \downarrow)_{bottom} = 2\varepsilon\sigma T_s^4$$

$$R_n = H + \lambda E = Q_{in} - 2\varepsilon\sigma T_{sfc}^4$$

$$H = 2\rho C_p g_h (T_s - T_a)$$

For Amphistomatous leaves, the stomatal resistance or conductance is defined as $r_{s,leaf}$, which is the sum of the parallel resistors of the top and bottom. If we assume r_{top} equals r_{bottom}

$$r_{s,leaf} = \frac{r_{top} r_{bottom}}{r_{top} + r_{bottom}} = \frac{r_s}{2} = \frac{2}{g_{s,leaf}}$$

$$\frac{1/g_{top} 1/g_{bottom}}{1/g_{top} + 1/g_{bottom}} = \frac{\dfrac{1}{g_{top} g_{bottom}}}{g_{top} + g_{bottom} \Big/ g_{top} g_{bottom}}$$

$$= \frac{g_{top} g_{bottom}}{g_{top} g_{bottom}} \frac{1}{g_{top} + g_{bottom}} = \frac{1}{2 g_s}$$

Also the boundary layer and stomatal resistances of the top and bottom are in series and r_{bottom} equal r_{top} equal r_{stom}.

$$r_{w,leaf} = \frac{(r_{top} + r_a)(r_{bottom} + r_a)}{(r_{top} + r_a) + (r_{bottom} + r_a)} = \frac{(r_{stom} + r_a)}{2}$$

$$g_{w,leaf} = \frac{2}{(r_{stom} + r_a)} = \frac{2 g_{stom} g_a}{g_{stom} + g_a}$$

$$g_{w,leaf} = \frac{2}{(1/g_{s1side} + 1/g_a)} =$$

$$\frac{2}{(g_a / g_a g_{s1side} + g_{s1side} / g_{s1side} g_a)} =$$

$$\frac{2}{\dfrac{g_{s1side} + g_a}{g_{s1side} g_a}} = \frac{2 g_{s1side} g_a}{g_{s1side} + g_a}$$

Be careful and not to put factor of 2 twice, in gw and in LE..

$$\lambda E = \frac{0.622 \lambda \rho_a g_w (e_s(T_l) - e_a)}{P} = \frac{0.622 \lambda \rho_a (e_s(T_l) - e_a)}{P \cdot r_{w,leaf}} = 2 \frac{0.622 \lambda \rho_a (e_s(T_l) - e_a)}{P(r_s + r_a)}$$

Solve for Surface Temperature, Ts

From Ohm's Law

$$T_s = \frac{H}{2\rho C_p g_h} + T_a$$

From Energy Balance of two-sided leaf by simplifying common denominator

$$H = Q - \lambda E - 2\varepsilon\sigma T_s^4$$

$$H = 2\rho C_p g_h (T_s - T_a) = Q - \lambda E - 2\varepsilon\sigma T_s^4$$

Taylor's Expansion for T^4

1st order

$$\varepsilon\sigma T_s^4 = \varepsilon\sigma T_a^4 + 4\varepsilon\sigma T_a^3 (T_s - T_a)$$

$$2\rho C_p g_h (T_s - T_a) = Q - \lambda E - 2\varepsilon\sigma T_a^4 - 8\varepsilon\sigma T_a^3 (T_s - T_a)$$

$$T_s - T_a = \frac{Q - \lambda E - 2\sigma\varepsilon T_a^4}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$

2nd order

$$2\varepsilon\sigma T_s^4 = 2\varepsilon\sigma T_a^4 + 8\varepsilon\sigma T_a^3 (T_s - T_a) + 12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

$$H = Q - \lambda E - 2\varepsilon\sigma T_a^4 - 8\varepsilon\sigma T_a^3 (T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

$$H = 2\rho C_p g_h (T_s - T_a) = Q - \lambda E - 2\varepsilon\sigma T_a^4 - 8\varepsilon\sigma T_a^3 (T_s - T_a) - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

Solve for delta T, as the equation above shows that delta T is a function of delta T

$$2\rho C_p g_h (T_s - T_a) + 8\varepsilon\sigma T_a^3 (T_s - T_a) = Q - \lambda E - 2\varepsilon\sigma T_a^4 - 12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

Insert Term for Latent Heat Exchange

$$\lambda E = \frac{\rho m_v}{P m_a} \lambda g_w [e_s(T_a) + \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{1}{2}\frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 - e_a]$$

$$\lambda E = \frac{\rho m_v}{P m_a} \lambda g_w (e_s(T_a) - e_a) + \frac{\rho m_v}{P m_a} \lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{\rho m_v}{2 P m_a} \lambda g_w \frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2$$

$$2\rho C_p g_h (T_s - T_a) + 8\varepsilon\sigma T_a{}^3 (T_s - T_a) = Q$$
$$-\frac{\rho m_v}{P m_a} \lambda g_w (e_s(T_a) - e_a) - \frac{\rho m_v}{P m_a} \lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) - \frac{\rho m_v}{2 P m_a} \lambda g_w \frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2$$
$$-2\varepsilon\sigma T_a{}^4$$
$$-12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$

Set terms to zero

$$2\rho C_p g_h (T_s - T_a) + 8\varepsilon\sigma T_a{}^3 (T_s - T_a) + \frac{\rho m_v}{P m_a} \lambda g_w (e_s(T_a) - e_a)$$
$$+\frac{\rho m_v}{P m_a} \lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a) + \frac{\rho m_v}{2 P m_a} \lambda g_w \frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2$$
$$+2\varepsilon\sigma T_a{}^4 + 12\varepsilon\sigma T_a^2 (T_s - T_a)^2 - Q = 0$$

Re-arranged to form Quadratic Equation in terms of Ts-Ta

$$+\frac{\rho m_v}{2 P m_a} \lambda g_w \frac{d^2 e_s(T_a)}{dT^2}(T_s - T_a)^2 + 12\varepsilon\sigma T_a^2 (T_s - T_a)^2$$
$$2\rho C_p g_h (T_s - T_a) + 8\varepsilon\sigma T_a{}^3 (T_s - T_a) + \frac{\rho m_v}{P m_a} \lambda g_w \frac{de_s(T_a)}{dT}(T_s - T_a)$$
$$+\frac{\rho m_v}{P m_a} \lambda g_w (e_s(T_a) - e_a) + 2\varepsilon\sigma T_a{}^4 - Q = 0$$

$$a(T_s - T_a)^2 + b(T_s - T_a) + c = 0$$

$$a = \frac{\rho m_v}{2 P m_a} \lambda g_w \frac{d^2 e_s(T_a)}{dT^2} + 12\varepsilon\sigma T_a^2$$

$$b = 2\rho C_p g_h + 8\varepsilon\sigma T_a{}^3 + \frac{\rho m_v}{P m_a} \lambda g_w \frac{de_s(T_a)}{dT}$$

$$c = \frac{\rho m_v}{P m_a} \lambda g_w (e_s(T_a) - e_a) + 2\varepsilon\sigma T_a^{\ 4} - Q$$

The leaf energy balance can also be used to derive a quadratic equation for latent heat exchange (λE; W m^{-2}):

$$a\ LE^2 + b\ LE + c = 0$$

We can solve directly for latent heat flux density

$$\lambda E = g_w \frac{0.622 \lambda \rho_a (e_s(T_l) - e_a)}{P} = 2 \frac{g_s g_a}{g_a + g_s} \frac{0.622 \lambda \rho_a (e_s(T_l) - e_a)}{P} = 2 \frac{0.622 \lambda \rho_a (e_s(T_l) - e_a)}{P(r_s + r_a)}$$

$$\lambda E = \frac{0.622 \lambda \rho_a g_w}{P} [(e_s(T_a) + s(T_a)(T_l - T_a) + \frac{e_s(T_a)''}{2} (T_l - T_a)^2 - e_a]$$

Just to make sure I don't miss any multiplication factors and signs, I factored this out

$$\lambda E = \frac{0.622 \lambda \rho_a g_w}{P} [(e_s(T_a) - e_a] + \frac{0.622 \lambda \rho_a g_w}{P} s(T_a)(T_l - T_a) + \frac{0.622 \lambda \rho_a g_w}{2P} e_s(T_a)''(T_l - T_a)^2$$

Substituting

For Algebraic simplicity we will introduce a truncated version of the second order expansion of surface energy balance and temperature into the equation for latent heat exchange, for the amphistomatous leaf

$$T_s - T_a = \frac{Q - \lambda E - 2\sigma\varepsilon T_a^4}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$

Be careful if factor of 2 on gw don't need to multiply again by 2

$$\lambda E = \frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a] + \frac{0.622\lambda\rho_a g_w}{P} s(T_a)\frac{Q - \lambda E - 2\sigma\varepsilon T_a^4}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$
$$+\frac{0.622\lambda\rho_a g_w}{2P} e_s(T_a)''(\frac{Q - \lambda E - 2\sigma\varepsilon T_a^4}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)})^2$$

Multiply LE by $2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3$

$$(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\lambda E = (2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a] + \frac{0.622\lambda\rho_a g_w}{P} s(T_a)(Q - \lambda E - 2\sigma\varepsilon T_a^4)$$
$$+\frac{0.622\lambda\rho_a g_w}{2P} e_s(T_a)''\frac{(Q - \lambda E - 2\sigma\varepsilon T_a^4)^2}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$

The squared term in the numerator is:

$$(Q - \lambda E - 2\varepsilon\sigma T_a^{\ 4})(Q - \lambda E - 2\varepsilon\sigma T_a^{\ 4}) =$$
$$QQ - Q\lambda E - Q2\varepsilon\sigma T_a^{\ 4}$$
$$-Q\lambda E + \lambda E\,\lambda E + 2\lambda E\,\varepsilon\sigma T_a^{\ 4}$$
$$-Q2\varepsilon\sigma T_a^{\ 4} + 2\lambda E\,\varepsilon\sigma T_a^{\ 4} + 4\varepsilon\sigma T_a^{\ 4}\,\varepsilon\sigma T_a^{\ 4})$$

$$(Q - \lambda E - 2\varepsilon\sigma T_a^{\ 4})(Q - \lambda E - 2\varepsilon\sigma T_a^{\ 4}) =$$
$$QQ - 2Q\lambda E - 4Q\varepsilon\sigma T_a^{\ 4}$$
$$+\lambda E\,\lambda E + 4\lambda E\,\varepsilon\sigma T_a^{\ 4}$$
$$+\ 4\varepsilon\sigma T_a^{\ 4}\,\varepsilon\sigma T_a^{\ 4})$$

Insert the squared term into the main equation

$$QQ - 2Q\lambda E - 4Q\varepsilon\sigma T_a^{\ 4} + \lambda E\,\lambda E + 4\lambda E\,\varepsilon\sigma T_a^{\ 4} + 4\varepsilon\sigma T_a^{\ 4}\,\varepsilon\sigma T_a^{\ 4}$$

$$(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\lambda E = (2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a] + \frac{0.622\lambda\rho_a g_w}{P} s(T_a)(Q - \lambda E - 2\sigma\varepsilon T_a^4)$$
$$+\frac{0.622\lambda\rho_a g_w}{2P} e_s(T_a)'' \frac{(QQ - 2Q\lambda E - 4Q\varepsilon\sigma T_a{}^4 + \lambda E\,\lambda E + 4\lambda E\,\varepsilon\sigma T_a{}^4 + 4\varepsilon\sigma T_a{}^4\,\varepsilon\sigma T_a{}^4)}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$

Reorganize the Main Equation and collect terms

$$0 = (2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a] + \frac{0.622\lambda\rho_a g_w}{P} s(T_a)(Q - \lambda E - 2\sigma\varepsilon T_a^4)$$
$$+\frac{0.622\lambda\rho_a g_w}{2P} e_s(T_a)'' \frac{(QQ - 2Q\lambda E - 4Q\varepsilon\sigma T_a{}^4 + \lambda E\,\lambda E + 4\lambda E\,\varepsilon\sigma T_a{}^4 + 4\varepsilon\sigma T_a{}^4\,\varepsilon\sigma T_a{}^4)}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)} - (2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\lambda E$$

Collect terms for the quadratic function

$$+\lambda E\,\lambda E \frac{0.622\lambda\rho_a g_w}{2P(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)} e_s(T_a)''$$
$$-\lambda E \frac{0.622\lambda\rho_a g_w}{P} s(T_a) - (2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\lambda E$$
$$\frac{-2Q\lambda E}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)} \frac{0.622\lambda\rho_a g_w}{P(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)} e_s(T_a)'' + \frac{4\lambda E\,\varepsilon\sigma T_a{}^4}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)} \frac{0.622\lambda\rho_a g_w}{P(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)} e_s(T_a)''$$
$$+(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a) - e_a] + \frac{0.622\lambda\rho_a g_w}{P} s(T_a)(Q - 2\sigma\varepsilon T_a^4)$$
$$+\frac{0.622\lambda\rho_a g_w}{2P} e_s(T_a)'' \frac{(QQ - 4Q\varepsilon\sigma T_a{}^4 + 4\varepsilon\sigma T_a{}^4\,\varepsilon\sigma T_a{}^4)}{(2\rho_a C_p g_h + 8\varepsilon\sigma T_a^3)}$$

$$+\lambda E\,\lambda E\frac{0.622\lambda\rho_a g_w}{2P(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}e_s(T_a)''$$

$$\lambda E[-\frac{0.622\lambda\rho_a g_w}{P}s(T_a)-(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)$$

$$-\frac{2Q}{(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}\frac{0.622\lambda\rho_a g_w}{2P(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}e_s(T_a)''+\frac{4\,\varepsilon\sigma T_a^{\,4}}{(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}\frac{0.622\lambda\rho_a g_w}{2P(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}e_s(T_a)'']$$

$$+(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a)-e_a]+\frac{0.622\lambda\rho_a g_w}{P}s(T_a)(Q-2\sigma\varepsilon T_a^4)$$

$$+\frac{0.622\lambda\rho_a g_w}{2P}e_s(T_a)''\frac{(QQ-4Q\varepsilon\sigma T_a^{\,4}+\,4\varepsilon\sigma T_a^{\,4}\,\varepsilon\sigma T_a^{\,4})}{(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}$$

$$a=\frac{0.622\lambda\rho_a g_w}{2P(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}e_s(T_a)''$$

$$b=[-\frac{0.622\lambda\rho_a g_w}{P}s(T_a)-(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)$$

$$-\frac{Q}{(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}\frac{0.622\lambda\rho_a g_w}{P(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}e_s(T_a)''+\frac{2\,\varepsilon\sigma T_a^{\,4}}{(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}\frac{0.622\lambda\rho_a g_w}{P(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}e_s(T_a)'']$$

$$c=+(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)\frac{0.622\lambda\rho_a g_w}{P}[(e_s(T_a)-e_a]+\frac{0.622\lambda\rho_a g_w}{P}s(T_a)(Q-2\sigma\varepsilon T_a^4)$$

$$+\frac{0.622\lambda\rho_a g_w}{2P}e_s(T_a)''\frac{(QQ-4Q\varepsilon\sigma T_a^{\,4}+\,4\varepsilon\sigma T_a^{\,4}\,\varepsilon\sigma T_a^{\,4})}{(2\rho_a C_p g_h+8\varepsilon\sigma T_a^3)}$$