On the Jacobian polygon and Łojasiewicz exponent of isolated complex hypersurface singularities

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Abstract

Given a hypersurface singularity $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ defined by a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, we introduce an alternating version of Teissier's Jacobian Newton polygon, associated with a complex isolated hypersurface singularity, and prove formulas for both these invariants in terms of an embedded resolution of $(X, 0)$. The formula for the alternating version has an advantage, in that for Newton nondegenerate functions, it can be calculated in terms of volumes of faces of the Newton diagram, whereas a similar formula for the original nonalternating version includes mixed volumes.

The Milnor fiber can be given a handlebody decomposition, with handles corresponding to intersection points with the polar curve in generic plane sections of the singularity. This way we obtain a Morse-Smale complex. Teissier associates with each branch of the polar curve a vanishing rate, and we show that this induces a filtration of the Morse-Smale complex. We apply this result in order to calculate the Łojasiewicz exponent in terms of the alternating Jacobian polygon, but we expect it to be of further independent interest. In the case of a Newton nondegenerate hypersurface, our result produces a formula for the Łojasiewicz exponent in terms of Newton numbers of certain subdiagrams.

This statement is related to a conjecture by Brzostowski, Krasiński and Oleksik, for which we provide a counterexample. Our formula for the Łojasiewicz exponent is based on a global calculation over the Newton diagram, rather than locally specifying a subset of the facets to consider, as in this conjecture. We conjecture a similar statement, which is based on our formula and inspired by the nonnegativity of local h-vectors.

Contents

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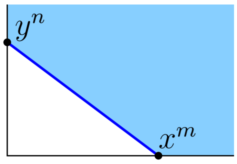
1 Introduction

1.1. In [Tei77], Teissier associates a Newton polygon with a complex isolated hypersurface singularity $(X,0) \subset (\mathbb{C}^{n+1},0)$, defined as the zero set of a holomorphic function $f \in \mathbb{C}\{x_0,x_1,\ldots,x_n\}$. This is the Newton polygon of *the Cerf diagram*, the image of the polar curve under the map $(f,g) : (\mathbb{C}^{n+1},0) \to (\mathbb{C}^2,0)$, where $g : \mathbb{C}^{n+1} \to \mathbb{C}$ is a generic linear function. We will refer to this polygon as the Jacobian polygon, and denote it $\mathrm{J}(f,0)$. The set of Newton polygons $\mathfrak{N}$ is a commutative and cancellative semigroup, generated by diagrams of the form

$$\left\{\frac{m}{n}\right\} = \Gamma_+(x^m + y^n) \subset \mathbb{R}^2_{\geqslant 0}. \tag{1.2}$$



Of particular interest is the highest number m/n appearing in $\mathrm{J}(f,0)$, which we will call its *degree*. By [Tei77, LJT08], this number coincides with the *Łojasiewicz exponent*. This work is motivated by the search of an identification of the Łojasiewicz exponent of a Newton nondegenerate function in terms of its Newton diagram, which has been conducted in [BKO12, BKO23, Fuk91, Lic81, Len98]. This problem—and in particular, the conjecture of Brzostowski, Krasiński and Oleksik, counter to which we give example 9.4—is discussed further below. This this problem can be seen as a version of one of Arnold's problems [Arn04, 1975-1].

We introduce *the alternating Jacobian polygon* $\mathrm{AJ}(f,0)$ (definition 4.3(iii)) which lives in the group $K\,\mathfrak{N}$ of formal differences of Newton polygons, and show that this invariant can be calculated from an embedded resolution. The alternating Jacobian polygon is the alternating sum of Jacobian polygons $\mathrm{J}\left(f^{(d+1)},0\right)$, where $f^{(d+1)}$ is the restriction of f to a generic linear subspace of dimension $d+1$. As a result, one recovers $\mathrm{J}\left(f,0\right) = \mathrm{AJ}\left(f^{(n+1)},0\right) + \mathrm{AJ}\left(f^{(n)},0\right)$. At any smooth point of the total transform of X in an embedded resolution of $(X,0)$, we can define a number m, the order of vanishing of the pullback of f, and similarly n, the order of vanishing of a generic linear function. The formula eq. 4.12 for $\mathrm{AJ}(f,0)$ can be seen as an integral of the additive Newton polygon eq. 1.2 with respect to the Euler characteristic, in a similar way that A'Campo's formula [A'C75] calculates the monodromy zeta function as an integral of the multiplicative polynomial $(t^m-1)^{-1}$. The strong relationship between the Jacobian and alternating Jacobian polygons means that we recover a similar formula for Teissier's Jacobian polygon as well.

Theorem A (4.10)**.** *Let $(Y,D) \to (\mathbb{C}^{n+1},0)$ be an embedded resolution of $(X,0)$. For each irreducible component $D_i \subset D$ of the exceptional divisor, denote by m_i and n_i the order of vanishing along D_i of the pullback of f and a generic linear function, respectively, and denote by $H \subset \mathbb{C}^{n+1}$ a generic linear hyperplane. The Jacobian and alternating Jacobian polygons can be computed from an embedded resolution of $(X,0)$ as*

$$\mathrm{J}(f,0) = (-1)^n \sum_{i\in I} \chi(D_i^\circ \backslash (\tilde{X} \cup \tilde{H})) \left\{\frac{m_i}{n_i}\right\}$$

and

$$\mathrm{AJ}(f,0) = (-1)^n \sum_{i\in I} \chi(D_i^\circ \backslash \tilde{X}) \left\{\frac{m_i}{n_i}\right\},$$

where $\tilde{X}$ and $\tilde{H}$ denote the strict transforms of X and H.

Remark 1.3. This statement, in the case of plane curves, $n = 1$, can be seen to follow from a more general result of Michel [Mic08], where she considers arbitrary finite holomorphic maps $(X, p) \to (\mathbb{C}^2, 0)$ in place of our map $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, where (X, p) is a normal surface singularity. We expect a similar generalization to hold for finite maps $(X, p) \to \mathbb{C}^2$ with (X, p) isolated of any dimension.

1.4. As an application of this theorem, we find a formula for the alternating Jacobian polygon in terms of volumes of faces of the Newton diagram, in the case of a Newton nondegenerate hypersurface, theorem 7.12(i), reflecting Varchenko's formula [Var76] for the monodromy zeta function. We also calculate the alternating Jacobian polygons associated with a generic plane section of any codimension in terms of mixed volumes. As a result, we find a formula for the Jacobian polygon associated with any plane section of $(X, 0)$, reflecting Oka's work on principal zeta functions [Oka90]. Here, a facet of a Newton diagram $\Gamma \in \mathbb{R}^{n+1}$ is a face of dimension n, and a coordinate facet is a facet of a diagram obtained by intersecting Γ with a coordinate subspace.

Theorem B (7.12, 7.15)**.** *(i) Let $f : (\mathbb{C}^{(n+1)}, 0) \to (\mathbb{C}, 0)$ be a Newton nondegenerate holomorphic function with an isolated singularity. Then, the alternating Jacobian polygon $\mathrm{AJ}(f, 0)$ can be calculated in terms of volumes of coordinate facets of the Newton diagram $\Gamma(f)$.*

(ii) The alternating Jacobian polygons $\mathrm{AJ}(f^{(d+1)}, 0)$ and the Jacobian polygons $\mathrm{AJ}(f^{(d+1)}, 0)$ of the restriction of f to generic linear subspaces of dimension $d + 1$ can be calculated in terms of mixed volumes of coordinate facets of the Newton diagrams $\Gamma(f)$ and $\Gamma(g)$, where $g : \mathbb{C}^{n+1} \to \mathbb{C}$ is a generic linear function.

1.5. The *Łojasiewicz exponent* $\mathcal{L}(f, 0)$ of f at 0 is, by definition, the infimum of all $\theta > 0$ such that there exists a $C > 0$ such that the inquality

$$\|x\|^\theta \leqslant C\|\nabla f\|$$

holds near $0 \in \mathbb{C}^{n+1}$. Teissier proved in [Tei77] that this invariant can be calculated directly as the degree of the Jacobian polygon minus one (see definition 2.6). See [LJT08] for more on this topic. We prove that the degree of the alternating Jacobian polygon coincides with that of the Jacobian polygon. The proof of this statement requires an analysis of the topology of the Milnor fibration, along with that of generic plane sections, which we expect to be of independent interest. We study the Morse-Smale complex of a certain Morse function on the Milnor fiber. It is known that the Milnor fiber can be constructed as a handlebody, whose number of handles of index d is $\mu^{(d+1)} + \mu^{(d)}$. Our Morse function has the same number of Morse points. In [Tei77], Teissier associates a rate of vanishing to each branch of the polar curve. Following this construction, we define a filtration on the Morse-Smale complex as an Abelian group, and we show that this is a filtration of the complex.

Theorem C (5.3 and 5.6)**.** *The vanishing rates associated to each branch of the polar curve induce an increasing filtration on the Morse-Smale complex on the Milnor fiber, generated by intersection points with polar curves in generic plane sections.*

1.6. This result allows us to conclude that the Jacobian and alternating Jacobian polygons have the same degree. In particular, we can calculate the Łojasiewicz exponent purely in terms of the alternating Jacobian polygon. In the Newton nondegenerate case, it can be computed in terms of volumes of coordinate facets. In fact, if F is a coordinate facet, then it has a unique primitive integral normal vector

$$v = (v_0, v_1, \ldots, v_n) \in (\mathbb{Z}_{>0} \cup \{+\infty\})^{n+1}.$$

Define the *maximal axial number* associated with v and f as the weight of f with respect to v, divided by that of a generic linear function. The maximal axial number of a coordinate facet is that of its normal vector. For facets, this invariant of a Newton diagram been studied in [BKO12]. By defining $s_\alpha(\Gamma(f)) \subset \Gamma(f)$ as the union of faces supported by weight vectors having maximal axial number $\leqslant \alpha$, for $\alpha \in \mathbb{Q}$, we have an identification of the Łojasiewicz exponent in terms of Newton numbers of these subdiagrams.

Theorem D (8.4)**.** *With the possible exception of Morse points in an even number of variables, the Łojasiewicz exponent of an isolated Newton nondegenerate singularity is the maximal axial number of some coordinate facet of its Newton diagram. It can be recovered as*

$$\mathcal{L}(f,0) = \min\left\{\alpha - 1 \in \mathbb{Q} \,|\, \nu(s_\alpha(\Gamma(f))_-) = \nu(\Gamma_-)\right\}.$$

where ν is the Newton number.

Remark 1.7. Let F be the Milnor fiber of an isolated singularity $(X,0) \subset (\mathbb{C}^{n+1})$ defined by $f \in \mathbb{C}\{x_0,\ldots,x_n\}$. Then we have $(-1)^{n+1}\chi(F) \geqslant 0$, with equality if and only if $(X,0)$ is Morse, i.e. the Hessian matrix of f at 0 is invertible, and $n+1$ is even.

1.8. In section 9 we recall the conjecture of Brzostowski, Krasiński and Oleksik [BKO12] on the Łojasiewicz exponent of a Newton nondegenerate singularity f which greatly motivated this manuscript. As pointed out by the same authors in [BKO23], Arnold suggests in [Arn04, 1975-1] that *"Every interesting discrete invariant of a generic singularity with Newton polyhedron Γ is an interesting function of the polyhedron."* In the current case of study, the discrete invariant is the Łojasiewicz exponent. The conjecture says that if $\Gamma(f)$ has a *nonexceptional* facet (see definition 9.2), then the Łojasiewicz exponent of f is the largest maximal axial number of a nonexceptional facet minus one. Stated differently, for any facet $F \subset \Gamma(f)$ of the Newton diagram, the conjecture gives a simple condition, which only depends on F, and not $\Gamma(f)$, determining whether F should be discarded or not. Unless all facetes have been discarded this way, the conjecture says that the Łojasiewicz exponent is the largest of the maximal axial numbers of remaining facets, minus one. They proved the conjecture in the case $n = 2$. We give a counterexample to this conjecture with $n = 3$. In light of theorem D, we ask what condition characterizes those *coordinate facets* of $\Gamma(f)$ (not just *facets*) to include or exclude in determining the Łojasiewicz exponent. Motivated by the theory of local h-polynomials [Sta92, Sel24], we define a subset $\mathcal{F}_{\mathrm{ne}}^{\mathcal{T}}$ of the set $\mathcal{F}(\Gamma(f))$ of coordinate facets of $\Gamma(f)$ which depends on a triangulation $\mathcal{T}$ of $\Gamma(f)$.

Conjecture E. *Let $f \in \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ be Newton nondegenerate, defining an isolated singularity, and let $\mathcal{T}$ be a triangulation of its Newton diagram $\Gamma(f)$. Then the formula*

$$\mathcal{L}(f,0) = \max_{F \in \mathcal{F}_{\mathrm{ne}}^{\mathcal{T}}} \mathcal{M}(F) - 1.$$

holds, with the possible exception of a Morse point in an even number of variables.

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1.9. In section 2, we review the notation for Newton polygons introduced by Teissier, along with some basic definitions.

In section 3, we prove a technical result on resolutions which we apply in proofs in the following sections, based on the notion of toroidal embeddings [KKMSD73].

In section 4 we review Teisser's definition of the Jacobian polygon, and fix related notation for the polar curve. We also define the alternating Jacobian polygon, and prove a formula for both polygons in terms of an embedded resolution and a generic linear function.

In section 5 we construct a Morse function on the Milnor fiber, whose critical points coincide with intersection points with polar curves. The main technical result of this section, lemma 5.3, shows that we can endow the associated Morse-Smale complex with an increasing filtration, using Teissier's *vanishing rate* [Tei77], which is a rational number associated with each polar branch,

In section 6 we obtain a string of corollaries of the above result. Of particular interest is corollary 6.6(iv), which says that the Jacobian and alternating Jacobian have the same degree. As a result, the alternating Jacobian polygon determines the Łojasiewicz exponent.

In section 7, we consider the case of a Newton nondegenerate singularity with Newton diagram $\Gamma(f)$. Using the results in section 4 we give formulas for the Jacobian and alternating Jacobian polygons in terms of the Newton diagram.

In section 8, we obtain a formula for the Łojasiewicz exponent for a Newton nondegenerate singularity in terms Newton numbers of subdiagrams (the Newton number was introduced by Kouchnirenko in [Kou76], see definition 8.3).

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2 The group of Newton polygons

2.1. In this section we fix notation for Newton polygons, seen as additive invariants, as in [Tei77, Tei12].

2.2. We will denote by $\mathfrak{N}$ the set of Newton polygons in $\mathbb{R}^2$. Thus, an element of $\mathfrak{N}$ is any integral polygon contained in $\mathbb{R}^2_{\geqslant 0}$, whose recession cone is $\mathbb{R}^2_{\geqslant 0}$. In particular, if m, n are any positive two numbers, then we denote by

$$\left\{\frac{m}{n}\right\} \tag{2.3}$$

the Newton polygon of the polynomial $x^m + y^n$. Extending either a or b to infinity, and imagining that $x^\infty = y^\infty = 0$, we define the polygons

$$\left\{\frac{\infty}{n}\right\}, \qquad \left\{\frac{m}{\infty}\right\} \tag{2.4}$$

associated with the polynomials y^n and x^m. The set $\mathfrak{N}$ has an operation which we denote by $+$, the Minkowski sum, given by

$$\Gamma + \Delta = \left\{ g + d \in \mathbb{R}^2 \,\middle|\, g \in \Gamma,\ d \in \Delta \right\}.$$

This way, we have a cancellative and commitative semigroup $(\mathfrak{N}, +)$, generated by the symbols eqs. (2.3) and (2.4), modulo the relations

$$\left\{\frac{cm}{cn}\right\} = c\left\{\frac{m}{n}\right\}, \qquad c \in \mathbb{Z}_{>0}.$$

In particular, as a commutative semigroup, $\mathfrak{N}$ is freely generated by the symbols eq. 2.3 for $m, n \in \mathbb{Z}_{>0}$ and $\gcd(m,n) = 1$, as well as the two elements eq. 2.4 with $m = 1$ and $n = 1$. Note that we simply set $c\infty = \infty$ for any positive integer c.

We denote by $K\mathfrak{N}$ the Grothendieck group of $\mathfrak{N}$. This group then has the same description as $\mathfrak{N}$ in terms of generators and relations, only as an Abelian group, rather than a commutative semigroup. In particular, any element $N \in K\mathfrak{N}$ has a unique presentation as

$$N = \sum_{\alpha \in \overline{\mathbb{Q}}_{\geqslant 0}} a_\alpha \{\alpha\}, \tag{2.5}$$

where the family of integers

$$(a_\alpha)_{\alpha\in\overline{\mathbb{Q}}_{\geqslant 0}}, \qquad \overline{\mathbb{Q}}_{\geqslant 0} = \mathbb{Q}_{\geqslant 0} \cup \infty$$

has finite support, and we set

$$\{\alpha\} = \left\{\frac{m}{n}\right\}, \qquad \text{if} \qquad \alpha = \frac{m}{n} \in \mathbb{Q}_{>0} \qquad \text{with} \qquad \gcd(m,n) = 1$$

and

$$\{0\} = \left\{\frac{1}{\infty}\right\}, \qquad \{\infty\} = \left\{\frac{\infty}{1}\right\}.$$

Definition 2.6. ❀ The *support* of an element $N \in K\mathfrak{N}$ expanded as in eq. 2.5 is

$$\operatorname{supp}(N) = \left\{\alpha \in \overline{\mathbb{Q}}_{\geqslant 0} \,\middle|\, a_\alpha \neq 0\right\} \subset \overline{\mathbb{Q}}_{\geqslant 0}.$$

❀ The *degree* of an element $N \neq 0$ is

$$\deg(N) = \begin{cases} \max\operatorname{supp}(N) & N \neq 0, \\ -\infty & N = 0. \end{cases}$$

❀ If $N \neq 0$, then the *leading coefficient* of N is

$$\operatorname{lc}(N) = a_\alpha.$$

where $\alpha = \deg(N)$.

❀ If $\alpha \in \overline{\mathbb{Q}}_{\geqslant 0}$, then the *truncation of N at α* is

$$N_{\geqslant \alpha} = \sum_{\beta \geqslant \alpha} a_\beta \{\beta\} \in K\mathfrak{N}.$$

❀ The *height* $h(N) \in \overline{\mathbb{Q}}_{\geqslant 0}$ and *length* $\ell(N) \in \overline{\mathbb{Q}}_{\geqslant 0}$ of N are defined by extending linearly the functions

$$h\left(\left\{\frac{m}{n}\right\}\right) = n, \qquad \ell\left(\left\{\frac{m}{n}\right\}\right) = m.$$

Note that this way, we have $h(N) = \pm\infty$ if and only if $a_0 \neq 0$, and $\ell(N) = \pm\infty$ if and only if $a_\infty \neq 0$.

2.7. In the above language, the vertices of a polygon $N \in \mathfrak{N}$ are precisely the points

$$\left(\ell(N_{<\alpha}), h(N_{\geqslant\alpha})\right), \qquad \alpha \in \mathbb{Q}_{>0}$$

with adjacent points joined by an edge. If $N \in K\,\mathfrak{N}$, then these points and edges are the *virtual vertices* and *virtual edges* of N.

3 Embedded resolutions, coordinates and polar

3.1. In this section we assume that $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ is an analytic function germ with an isolated critical point at the origin. We describe some notation and conditions on an embedded resolution of f. Lemma 3.13 is a technical result which is used in the following sections.

3.2. Let $x_0, x_1, \ldots, x_n$ be linear coordinates in $\mathbb{C}^{n+1}$, i.e. a basis of the dual space. For $d = 0, 1, \ldots, n$, define a linear subspace

$$H^{(d+1)} = \left\{ x \in \mathbb{C}^{n+1} \,\middle|\, x_{d+1} = \ldots = x_n = 0 \right\}.$$

and set

$$f^{(d+1)} = f|_{H^{(d+1)}}, \qquad g^{(d+1)} = x_d|_{H^{(d+1)}}.$$

This way we have hypersurfaces $(X^{(d+1)}, 0) \subset (H^{(d+1)}, 0)$ defined by $f^{(d+1)}$, along with linear functions $g^{(d+1)}$.

If $\pi : Y \to \mathbb{C}^{n+1}$ is an embedded resolution of f, then we denote by $Y^{(d+1)}$ the strict transform of $H^{(d+1)}$ in Y, and by

$$\pi^{(d+1)} : Y^{(d+1)} \to H^{(d+1)}$$

the modification induced by π. Denote by and $D^{(d+1)} \subset Y^{(d+1)}$ its exceptional divisor. In particular, we have $\mathbb{C}^{n+1} = H^{(n+1)}$ and $Y = Y^{(n+1)}$ and $\pi = \pi^{(n+1)}$. Denote also by $\tilde{X}^{(d+1)}$ the strict transform of $X^{(d+1)}$ in $Y^{(d+1)}$. Note that $\tilde{X}^{(d+1)}$ is not necessarily smooth, under the conditions considered so far.

For $d = 0, 1, \ldots, n$, denote by $P^{(d+1)}$ the polar curve of $f^{(d+1)}$ with respect to the hyperplane $H^{(d)} \subset H^{(d+1)}$, that is

$$P^{(d+1)} = P_1^{(d+1)} \cup \ldots \cup P_{l^{(d+1)}}^{(d+1)} = \left\{ x \in H^{(d+1)} \,\middle|\, \partial_{x_0} f = \ldots = \partial_{x_{d-1}} f = 0 \right\}.$$

If $P^{(d+1)}$ is a curve, denote its branches by $P_1^{(d+1)}, \ldots, P_{l^{(d+1)}}^{(d+1)}$. We also set $P = P^{(n+1)}$.

By [Tei77], there exist numbers

$$\mu^{(d+1)}(f, 0) \in \mathbb{Z}_{>0}, \qquad d = 0, 1, \ldots, n$$

such that for a generic choicie of coordinates, $P^{(d+1)}$ is a reduced curve, and

$$\mu(f|_{H^{(d+1)}}, 0) = \mu^{(d+1)}(f, 0).$$

Let us say that such coordinates are *generic with respect to* f.

Definition 3.3. If $D_i \subset Y$ is a component of the exceptional divisor of a modification $\pi : Y \to \mathbb{C}^{n+1}$, and $h \in \mathcal{O}_{\mathbb{C}^{n+1},0}$, then we define $\mathrm{ord}_{D_i}(h)$ as the order of vanishing of $\pi^* h$ along D_i. If $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^{n+1},0}$ is an ideal, then we set

$$\mathrm{ord}_{D_i^{(d+1)}}(\mathfrak{a}) = \min \left\{ \mathrm{ord}_{D_i^{(d+1)}}(h) \,\middle|\, h \in \mathfrak{a} \right\}.$$

Definition 3.4. Let $\pi : (Y, D) \to (\mathbb{C}^{n+1}, 0)$ be a point modicfication of $\mathbb{C}^{(n+1)}$ at 0, whose exceptional divisor consists of components D_i, $i \in I$. Let $x_0, \ldots, x_n$ and $H^{(d+1)}$ be as in 3.2.

(i) A function $h \in \mathfrak{m}_{\mathbb{C}^{n+1},0}$ is *generic with respect to* π if

$$\forall i \in I \, : \, \mathrm{ord}_{D_i}(h) = \mathrm{ord}_{D_i}(\mathfrak{m}_{\mathbb{C}^{n+1},0}).$$

(ii) The coordinates $x_0, x_1, \ldots, x_n$ in $\mathbb{C}^{n+1}$ are generic with respect to π if for all $d = 0, 1, \ldots, n$, the function $g^{(d+1)} \in \mathfrak{m}_{H^{(d+1)},0}$ is generic with respect to $\pi^{(d+1)}$, and for $d > 0$ the divisor

$$D^{(d+1)} \cup \tilde{X}^{(d+1)} \cup Y^{(d)} \subset Y^{(d+1)}$$

is a normal crossing divisor.

Lemma 3.5. *If π factors through the blow-up of $\mathbb{C}^{n+1}$ at 0, then the set of coordinates generic with respect to π contains a dense Zariski open subset of* $\mathrm{GL}(\mathbb{C}^{n+1})$.

Proof. This follows by applying Bertini's theorem to the linear system of hyperplanes. ∎

3.6. We will assume from now on that $Y \to \mathbb{C}^{n+1}$ is an embedded resolution of f, and that $x_0, \ldots, x_n$ are coordinates in $\mathbb{C}^{n+1}$ which are generic with respect to f and π. Decompose the exceptional divisor into irreducible components as $D = \cup_{i \in I} D_i$, and set

$$D_J = \bigcap_{j \in J} D_j, \qquad D_J^\circ = D_J \backslash \bigcup_{j \notin J} D_j.$$

for any $J \subset I$. For any $d = 0, 1, \ldots, n$, denote by

$$D^{(d+1)} = \bigcup_{i \in I^{(d+1)}} D_i$$

the exceptional divisor of $\pi^{(d+1)}$, decomposed in its irreducible components. We will assume that the sets $I^{(d+1)}$, for different d, are chosen disjoint, so that we can freely refer to $D_i \subset Y^{(d+1)}$, with d depending on i. We then define $D_J^{(d+1)}$ and $D_J^{(d+1)\circ}$ for $J \subset I^{(d+1)}$ similarly. Since $D^{(d+1)} \cup Y^{(d)} \subset Y^{(d+1)}$ is a normal crossing divisor, we have well defined maps

$$\omega^{(d+1)} : I^{(d)} \to I^{(d+1)},$$

definde by the condition that

$$D_i^\circ \subset D_{\omega(i)}^\circ.$$

If $h \in \mathcal{O}_{\mathbb{C}^{n+1},0}$, then we define

$$\mathrm{ord}_{D_i}(h)$$

the order of vanishing of $\pi^*(h)$ along D_i. For any $d = 0, 1, \ldots, n$ and $i \in I^{(d+1)}$, denote by m_i, n_i the orders of vanishing of $\pi^* f, \pi^* g$ along D_i, respectively, and set $\alpha_i = m_i/n_i$. As divisors, we then have

$$(\pi^{(d+1)*} f^{(d+1)}) = \tilde{X}^{(d+1)} + \sum_{i \in I^{(d+1)}} m_i D_i, \qquad (\pi^{(d+1)*} g^{(d+1)}) = \tilde{Y}^{(d)} + \sum_{i \in I^{(d+1)}} n_i D_i. \tag{3.7}$$

Note that, since $D^{(d+1)} \cup \tilde{X}^{(d+1)} \cup Y^{(d)}$ is a normal crossing divisor in $Y^{(d+1)}$, we have, for any $i \in I^{(d)}$

$$m_i = m_{\omega^{(d+1)}(i))}, \qquad n_i = n_{\omega^{(d+1)}(i))}.$$

If $J \subset I^{(d+1)}$, then we also set

$$m_J = \gcd\left(\{m_j \,|\, j \in J\}\right), \qquad n_J = \gcd\left(\{n_j \,|\, j \in J\}\right).$$

Definition 3.8. Let $\alpha \in \mathbb{Q}_{>0}$ be given as a reduced fraction $\alpha = m/n$ with $m, n > 0$, and $d \in \{0, 1, \ldots, n\}$. We define the memomorphic germ $\phi_\alpha^{(d+1)}$ at $(H^{(d+1)}, 0)$ as the fraction

$$\phi_\alpha^{(d+1)} = \frac{\left(g^{(d+1)}\right)^m}{\left(f^{(d+1)}\right)^n}.$$

Definition 3.9. Let $A \subset \mathbb{Q}_{>0}$ be a finite set of postitive rational numbers. The resolution π is *A-separating* if for every $d = 0, 1, \ldots, n$, and every $\alpha \in A$, the indeterminacy locus of the meromorphic function

$$\left(\pi^{(d+1)}\right)^* \phi_\alpha^{(d+1)}$$

is contained in $Y^{(d)} \cap \tilde{X}^{(d+1)}$.

3.10. We end this section by recalling a construction from [KKMSD73, Chapter II], which allows us to assume A-separatedness for any finite set $A \subset \mathbb{Q}_{>0}$, and some properties. Let $\pi : Y \to \mathbb{C}^{n+1}$ be a given resolution of f, which factors through the origin, and assume that the coordinates $x_0, x_1, \ldots, x_n$ in $\mathbb{C}^{n+1}$ are generic with respect to π_0 and f. Use the notation introduced so far in this section for π. Denote by $\triangle$ and $\hat{\triangle}$ the *conical polyhedral complexes* associated with the toroidal embeddings

$$Y \backslash (D \cup \tilde{X}) \subset Y, \qquad Y \backslash (D \cup \tilde{X} \cup \tilde{Y}^{(n)}) \subset Y,$$

respectively, as in [KKMSD73, Definition 5, II§1, p. 69]. What this means is the following: Choose distinct elements i_X, i_H not in in $I^{(d+1)}$ for any d, and set $D_{i_X} = \tilde{X} \subset Y$ and $D_{i_H} = Y^{(n)} \subset Y_0$. For any $J \subset I \cup \{i_X, i_H\}$ such that $D_J \neq \varnothing$, we have objects

$$N_J = \mathbb{Z}^J, \qquad N_{J,\mathbb{R}} = \mathbb{R}^J, \qquad \sigma_J = \mathbb{R}^J_{\geqslant 0},$$

along with natural inclusions between them whenever $K \subset J$. Then $\hat{\triangle}$ is the conical polyhedral complex consisting of all the cones σ_J for such J, whereas $\triangle$ is the subcomplex consisting of cones σ_J with $i_H \notin J$. We also define linear functions

$$u_{J,f}, u_{J,g} : N_{J,\mathbb{R}} \to \mathbb{R}$$

by the requirements

$$u_{J,f}(i) = m_i, \qquad u_{J,g}(i) = n_i, \qquad i \in J.$$

Note that these are compatible with inclusions $K \subset J$.

Given any subdivision $\triangle_A$ of $\triangle$ into cones, Mumfords describes a map $Y_A \to Y$ which, in local coordinates compatible with the divisor $D \cup \tilde{X} \cup \tilde{Y}^{(n)}$, is a toric map [KKMSD73]. Since this map is an isomorphism outside $\cup_{i \neq j}(D_i \cap D_j)$, the composed map $\pi : Y_A \to Y_0 \to \mathbb{C}^{n+1}$ remains an isomorphism outside $0 \in \mathbb{C}^{n+1}$. Furthermore, if π_A is constructed in this way, and $g = x_n$ is generic with respect to π, then, since $Y^{(n)}$ is transverse to all the strata D_J, with $i_H \notin J$, no exceptional component of π_A in Y_A maps into $Y^{(n)}$ via the map $Y_A \to Y$, and we find that $g = x_n$ is generic with respect to π_A as well. Finally, since the restriction of π_A to the strict transform $Y_A^{(n)}$ of $H^{(n)}$ in Y_A is in fact obtained by the same construction, we find a finite iteration that $x_0, \ldots, x_n$ are generic coordinates with respect to π_A.

Note that irreducible components D_i of the exceptional divisor of π correspond to rays, or one dimensional cones, in $\triangle$, except for the one generated by i_X. Similarly, irreducible exceptional components $D_{i,A}$ of π_A, indexed by a set I_A, correspond to rays in $\triangle$. Thus, we have an inclusion $I \hookrightarrow I_A$, such that if $i \in I$ then the map $Y_A \to Y$ induces a birational morphism $D_{i,A} \to D_i$. If $i \in I_A$ does not generate a ray in $\triangle$, then there is a smallest $J \subset I$ such that $\sigma_{\{i\}} \subset \sigma_J$, and in this case, the image of $D_{i,A}$ in Y is D_J, which has dimension $n + 1 - |J| < n$. In either case, the induced map $D^\circ_{i,A} \backslash \tilde{X}_A \to D^\circ_J \backslash \tilde{X}$, where $\tilde{X}_A$ is the strict transform of $\tilde{X}$ in Y_A, is a locally trivial fiber bundle with fiber $(\mathbb{C}^*)^{n+1-|J|}$, which gives

$$\chi\left(D^\circ_{i,A} \backslash \tilde{X}_A\right) = \begin{cases} \chi\left(D^\circ_i \backslash \tilde{X}\right) & |J| = 1, \\ 0 & |J| > 1. \end{cases}$$

Given a regular subdivision of $\hat{\triangle}_0$, we obtain a similar map $\hat{Y}_A \to Y$ which induces a map $\hat{\pi}_A : \hat{Y}_A \to \mathbb{C}^{n+1}$. This map may modify the intersection $X \cap H^{(n)}$, so it is not necessarily a point modification. Also, we will not consider any modification of $Y^{(d+1)}$ for $d < n$ induced by this map, as in the case of $\triangle_A$ above. For any $J \subset \hat{I}$, we will denote by

$$\hat{D}^\circ_J \subset \hat{Y}_A$$

the corresponding stratum.

Now, given any finite subset $A \subset \mathbb{Q}_{>0}$, we can choose a regular subdivision of $\triangle_A$ of $\triangle$ which, for any $\alpha \in A$, and $\sigma_J \in \hat{\triangle}$, refines the cone

$$\{v \in \sigma_J \mid \alpha u_{J,g}(v) = u_{J,f}(v_i)\} \subset \sigma_J. \tag{3.11}$$

Let Y_A be the induced space. Let $\hat{\triangle}_A$ be the subdivision of $\hat{\triangle}$, obtained as follows:

- The cone σ generated by i_X and i_H is subdivided so that each cone eq. 3.11 is an element of this subdivision.
- If $i_X \notin J$ and $i_H \notin J$, then the cone σ_J is subdivided as in $\triangle_A$.
- Any cone generated by i_X, i_H and another nonempty set J not containing i_X or i_H is subdivided as the join of the two subdivisions above, which induces a subdivision of cones generated by i_X or i_H and J.

The morphism $Y_A \to \mathbb{C}^{n+1}$ is then a point modification, and we use the notation in 3.6, with $D_A = \cup_{i \in I_A} D_i$. For each exceptional divisor D_i, with $i \in I_A$, denote by $\hat{D}_i$ the corresponding divisor in $\hat{Y}_A$, i.e. the strict transform of D_i in $\hat{Y}_A$, and set

$$\hat{D}_i^\circ = \hat{D}_i \backslash \bigcup_{j \in I_A \backslash \{i\}} \hat{D}_j$$

Since the map $\hat{D}_i \to D_i$ only modifies strata of codimension $\geqslant 2$, we see that the map

$$\hat{D}_i^\circ \backslash (\hat{X}_A \cup \hat{H}_A) \xrightarrow{\cong} D_i^\circ \backslash (\tilde{X}_A \cup \tilde{H}_A) \tag{3.12}$$

is an isomorphism. Here $\hat{X}_A, \hat{H}_A \subset \hat{Y}_A$ and $\tilde{X}_A, \tilde{H}_A \subset Y_A$ are the strict transforms of X, H.

Lemma 3.13. *Assume the notation introduced in 3.10.*

(i) The composed map $\pi_A : Y \to \mathbb{C}^{n+1}$ is an A-separating embedded resolution which factors through the blow-up of $\mathbb{C}^{n+1}$ at 0.

(ii) For any $\alpha \in A$, the map $\hat{\pi}$ resolves the indeterminacy locus of ϕ_α. In other words, pulling back via $\hat{\pi}$ gives a genuine map

$$\hat{\pi}_A^* \phi_\alpha : \hat{Y}_A \to \mathbb{CP}^1.$$

(iii) If the coordinates $x_0, x_1, \ldots, x_n$ are generic with respect to π_0, then they are generic with respect to π.

(iv) For every $i \in I_A$, we have

$$\chi(D_{i,A}^\circ \backslash \tilde{X}) = \begin{cases} \chi(D_{i,0}^\circ \backslash \tilde{X}_0) & i \in I, \\ 0 & i \notin I. \end{cases}$$

Proof. (i) and (ii) follow from the requirement eq. 3.11, which guarantees that irreducible component of the zero set and the poles of ϕ_α do not intersect, with the possible exception of $\tilde{X}_A$ and $\tilde{H}_A$ in Y_A. We have already seen (iii) and (iv). ∎

4 The Jacobian and alternating Jacobian polygons

4.1. In this section we introduce and discuss an invariant of an isolated hypersurface singularity, defined by Teissier [Tei77, Tei12]. We give a formula for the Jacobian polygon in terms of an embedded resolution, which induces a natural splitting into terms which can be seen as the integral of the Hironaka function over the Milnor fiber at radius zero with respect to the Euler characteristic. These terms are the *alternating Jacobian polygons*.

In [Mic08], Michel proves a formula similar to eq. 4.12 in the context of a finite morphism $(f,g) : (X,p) \to (\mathbb{C}^2, 0)$, where X is a surface with an isolated singularity at p. This statement reduces to our formula with $n = 1$ if $(X,p) = (\mathbb{C}^2, 0)$ is smooth, and g is a generic linear function.

We will assume that $\pi : Y \to \mathbb{C}^{n+1}$ is an embedded resolution of f, that $x_0, x_1, \ldots, x_n$ are generic coordinates with respect to f and π.

Definition 4.2. With the polar curve $P = P_1 \cup \cdots \cup P_l$ with respect to f and $g = x_n$, set

$$m_q = (X, P_q)_{\mathbb{C}^{n+1},0}\,, \qquad n_q = (H, P_q)_{\mathbb{C}^{n+1},0}\,, \qquad \alpha_q = \frac{m_q}{n_q}.$$

For any $d = 0, 1, \ldots, n$ and $q = 1, 2, \ldots, l^{(d+1)}$, define similarly

$$m_q^{(d+1)} = \left(X^{(d+1)}, P_q^{(d+1)}\right)_{H^{(d+1)},0}\,, \qquad n_q^{(d+1)} = \left(H^{(d)}, P_q^{(d+1)}\right)_{H^{(d+1)},0}\,, \qquad \alpha_q^{(d+1)} = \frac{m_q^{(d+1)}}{n_q^{(d+1)}}.$$

Definition 4.3. Let $x_0, x_1, \ldots, x_n$ be generic coordinates in $\mathbb{C}^{n+1}$ with respect to f.

(i) The Jacobian polygon associated with $(X, 0)$ is

$$\mathrm{J}(f,0) = \sum_{q=1}^{l} \left\{ \frac{m_q}{n_q} \right\} \in \mathfrak{N}\,. \tag{4.4}$$

(ii) For $0 \leqslant d \leqslant n$, we define

$$\mathrm{J}^{(d+1)}(f,0) = \mathrm{J}\left(f|_{H^{(d+1)}}, 0\right) = \sum_{q=1}^{l^{(d+1)}} \left\{ \frac{m_q^{(d+1)}}{n_q^{(d+1)}} \right\} \in \mathfrak{N}\,.$$

(iii) We define the *alternating Jacobian polygon of f* as

$$\mathrm{AJ}(f,0) = \sum_{d=0}^{n} (-1)^{n-d}\, \mathrm{J}^{(d+1)}(f,0) \in K\mathfrak{N}\,. \tag{4.5}$$

(iv) For $d = 0, 1, \ldots, n$, set

$$\mathrm{AJ}^{(d+1)}(f,0) = \mathrm{AJ}(f|_{H^{(d+1)}}, 0) \in K\,\mathfrak{N}$$

and $\mathrm{AJ}(f,0)^{(0)} = 0$.

Remark 4.6. (i) The set of coordinates generic with respect to f forms an open subset in the set of linear maps $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, over which we have a deformation for each $d = 0, 1, \ldots, n$ with fibers $X^{(d+1)}$, and these deformations are (c)-equisingular (see [Tei77] for the definition of (c)-equisingularity). As a result, the Jacobian polygons are independent of the choice of generic coordinates.

(ii) In the case $n = 0$, there exists a unit $u \in \mathbb{C}\{x_0\}$ such that $f(x_0) = x_0^e \cdot u(x_0)$, where e is the multiplicity of X at 0. In this case, $l = 1$, $(X, P_1)_0 = e$, $(H, P_1)_0 = 1$ and so

$$\mathrm{AJ}(f, 0) = \mathrm{J}(f, 0) = \left\{ \frac{e}{1} \right\}.$$

As a result, if e is the multiplicity of the hypersurface X at 0, then

$$\mathrm{J}^{(1)}(f, 0) = \left\{ \frac{e}{1} \right\}.$$

(iii) It follows immediately from eq. 4.5 that

$$\mathrm{J}^{(d+1)}(f, 0) = \mathrm{AJ}^{(d+1)}(f, 0) + \mathrm{AJ}^{(d)}(f, 0). \tag{4.7}$$

(iv) As a result of [Tei77, 1.4 Remarque], we have

$$\ell\left(\mathrm{J}^{(d+1)}(f, 0)\right) = \mu^{(d+1)} + \mu^{(d)}, \qquad h\left(\mathrm{J}^{(d+1)}(f, 0)\right) = \mu^{(d)}$$

for $d = 0, 1, \ldots, n$. Note that here, by convention, we set $\mu^{(0)} = 1$.

(v) By the previous remark, we get a telescopic series

$$\ell(\mathrm{AJ}(f, 0)) = \sum_{d=0}^{n} (-1)^{n-d} \left(\mu^{(d+1)} + \mu^{(d)}\right) = \mu^{(n+1)} + (-1)^n.$$

(vi) As a consequence of the previous point, we have $\ell(\mathrm{AJ}(f, 0)) \neq 0$ unless n is odd and $\mu = 1$, which happens precisely when $(X, 0)$ is a Morse point, i.e. when the Hessian matrix of f at 0 is invertible. In this case one readily verifies that $(X^{(d+1)}, 0)$ is a Morse point for all d, and that

$$\mathrm{J}^{(d+1)}(f, 0) = \left\{ \frac{2}{1} \right\}, \qquad d = 0, 1, \ldots, n,$$

and we find $\mathrm{AJ}(f, 0) = 0$ in this case. We have proved that the following are equivalent

- ❀ $\mathrm{AJ}(f, 0) = 0$,
- ❀ $\ell\left(\mathrm{AJ}(f, 0)\right) = 0$,
- ❀ n is odd and f is Morse.

(vii) By [Tei77, 1.7 Corollaire 2], the Jacobian polygon contains the Łojasiewicz exponent:

$$\mathcal{L}(f, 0) = \deg \mathrm{J}(f, 0) - 1. \tag{4.8}$$

(viii) If $\alpha \in \mathrm{supp}(\mathrm{J}(f, 0))$, then $\alpha \geqslant \mathrm{mult}(X, 0)$. Indeed, we have

$$\alpha = \frac{(X, P_q)_0}{(H, P_q)_0} \tag{4.9}$$

for some $q = 1, 2, \ldots, l$, and by [Tei77, 1.2 Théorème 1], we have $(H, P_q)_0 = \mathrm{mult}(P_q, 0)$ for each $q = 1, \ldots, l$. Therefore,

$$\alpha = \frac{(X, P_q)_0}{(H, P_q)_0} \geqslant \frac{\mathrm{mult}(X, 0) \cdot \mathrm{mult}(P_q, 0)}{\mathrm{mult}(P_q, 0)} = \mathrm{mult}(X, 0).$$

Theorem 4.10. *Let $\pi : Y \to \mathbb{C}^{n+1}$ be an embedded resolution of f which factors through the first blow-up. Then*

$$\mathrm{J}(f,0) = (-1)^n \sum_{i\in I} \chi(D_i^\circ \backslash (\tilde{X} \cup \tilde{H})) \left\{ \frac{m_i}{n_i} \right\} \tag{4.11}$$

and

$$\mathrm{AJ}(f,0) = (-1)^n \sum_{i\in I} \chi(D_i^\circ \backslash \tilde{X}) \left\{ \frac{m_i}{n_i} \right\}. \tag{4.12}$$

Proof. Using eqs. (4.4) and (4.5) and additivity of the Euler characteristic, we see that eq. 4.12 follows from eq. 4.11, so we only prove eq. 4.11. For any rational number $\alpha = m_\alpha/n_\alpha$, with $m_\alpha, n_\alpha > 0$ and $\gcd(m_\alpha, n_\alpha) = 1$, set

$$I_\alpha = \left\{ i \in I \,\middle|\, \frac{m_i}{n_i} = \alpha \right\}.$$

We have to show that

$$\sum_{\alpha_q = \alpha} (X, P_q)_{\mathbb{C}^{n+1},0} = (-1)^n \sum_{i \in I_\alpha} m_i \chi(D_i^\circ \backslash (\tilde{X} \cup \tilde{H})) \tag{4.13}$$

for all α. Indeed, the left hand side here is that of eq. 4.11, multiplied with m_α, and a similar statement holds for the two left hand sides. Note that in eq. 4.11, each side is a finite sum. Therefore we know, a priori, that eq. 4.13 holds for all but a finite list of rational numbers, those of the form $\alpha = m_i/n_i$ for an exceptional divisor D_i, $i \in I$ or $\alpha = m_q/n_q$ for a polar branch P_q.

For any reduced fraction $\alpha = m_\alpha/n_\alpha$ as before, define the function

$$\phi_\alpha = \frac{g^{m_\alpha}}{f^{n_\alpha}}.$$

Fixing an α, as well as a polar branch P_q, the order of the pullback of ϕ_α to the normalization of P_q is a positive multiple of $n_q m_\alpha - m_q n_\alpha$, and so,

$$\lim_{x\to 0} \phi_\alpha|_{P_q} \begin{cases} = \infty & \alpha_q > \alpha, \\ \in \mathbb{C}^* & \alpha_q = \alpha, \\ = 0 & \alpha_q < \alpha. \end{cases}$$

As a result, if

$$M = M_{\varepsilon,\eta} = B_\varepsilon^{2n+2} \cap f^{-1}(D_\eta)$$

is a small Milnor tube, i.e. $0 \ll \eta < \varepsilon \ll 1$, then we can find positive real constants constants a, b satisfying the following conditions:

(i) For any branch P_q of the polar curve, we have

$$a < \left|\phi_{\alpha_q}|_{P_q \cap M}\right| < b.$$

(ii) For any two branches P_q, P_r of the polar curves such that $\alpha_r < \alpha_q$, we have

$$\left|\phi_{\alpha_q}|_{P_r \cap M}\right| < a \qquad \text{and} \qquad b < \left|\phi_{\alpha_r}|_{P_q \cap M}\right|.$$

(iii) If $\varnothing \neq J \subset I_\alpha$, for some α, and c is a nonregular value of the restriction $\pi^*\phi_\alpha|_{D_J^\circ}$, then $a < c < b$.

Set

$$U = \{y \in \mathbb{C} \,|\, a \leqslant |y| \leqslant b\}\,.$$

With small $0 < \eta \ll \varepsilon$, denote in this proof

$$F = F_{\alpha,\varepsilon,z} = f^{-1}(z) \cap B^{2n+2}_{\varepsilon} \cap \phi_\alpha^{-1}(U).$$

Then, for $0 < |z| < \eta$, we have

$$\sum_{\alpha_q=\alpha} (X, P_q)_{\mathbb{C}^{n+1},0} = (-1)^n \chi(F_{\alpha,\varepsilon,z}) \tag{4.14}$$

Indeed, the restriction $\phi_\alpha|_F$ of ϕ_α to $F = F_{\alpha,\varepsilon,z}$ coincides with that of $g^{m_\alpha}/|z|^{n_\alpha}$. Therefore, $\phi_\alpha|_F$ is a submersion everywhere, except for at intersection points $F \cap P$. It follows from construction that the branch P_q intersects F in exactly $(X, P_q)_0$ points if $\alpha_q = \alpha$, and in no points otherwise. As a result, the restriction of ϕ_α to $F \to U$ is a submersion, except for at $\sum_{\alpha_q=\alpha}(X, P_q)_0$ points, where it has an A_1 singularitiy. If G is the fiber over a general point of this map, then we find

$$\chi(F) = \chi(G)\chi(U) + (-1)^n \sum_{\alpha_q=\alpha} (X, P_q)$$

proving eq. 4.14, since $\chi(U) = 0$.

Now, with $\alpha = m/n$ fixed, consider the subset $A = \{\alpha\} \subset \mathbb{Q}$, and let π_A and $\hat{\pi}_A$ be as in 3.10. By lemma 3.13, it suffices to prove the lemma for π_A, so in order to save effort on notation, let us assume that $\pi = \pi_A$. We identify $F_{\alpha,\varepsilon,z}$ with its preimage in $\hat{Y} = \hat{Y}_A$ by $\hat{\pi}$. Choose any metric on $\hat{Y}$, and consider a gradient-like vector field for the function $|\hat{\pi}^* f|^2$ which is tangent to the preimages $\phi_\alpha^{-1}(a)$ and $\phi_\alpha^{-1}(b)$. Note that by construction, a and b are regular values of ϕ_α. Integrating this vector field induces a map

$$F_{\alpha,\varepsilon,\eta} \to \hat{D} \cap \phi^{-1}([a, b]),$$

whose fiber over a point in $\hat{D}^\circ_J \cap \phi_\alpha^{-1}([a, b]))$ consists of $m_J = \gcd\{m_j \,|\, j \in J\}$ disjoint copies of $(S^1)^{|J|-1}$. As a result,

$$\chi(F_{\alpha,\varepsilon,\eta}) = \sum_{\varnothing \neq J \subset I_\alpha} \chi(\hat{D}^\circ_J \cap \phi_\alpha^{-1}([a, b])) \cdot m_J \cdot \chi((S^1)^{|J|-1}) = \sum_{i \in I_\alpha} m_i \chi(\hat{D}^\circ_i \cap \phi_\alpha^{-1}([a, b])) \tag{4.15}$$

Now, by using a gradient-like vector field for $\pi^*\phi_\alpha|_{D^\circ_i}$ for $i \in J_\alpha$, satisfying $m_i/n_i = \alpha$, and $\hat{D}_i \cong D_i$, we find that the natural homotopy equivalence

$$\hat{D}^\circ_i \cap \pi^*_\alpha \phi_\alpha^{-1}([a, b]) \hookrightarrow D^\circ_i \backslash (\tilde{X} \cup \tilde{H}) \tag{4.16}$$

In particular, the two spaces have the same Euler characteristic. As a result, eq. 4.13 follows from eq. 4.14 and eq. 4.15. ■

5 The vanishing rate filtration

5.1. In this section we describe a Morse function and a gradient-like vector field for it on the Milnor fiber. The critical points of this Morse function are intersection points with polar curves in subspaces of varying dimension. As a result, we have a corrsponding Morse-Smale complex $(C_\cdot, \partial_\cdot)$ which computes the homology of the Milnor fiber, as in [Mil65, Theorem 7.4]. This construction can be made in such a way that all but $\mu = \mu^{(n+1)}$ handles of index n combine to form a ball $B^{(2n)}$, as in [LP79]. Now, to each polar curve, we associate it's *vanishing rate*, following Teissier [Tei75, 3.6.4]. This induces a grading on the Abelian group $C_\cdot$, which induces an increasing filtration of the complex $(C_\cdot, \partial_\cdot)$, which we call the *vanishing rate filtration*.

5.2. In order to describe the Milnor fiber as a handle body, we construct a Morse function $\psi_{\varepsilon,\eta}^{(d+1)}$ on the Milnor fiber $F_{\varepsilon,\eta}^{(d+1)}$, as well as a gradient like vector field $\xi_{\varepsilon,\eta}^{(d+1)}$ for it. As a result, we obtain the Morse-Smale complex generated by the critical values of $\psi_{\varepsilon,\eta} = \psi_{\varepsilon,\eta}^{(n+1)}$, whose differential counts trajectories of $\xi_{\varepsilon,\eta} = \xi_{\varepsilon,\eta}^{(d+1)}$. We will skip the indices ε, η when they are clear from context.

First, we construct a vector field ζ on $Y^{(d+1)}$ For any point $x \in Y^{(d)} \cap D^{(d+1)}$, let $J \subset I^{(d+1)}$ be the set such that $x \in D_J^\circ$. If $x \notin \tilde{X}$, then there exist holomorphic coordinates $u_0, u_1, \ldots, u_d$ centered at x in a chart $U_x \ni x$ with the property that for each $i \in J$ there is a $k_i \in \{1, 2, \ldots, |J|\}$ such that $D_i \cap U_x = \{u_{k_i} = 0\}$ and

$$\pi^* f|_{U_x} = \prod_{i \in J} u_{k_i}^{m_i}, \qquad \pi^* g^{(d+1)}|_{U_x} = u_0 \prod_{i \in J} u_{k_i}^{n_i}.$$

If $x \in \tilde{X}$, then we can find a similar coordinate chart such that

$$\pi^* f|_{U_x} = u_1 \prod_{i \in J} u_{k_i}^{m_i}, \qquad \pi^* g^{(d+1)}|_{U_x} = u_0 \prod_{i \in J} u_{k_i}^{n_i}.$$

In either case, define a vector field in U_x

$$\zeta_x = u_0 \frac{\partial}{\partial u_0}$$

By compactness, we can cover $Y^{(d)} \cap D^{(d+1)} = D^{(d)}$ with finitely many such U_x, say $H \cap D^{(d)} \subset \cup_{x \in T} U_x$, with $|T| < \infty$. We will assume that ε is chosen small enough that $\pi^{-1}(B_\varepsilon \cap H^{(d)}) \subset \cup_{x \in T} U_x$. In fact, if we fix some Riemannian metric on $Y^{(d+1)}$, inducing a metrix d, then, if $\kappa > 0$ is small enough, then the set

$$N_\kappa = \left\{ x \in Y^{(d+1)} \cap \pi^{-1}(B_\varepsilon) \,\middle|\, d(x, Y^{(d)}) < \kappa \right\}$$

is a tubular neighborhood of $Y^{(d)} \cap \pi^{-1}(B_\varepsilon)$, and we have $\overline{N}_\kappa \subset \cup_{x \in T} U_x$ for κ small enough. Now, set

$$U_0 = \pi^{-1}(B_\varepsilon) \cap Y^{(d+1)} \backslash \overline{N}_\kappa.$$

We have the vector field

$$\zeta_0^{(d+1)}(x) = \nabla \left| \pi^* \left(g^{(d+1)}|_{F_{\varepsilon,z}} \right) \right|^2 (x).$$

for $x \in U_0$, where $z = \pi^* f(x)$. We define the vector field ζ on $Y^{(d+1)} \cap \pi^{-1}(B_\varepsilon)$ by gluing together ζ_0 and the ζ_x for $x \in T$ via a partition of unity. Observe that

- The vector field ζ is tangent to the Milnor fibers $F_{\varepsilon,z}^{(d+1)}$ (which we identify with their preimages in $Y^{(d+1)}$).
- For any $z \in D_\eta^*$, the vector field $\zeta_{F_{\varepsilon,z}^{(d+1)}}$ has nondegenerate singular points at intersection points $P^{(d+1)} \cap F_{\varepsilon,z}^{(d+1)}$ of index d.
- ζ is defined in a neighborhood around $Y^{(d)}$, vanishes along $Y^{(d)}$ in a nondegenerate way. In fact, the Hessian of ζ restricted to the normal bundle of $Y^{(d)} \subset Y^{(d+1)}$ is positive definite.

As a result of the last item, we can, and will, assume that κ is chosen small enough so that there exists a trivialization

$$\iota^{(d+1)} : Y^{(d)} \times D_\kappa \cong N_\kappa^{(d+1)}$$

of N_κ, with the property that for any $x \in Y^{(d)}$, the set $\iota^{(d+1)}(\{x\} \times D_\kappa)$ is the unstable manifold of ζ at x.

Next, for a fixed z, we construct a Morse function

$$\psi_{\varepsilon,z}^{(d+1)} : F_{\varepsilon,z}^{(d+1)} \to \mathbb{R}$$

and a gradient-like vector field $\xi_{\varepsilon,z}^{(d+1)}$ recursively for $d = 0, 1, \ldots, n$. Starting with $d = 0$, we set

$$\psi^{(1)} = 0, \qquad \xi^{(1)} = 0.$$

Assuming that we have defined $\psi_{\varepsilon,\eta}^{(d)}$ and $\xi_{\varepsilon,\eta}^{(d)}$ for some $d > 0$, we then construct $\psi^{(d+1)}$ and $\xi^{(d+1)}$. Take a $\mathcal{C}^\infty$ function $B : \mathbb{R} \to \mathbb{R}$ satisfying $B|_{(-\infty,1/3]} = 1$ and $B|_{[2/3,\infty)} = 0$, and set

$$\tilde{\psi}^{(d)}(\tilde{x}) = \begin{cases} B\left(|g^{(d+1)}(x)|^2/\rho\right)\psi^{(d)}(x') & \text{if } x = \iota(x', t), \\ 0 & \text{else.} \end{cases}$$

where $\rho > 0$ is so small that if $x \in F_{\varepsilon,z}^{(d+1)}$ and $|g^{(d+1)}(x)| < \rho$ then $x \in N_\kappa^{(d+1)}$. Then, the function

$$\psi_{\varepsilon,\eta}^{(d+1)} = |g^{(n+1)}|^2 + \nu\tilde{\psi}^{(d)}$$

is Morse on $F_{\varepsilon,\eta}^{(d+1)}$ for ν small enough. In fact, since $g^{(d+1)}|_{F_{\varepsilon,\eta}^{(d+1)}}$ has A_1 singularities at intersection points with $P^{(d+1)}$, $\psi^{(d+1)}$ has nondegenerate singularities there of index d. Furthermore, the Hessian of $|g^{(d+1)}|^2$ restricted to the normal bundle of $F^{(d)}$ in $F^{(d+1)}$ is positive definite. Therefore, we see by induction that the index of $\psi^{(n+1)}$ at an intersection point in $F^{(n+1)} \cap P^{(d+1)}$ has index d.

By extending ξ similarly as above, using the product structure on the image of $\iota^{(d+1)}$ to a vector field $\tilde{\xi}^{(d)}$ on $F^{(d+1)}$ with support near $F^{(d)}$ we set

$$\xi^{(d+1)} = \nabla|g^{(d+1)}|^2 + \nu\tilde{\xi}^{(d)}.$$

Then for ν small enough, $\xi^{(d+1)}$ is gradient like for $\psi^{(d+1)}$.

Lemma 5.3. *Fix polar curve branches $P_q^{(d+1)}$ and $P_r^{(d+1)}$ for some d. Then, if, for η arbitrarily small, there exists a $z \in D_\eta^*$ and a trajectory $\gamma : \mathbb{R} \to F_{\varepsilon,z}^{(d+1)}$ of $\xi^{(d+1)}$ satisfying*

$$\lim_{t\to-\infty} \gamma(t) \in P_r^{(d)}, \qquad \lim_{t\to+\infty} \gamma(t) \in P_q^{(d+1)},$$

then

$$\alpha_r^{(d)} \leqslant \alpha_s^{(d+1)}. \tag{5.4}$$

Proof. We start by making some assumptions on the resolution π. Similarly as in the proof of theorem 4.10, let $\triangle$ be the conical polyhedral complex associated with the divisor

$$D \cup \tilde{X} \subset Y.$$

Note that we do not include the divisor $Y^{(n)} \subset Y$. Set

$$\alpha = \alpha_q^{(d+1)}, \qquad \beta = \alpha_r^{(d)}, \qquad A = \{\alpha, \beta\}.$$

Assume, then, that $\pi = \pi_A$ is an embedded resolution of f which satisfies the conclusion of lemma 3.13. In particular, the indeterminacy locus of the meromorphic function

$$\pi^*\phi_\alpha^{(d+1)} = \frac{\left(g^{(d+1)}\right)^m}{\left(f^{(d+1)}\right)^n}, \qquad \alpha = \frac{m}{n}$$

is contained in $Y^{(d)}$. Similarly, the indeterminacy locus of the meromorphic function

$$\pi^* \phi_\alpha^{(d)} = \frac{\left(g^{(d)}\right)^m}{\left(f^{(d)}\right)^n}, \qquad \alpha = \frac{m}{n}$$

is contained in $Y^{(d-1)}$.

Let us assume that z_k is a sequence of small numbers $z_k \in \mathbb{C}$ converging to zero such that there exist trajectories γ_k in $F^{(d+1)}_{z_k,\varepsilon}$ as described in the statement of the lemma. We then want to prove that $\beta \leqslant \alpha$. For each k, the vector field $\xi^{(d+1)}_{\varepsilon,z_k}$ points outwards along the boundary of $N_\kappa^{(d+1)} \cap F^{(d+1)}_{\varepsilon,z_k}$, and so the trajectory γ_k intersects this boundary in exactly one point. By reparametrization, we will assume that this point, say p_k, is the value of γ_k at 0.

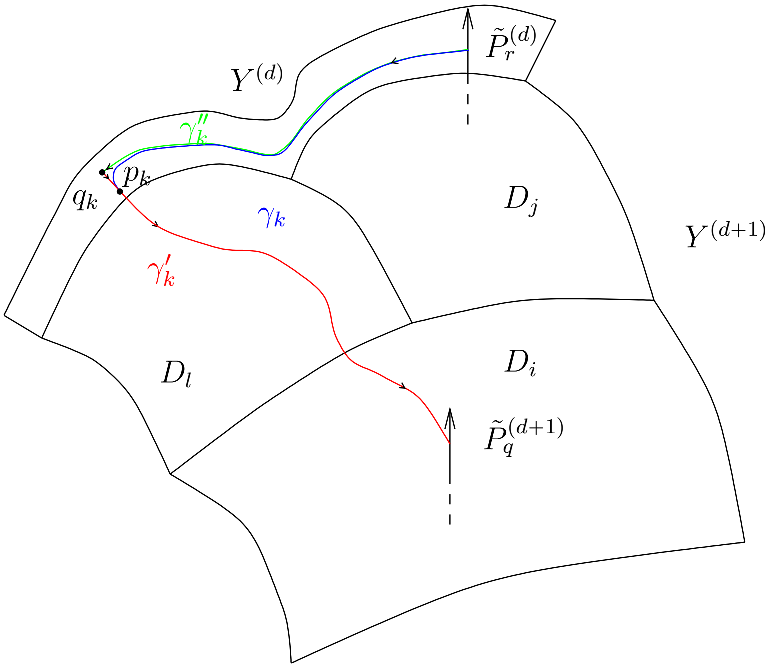

Figure 5.1: The trajectories $\gamma_k, \gamma_k', \gamma_k''$ in $Y^{(d+1)}$.

The restriction of γ_k to the positive real axis is then a trajectory of $\zeta^{(d+1)}$ as well. Denote by γ_k' the trajectory of $\zeta^{(d+1)}$ which coincides with γ_k along $\mathbb{R}_{\geqslant 0}$. We then have $\gamma_k(0) = \iota^{(d+1)}(q_k, \theta_k)$ for some $q_k \in Y^{(d+1)}$ and $\theta_k \in \partial D_\kappa$. The continuation of γ_k' to the negative axis then follows the segment from (q_k, θ_k) towards $(q_k, 0)$, and we have

$$\lim_{t \to +\infty} \gamma_k'(t) = q_k \in Y^{(d)}.$$

Since $\phi_\alpha(\gamma'(t))$ is increasing, and by choosing $a, b \in \mathbb{R}_{>0}$ as in the proof of theorem 4.10, we find

$$|\pi^*\phi_\alpha^{(d+1)}(p_k)| = |\pi^*\phi_\alpha^{(d+1)}(\gamma_k(0))| < \lim_{t\to+\infty} |\pi^*\phi_\alpha^{(d+1)}(\gamma_k(t))| \leqslant b$$

By compactnes, the sequence p_k has an accumulation point $p \in Y^{(d+1)}$. By taking a subsequence, we will assume that, in fact, $p_k \to p$ when $k \to \infty$. By continuity of $\pi^*\phi_\alpha^{(d+1)}$ away from $Y^{(d)}$, we then have

$$|\pi^*\phi_\alpha(p)| \leqslant b \qquad \text{and} \qquad \pi^* f(p) = 0.$$

In particlar, we have $p \in D^{(d+1)} \cup \tilde{X}^{(d+1)}$. Furthermore, $p \notin \tilde{X}^{(d+1)}$ and $p \notin D_i$ for any $i \in I^{(d+1)}$ satisfying $m_i/n_i > \alpha$. As a result, as for any point on $D^{(d+1)}$, there exists a unique $J \subset I^{(d+1)}$ such that $q \in D_J^\circ$, but we also have $m_i/n_i \leqslant \alpha$ for any $i \in J$.

Now, since p_k converges to p, there exists a $q \in Y^{(d)}$ such that $q_k \to q$. Since the vector field ζ is tangent to any D_i for $i \in I^{(d+1)}$, we find that

$$q \in D_J^\circ.$$

Since $q \in Y^{(d)}$, there exists a unique $K \subset I^{(d)}$ such that $q \in D_K^\circ$, and the map $\omega^{(d+1)} : I^{(d)} \to I^{(d+1)}$ induces a bijection $K \to J$.

Next, we restrict our attention to the part of γ_k contained in $N_k^{(d+1)}$, which has a product structure via $\iota^{(d+1)}$. For $t < 0$, write

$$\gamma_k(t) = \iota^{(d+1)}(\gamma_k''(t), \vartheta(t)).$$

By construction, $\gamma_k''(t)$ is then a reparametrization of a trajectory of $\xi_{\eta,z_k}^{(d)}$, and ϑ is a parametrization of the open segment from $0 \in D_\eta$ to $\theta_k \in \partial D_\kappa$. Similarly as above, we have for any k

$$a \leqslant \lim_{t \to -\infty} \left| \pi^* \phi_\beta^{(d)} \left(\gamma''(t)\right) \right| \leqslant \left| \pi^* \phi_\beta^{(d)} \left(q_k\right) \right|$$

and so $a \leqslant \phi_\beta^{(d)}(q)$. It follows that for any $i \in K \subset I^{(d)}$ we have $\beta \leqslant m_i/n_i$. Therefore,

$$\beta \leqslant \frac{m_i}{n_i} = \frac{m_{\omega^{(d+1)}(i)}}{n_{\omega^{(d+1)}(i)}} \leqslant \alpha. \qquad \blacksquare$$

5.5. Denote by

$$\Sigma^{(d+1)} = P^{(d+1)} \cap F_{\varepsilon,\delta}$$

the set of critical points of $\psi^{(d+1)}$ having index d, which is partitioned by setting $\Sigma_q^{(d+1)} = \Sigma^{(d+1)} \cap P_q^{(d+1)}$ for any q. Let $(C_\cdot, \partial_\cdot)$ be the Morse-Smale complex associated with $\psi = \psi^{(n+1)}$ and $\xi = \xi^{(n+1)}$ so that

$$C_d = \mathbb{Z}\langle \Sigma^{(d+1)} \rangle, \qquad C_\cdot = \bigoplus_{d=0}^{n} C_d.$$

Thus, if $x \in \Sigma^{(d+1)}$ and $y \in \Sigma^{(d)}$, then the coefficient in front of y in $\partial_d(x)$ is a signed count of trajectories γ of $\xi^{(n+1)}$ (or a small perturbation of $\xi^{(d+1)}$) satisfying

$$\lim_{t \to -\infty} \gamma(t) = y, \qquad \lim_{t \to +\infty} \gamma(t) = x.$$

If $p \in \Sigma_q^{(d+1)}$, the we define the *vanishing rate* of p as $\alpha(p) = \alpha_q^{(d+1)}$. The *vanishing rate filtration* on $C_\cdot$ is the filtration $F_\cdot$ given by

$$F_\alpha C_d = \mathbb{Z} \left\langle p \in F^{(n+1)} \cap P^{(d+1)} \,\middle|\, \alpha(p) < \alpha \right\rangle \subset C_d,$$
$$F_\alpha C_\cdot = \sum_{d=0}^{n} F_\alpha C_d \subset C$$

for any $\alpha \in \mathbb{Q}$. For $d = 0, 1, \ldots, n$, we define $(C_\cdot^{(d+1)}, \partial_\cdot^{(d+1)})$ similarly as the Morse-Smale complex associated with $\psi^{(d+1)}$ and $\xi^{(d+1)}$. The Abelian groups $C_\cdot^{(d+1)}$ are filtered in the same way by the vanishing rates. By the construction of the Morse-Smale complex, lemma 5.3 immediately implies

Corollary 5.6. *The filtration $F_\cdot$ is a filtration of $(C_\cdot, \partial_\cdot)$ as a complex, that is,*

$$\partial_d(F_\alpha C_d) \subset F_\alpha C_{d-1}, \qquad \alpha \in \mathbb{Q}. \qquad \blacksquare$$

6 Nonnegativity for the alternating Jacobian polygon

6.1. In this section, we prove theorem 6.2, as well as several corollaries. This result shows that, although the alternating Jacobian polygon does not always have nonnegative coefficients like Teissier's Jacobian polygon, the length of any of its truncations is nonnegative.

The proof of this result uses the filtration on the Morse-Smale complex introduced in the previous section. This argument is sketched out informally in 6.24.

Theorem 6.2. *For any $\alpha \in \mathbb{Q}_{>0}$ and $d = 1, \dots, n$, we have*

$$h\left(\mathrm{J}^{(d+1)}_{\geqslant\alpha}(f,0)\right) \geqslant \ell\left(\mathrm{J}^{(d)}_{\geqslant\alpha}(f,0)\right) - h\left(\mathrm{J}^{(d)}_{\geqslant\alpha}(f,0)\right). \tag{6.3}$$

As a result,

$$\ell\left(\mathrm{J}^{(d+1)}_{\geqslant\alpha}(f,0)\right) \geqslant (\alpha-1)\ell\left(\mathrm{J}^{(d)}_{\geqslant\alpha}(f,0)\right). \tag{6.4}$$

with a strict inequality if $\alpha < \deg \mathrm{J}^{(d+1)}(f,0)$.

Corollary 6.5. *The virtual vertices of $\mathrm{AJ}(f,0)$ (see 2.7) lie in the closed upper halfplane.*

Proof. Since f is singular, we have $m_q^{(d+1)} \geqslant 2n_q^{(d+1)}$ for all d and q. Therefore, eq. 6.3 gives

$$h\left(\mathrm{J}^{(n+1)}_{\geqslant\alpha}(f,0)\right) \geqslant h\left(\mathrm{J}^{(n)}_{\geqslant\alpha}(f,0)\right) \geqslant \cdots \geqslant h\left(\mathrm{J}^{(0)}_{\geqslant\alpha}(f,0)\right) = 1$$

As a result, we have a nonnegative alternating sum

$$h\left(\mathrm{AJ}_{\geqslant\alpha}(f,0)\right) = \sum_{d=0}^{n}(-1)^{n-d}h\left(\mathrm{J}^{(d+1)}_{\geqslant\alpha}(f,0)\right) \geqslant 0.$$

■

Corollary 6.6. *Let r be the rank of the Hessian matrix of f at 0 and assume that either*

$$d+1 > r.$$

or that d is even (see remark 4.6(vi)). Then, the following inequalities hold for the alternating Jacobian polygon:

(i) For $\alpha \in \mathbb{Q}$, we have

$$\ell\left(\mathrm{AJ}^{(d+1)}_{\geqslant\alpha}(f,0)\right) \geqslant 0$$

with equality if and only if $\alpha > \deg \mathrm{J}^{(d+1)}(f,0)$.

(ii) The leading coefficient of the alternating Jacobian polygon is positive

$$\mathrm{lc}\,\mathrm{AJ}^{(d+1)}(f,0) > 0. \tag{6.7}$$

(iii) The degree of the Jacobian polygon increases with dimension

$$\deg \mathrm{J}^{(d+1)}(f,0) \geqslant \deg \mathrm{J}^{(d)}(f,0), \qquad d = 1, \dots, n. \tag{6.8}$$

(iv) The Jacobian and alternating Jacobian polygons have the same degree

$$\deg \mathrm{AJ}^{(d+1)}(f,0) = \deg \mathrm{J}^{(d+1)}(f,0). \tag{6.9}$$

Proof. To prove (i), consider first the case $\alpha \leqslant 2$. Since $\operatorname{supp} \mathrm{J}^{(d+1)}(f,0) \subset [2,\infty)$ by remark 4.6(viii), we have $\mathrm{J}^{(d+1)}_{\geqslant\alpha}(f,0) = \mathrm{J}^{(d+1)}(f,0)$, and so by remark 4.6(iv) and (v)

$$\ell\left(\mathrm{J}^{(d+1)}_{\geqslant\alpha}(f,0)\right) = \mu^{(d+1)} + (-1)^d > 0.$$

Consider next the case $2 < \alpha \leqslant \deg \mathrm{J}^{(d+1)}(f,0)$. Then eq. 6.4 gives

$$\ell\left(\mathrm{J}^{(d+1)}_{\geqslant\alpha}(f,0)\right) \geqslant \ell\left(\mathrm{J}^{(d)}_{\geqslant\alpha}(f,0)\right) \geqslant \cdots \geqslant \ell\left(\mathrm{J}^{(0)}_{\geqslant\alpha}(f,0)\right) = 0 \tag{6.10}$$

with the first inequality strict. Indeed, we have $\alpha - 1 > 1$ since $\alpha > 2$, and $\ell\left(\mathrm{J}^{(d+1)}_{\geqslant\alpha}(f,0)\right) > 0$ since $\alpha \leqslant \deg \mathrm{J}^{(d+1)}(f,0)$. As a result, the alternating sum

$$\ell\left(\mathrm{AJ}^{(d+1)}_{\geqslant\alpha}(f,0)\right) = \sum_{i=0}^{d+1}(-1)^i\left(\mathrm{J}^{(d+1-i)}_{\geqslant\alpha}(f,0)\right)$$

is strictly positive. Next, consider the case $\alpha > \deg \mathrm{J}^{(d+1)}$. Then all the inequalities in eq. 6.10 are equalities, and all the terms zero, proving (i), which immediately gives (ii).

Now, (iii) follows immediately from eq. 6.4. Along with the definition of the alternating Jacobian polygon, this gives $\leqslant$ in eq. 6.9. But $\geqslant$ follows from eq. 6.7 and eq. 4.7 ■

Corollary 6.11. *The Łojasiewicz exponent increases with dimenson*

$$\mathcal{L}\left(f^{(d+1)},0\right) \geqslant \mathcal{L}\left(f^{(d)},0\right), \qquad d = 1,2,\ldots,n. \quad ■$$

Corollary 6.12. *The Łojasiewicz exponent of f at 0 is given by the degree of the alternating Jacobian polygon minus one*

$$\mathcal{L}(f,0) = \deg \mathrm{AJ}(f,0) - 1$$

unless n is odd and f has a Morse point at 0. ■

6.13. Let Δ be the *Cerf diagram* associated with the functions f and g, that is, the image of the polar curve under the map

$$\Phi = (g,f) : (\mathbb{C}^{n+1},0) \to (\mathbb{C}^2,0).$$

We will use coordinates u,v in $\mathbb{C}^2$, so that this map is given by $u = g$ and $v = f$. For any $a,b,c \in \mathbb{C}$, set

$$L_{a,b,c} = \left\{(u,v) \in \mathbb{C}^2 \,\middle|\, au + bv = c\right\}.$$

Writing the polar curve as a union of branches $P = P_1 \cup \ldots \cup P_l$, denote by Δ_q the image of P_q under Φ. Then

$$(X,P_q)_{\mathbb{C}^{n+1},0} = (L_{0,1,0},\Delta_q)_{\mathbb{C}^2,0}, \qquad (H,P_q)_{\mathbb{C}^{n+1},0} = (L_{1,0,0},\Delta_q)_{\mathbb{C}^2,0}.$$

Furthermore, we have the Milnor fiber

$$F_{\varepsilon,\eta} = \Phi^{-1}\left(L_{0,1,\eta}\right) \cap B^{2n+2}_{\varepsilon}$$

and the map $F_{\varepsilon,\eta} \to L_{0,1,\eta}$ has critical set $\Sigma = P \cap F_{\varepsilon,\eta}$. For each $q = 1,\ldots,l$, we can write

$$\Delta_q \cap L_{0,1,\eta} = \{d_{q,1},\ldots,d_{q,m_q}\}, \qquad P_q \cap F_{\varepsilon,\eta} = \{c_{q,1},\ldots,c_{q,m_q}\}$$

so that $\Phi(c_{q,j}) = d_{q,j}$ (recall $m_q = (X,P_q)_0$). We can lift these points to continuous paths $c_{q,j} : [0,1] \to \mathbb{C}^2$ and $d_{q,j} \to \mathbb{C}^{n+1}$, such that that for every $t \in [0,1]$

$$\Delta_q \cap L_{-t\eta,1,(1-t)\eta} = \{d_{q,1}(t),\ldots,d_{q,m_q}(t)\},$$
$$P_q \cap \Phi^{-1}(L_{-t\eta,1,(1-t)\eta}) \cap B^{2n+2}_{\varepsilon} = \{c_{q,1}(t),\ldots,c_{q,m_q}(t)\}$$

and $c_{q,j}(0) = c_{q,j}$ and $d_{q,j}(0) = d_{q,j}$.

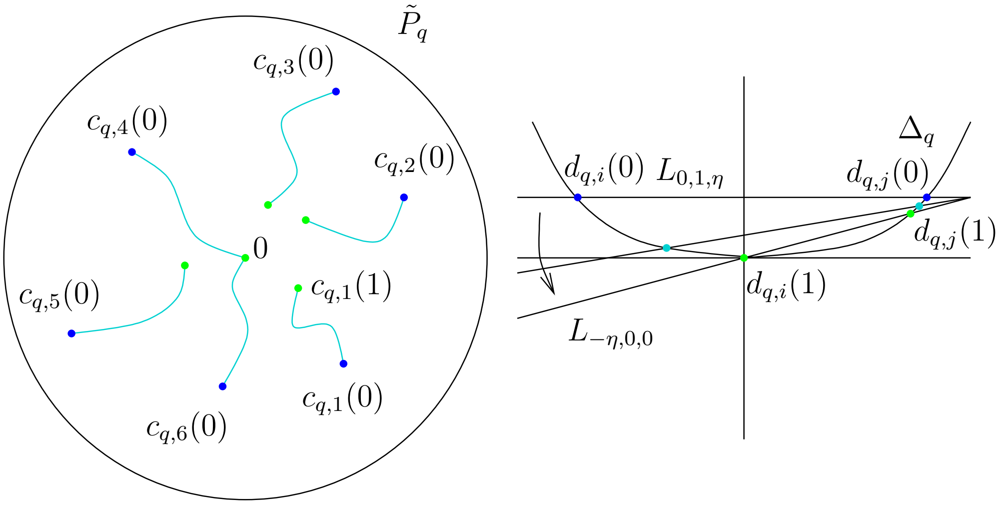

Figure 6.1: In this example we have $m_q = 6$ and $n_q = 2$.

Definition 6.14. (i) For $q = 1, \ldots, l$, set

$$K_q^{(d+1)} = \left\{ d_{q,l}^{(d+1)} \in \sigma^{(d+1)} \,\middle|\, c_{q,l}^{(d+1)}(1) = 0 \right\}$$

(ii) Denote by $(\hat{C}_\cdot^{(d+1)}, \hat{\partial}_\cdot)$ the subcomplex of $(C_\cdot, \partial_\cdot)$ generated by $\Sigma^{(i)}$ in degree $i \leqslant d$, and by $K^{(d+1)}$ in degree d.

(iii) The complex $(\hat{C}_\cdot, \hat{\partial}_\cdot)$ is a filtered complex by setting

$$F_\alpha \hat{C}_d = F_\alpha C_d \cap \hat{C}_d.$$

Lemma 6.15. *For every q, we have*

$$\left| K_q^{(d+1)} \right| = n_q^{(d+1)}.$$

Proof. Recall $n_q^{(d+1)} = (H^{(d)}, P_q^{(d+1)})_{H^{(d+1)},0}$ by definition 4.2. For $t \neq 0$, the lines $L_{-t\eta,1,(1-t)\eta}$ are not parallel to a tangent of Δ, i.e. the u-axis. As a result, if $0 \ll t < 1$, then, in a neighborhood of $0 \in \mathbb{C}^2$, there are precisely

$$\left(L_{-\eta,1,0}, \Delta_q^{(d+1)} \right)_{\mathbb{C}^2,0} = \operatorname{mult}(\Delta_q^{(d+1)}, 0) = n_q^{(d+1)}$$

points in the intersection $L_{-\eta,1,0} \cap \Delta_q^{(d+1)})_{\mathbb{C}^2,0}$, which corresopnd precisely the points $c_{q,j}(t)$ which then converge to zero when $t \to 1$. ∎

Lemma 6.16. *The complex $(\hat{C}_\cdot^{(d+1)}, \hat{\partial}_\cdot)$ has the homology of a ball.*

Proof. Choose $\kappa > 0$ small, and $t \in [0,1]$ near 1, such that $|v(d_{q,j}(t))| < \kappa$ if and only if $d_{q,j} \in K^{(d+1)}$. Then the set

$$\left(\Phi^{(d+1)} \right)^{-1} \left(L_{-t\eta,1,(1-t)\eta} \cap \{|v| < \kappa\} \right) \cap B_\varepsilon^{2n+2}$$

is homeomorphic to a Milnor fiber for $g^{(d+1)}$, which is a ball. The complex $\hat{C}_\cdot^{(d+1)}$ is then the Morse-Smale complex for a Morse function and gradient-like vector field constructed for this set, similarly as in 5.5. ∎

Lemma 6.17. *Assume that $1 \leqslant d \leqslant n$. Then*

(i) The images $\operatorname{im}\partial_d^{(d+1)}$ and $\operatorname{im}\hat{\partial}_d^{(d+1)}$ coincide.

(ii) For $\alpha \in \mathbb{Q}$, we have

$$\operatorname{rk}\ker\left(\frac{\hat{C}_{d-1}^{(d+1)}}{F_\alpha \hat{C}_{d-1}^{(d+1)}} \xrightarrow{\partial} \frac{\hat{C}_{d-2}^{(d+1)}}{F_\alpha \hat{C}_{d-2}^{(d+1)}}\right) \geqslant \sum_{\alpha_q^{(d+1)} \geqslant \alpha} m_q^{(d)} - n_q^{(d)}. \tag{6.18}$$

Proof. First, we prove (i). If $d = 1$, then the homology of either complex in degree 0 is free of rank one, and we have

$$\operatorname{im}\partial_1^{(2)} = \operatorname{im}\hat{\partial}_1^{(2)} = \left\{ \sum_{p \in \Sigma^{(1)}} a_p p \in C_0 \,\middle|\, \sum_{p \in \Sigma^{(1)}} a_p = 0 \right\}$$

If $d > 1$, then the complexes $(C_\cdot^{(d+1)}, \partial_\cdot^{(d+1)})$ and $(\hat{C}_\cdot^{(d+1)}, \hat{\partial}_\cdot^{(d+1)})$ are exact in degree $d-1$, and so both images coincide with the kernel of $\partial_{d-1}^{(d+1)} = \hat{\partial}_{d-1}^{(d+1)}$

$$\operatorname{im}\partial_d^{(d+1)} = \ker\partial_{d-1}^{(d+1)} = \ker\hat{\partial}_{d-1}^{(d+1)} = \operatorname{im}\hat{\partial}_d^{(d+1)}.$$

Next, we prove (ii) using (i). For every element, $x \in \Sigma^{(d+1)} \backslash K^{(d+1)}$, there exists a linear combination y_x of elements of $K^{(d+1)}$ such that $\hat{\partial}(y_x) = \partial(x)$. As a result, the elements $x - y_x$ for $x \in P_q^{(d)}$ satisfying $\alpha_q^{(d)} \geqslant \alpha$ form a linearly independent set of vectors modulo F_α in the kernel of the map ∂ on the left in eq. 6.18. For each q such that $\alpha_q^{(d)} \geqslant \alpha$, there are precisely $m_q^{(d)} - n_q^{(d)}$ of them. ∎

Lemma 6.19. *For $d = 1, \ldots, n$, the morphism*

$$\hat{C}_d^{(d+1)} \to \frac{\hat{C}_{d-1}^{(d+1)}}{\hat{C}_{d-1}^{(d)}} \tag{6.20}$$

induced by $\hat{\partial}_d^{(d+1)}$ is an isomorphism.

Proof. The inclusion $\hat{C}_\cdot^{(d)} \subset \hat{C}_\cdot^{(d+1)}$ corresponds to the inclusion of a ball in another ball. In particular, it induces an isomorphism on homology, and so the corresponding quotient complex is acyclic. In particular, we have an exact sequence

$$\to \frac{\hat{C}_{d+1}^{(d+1)}}{\hat{C}_{d+1}^{(d)}} \to \frac{\hat{C}_{d}^{(d+1)}}{\hat{C}_{d}^{(d)}} \to \frac{\hat{C}_{d-1}^{(d+1)}}{\hat{C}_{d-1}^{(d)}} \to \frac{\hat{C}_{d-2}^{(d+1)}}{\hat{C}_{d-2}^{(d)}} \to$$

The term to the left is 0, since $C_{d+1}^{(d+1)} = 0$, and the term to the right is 0, since $\hat{C}_{d-2}^{(d+1)} = C_{d-2} = \hat{C}_{d-2}^{(d)}$. As a result, the middle morphism is an isomorphism, which shows that eq. 6.20 is an isomorphism, since $\hat{C}_d^{(d)} = 0$. ∎

Proof of theorem 6.2. For a fixed α, the height on the left hand side of eq. 6.3 consists of terms of the type $n_q^{(d+1)}$ with $\alpha_q^{(d+1)} \leqslant \alpha$. As a result,

$$\ell\left(\mathrm{J}_{\geqslant 0}^{(d+1)}(f,0)\right) \geqslant \alpha h\left(\mathrm{J}_{\geqslant 0}^{(d+1)}(f,0)\right).$$

Using the same argument for the height on the right hand side, eq. 6.4 therefore results from multiplying eq. 6.3 by α. Thus, we focus on eq. 6.3.

Assuming first that $d = 1$, set $e = \operatorname{mult}(X, 0)$. If $\alpha > e$, then, by remark 4.6(ii), the right hand side of eq. 6.3 vanishes. Since the right hand side is nonnegative, the inequality holds. If, however, $\alpha \leqslant e$, then eq. 6.3 reads as

$$\mu^{(1)} \geqslant \mu^{(1)} + \mu^{(0)} - \mu^{(0)}$$

by remark 4.6(viii) and (iv), which clearly holds.

For the rest of this proof, we assume that $d > 1$. By lemma 6.19, for any $\alpha \in \mathbb{Q}$, the induced map

$$\frac{\hat{C}_d^{(d+1)}}{F_\alpha \hat{C}_d^{(d+1)}} \to \frac{\hat{C}_{d-1}^{(d+1)}}{\hat{C}_{d-1}^{(d)} + F_\alpha \hat{C}_{d-1}^{(d+1)}}$$

is surjective. As a result,

$$\operatorname{rk} \frac{\hat{C}_d^{(d+1)}}{F_\alpha \hat{C}_d^{(d+1)}} \geqslant \operatorname{rk} \frac{\hat{C}_{d-1}^{(d+1)}}{\hat{C}_{d-1}^{(d)} + F_\alpha \hat{C}_{d-1}^{(d+1)}} \tag{6.21}$$

On one hand, the left side of eq. 6.21 is freely generated by $p \in K_q^{(d+1)}$, and so

$$\operatorname{rk} \frac{\hat{C}_d^{(d+1)}}{F_\alpha \hat{C}_d^{(d+1)}} = \sum_{\alpha_q^{(d+1)} \geqslant \alpha} n_q^{(d+1)} = h\left(\mathrm{J}_{\geqslant \alpha}^{(d+1)}(f, 0)\right). \tag{6.22}$$

On the other hand, the right side of eq. 6.21 is freely generated by $p \in \Sigma_q^{(d)} \backslash K_q^{(d)}$, and so

$$\operatorname{rk} \frac{\hat{C}_{d-1}^{(d+1)}}{\hat{C}_{d-1}^{(d)} + F_\alpha \hat{C}_{d-1}^{(d+1)}} = \sum_{\alpha_q^{(d)} \geqslant \alpha} m_q^{(d)} - n_q^{(d)} = \ell\left(\mathrm{J}_{\geqslant \alpha}^{(d)}(f, 0)\right) - h\left(\mathrm{J}_{\geqslant \alpha}^{(d)}(f, 0)\right). \tag{6.23}$$

■

6.24. In the above proof, we prove the key inequality

$$\sum_{\alpha_q^{(d+1)} \geqslant \alpha} n_q^{(d+1)} \geqslant \sum_{\alpha_q^{(d)} \geqslant \alpha} m_q^{(d)} - n_q^{(d)}. \tag{6.25}$$

We provide here a purely visual argument for this equality, sketched out in fig. 6.2. What lies below a black horizontal line is a Milnor fiber $F^{(d+1)}$. The blue and red points are generators of the Morse-Smale complex $(C_\cdot, \partial_\cdot)$, the red points correspond to the elements of $K_q^{(d+1)}$ or $K_r^{(d)}$, and the blue points to elements of $\Sigma_q^{(d+1)} \backslash K_q^{(d+1)}$ or $\Sigma_r^{(d)} \backslash K_r^{(d)}$. Thus, the blue points correspond to handles freely generating the homology of $F^{(d+1)}$ or $F^{(d)}$, whereas the red points are handles which cancel out blue handles below. Thus, what lies below the yellow line is a ball.

The further to the right, the higher the vanishing rate of the generators. The blue points in $F^{(d)}$ must be cancelled by the red points in $F^{(d+1)}$. This means that there must be enough trajectories passing up from the blue points to the red points. Since trajectories going up cannot go to the left by lemma 5.3, the number of red points in the green box must be greater or equal to the number of blue points in the magenta box, and this is precisely eq. 6.25.

7 Newton nondegenerate functions

7.1. In this section, we introduce notation related to the Newton polyhedron associated with a power series. We will use freely the notion of being *Newton nondegenerate*, introduced in [Kou76]. We use coordinates $z_0, z_1, \ldots, z_n$ in $\mathbb{C}^{n+1}$ which are *not* generic in the sense of the previous sections.

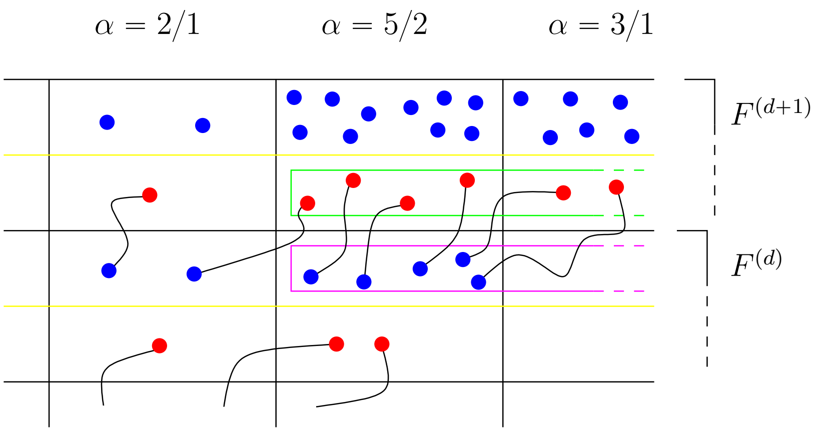

Figure 6.2: A visualization of the Morse-Smale complex.

The proof of theorem 7.12(i) is similar to that of Varchenko's formula for the monodromy zeta function [Var76], with theorem 4.10 playing the role of A'Campos formula. The more general theorem 7.12(ii) is similar to Oka's formula for principal zeta-functions [Oka90]. By taking lengths on both sides of eq. 7.13 and using remark 4.6(v), we recover Kouchnirenko's formula [Kou76].

We use the notion of *sedentarity*, a concept from tropical geometry, used in e.g. [MZ14, IKMZ19].

7.2. Let $f \in \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ be a convergent power series in $n+1$ variable:

$$f(z) = \sum_{i \in \mathbb{Z}^{n+1}_{\geqslant 0}} a_i z^i.$$

We will assume that f is not identically zero, but vanishes at zero, with at most an isolated critical point. Furthermore, we assume that f is *Newton nondegenerate*, as follows. The *support* of f is the set

$$\operatorname{supp}(f) = \left\{ i \in \mathbb{Z}^{n+1} \,\middle|\, a_i \neq 0 \right\}$$

and its *Newton polyhedron* is the convex closure of the union of positive orthants translated by the support:

$$\Gamma_+(f) = \operatorname{conv}\left(\bigcup_{i \in \operatorname{supp}(f)} i + \mathbb{R}^{n+1}_{\geqslant 0}. \right)$$

Using the interval

$$\overline{\mathbb{R}}_{\geqslant 0} = [0, +\infty]$$

any $v = (v_0, \ldots, v_n) \in \overline{\mathbb{R}}^{n+1}_{\geqslant 0}$ will be identified with the function

$$v : \mathbb{R}^{n+1}_{\geqslant 0} \to \overline{\mathbb{R}}_{\geqslant 0}, \qquad (u_0, u_1, \ldots, u_n) \mapsto \sum_{i=0}^{n} v_i u_i.$$

The *refined sedentarity* of such a vector, or function, is

$$\operatorname{sed}(v) = \{ i \in \{0, \ldots, n\} \,|\, v_i = +\infty \}.$$

The *sedentarity* of v is $|\operatorname{sed}(v)|$. and we define

$$\wedge_f v = \min_{\Gamma_+(f)} v, \qquad K_f(v) = \begin{cases} \{ u \in \Gamma_+(f) \,|\, v(u) = \wedge_f(v) \} & \wedge_f v < +\infty, \\ \varnothing & \wedge_f v = +\infty. \end{cases}$$

A *face* of Γ_+ is a nonempty set of the form $K_f(v)$. We denote by $K_f^\circ(v)$ its relative inerior, i.e. the topological interior of the face seen as a subset of the affine subspace it generates in $\mathbb{R}^{n+1}$. Such a face is compact if and only if all the coordinates of the vector v are positive, i.e. positive real numbers or $+\infty$. The *Newton diagram* of f is the union of all compact faces of $\Gamma_+(f)$

$$\Gamma(f) = \bigcup_{v\in\overline{\mathbb{R}}_{>0}^{n+1}} K_f(v),$$

and we define $\Gamma_-(f)$ as the union of segments joining the origin $0 \in \mathbb{R}^{n+1}$ and some point in $\Gamma(f)$. If $K = K_f(v)$ is a compact face of the Newton diagram, then there exists a unique $I_K \subset \{0, \ldots, n\}$ such that

$$K^\circ \subset \overline{\mathbb{R}}_{>0}^{I_K} = \left\{ (u_0, \ldots, u_n) \in \overline{\mathbb{R}}_{\geqslant 0}^{n+1} \,\middle|\, u_i = 0 \Leftrightarrow i \in I_K \right\}.$$

If $\dim(K) = |I_K| - 1$, then K is a *coordinate facet* of the diagram $\Gamma(f)$. A coordinate facet of dimension n is simply a facet. If K is a coordinate facet, then there exists a unique vector v_K called *the (primitive integral) normal vector* of K, satisfying

$$v_K \in \overline{\mathbb{Z}}_{\geqslant 0}^{n+1}, \qquad \gcd\left(\{v_{K,i} \,|\, i \notin \operatorname{sed}(v)\}\right) = 1, \tag{7.3}$$

such that $K = K(v_K)$ and $\operatorname{sed}(v_K) = \{0, 1, \ldots, n\} \backslash I_K$.

Definition 7.4. (i) Denote by $\mathcal{F}(\Gamma)$ the set of coordinate facets of the Newton diagram Γ, and by $\mathcal{N}(\Gamma)$ the set of their primitive integral normal vectors. If F is a coordinate facet corresponding to the primitive integral normal vector v, then we set

$$m_F = m_v = \wedge_f v.$$

(ii) We will denote by $g \in \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ a *generic linear function*, that is,

$$g(z_0, z_1, \ldots, z_n) = b_0 z_0 + b_1 z_1 + \ldots + b_n z_n$$

for some generic cofficients $b_i \in \mathbb{C}$. We will, furthermore, always assume that none of the b_i vanish. As a result, the Newton diagram of g is the standard n-simplex in $\mathbb{R}^{n+1}$, i.e. the convex hull of the natural basis. We set

$$n_F = n_v = \wedge_g v.$$

Definition 7.5. Denote by V_s the *generalized mixed s-volume* in $\mathbb{R}^n$. See e.g. [Ber76, Oka90] for further details. If $K_1, \ldots, K_s$ are convex bodies in $\mathbb{R}^n$, and there exist translates K_i' of K_i each contained in the same rational s-plane $L \subset \mathbb{R}^n$, then

$$V_s(K_1, \ldots, K_s) \in \mathbb{R}$$

is the mixed volume of the convex bodies K_i' with respect to the Lebesuge measure Vol_s on L, normalized so that the parallelpiped spanned by an integral basis of $L \cap \mathbb{Z}^n$ has volume 1. If $k_1, \ldots, k_c$ are nonnegative integers that sum up to s, then $V_s(K_1^{k_1}, \ldots, K_c^{k_c})$ means that each K_i is repeated k_i times. As in [Oka90], but not as in [Ber76], the mixed volume is normalized in such a way that if K is a convex set in an s dimensional affine subspace of $\mathbb{R}^{n+1}$, then

$$V_s(K^s) = \mathrm{Vol}_s(K). \tag{7.6}$$

Definition 7.7. Let $v \in \overline{\mathbb{Z}}_{>0}^{n+1}$ be primitive, and $d \in \{0, 1, \ldots, n\}$. With $s = n - |\operatorname{sed}(v)|$ and $c = n - d$, define

$$W^{(d+1)}(v) = \sum_{k_0, k_1, \ldots, k_c} s! V_s\left(K_f(v)^{k_0}, K_g(v)^{k_1}, \ldots, K_g(v)^{k_c}\right) \tag{7.8}$$

where the sum runs through $c+1$-tuples of integers

$$(k_0, k_1, \dots, k_c) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}^c_{>0}, \qquad \sum_{j=0}^{c} k_j = s.$$

Remark 7.9. (i) In the case $n = d$ we have $c = 0$, and so by eq. 7.6, if $v \in \overline{\mathbb{Z}}^{n+1}_{>0}$ with $s = n - |\operatorname{sed}(v)|$, then

$$W^{(n+1)}(v) = s! \operatorname{Vol}_s(K_f(v)). \tag{7.10}$$

In particular, $W^{(n+1)}(v) = 0$ unless $v \in \mathcal{N}(\Gamma(f))$.

(ii) The polytopes $K_f(v)$ and $K_g(v)$ are contained in an s dimensional affine subspace of $\mathbb{R}^{n+1}$. Using the normalized volume on this subspace, along with the notation used in [Kho78], we have

$$W^{(d+1)}(v) = (-1)^{s-c} \left(\frac{1}{1 + K_f(v)} \right) \left(\frac{K_g(v)}{1 + K_g(v)} \right)^c_s$$

What this means is the following: Treat $K_f(v)$ and $K_g(v)$ as variables, and expand the rational function at the origin. For every homogeneous monomial $K_f(v)^a K_g(v)^b$, with $a + b = s$, evaluate the mixed volume $V_s(K_f(v)^a, K_g(v)^b)$. The right hand side above is the sum of all such terms. Counting up the number of possible $k_1, k_2, \dots, k_c$, we find, if $d < n$,

$$W^{(d+1)}(v) = \sum_{k=c}^{s} \binom{k-1}{c-1} s! V_s \left(K_f(v)^{s-k}, K_g(v)^k \right) \tag{7.11}$$

(iii) If $\dim(K_f(v) + K_g(v)) < s$, then every term in eq. 7.8 vanishes. As a result, we have $W^{(d+1)}(v) = 0$ unless $v \in \mathcal{N}(\Gamma(fg))$. In particular, $W^{(d+1)}(v) \neq 0$ for all but finitely many primitive vectors $v \in \overline{\mathbb{Z}}^{n+1}_{>0}$.

Theorem 7.12. *(i) We have*

$$\operatorname{AJ}(f, 0) = \sum_{F \in \mathcal{F}} (-1)^{n-s} s! \operatorname{Vol}_s F \left\{ \frac{m_F}{n_F} \right\}. \tag{7.13}$$

where for any $F \in \mathcal{F}$, we set $s = \dim(F)$.

(ii) For any $d = 0, 1, \dots, n$, we have $W^{(d+1)}(v) = 0$ for all but finitely many primitive integral vectors $v \in \overline{\mathbb{Z}}^{n+1}_{>0}$, and

$$\operatorname{AJ}^{(d+1)}(f, 0) = \sum_{\substack{v \in \overline{\mathbb{Z}}^{n+1}_{>0} \\ \text{primitive}}} (-1)^{n-s} W^{(d+1)}(v) \left\{ \frac{m_v}{n_v} \right\}. \tag{7.14}$$

Corollary 7.15. *The Jacobian polygons associated with f are given by*

$$\operatorname{J}^{(d+1)}(f, 0) = \sum_{\substack{v \in \overline{\mathbb{Z}}^{n+1}_{>0} \\ \text{primitive}}} (-1)^{n-s} \left(W^{(d+1)}(v) + W^{(d)}(v) \right) \left\{ \frac{m_v}{n_v} \right\}. \tag{7.16}$$

Proof. This follows from the above theorem, and remark 4.6(iii) ∎

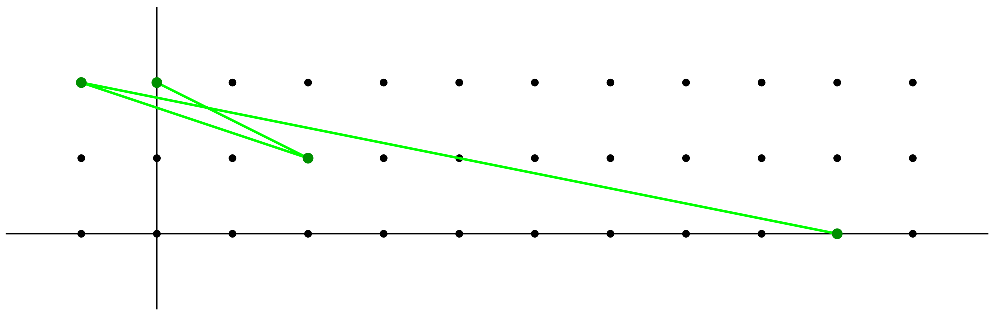

Figure 7.1: The alternating Jacobian polyhedron of the E_8 surface singularity.

Example 7.17. (i) The E_8 singularity $f(x, y, z) = x^2 + y^3 + z^5$ is Newton nondegenerate. By theorem 7.12, its alternating Jacobian polyhedron is

$$\mathrm{AJ}(f, 0) = 2\left\{\frac{5}{1}\right\} - \left\{\frac{3}{1}\right\} + \left\{\frac{2}{1}\right\}.$$

In fig. 7.1, we see its virtual vertices and edges (see 2.7). As predicted by corollary 6.5, the vertices lie in the upper halfplane. One vertex, however, strays over to the left halfplane.

(ii) In [Fuk91, Example (3.7) and (3.8)], Fukui considers the examples

$$f_\delta(x_1, x_2, x_3, x_4) = x_1^2 x_2^2 x_3^2 x_4^2 + x_1 x_3^8 + x_2 x_4^8 + x_1^8 x_4 + x_2^8 x_3 + \delta x_2^4 x_3^4$$

where $\delta = 0, 1$. One checks that every face of this diagram is a simplex, and so the functions are Newton nondegenerate. Using theorem 7.12, we find

$$\mathrm{AJ}^{(4)}(f_0, 0) = 8\left\{\frac{455}{47}\right\} \qquad \mathrm{AJ}^{(4)}(f_1, 0) = 57\left\{\frac{28}{3}\right\} + 4\left\{\frac{455}{47}\right\}$$

$$\mathrm{AJ}^{(3)}(f_0, 0) = \left\{\frac{8}{1}\right\} + 40\left\{\frac{9}{1}\right\} + 2\left\{\frac{28}{3}\right\} \qquad \mathrm{AJ}^{(3)}(f_1, 0) = 4\left\{\frac{8}{1}\right\} + 29\left\{\frac{9}{1}\right\} + 4\left\{\frac{28}{3}\right\}$$

$$\mathrm{AJ}^{(2)}(f_0, 0) = 2\left\{\frac{8}{1}\right\} + 4\left\{\frac{9}{1}\right\} \qquad \mathrm{AJ}^{(2)}(f_1, 0) = 4\left\{\frac{8}{1}\right\} + 2\left\{\frac{9}{1}\right\}$$

$$\mathrm{AJ}^{(1)}(f_0, 0) = \left\{\frac{1}{8}\right\} \qquad \mathrm{AJ}^{(1)}(f_0, 0) = \left\{\frac{1}{8}\right\}$$

Fukui showed that in either case, the inequality

$$\mathcal{L}(f_\delta, 0) \leqslant 8 + \frac{32}{47}$$

holds. By our computations and corollary 6.12, in fact, equality holds.

Proof of theorem 7.12. It follows from remark 7.9(i) that (ii) implies (i). Furhtermore, if $v \in \overline{\mathbb{Z}}^{n+1}$ is primitive, then we have $W^{(d+1)}(v) = 0$ unless $v \in \mathcal{N}(fg)$ by remark 7.9(iii). We are therefore left with proving eq. 7.14.

Choose a regular subdivision $\triangle$ of the cone $\mathbb{R}^{n+1}_{\geqslant 0}$ which refines the natural subdivisions $\triangle_f, \triangle_g$ dual to $\Gamma(f)$ and $\Gamma(g)$. Note that as a result, any normal vector to a facet of $\Gamma(fg)$ therefore generates a ray in $\triangle$. This way, $\triangle$ is a fan which induces a toric variety Y and a toric map $\pi_\triangle : Y \to \mathbb{C}^{n+1}$. That $\triangle$ refines $\triangle_f$ implies that $\pi_\triangle$ is an embedded resolution of f. The requirement that $\triangle$ refines $\triangle_g$ is equivalent to requiring that $\pi_\triangle$ factors through the blow-up of $\mathbb{C}^{n+1}$ at 0. We use the notation inroduced in section 3 for $\pi = \pi_\triangle$.

There is a natural map $v : I \to \mathbb{Z}^{n+1}_{>0}$ such that if $D_i \subset Y$ is an irreducible exceptional component, then $v(i) \in \mathbb{Z}^{n+1}_{>0}$ is prime, and D_i equals the orbit closure $\overline{O}_{v(i)}$. The subset D_i° is the union of those orbits O_σ coresponding to cones σ generated by $v(i)$ and some subset S of the natural basis $e_0, e_1, \ldots, e_n$ of $\mathbb{Z}^{n+1}$. For such a σ, denote by v_σ the primitive vector in $\overline{\mathbb{Z}}^{n+1}_{>0}$ obtained by replacing the jth coordinate of v by ∞ for $j \in S$, and dividing by the greatest common divisor of the remaining coordinates, e.g.

$$v_i = (2,3,6), \quad S = \{0\} \qquad \rightsquigarrow \qquad v_\sigma = (\infty, 1, 2).$$

In this case, set $O_v = O_\sigma$. Any vector in $\mathcal{N}(fg)$ arises in this fashion from some $\sigma \in \triangle$. Thus, applying theorem 4.10, we find

$$\begin{aligned} \mathrm{AJ}^{(d+1)}(f,0) &= (-1)^d \sum_{i\in I} \chi\left(D_i^\circ \cap Y^{(d+1)} \backslash \tilde{X}\right) \left\{ \frac{m_i}{n_i} \right\} \\ &= (-1)^d \sum_{\substack{v \in \overline{\mathbb{Z}}^{n+1}_{>0} \\ \text{primitive}}} \chi\left(O_v \cap Y^{(d+1)} \backslash \tilde{X}\right) \left\{ \frac{m_v}{n_v} \right\} \end{aligned} \tag{7.18}$$

For any such $v = v_\sigma$, we have

$$\chi\left(O_v \cap Y^{(d+1)} \backslash \tilde{X}\right) = \chi\left(O_v \cap Y^{(d+1)}\right) - \chi\left(O_v \cap Y^{(d+1)} \cap \tilde{X}\right). \tag{7.19}$$

In [Oka90, §6], Oka proves that

$$\chi\left(O_v \cap Y^{(d+1)}\right) = \left(\frac{K_g(v)}{1+K_g(v)}\right)^c_s$$

and

$$\chi\left(O_v \cap Y^{(d+1)} \cap \tilde{X}\right) = \left(\frac{K_f(v)}{1+K_f(v)}\right)\left(\frac{K_g(v)}{1+K_g(v)}\right)^c_s.$$

Thus, we have

$$\begin{aligned} \chi\left(O_v \cap Y^{(d+1)} \backslash \tilde{X}\right) &= \left(1 - \frac{K_f(v)}{1+K_f(v)}\right)\left(\frac{K_g(v)}{1+K_g(v)}\right)^c_s \\ &= \left(\frac{1}{1+K_f(v)}\right)\left(\frac{K_g(v)}{1+K_g(v)}\right)^c_s \\ &= (-1)^{s-c} W^{(d+1)}(v). \end{aligned}$$

Since $(-1)^d(-1)^{s-c} = (-1)^{n-s}$, eq. 7.18 gives eq. 7.14. ■

8 The Łojasiewicz exponent from the Newton diagram

8.1. In this section we give a formula for the Łojasiewicz exponent of a Newton nondegenerate function in terms of its Newton diagram, theorem 8.4, which is a direct result of the previous results of this paper. As a corollary, we recover a result of Brzostowski [Brz19].

Definition 8.2. Let $\Gamma, \Xi \subset \mathbb{R}^{n+1}_{\geqslant 0}$ be Newton diagrams, i.e. the union of compact faces of a Newton polyhedron. Denote by Λ the Newton diagram of a generic linear function

$$\Lambda = \Gamma\left(\sum_{k=0}^n b_k z_k\right), \qquad b_0, b_1, \ldots, b_n \in \mathbb{C}^*.$$

(i) If $F \in \mathcal{F}(\Gamma)$ is a coordinate facet, corresponding to a primitive normal vector $v \in \mathcal{N}(\Gamma)$, then the *maximal axial intersection* of F, or v, is

$$\mathcal{M}(F) = \mathcal{M}(v) = \frac{\wedge_\Gamma v}{\wedge_\Lambda v}.$$

(ii) The *maximal axial intersection* of Γ is

$$\mathcal{M}(\Gamma) = \max\left\{ \frac{\wedge_\Gamma v}{\wedge_\Lambda v} \,\middle|\, v \in \mathcal{N}(\Gamma) \right\}.$$

(iii) For $\alpha \in \mathbb{Q}$, let

$$s_\alpha(\Gamma) = \bigcup \left\{ K_f(v) \,\middle|\, v \in \overline{\mathbb{R}}^{n+1}_{>0},\ \mathcal{M}(v) \leqslant \alpha \right\} \subset \Gamma.$$

Definition 8.3 ([Kou76])**.** (i) If $S \subset \mathbb{R}^{n+1}$ is a measurable subset, define the *Newton number*

$$\nu(S) = \sum_{J \subset \{0,1,\ldots,n\}} |J|!\,\mathrm{Vol}_{|J|}\left(S \cap \mathbb{R}^J\right)$$

(ii) If $\Gamma \subset \mathbb{R}^{n+1}$ is a Newton diagram, denote by Γ_- the union of segments joining any point in Γ with the origin.

Theorem 8.4. *Let $f \in \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ have a Newton nondegenerate isolated singularity at 0, with Newton diagram $\Gamma = \Gamma(f)$. Assume that either n is even, or that f does not have a Morse point at 0. Then, there exists a coordinate facet $F \subset \Gamma$ such that*

$$\mathcal{L}(f, 0) = \mathcal{M}(F) - 1. \tag{8.5}$$

For any $\alpha \in \mathbb{Q}$, we have

$$\nu(\Gamma_-) \geqslant \nu(s_\alpha(\Gamma)_-) \tag{8.6}$$

with an equality if and only if $\alpha - 1 \geqslant \mathcal{L}(f, 0)$. In particular,

$$\mathcal{L}(f, 0) = \min\left\{\alpha - 1 \in \mathbb{Q} \,|\, \nu(s_\alpha(\Gamma(f))_-) = \nu(\Gamma)\right\}. \tag{8.7}$$

Proof. The existence of F follows immediately from theorem 7.12(i) and corollary 6.12. The rest follows from corollary 6.6(i). ■

Corollary 8.8 ([Brz19])**.** *If $f, f' \in \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ are Newton nondegenerate and $\Gamma(f) = \Gamma(f')$, then*

$$\mathcal{L}(f, 0) = \mathcal{L}(f', 0).$$ ■

Remark 8.9. If f has a Morse point at 0, then $\mathcal{L}(f, 0) = 1$. In the case when n is odd, the function

$$f(z_0, \ldots, z_n) = z_0 z_1 + z_2 z_3 + \ldots + z_{n-1} z_n$$

is Newton nondegenerate and has an isolated singularity, but $\Gamma(f)$ has no coordinate facet.

Question 8.10. (i) If $\alpha = \mathcal{L}(f, 0) + 1$, and $f' \in \mathbb{C}\{z_0, \ldots, z_n\}$ is Newton nondegenerate such that $\Gamma(f') = s_\alpha(\Gamma(f))$, does f' then have an isolated singularity? Note that by Kouchnirenko's criterion [Kou76], this property only depends on $\Gamma(f')$.

(ii) More generally, assume that f' is Newton nondegenerate and that $\Gamma(f') \subset \Gamma(f)$. Does f' then have an isolated singularity, if the two diagrams give rise to the same Newton number, i.e. if $\nu(\Gamma_-(f)) = \nu(\Gamma_-(f'))$?

(iii) Conversely, if we assume that f and f' are Newton nondegenerate, have isolated singularities, and that $\Gamma(f') \subset \Gamma(f)$, does it then follow that f and f' have the same Milnor number? By semicontinuity of the Milnor number, we know that in this case, we have $\mu(f', 0) \geqslant \mu(f, 0)$. Can it happen that

$$\nu(\Gamma_-(f')) > \nu(\Gamma_-(f))\ ?$$

9 Past, present and future

9.1. In this section we start by recalling a conjecture by Brzostowski, Krasiński and Oleksik [BKO12] which greatly motivated this manuscript. We then give a counterexample to this conjecture. Finally, we give a similar conjecture which takes inspiration from the nonnegativity of Stanley's local h-vector [Sta92] and a formula for the Newton number [Sel24].

In definition 9.10, we use rational coefficients in $K\mathfrak{N}$, i.e. we work implicitly in the group $K\mathfrak{N}\otimes_{\mathbb{Z}}\mathbb{Q}$. Given that $K\mathfrak{N}$ is a free Abelian group, this should not cause any confusion.

Definition 9.2 ([BKO12])**.** Let $f \in \mathbb{C}\{x_0, \dots, x_n\}$. A facet $F \subset \Gamma(f)$ of its Newton diagram is *exceptional* if there exist $i \neq j$ such that all but one vertex of F lies on the j-th coordinate hyperplane, and that this one vertex corresponds to a monomial of the form $x_j x_i^k$. Denote by E_f the set of exceptional facets of $\Gamma(f)$.

Conjecture 9.3 ([BKO12])**.** *Let $f \in \mathbb{C}\{x_0, \dots, x_n\}$ be Newton nondegenerate, and assume that the diagram $\Gamma(f)$ contains a nonexceptional facet $F \in \mathcal{F}\backslash E_f$, $\dim F = n$. Then*

$$\mathcal{L}(f,0) = \max\left\{\mathcal{M}(F) - 1 \in \mathbb{Q} \,\middle|\, F \in \mathcal{F}\backslash E_f, \dim F = n\right\}.$$

Example 9.4. Consider the function $f \in \mathbb{C}\{x, y, z, w\}$ with generic coefficients in front of the monomials

$$x^2,\ y^2,\ xz,\ xw,\ yz,\ yw,\ z^3,\ w^3.$$

In particular, we assume that f is Newton nondegenerate. A computation shows that the Hessian of f at 0 is nonzero, and so f has a Morse point at 0. In particular, we have

$$\mathcal{L}(f,0) = 1.$$

The Newton diagram $\Gamma(f)$ has two facets, say, F_1 with normal vector $v_1 = (1,1,1,1)$, whose vertices correspond to the monomials

$$x^2,\ y^2,\ xz,\ xw,\ yz,\ yw$$

and F_2, with normal vector $(2,2,1,1)$, whose vertices correspond to the monomials

$$xz,\ xw,\ yz,\ yw,\ z^3,\ w^3.$$

Neither facet is exceptional by definition 9.2, and we find

$$\mathcal{M}(F_1) = 2, \qquad \mathcal{M}(F_2) = 3.$$

As a result, f is a counterexample to conjecture 9.3.

9.5. In order to improve conjecture 9.3, it is tempting to weaken the condition of being exceptional as follow: F *is weakly exceptional* if it has a triangulation consisting of exceptional triangles. In example 9.4, the facet F_2 is then weakly exceptional, a triangulation is described in fig. 9.1. Note that the indices i, j in the definition of exceptional cannot be chosen the same for each simplex in this decomposition. However, if f is a function with generic coefficients in front of the monomials

$$x^3, y^3, xz^2, xw^2, yz^2, yw^2, w^4, z^4, \tag{9.6}$$

then we can find a similar decomposition which includes the simplex with vertices

$$xzw, yzw, z^3w, z^3w$$

which is not exceptional. Thus, the following question remains:

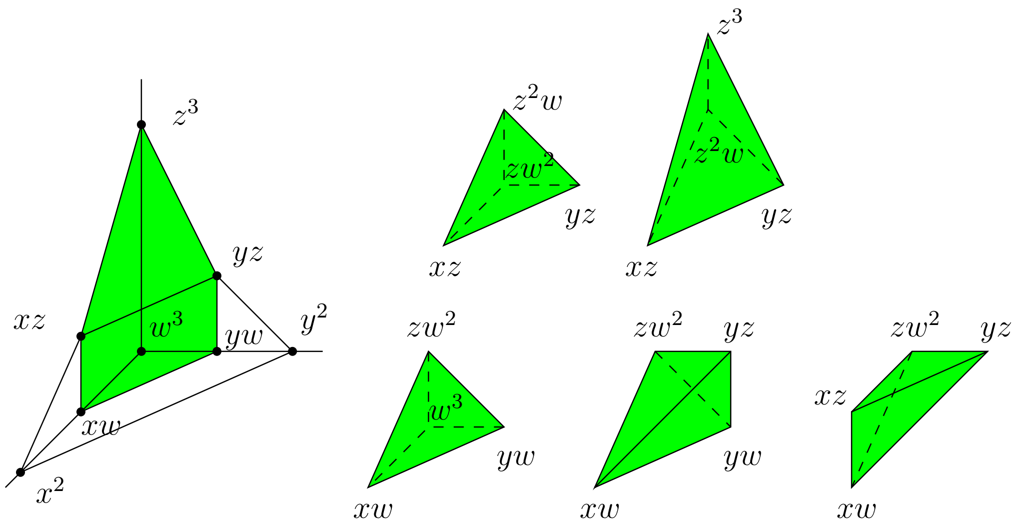

Figure 9.1: A decomposition of the facet F_2. The picture shows a projection of the Newton diagram of f to the xyz-coordinate space along the w-axis.

Question 9.7. Given only a facet F of a Newton diagram $\Gamma = \Gamma(f)$, how can one identify whether F should be considered *exceptional* (without knowing Γ), in such a way that the statement in conjecture 9.3 is true?

9.8. We sketch a possible answer to question 9.7, relative to a triangulation $\mathcal{T}$ of the Newton diagram, following methods from [Sel24, 6.1]. Assume that we have chosen a triangulation $\mathcal{T}$ of $\Gamma(f)$. If $S \subset \mathbb{R}^{n+1}$ is a simplex whose affine hull does not contain the origin, denote by S_- the convex hull of $S \cup \{0\}$. If $v_0, v_1, \ldots, v_s$ are the vertices of S, set (see [Sel24, Notation 2.21])

$$\mathrm{Cap}(S) = \left| \left\{ \sum_{i=0}^{s} \lambda_i v_i \,\middle|\, 0 < \lambda_i < 1 \right\} \cap \mathbb{Z}^{n+1} \right|.$$

We also set

$$\mathrm{Cap}(\varnothing) = 1.$$

If $S \in \mathcal{T}$ is a coordinate simplex contained in a coordinate facet F of the same dimension, then we set $m_S = m_F$ and $n_S = n_F$. This way, we have

$$m_S s!\,\mathrm{Vol}_s(S) = (s+1)!\,\mathrm{Vol}_{s+1}(S_-) = \sum_{\substack{T \in \mathcal{T} \cup \{\varnothing\} \\ T \subset S}} \mathrm{Cap}(T).$$

Denote by $\mathcal{T}_c \subset \mathcal{T}$ the set of coordinate simplices. Then, using $s = \dim(S)$, eq. 7.13 reads

$$\begin{aligned} \mathrm{AJ}(f,0) &= \sum_{S \in \mathcal{T}_c} \frac{(-1)^{n-s}}{m_S} \sum_{\substack{T \in \mathcal{T} \cup \{\varnothing\} \\ T \subset S}} \mathrm{Cap}(T) \left\{ \frac{m_S}{n_S} \right\} \\ &= \sum_{T \in \mathcal{T} \cup \{\varnothing\}} \mathrm{Cap}(T) \sum_{\substack{S \in \mathcal{T}_c \\ S \supset T}} \frac{(-1)^{n-s}}{m_S} \left\{ \frac{m_S}{n_S} \right\} \end{aligned} \tag{9.9}$$

Definition 9.10. With $\mathcal{T}$ as above, and $T \in \mathcal{T}$, or $T = \varnothing$, we define the *relative combinatorial Newton polyhedron* of $\mathcal{T}$ as

$$\mathrm{CN}_{\mathfrak{R}}(\mathcal{T}/T) = \sum_{\substack{S \in \mathcal{T}_c \\ S \supset T}} \frac{(-1)^{n-s}}{m_S} \left\{ \frac{m_S}{n_S} \right\}$$

9.11. The above definition lifts the definition of the combinatorial Newton number (see e.g. [Sel24, Definition 2.18]) from an integral invariant of a triangulation, to a Newton polygon, up to the inclusion of the empty set in a triangulation. Assuming first that $T = \varnothing$, then $\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T)$ is related to the combinatorial Newton number $\mathrm{CN}(\mathcal{T})$ by the equality

$$(-1)^{n+1} + \ell(\mathrm{CN}_{\mathfrak{N}}(\mathcal{T})) = \sum_{S \in \mathcal{T}_c \cup \{\varnothing\}} (-1)^{n-s} = \mathrm{CN}(\mathcal{T}).$$

where we set $s = -1$ if $S = \varnothing$. If $T \neq \varnothing$, then the length of $\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T)$ equals the combinatorial Newton number $\mathrm{CN}(\mathcal{T}/T)$ of the link of T in $\mathcal{T}$. Assuming that $\Gamma(f)$ is convenient, the Newton number $\mathrm{CN}(\mathcal{T})$ is nonnegative by [Sel24]. In fact, it equals the value of a local h-polynomial at 1, and this polynomial has nonnegative coefficients [Sta92]. Note, however, that $\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T)$ does not always have nonnegative coefficients.

Since the alternating Jacobian polygon can be calculated in terms of these combinatorial Newton polygons by eq. 9.9, we would like to use this formula to obtain the Łojasiewicz exponent of f. Indeed, the degree on the left hand side of eq. 9.9 is bounded above by the maximal degree of the terms on the right hand side, i.e.

$$\deg \mathrm{AJ}(f,0) \leqslant \max \left\{ \deg \mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T) \,\middle|\, \begin{array}{c} T \in \mathcal{T} \cup \{\varnothing\}, \\ \mathrm{Cap}(T) \neq 0, \\ \mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T) \neq 0 \end{array} \right\}. \tag{9.12}$$

If equalith holds here, then we do not have to include the case $T = \varnothing$. This is because, for any $S \in \mathcal{T}_c$, we have $\chi(S) = 1$, and so we find

$$\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/\varnothing) = \sum_{T \in \mathcal{T}} (-1)^{\dim T} \mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T).$$

This is to say that, if equality holds in eq. 9.12, then the same equality holds with the condition $T \in \mathcal{T} \cup \{\varnothing\}$ replaced by $T \in \mathcal{T}$. If this is true, then we find that if $F \subset \Gamma(f)$ is a coordinate facet satisfying $\mathcal{M}(F) - 1 > \mathcal{L}(f,0)$, then

$$\mathcal{M}(F) > \deg(\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T))$$

for all $T \in \mathcal{T}$, $T \subset F$. Furthermore, there would necessarily exists some coordinate facet F, and a $T \in \mathcal{T}$, $T \subset F$ with $\mathcal{M}(F) = \deg(\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T))$.

Definition 9.13. Let $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ be a Newton nondegenerate function germ having an isolated singularity at 0, and $\mathcal{T}$ a triangulation of $\Gamma(f)$.

(i) Denote by $\mathcal{T}_{\mathrm{ne}}$ the set of simplices in $\mathcal{T}$ satisfying

$$\mathrm{Cap}(T)\,\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T) = 0.$$

(ii) Denote by $\mathcal{F}_{\mathrm{ne}}^{\mathcal{T}}$ the set of those coordinate facets $F \in \mathcal{F}$ which contain a simplex $T \in \mathcal{T}_{\mathrm{ne}}$ satisfying

$$\deg(\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T)) \geqslant \mathcal{M}(F).$$

Conjecture 9.14. *Let $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ be a Newton nondegenerate function germ having an isolated singularity at 0. Assume that f does not have a Morse point at 0, or that n is even. If $\mathcal{T}$ is a triangulation of $\Gamma(f)$, then*

$$\mathcal{L}(f,0) = \max_{T \in \mathcal{T}_{\mathrm{ne}}} \deg(\mathrm{CN}_{\mathfrak{N}}(\mathcal{T}/T)) - 1 = \max_{F \in \mathcal{F}_{\mathrm{ne}}^{\mathcal{T}}} \mathcal{M}(F) - 1.$$

Example 9.15. Consider the Morse point in three variables

$$f(x, y, z) = xy + xz + 2yz + z^2$$

and the two triangulations of the Newton diagram $\Gamma(f)$ seen in fig. 9.2, with vertices

$$A : yz, \quad B : z^2, \quad C : xz, \quad D : xy.$$

On the left hand side, the set $\mathcal{T}_{\text{ne}}$ contains only the blue triangle ACD. As a result, $F = ABCD$ is the only element of $\mathcal{F}_{\text{ne}}$. On the right hand side, however, $\mathcal{T}_{\text{ne}}$ consists of only the vertex B, and $\mathcal{F}_{\text{ne}}$ contains all coordinate facets.

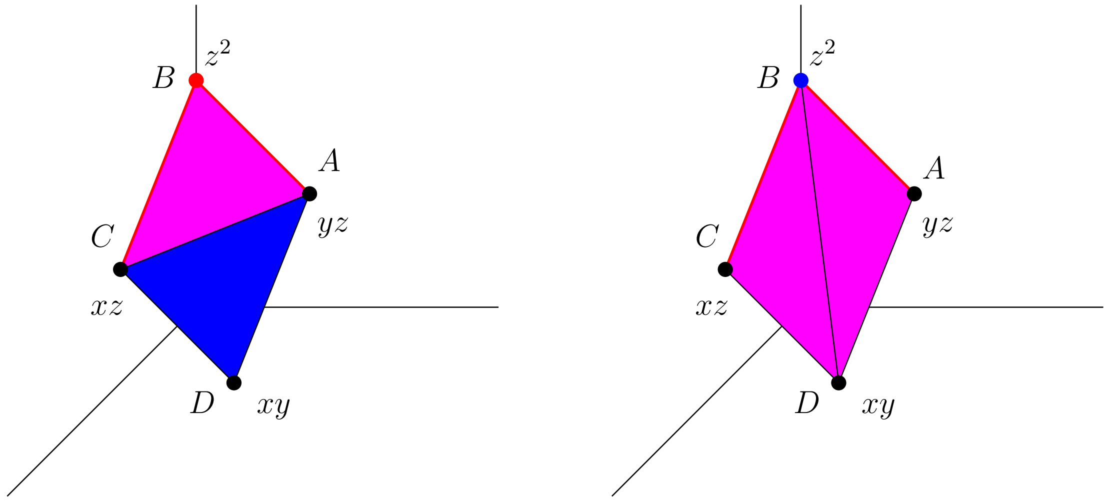

Figure 9.2: Two triangulations of the Newton diagram $\Gamma(f)$.

10 References

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