The boundary of the Milnor fiber of the singularity $f(x,y) + zg(x,y) = 0$

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Abstract

Let $f, g \in \mathbb{C}\{x,y\}$ be germs of functions defining plane curve singularities without common components in $(\mathbb{C}^2, 0)$ and let $\Phi(x,y,z) = f(x,y) + zg(x,y)$. We give an explicit algorithm producing a plumbing graph for the boundary of the Milnor fiber of Φ in terms of a common resolution for f and g.

1 Introduction

It is known that the boundary of any hypersurface singularity in $(\mathbb{C}^3, 0)$ is a plumbed manifold. This was proved by Némethi and Szilárd [8]. A stronger statement for certain real analytic map germs was proved by de Bobadilla and Neto [4]. As these theorems rely on resolution of singularities, they do not easily provide an explicit description of a plumbing graph describing the boundary. In the case of a hypersurface singularity given by the equation

$$\Phi(x,y,z) = f(x,y) + zg(x,y) = 0,$$

where f, g are singular germs with no common factors (but not necessarily reduced), we give an explicit algorithm producing a plumbing graph for the boundary of the Milnor fiber in terms of the graph associated with an embedded resolution of the plane curve singularities defined by f and g. This is obtained from an explicit description of the Milnor fiber by the author [10]. Singularities of the form $f(x,y) + zg(x,y)$ are closely related to the deformation theory of sandwiched singularities, see [5]. The article is organized as follows.

In section 2 we recall the results of [10] and fix notation related to the resolution graph of f and g.

In section 3 we define plumbed manifolds and prove some useful lemmas related to them.

In section 4 we introduce families of multiplicities and dual multiplicities assigned to a complex valued function on a plumbed 3-manifold, satisfying certain conditions. In the case of a fibration over S^1, these multiplicities coincide with the multiplicities used in [3, 8].

In section 5 we prove a useful lemma relating the negative continued fraction expansion of a rational number to a plumbing construction.

In section 6 and section 7 we provide the details of the construction of the plumbing graph for the boundary of the Milnor fiber of Φ and the families of multiplicities and dual multiplicities for the coordinate function z. These statements can be read after only reading definition 2.1.

In section 8 we provide some examples. First we give the simple plumbing graph describing boundary of the Milnor fiber of a $T_{a,b,*}$ singularity given by the equation $x^a + y^b + xyz = 0$. This example is discussed in [8] 22.2.

Section 9 contains proofs of theorem 6.3 and theorem 7.1.

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1.1 Notation and conventions. (i) We denote by $D \subset \mathbb{C}$ the open unit disk and by $\overline{D}$ the closed unit disk. We also set $S^1 = \partial\overline{D}$. For any $r > 0$, let $D_r, \overline{D}_r, S^1_r$ be the corresponding disks and circle with radius r.

(ii) If X is a manifold, and $C \subset X$ is a submanifold of dimension d, then we denote by $[C] \in H_d(M, \mathbb{Z})$ the associated homology class. If X is a compact oriented compact manifold, possibly with boundary, we denote by $(\cdot, \cdot)_X$ the intersection pairing between $H_i(X, \mathbb{Z})$ and $H_{n-i}(X, \partial X, \mathbb{Z})$, where $n = \dim X$. In particular, if $\partial X = \emptyset$ and $i = n/2$, then $(\cdot, \cdot)_X$ is the intersection form on the middle homology.

(iii) The boundary of an oriented manifold is oriented by the usual *outward-pointing-vector first* rule. Note that if a codimension one submanifold $N \subset M$ splits M into two pieces, this rule induces opposite orientations according to which piece is chosen.

(iv) A locally trivial differential fiber bundle with a chosen orientation on the total space and the base space induces an orientation on each fiber by the following requirement. A lifting of a positive basis of the tangent space of the base space, followed by a positive basis of the tangent space of the fiber yields a positive orientation of the total space. In fact, this rule induces an orientation on the fibers, the total space or the base space, given orientations on the other two.

2 The Milnor fiber

In [10] the author gives a description of the Milnor fiber of the singularity $f(x, y) + zg(x, y) = 0$. We will now recall that result and fix some notation.

2.1 Definition. Let $\phi : V \to \mathbb{C}^2$ be a common resolution of the functions f and g with exceptional divisor E, decomposing into irreducible components as $E = \cup_{v \in \mathcal{W}} E_v$ and denote by Γ the associated embedded resolution graph. The set of vertices in Γ is $\mathcal{V} = \mathcal{W} \amalg \mathcal{A}$, where $\mathcal{W}$ corresponds to components of the exceptional divisor, while elements of $\mathcal{A}$ are *arrowheads*, corresponding to components of the strict transforms of f and g. For any $a \in \mathcal{A}$ there is a $w_a \in \mathcal{W}$ so that $\{w_a, a\}$ is an edge in Γ. Write $\mathcal{A} = \mathcal{A}_f \amalg \mathcal{A}_g$, where elements of $\mathcal{A}_f$ and $\mathcal{A}_g$ correspond to components of the strict transform of f and g. For $v \in \mathcal{V}$, we denote by m_v and l_v the multiplicities of f and g, respectively. In particular, $m_v = 0$ if and only if $v \in \mathcal{A}_g$ and, similarly, $l_v = 0$ if and only if $v \in \mathcal{A}_f$.

   Define $\mathcal{W}_1 = \{w \in \mathcal{W} \,|\, m_w \leq l_w\}$ and $\mathcal{W}_2 = \{w \in \mathcal{W} \,|\, m_w > l_w\}$. Write $\mathcal{A}_i = \{a \in \mathcal{A} \,|\, w_a \in \mathcal{W}_i\}$ for $i = 1, 2$. Similarly, take $\mathcal{A}_{f,i}, \mathcal{A}_{g,i} \subset \mathcal{A}_i$ so that $\mathcal{A}_f = \mathcal{A}_{f,1} \amalg \mathcal{A}_{f,1}$ and $\mathcal{A}_g = \mathcal{A}_{g,1} \amalg \mathcal{A}_{g,2}$.

2.2 Definition. For $w \in \mathcal{W}$, let T_w be a tubular neighbourhood around E_w in V and let $T = \cup_{w \in \mathcal{W}} T_w$. Set also $T_i = \cup_{w \in \mathcal{W}_i} T_w$ for $i = 1, 2$. For a given $0 < \varepsilon \ll 1$, let $F_f = f^{-1}(\varepsilon)$ be the Milnor fiber of f, and $F'_f = \phi^{-1}(F_f)$ its pullback to V. Let T_ε be a small tubular neighbourhood around F'_f in T. We also choose tubular neighbourhoods $T_a \subset T$ around E_a for any $a \in \mathcal{A}$. With

these choices fixed, choose a small tubular neighbourhood $T' = \cup_{w \in \mathcal{W}} T'_w$ around the exceptional divisor inside T. This is chosen small enough that $T' \cap T_\varepsilon = \emptyset$.

Set $T_{f,i} = \cup_{a \in \mathcal{A}_{f,i}} T_a$, and $T_{g,i} = \cup_{a \in \mathcal{A}_{g,i}} T_a$ for $i = 1, 2$ and let

$$\overline{T}_{f,g} = (\overline{T}_{f,1} \setminus T') \cup \overline{T}_\varepsilon \cup (\overline{T}_2 \setminus (T' \cup T_{g,2})).$$
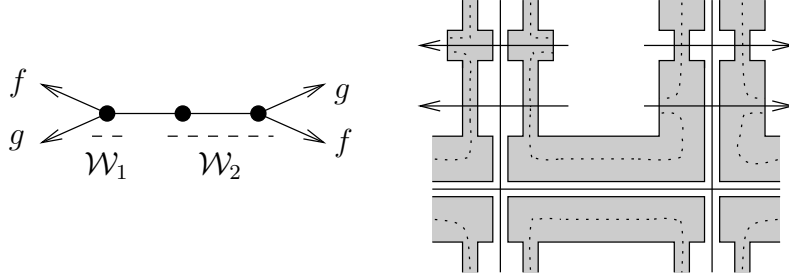

Figure 1: A schematic picture showing $\overline{T}_{f,g}$. The Milnor fiber of f is shown as a dotted curve. The Milnor fiber of Φ is obtained by twisting along the strict transform of g.

2.3 Definition. Let X be a four dimensional manifold with boundary and $\iota : \overline{D} \to X$ an embedding of the closed disk into X such that ι sends $S^1 = \partial\overline{D}$ to ∂X and the image of $\overline{D}$ is transversal to ∂X. Then there exists a map $\psi : \overline{D} \times \overline{D} \to X$ parametrizing a tubular neighbourhood of $\iota(D)$ in X so that $\psi(0, z) = \iota(z)$ for $z \in \overline{D}$ and $\psi(x, z) \in \partial X$ for $x \in \overline{D}$ and $z \in S_1 = \partial D$. For $k \in \mathbb{Z}$, the kth *twist* along ι is defined as $(X \setminus \psi(D \times \overline{D})) \amalg_{t_k} \overline{D} \times \overline{D}$ where the glueing map $t_k : S^1 \times \overline{D} \to (X \setminus \psi(D \times \overline{D}))$ is defined by $t_k(x, z) = \psi(x, x^k z)$ and is denoted by $X_{\iota,k}$. We also say that $X_{\iota,k}$ is obtained from X by *twisting X k times along $\iota(\overline{D})$*.

2.4 Definition. In [10], the author shows that for any $a \in A_g$, the intersection $E_a \cap \overline{T}_{f,g}$ is a disjoint union of m_{w_a} disks embedded in $T_{f,g}$. Let $F_{f,g}$ be the manifold obtained from $\overline{T}_{f,g}$ by twisting each of these disks l_a times for all a.

2.5 Theorem ([10]). *The Milnor fiber F_Φ is diffeomorphic to $F_{f,g}$.* ∎

2.6 Definition. Let $M_{f,g} = \partial F_{f,g}$. We also set $M'_{f,g} = \partial \overline{T}_{f,g}$.

3 Plumbed 3-manifolds

In this section we give an introduction to plumbed three manifods and plumbing graphs, along with some useful properties. Throughout this text, an S^1-bundle will mean a principal S^1-bundle. In particular, we assume that there is a consistent choice for orientation on each fiber. In fact, all our S^1-bundles will have as base space an *oriented* real surface. This determines a consistent choice of orientation on fibers as described in 1.1(iii).

We note that apart from our restriction on orientability, our definition of a plumbed manifold is equivalent to the definition in [9]. This can be seen from lemma 3.7. We note, however, that our construction differs slightly to the

standard one. This is explicated in remark 3.8. The main reason for this is that in our construction in section 9, we identify the three dimensional plumbed pieces directly, but the result can in no natural way be seen as the boundary of a four dimensional plumbed manifold (as is the case for links of isolated surface singularities).

3.1 Definition. A *plumbed manifold* is a three dimensional compact manifold M, possibly with boundary, given as a union of submanifolds with boundary $M = \cup_{v \in \mathcal{W}} M_v$ having the following properties.

(i) For each $v, w \in \mathcal{W}$, $v \neq w$ we have an $e_{v,w} \in \mathbb{N}$ so that

$$M_v \cap M_w = \mathrm{II}_{i=1}^{e_{v,w}} S_{v,w,i},$$

with $S_{v,w,i}$ an embedded torus $M \supset S_{v,w,i} \cong S^1 \times S^1$. Thus, $S_{v,w,i}$ is a component of ∂M_v and inherits an orientation. Since $e_{v,w} = e_{v,w}$, we can assume that as sets, we have $S_{w,v,i} = S_{v,w,i}$ for $i = 1, \ldots, e_{v,w}$.

(ii) For each v we have a compact connected surface Σ_v (possibly with boundary) and a locally trivial S^1 bundle $\pi_v : M_v \to \Sigma_v$. If $1 \leq i \leq e_{v,w}$ for some $w \neq v$, then $S_{v,w,i} = \pi^{-1}(B_{v,w,i})$ where $B_{v,w,i} \cong S^1$ is a component of the boundary of Σ.

(iii) Assume that $1 \leq i \leq e_{v,w}$ for some $v \neq w$. The map

$$S_{v,w,i} \to B_{v,w,i} \times B_{w,v,i}, \quad p \mapsto (\pi_v(p), \pi_w(p))$$

is a diffeomorphism.

(iv) For each $v \in \mathcal{V}$, let $B_{v,1}, \ldots, B_{v,e_v}$ be the components of $\partial \Sigma$ not of the form $B_{v,w,i}$ for some w, i. We assume given a section $s_{v,i} : B_{v,i} \to \Sigma$ to the reduced bundle $\pi_v|_{B_{v,i}}$.

3.2. We orient $S_{v,w,i}$ by considering it as a subset of the boundary of M_v. This way, $S_{v,w,i} = S_{w,v,i}$ as sets, but $S_{v,w,i} = -S_{w,v,i}$ as oriented manifolds. We also orient the boundary of Σ_w by the same rule, for any $w \in \mathcal{W}$.

3.3. For a closed surface Σ, the *Euler number* classifies the S^1 bundles over Σ. However, every S^1 bundle $\pi : M \to \Sigma$ over a compact surface with nonempty boundary is trivial. But given a trivialization, or, equivalently, a section $s : \partial \Sigma \to \partial M$, over the boundary, a *relative Euler number* is well defined, and invariant under homotopy of the section. This is a complete invariant in the following sense. Let Σ be a compact surface with boundary and take two S^1 bundles $M, M' \to \Sigma$ with sections $s : \partial \Sigma \to M$ and $s' : \partial \Sigma \to M'$ and an isomorphism of bundles $\psi : M|_{\partial \Sigma} \to M'|_{\partial \Sigma}$ sending s to s'. Then ψ extends to an isomorphism of bundles $M \to M'$ if and only if the relative Euler numbers coincide. We will refer to the relative Euler number simply as the Euler number.

The relative Euler number is defined as follows. Let $D \subset \Sigma$ be an open disk. We can extend the section $s : \partial \Sigma \to \partial M$ to a section $\bar{s} : \Sigma \setminus D \to M \setminus \pi^{-1}(D)$. Given an orientation preserving diffeomorphism $\varphi : \partial D \to S^1$, there is a unique number $b \in \mathbb{Z}$ so that the twisted section $\bar{s}^b : \partial D \to \pi^{-1}(\partial D)$, $x \mapsto \varphi(x)^b \bar{s}(x)$ extends over the disk D. The relative Euler number is defined as $-b$.

3.4 Lemma. *Let $M \to \Sigma$ be an S^1 bundle over a compact surface with bound-ary. Let $-b$ be its the Euler number relative to a section $s : \partial \Sigma \to M$. Let $C \subset M$ be a fiber of the bundle and C' the image of s (as* oriented *submani-folds). Then, in $H_1(M, \mathbb{Z})$*

$$-b[C] = [C']. \tag{3.1}$$

Proof. This follows from the definition of the relative Euler number. Indeed, let $\bar{s} : \Sigma \setminus D \to M$ be a section as above. It follows that $[C'] - \bar{s}_*[\partial D] = 0$. The sign comes from the fact that ∂D is oriented as the boundary of the disk D, which is the opposite to the orientation inherited from the complement of the disk. Since the section $\bar{s}_*^b$ extends over D and D is null-homotopic, the map $\bar{s}^b : \partial D \to M$ is homotopic to a constant map $\partial D \to M$. It follows that $\bar{s}_*[\partial D] = -b[C]$. ∎

3.5 Remark. If $\partial \Sigma \neq \emptyset$, then eq. 3.1 can be taken as an alternative definition of the (relative) Euler number. Indeed, it follows from the Künneth formula that the $[C]$ is not a torsion element of $H_1(M, \mathbb{Z})$.

3.6 Definition. A *plumbing graph* is a decorated graph G (with no loops) with vertex set $\mathcal{V} = \mathcal{W} \amalg \mathcal{A}$, where each vertex $a \in \mathcal{A}$ has a unique neighbour w_a and w_a. We refer to vertices in $\mathcal{A}$ as *arrowhead vertices*. G is decorated as follows.

- ❊ For each $w \in \mathcal{W}$, we have integers $-b_w \in \mathbb{Z}$ and $g_v \in \mathbb{Z}_{\geq 0}$. These are referred to as the associated *Euler number* (or sometimes *selfintersection number*) and the *genus*.

- ❊ Each edge e connecting two vertices in $\mathcal{W}$ is given a sign $\varepsilon_e \in \{+, -\}$.

In a drawing of a graph, the genus g_v is written within square brackets as $[g_v]$ to be distinguished from the Euler number. If it is omitted, it is assumed to be 0. A negative edge will be indicated by the symbol $\ominus$, whereas if indication is omitted, the sign is assumed to be positive. An edge connecting $w \in \mathcal{W}$ and an arrowhead $a \in \mathcal{A}$ is drawn as a dashed edge, see e.g. fig. 2.

Let $M = \cup_{v \in \mathcal{W}} M_i$ be a plumbed manifold and use the notation introduced in definition 3.1. The *associated plumbing graph* G has vertex set $\mathcal{V} = \mathcal{W} \amalg \mathcal{A}$ where $\mathcal{A} = \amalg_{v \in \mathcal{W}} \mathcal{A}_v$, where the elements of $\mathcal{A}_v$ correspond to the boundary components $B_{v,1}, \ldots, B_{v,e_v}$ of Σ_v. It has $e_{v,w}$ edges connecting v and w if v, w are distinct elements of $\mathcal{W}$ and a single edge connecting any $a \in \mathcal{A}_w$ with w if $w \in \mathcal{W}$, and no other edges. Denote by $\mathcal{E}$ this set of edges.

The genus g_v is the genus of the surface Σ_v. The Euler number $-b_v$ is the Euler number of the S^1 bundle $M_v \to \Sigma_v$, trivialized on the boundary components $B_{v,i}$ by the given section, and on the components $B_{v,w,i}$ by any fiber of $S_{v,w,i} = S_{w,v,i} \to B_{w,v,i}$.

Any edge $e \in \mathcal{E}$ connecting $v, w \in \mathcal{W}$ corresponds to a component $S_{v,w,i} = S_{w,v,i}$ of the intersection $M_v \cap M_w$. Take fibers C_v and C_w of π_v and π_w, re-spectively, contained in $S_{v,w,i}$. The sign ε_e is defined as the intersection number of C_v and C_w in $S_{v,w,i}$, that is,

$$\varepsilon_e = ([C_v], [C_w])_{S_{v,w,i}}.$$

It follows from definition that this intersection number is ± 1. This sign depends on the orientation on $S_{v,w,i}$, which, we recall, is obtained by viewing $S_{v,w,i}$ as a subset of ∂M_v.

3.7 Lemma. *Let v, w be vertices connected by an edge e in a plumbing graph associated to a plumbed manifold M. Let C_w be a fiber of π_w contained in the torus $S_{v,w,i}$ corresponding to e. Then the sign ε_e is positive if and only if $-C_w$ is an oriented section to the map $S_{v,w,i} \to B_{v,w,i}$.*

Proof. Let $C_v \subset S_{v,w,i}$ be a fiber of π_v. We have $\varepsilon_e = ([C_v], [C_w])_{S_{v,w,i}}$. Therefore, if B is the oriented image of some section of $\pi_v|_{S_{v,w,i}}$, then it suffices to show that $([C_v], [B])_{S_{v,w,i}} = -1$. By construction, C_v and B intersect in a single point, say $x \in S_{v,w,i}$, and we can assume that this intersection is transverse. Let $c, b \in T_x S_{v,w,i}$ be tangent vectors inducing positive bases of $T_x C_v$ and $T_x B_v$. Let $a \in T_x M_v$ be an outward pointing tangent vector. By definition, $(\pi_v(a), \pi_v(b))$ is a positive basis of $T_{\pi_v(x)} B_{v,w,i}$. Therefore, (a, b, c) is a positive basis of $T_x M_v$, and so (b, c) is a positive basis of $T_x S_{v,w,i}$. This means that $([B], [C_v])_{S_{v,w,i}} = 1$ and so $([C_v], [B])_{S_{v,w,i}} = -1$. ∎

3.8 Remark. The above lemma may seem contrary to the usual definition of plumbing [9, 8]. There, the authors start with S^1-bundles over a closed surfaces. The glueing of two pieces, corresponding to an edge e, is made by removing a tubular neighbourhood around a fiber in each piece and identifying the boundaries by switching meridians and fibers, multiplied with a sign ε_e. The output of the two constructions is identical, but the submanifold B in the proof above, is a meridian, but with the opposite orientation to that of a standard meridian.

3.9 Example. [6, 8] Let $\tilde{X}$ be a smooth complex surface and let $E \subset \tilde{X}$ be a compact *normal crossing divisor*. This means that E is a compact reduced analytic subspace of pure dimension one, decomposing as $E = \cup_{v \in \mathcal{V}} E_v$ into irreducible components, with the condition that each E_v is a submanifold of $\tilde{X}$, that each E_v and E_w intersect transversally, and that any singularity of E is a double point. If $T \subset \tilde{X}$ is a suitable small neighbourhood of E, then $M = \partial T$ is a plumbed manifold, whose plumbing graph G is given by the intersection matrix of E, that is, G has vertex set $\mathcal{V}$, the genus g_v is the genus of E_v, the Euler number $-b_v$ is the selfintersection number (E_v, E_v), equivalently, it is the Euler number of the normal bundle of the embedding $E_v \hookrightarrow \tilde{X}$, and the number of edges between $v, w \in \mathcal{V}$ is the cardinality $|E_v \cap E_w|$. Furthermore, $\varepsilon_e = +$ for any edge e.

3.10. A plumbed manifold M can be recovered from its (decorated) plumbing graph G as follows. As before, denote by $\mathcal{V}$ and $\mathcal{E}$ the set of vertices and edges in G, and by g_v and $-b_v$ the genus and the selfintersecion number of a vertex v and by ε_e the sign of an edge. For each $v \in \mathcal{V}$, let Σ_v be a compact surface of genus g_v with $e_v + \sum_{w \in \mathcal{V} \setminus \{v\}} e_{v,w}$ boundary components, give names $B_{v,w,i}$, for $w \in \mathcal{V} \setminus \{v\}$ and $1 \le e_{v,w}$ and $B_{v,i}$ for $1 \le i \le e_v$. Let $\pi_v : M_v \to \Sigma_v$ be an S^1 bundle with sections $s_{v,w,i}$ and $s_{v,i}$ over the boundary inducing Euler number $-b_v$. The section $s_{v,w,i}$ induces a trivialization $\phi_{v,w,i} : S^1 \times S^1 \to \pi_v^{-1}(B_{v,w,i})$.

We then have $M \cong \amalg_{v \in \mathcal{V}} M_v / \sim$ where $\sim$ is the equivalence relation on $\amalg_{v \in \mathcal{V}} M_v$ generated by $\phi_{v,w,i}(\theta_1, \theta_2) \sim \phi_{w,v,i}(\theta_2^{-\varepsilon_e}, \theta_1^{-\varepsilon_e})$ where e is the ith edge connecting v and w. The negative sign in the exponents in the glueing map is explained by remark 3.8.

4 Multiplicities associated with complex valued functions

In this section we give a definition of multiplicities of a complex valued function on a plumbed manifold under some restrictions (see 4.1). This definition coincides with the multiplicities associated with fibred links in section 18 of [3], if the function is a fibration over S^1. These multiplicities are useful as they can be obtained by local computation, but can be used to determine Euler numbers, see lemma 4.3.

4.1. Let $M = \cup_{v \in \mathcal{V}} M_v$ be a plumbed manifold with graph G, with vertex set $\mathcal{V} = \mathcal{W} \cup \mathcal{A}$ and let $\zeta : M \to \mathbb{C}$ be a differentiable function having 0 as a regular value. Furthermore, assume that ζ does not vanish on ∂M_v for all $v \in \mathcal{W}$. Thus, $N_v = \zeta^{-1}(0) \cap M_v$ is a closed submanifold of M_v which does not intersect its boundary. Assume also that N_v is homologous to a multiple of $[C_v]$ in M_v, that is, $[N_v] = n_v[\pi_v^{-1}(p)]$ for some (well defined) $n_v \in \mathbb{Z}$.

For any $x \in \Sigma_v \setminus \pi_v(N_v)$, there is a unique $m_x \in \mathbb{Z}$ so that $\zeta_*([\pi^{-1}(x)]) = m_x[S^1] \in H_1(\mathbb{C}^*)$. This number is a locally constant function of x. In fact, let $\xi : [0,1] \to \sigma$ be a 1-chain connecting $x = \xi(0)$ and $y = \xi(1)$. We can assume that ξ is an embedding, and by a small perturbation, we can assume that the map $N_v \to \Sigma$, induced by π_v, is an immersion, transverse to ξ. At any intersection point of ξ and $\pi_v(N_v)$, one sees that $m_{\xi(.)}$ changes by ± 1, depending on the sign of the intersection. In particular, if $x, y \in \partial \Sigma_v$, then ξ is a cycle inducing an element $[\xi] \in H_1(\Sigma_v, \partial \Sigma_1, \mathbb{Z})$. It follows from the assumptions that we made that $\pi_{v,*}([N_v]) = 0 \in H_1(\Sigma, \mathbb{Z})$, and so

$$(\pi_{v,*}([N_v]), [\xi])_{\Sigma_v} = 0.$$

It follows that for $x, y \in \partial \Sigma$, the number $m_x = m_y$ is a number which well defined by the map ζ; we denote it by m_v.

4.2 Definition. Let $\zeta : M \to \mathbb{C}$ be as in 4.1. We refer to the families $(m_v)_{v \in \mathcal{V}}$ and $(n_w)_{w \in \mathcal{W}}$ (defined above) as the *family of multiplicities* and *dual family of multiplicities* associated with ζ, respectively. In a drawing of a plumbing graph, a multiplicity is written within parenthesis, whereas a dual multiplicity is written in parenthesis next to an arrow emanating from the vertex.

4.3 Lemma. *Let $\zeta : M \to \mathbb{C}$ be as in 4.1, and let $(m_v)_{v \in \mathcal{V}}$ and $(n_v)_{v \in \mathcal{W}}$ be the associated families of multiplicities and dual multiplicities. Let $w \in \mathcal{W}$. If $e \in \mathcal{E}_w$ connects w and v, set $m_e = m_v$. We then have*

$$-b_w m_w + \sum_{e \in \mathcal{E}_w} \varepsilon_e m_e = n_v.$$

Proof. Let C_w be a fiber of π_w. Since $M_w \cong \Sigma_w \times S^1$, the element $[C_w] \in H_1(M_w, \mathbb{Z})$ is nontorsion. It therefore suffices to show that

$$\left(-n_w - b_w m_w + \sum_{e \in \mathcal{E}_w} \varepsilon_e m_e \right)[C_w] = 0.$$

We can assume that ζ is transversal to the submanifold with boundary $\mathbb{R}_{\geq 0} \subset \mathbb{C}$ so that $\zeta|_{M_w}^{-1}(\mathbb{R}_{\geq 0})$ is a submanifold with boundary K_w in M_w. Furthermore, we can assume that K_w is transversal to ∂M_w. This way, $[\partial K_w] = -[N_w] +$

$\sum_{e \in \mathcal{E}_w}[K_w \cap S_e] \in H_1(M_w, \mathbb{Z})$. Let $e \in \mathcal{E}_w$, connecting w and $v \in \mathcal{V}$. Assume that the fiber C_w was chosen so that $C_w \subset S_e$. Furthermore, let C_v be a fiber of π_v contained in S_e if $v \in \mathcal{W}$, otherwise, let C_v be the image of s_v. It follows from definition that

$$([K_e \cap S_e], [C_w])_{S_e} = m_v, \quad ([K_e \cap S_e], [C_v])_{S_e} = m_v.$$

Since $[C_v]$ and $[C_w]$ form a basis of $H_1(S_e, \mathbb{Z})$, and we have

$$([C_w], [C_w])_{S_e} = ([C_v], [C_v])_{S_e} = 0, \quad ([C_w], [C_v])_{S_e} = \varepsilon_e,$$

we get $[K_e \cap S_e] = \varepsilon_e(m_v[C_w] - m_w[C_v])$. This yields

$$0 = [\partial K_e] = -n_w[C_w] + \sum_{e \in \mathcal{E}_w} \varepsilon_e(m_v[C_w] - m_w[C_v])$$

$$= \left(-n_w - b_w m_w + \sum_{e \in \mathcal{E}_w} \varepsilon_e m_v \right) [C_w].$$

Here, the variable v inside the sum depends on e. The last equality follows from lemma 3.4 ∎

4.4 Example. Let $\tilde{X}$ and $E = \cup_{v \in \mathcal{V}} E_v$ be as in example 3.9, and let $h : \tilde{X} \to \mathbb{C}$ be a holomorphic function. Decompose the divisor of h as $(h) = (h)_{\text{exc}} + (h)_{\text{str}}$ so that $(h)_{\text{exc}}$ is supported on E, and $(h)_{\text{str}}$ has no components with nonzero coefficient included in E. We can then write $(h)_{\text{exc}} = \sum_{v \in \mathcal{V}} m_v E_v$, and $(h)_{\text{str}} = \sum_D n_D D$ with $n_D = 0$ if $D = E_v$ for some $v \in \mathcal{V}$. Assume that the support of $(h)_{\text{str}}$ does not contain any intersection points in E, that is, if $n_D \neq 0$, then $D \cap E_v \cap E_w = \emptyset$ for $v, w \in \mathcal{W}$, $v \neq w$. If $T \subset \tilde{X}$ is a small tubular neighbourhood around E, then $M = \partial T$ is a plumbed manifold and $h|_M$ satisfies the conditions in 4.1. The associated family of multiplicities is $(m_v)_{v \in \mathcal{W}}$. Furthermore, the family $(n_v)_{v \in \mathcal{V}}$ of dual multiplicities is given as the intersection $n_v = (E_v, (h)_{\text{str}})$.

Note that here we do not assume $(h)_{\text{str}}$ to be smooth, only that its intersection points with E lie in the regular part of E.

5 Negative continued fractions

In this section we discuss negative continued fractions and a plumbing construction related to them. Some of the notation introduced in this section follows [2, III.5].

5.1. Let a, b be relatively prime integers, $b > 0$. The fraction a/b can be written in a unieque way as a (negative) continued fraction

$$\frac{a}{b} = k_1 - \frac{1}{k_2 - \frac{1}{\dots - \frac{1}{k_s}}} \tag{5.1}$$

where $k_i \geq 2$ for $i \geq 2$. Further, we have $k_1 \geq 2$ if and only if $a > b$ and $k_1 > 0$ if and only if $a > 0$.

5.2 Definition. The rational number a/b is called the *(negative) continued fraction* associated with the sequence $k_1, \dots, k_s$ and is denoted by $[k_1, \dots, k_s]$. The sequence $k_1, \dots, k_s$ is called the *(negative) continued fraction expansion* of the rational number a/b.

5.3. Given $a/b = [k_1, \ldots, k_s]$ as above, define integers μ_i and $\tilde{\mu}_i$ for $0 \le i \le s+1$ as follows. Start by setting

$$\mu_0 = 0, \quad \mu_1 = 1, \quad \tilde{\mu}_0 = -1, \quad \tilde{\mu}_1 = 0.$$

Then, assuming that we have defined $\mu_j, \tilde{\mu}_j$ for $0 \ge j \ge i$ for some $i > 0$, define

$$\mu_{i+1} = k_i \mu_i - \mu_{i-1}, \quad \tilde{\mu}_{i+1} = k_i \tilde{\mu}_i - \tilde{\mu}_{i-1}.$$

Using induction, one finds

$$\begin{vmatrix} \mu_i & \mu_{i+1} \\ \tilde{\mu}_i & \tilde{\mu}_{i+1} \end{vmatrix} = 1, \quad i = 0, \ldots, s. \tag{5.2}$$

Furthermore, the numbers μ_i and $\tilde{\mu}_i$ are positive for $i > 1$ if $a > 0$. A simple induction on s also proves $\mu_{s+1} = a$ and $\tilde{\mu}_{s+1} = b$.

5.4 Lemma. *Let a, b be positive integers with no common factors, and let $k_i, \mu_i, \tilde{\mu}_i$ be defined as above. The manifold $M = \bar{D} \times S^1$ is a plumbed manifold, given as $M = \cup_{i=1}^s M_i$ where*

$$M_1 = D_{\frac{1}{s}} \times S^1, \quad M_i = (\bar{D}_{\frac{i+1}{s}} \setminus D_{\frac{i}{s}}) \times S^1, \ i = 2, \ldots, s.$$

We set $\Sigma_1 = D_{\frac{1}{s}}$ and $\pi_1 : M_1 \to \Sigma_1$, $(rt_1, t_2) \mapsto rt_1$ where $r \in \mathbb{R}_{\ge 0}$ and $t_i \in S^1$, as well as $\Sigma_i = \bar{D}_{\frac{i}{s}} \setminus D_{\frac{i-1}{s}}$ for $i > 1$ and $\pi_i : M_i \to \Sigma_i$, $(rt_1, t_2) \mapsto rt_1^{\mu_i} t_2^{\tilde{\mu}_i}$ for $i > 1$. The section over $S^1 \subset \Sigma_s$ is given by $t \mapsto (t^{\tilde{\mu}_{s+1}}, t^{-\mu_{s+1}})$. The associated plumbing graph is shown in fig. 2.

$$-k_1 \qquad -k_2 \qquad\qquad\qquad -k_s$$

Figure 2: Plumbing representation of $S^1 \times \overline{D}$.

Proof. It is clear that the given components intersect in tori. Furthermore, eq. 5.2 gives $\gcd(\tilde{\mu}_i, \mu_i) = 1$. It follows that π_i is an S^1 fibration for all i. Another consequence of eq. 5.2 is that for $1 \le i < s$, the map $\pi_i \times \pi_{i+1} : M_i \cap M_{i+1} \to S^1 \times S^1$ is a diffeomorphism and that fibers of π_i and π_{i+1} intersect positively in the torus $M_i \cap M_{i+1}$. The same equation shows that the map $t \mapsto (t^{\tilde{\mu}_{s+1}}, t^{-\mu_{s+1}})$ is really a section:

$$\pi_s(t^{\tilde{\mu}_{s+1}}, t^{-\mu_{s+1}}) = t^{\mu_s \tilde{\mu}_{s+1} - \tilde{\mu}_s \mu_{s+1}} = t.$$

Therefore, M is a plumbed manifold with s components. What is left to show is that the Euler number for the ith vertex, call it v_i, in the graph is $-k_i$. To see this, consider the function $\zeta : M \to \mathbb{C}$, $(z, t) \mapsto t$. The function does not vanish on M, and so the dual set of multiplicities vanish. We have parametrizations $S^1 \to M_i$, $t \mapsto (rt^{-\tilde{\mu}_i}, t^{\mu_i})$ of a fiber of π_i for a suitable r. Thus, the multiplicities of ζ are given by $m_{v_i} = \mu_i$, and similarly, $m_a = \mu_{s+1}$, where a is the arrowhead. Thus, by lemma 4.3, we have $-b_{v_i} \mu_i + \mu_{i-1} + \mu_{i+1} = 0$ for $1 \le i \le s$. Since the same equation holds with b_{v_i} replaced with k_i (and $\mu_i \ne 0$), we get $-b_{v_i} = -k_i$. ∎

9

6 Construction

In this section we state our main result in details. We construct a plumbing graph G from the resolution graph Γ along with the multiplicities m_v and l_v of f and g. Theorem 6.3 says that this construction describes the boundary of the Milnor fiber of the hypersurface singularity given by $\Phi(x, y, z) = f(x, y) + zg(x, y)$.

6.1 Definition. (i) Let Γ' be a connected component of Γ_1. Let $\mathcal{V}(\Gamma')$ be the vertex set of Γ' and, for $v \in \mathcal{V}(\Gamma')$, let $\hat{\mathcal{E}}_v(\Gamma')$ be the set of edges connecting v and a vertex in $\mathcal{A}_{f,1} \cup \mathcal{W}_2$. Set also $\hat{\mathcal{E}}(\Gamma') = \cup_{v \in \mathcal{V}(\Gamma')} \hat{\mathcal{E}}_v(\Gamma')$. For any edge $e \in \hat{\mathcal{E}}(\Gamma')$ connecting $v \in \mathcal{V}(\Gamma')$ and $w \in \mathcal{A}_{f,1} \cup \mathcal{W}_2$, set $v_e = v$ and $w_e = w$ and $m_e = \gcd(m_v, m_w)$. For $v \in \mathcal{V}(\Gamma')$, let $\hat{\delta}_v$ be the number of edges connecting v and some vertex in $\mathcal{V}(\Gamma')$ or $\hat{\mathcal{V}}(\Gamma')$. Let $d_{\Gamma'} = \gcd_{v \in \mathcal{V}(\Gamma') \cup \hat{\mathcal{V}}(\Gamma')} m_v$ and define $g_{\Gamma'}, -b_{\Gamma'}$ by the equations

$$d_{\Gamma'}(2 - 2g_{\Gamma'}) = \sum_{v \in \mathcal{V}(\Gamma')} m_v(2 - \hat{\delta}_v) + \sum_{e \in \hat{\mathcal{E}}(\Gamma')} m_e, \tag{6.1}$$

$$d_{\Gamma'}(-b_{\Gamma'}) = \sum_{e \in \hat{\mathcal{E}}(\Gamma')} m_{v_e} l_{w_e} - m_{w_e} l_{v_e}. \tag{6.2}$$

Since $d_{\Gamma'} \neq 0$, these are well defined. As we will see later, we have $g_{\Gamma'}, -b_{\Gamma'} \in \mathbb{Z}$.

The graph $G_{\Gamma'}$ has vertex set $\mathcal{W}(G_{\Gamma'}) = \{v_{\Gamma',1}, \ldots, v_{\Gamma',d_{\Gamma'}}\}$, with each vertex decorated by the selfintersection number $-b_{\Gamma'}$ and genus $[g_{\Gamma'}]$. No two of these vertices are connected by an edge. Define G_1 as the disjoint union of the graphs obtained in this way.

(ii) Let $a \in \mathcal{A}_{f,1}$ and write $m_a/m_w = [k_1, \ldots, k_s]$. The graph G_a has $2s + 1$ vertices $v_{a,1,+}, \ldots, v_{a,s,+}, v_{a,1,-}, \ldots, v_{a,s,-}, v_{a,0}$. There is an edge with sign $\pm$ connecting $v_{a,i,\pm}$ and $v_{a,i,\pm}$ for each $1 \leq i \leq s-1$, as well as positive edges connecting $v_{a,0}$ and $v_{a,s,\pm}$. All these vertices have genus zero. The vertex $v_{a,i,\pm}$ has selfintersection $-b_{a,i,\pm} = \mp k_i$ and $v_{a,0}$ has selfintersection number $-b_{a,0} = 0$.

Define $G_{f,1}$ as the disjoint union of these graphs.

(iii) Let $v_1 \in \mathcal{W}_1$ and $v_2 \in \mathcal{W}_2$ be vertices of Γ connected by an edge e and write $m_{v_2}/m_{v_1} = [k_1, \ldots, k_s]$. The graph G_e has vertices $v_{e,1,+}, \ldots, v_{e,s,+}, v_{e,1,-}, \ldots, v_{e,s,-}, v_{e,0}$, each with genus zero. The vertex $v_{e,i,\pm}$ has the selfintersection number $-b_{e,i,\pm} = \mp k_i$ and $v_{e,0}$ has selfintersection $b_{e,0} = 0$. We have an edge with sign $\pm$ connecting $v_{e,i,\pm}$ and $v_{e,i+1,\pm}$, as well as positive edges connecting $v_{e,s,\pm}$ and $v_{e,0}$.

Let G_b be the disjoint union of graphs obtained in this way.

(iv) The graph G_2 is defined as follows. For each $w \in \mathcal{W}_2$, we have two vertices $v_{w,+}, v_{w,-}$ in G_2 and these are all the nonarrowhead vertices of G_2. They are decorated by genus zero and have selfintersection number $-b_{w,\pm} = \mp b_w$, where $-b_w$ is the selfintersection number of w in Γ.

(v) Let $a \in \mathcal{A}_{g,2}$. The graph G_a has nonarrowhead vertices $v_{a,0}, \ldots, v_{a,m_w}$ where $w = w_a$, each of genus zero. The vertex $v_{a,0}$ has selfintersection number

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$-b_{a,0} = 0$, whereas $v_{a,i}$ has selfintersection $-b_{a,i} = -l_a$. For each $1 \leq i \leq m_w$ there is a negative edge connecting $v_{a,0}$ and $v_{a,i}$.

Let $G_{g,2}$ be the disjoint union of graphs obtained in this way.

6.2 Definition. The graph G is the disjoint union of the graphs G_1, $G_{f,1}$, G_{b}, G_2, $G_{g,2}$, with the following additional edges.

(i) For $\Gamma' \subset \Gamma_1$ a connected component, $1 \leq i \leq d_{\Gamma'}$ and $a \in \mathcal{A}_{f,1}$, connect $v_{\Gamma',i}$ and $v_{a,0}$ with $m_e/d_{\Gamma'}$ negative edges, where m_e is as in definition 6.1(i).

(ii) Similarly, assuming that $\Gamma' \subset \Gamma_1$ is a connected component, $1 \leq i \leq d_{\Gamma'}$ and that $v_1 \in \mathcal{W}(\Gamma')$ and $v_2 \in \mathcal{W}_2$ are connected by an edge $e \in \mathcal{E}(\Gamma)$. Connect $v_{\Gamma',i}$ and $v_{e,0}$ with $m_e/d_{\Gamma'}$ negative edges and connect $v_{w,\pm}$ and $v_{a,1,\pm}$ with an edge with sign $\pm$.

(iii) Let $a \in \mathcal{A}_{g,2}$ and $w = w_a$. The vertex $v_{a,0}$ is connected to both $v_{w,+}$ and $v_{w,-}$ by a positive edge.

6.3 Theorem. *The boundary of the Milnor fiber of the singularity $f(x,y) + zg(x,y) = 0$ at the origin is a plumbed manifold with plumbing graph G.*

6.4 Remark. (i) Let $a \in \mathcal{A}_{g,2}$ and assume that $l_a = 1$. In this case, the vertices $v_{a,1}, \ldots, v_{a,m}$, where $w = w_a$ and $m = m_w$, blow down to simplify the graph (see [9] for blowing down). This operation removes these vertices, and replaces the Euler number $-b_{a,0} = 0$ with $-b_{a,0} = m$.

(ii) We can apply the operation R0(a) from [9] to the vertices $v_{\Gamma',i}$ for a connected component $\Gamma' \subset \Gamma_1$ as well as to $v_{a,i}$ for $a \in \mathcal{A}_{g,2}$ and $1 \leq i \leq m_{w_a}$. This way, all the edges adjacent to these vertices will be positive instead of negative. Note, however, that this also changes the sign of the corresponding multiplicities given in section 7.

7 A multiplicity system for z

In this section, we give multiplicities and dual multiplicities for the function z. For simplicity, the multiplicitiy and dual multiplicity for a vertex v_* constructed in definition 6.1 will be denoted by m_* and n_*. The proof of theorem 7.1 is given in section 9.

7.1 Theorem. *The restriction $z|_M$ of the coordinate function z to the boundary $M = \partial F_\Phi$ of the Milnor fiber of Φ satisfies the conditions given in 4.1. Furthermore, the associated families of multiplicities $(m_v)_{v \in \mathcal{V}(G)}$ and dual multiplicities $(n_v)_{v \in \mathcal{W}(G)}$ are is given as follows.*

(i) If $\Gamma' \subset \Gamma_1$ is a connected component, then $m_v = 1$ and $n_v = 0$ for $v = v_{\Gamma',i}$, $i = 1, \ldots, d_{\Gamma'}$.

(ii) Let $a \in \mathcal{A}_{1,f}$ be an arrowhead connected to $w \in \mathcal{W}_1$ in Γ and set $\tilde{m}_w = m_w/\gcd(m_w, m_a)$ and $\tilde{m}_a = m_a/\gcd(m_w, m_a)$. Write $m_w/m_a = \tilde{m}_w/\tilde{m}_a = [k_1, \ldots, k_s]$ and define $\mu_i, \tilde{\mu}_i$ as in 5.1. The multiplicities of z are given by

$$m_{a,i,+} = (m_w - l_w)\mu_i - m_a\tilde{\mu}_i, \quad m_{a,i,-} = l_w\mu_i$$

for $i = 1, \ldots, s$ and $m_{a,0} = -\tilde{m}_a l_w$. The dual multiplicities for these vertices are given by $n_{a,1,+} = m_a$, and 0 otherwise.

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(iii) Let v_1, v_2 and e be as in definition 6.1(iii). Let $\tilde{m}_i = m_{v_i}/\gcd(m_{v_1}, m_{v_2})$ for $i = 1, 2$. Write $m_{v_2}/m_{v_1} = \tilde{m}_2/\tilde{m}_1 = [k_1, \ldots, k_s]$ and define $\mu_i, \tilde{\mu}_i$ for $i = 1, \ldots, s$ as in 5.1. Then $m_{e,0} = \tilde{m}_1 l_{v_2} - \tilde{m}_2 l_{v_1}$ and

$$m_{e,i,+} = (m_{v_1} - l_{v_1})\mu_i - (m_{v_2} - l_{v_2})\tilde{\mu}_i, \quad m_{e,i,-} = l_{v_1}\mu_i - l_{v_2}\tilde{\mu}_i$$

The dual multiplicities vanish on these vertices.

(iv) Let $w \in \mathcal{W}_2$. Then $m_{w,+} = m_w - l_w$ and $m_{w,-} = l_w$. The dual multiplicities are given by $n_{w,+} = \sum_{w_a = w} m_a$ and $n_{w,-} = 0$.

(v) Let $a \in \mathcal{A}_{g,2}$ and set $w = w_a$. Then $m_{a,0} = -l_a$ and $m_{a,i} = 1$ for $i = 1, \ldots, m_w$. The dual multiplicities associated with $v_{a,i}$ vanish.

7.2 Remark. Let e, v_1 be as in theorem 7.1(iii). One proves that, in fact, $m_{e,1,+} = m_{v_1} - l_{v_1}$ and $m_{e,1,-} = l_{v_1}$.

8   Examples

8.1 Example. The singularity $T_{a,b,\infty}$ is the singularity at the origin of the hypersurface given by $x^a + y^b + xyz = 0$. In the case $b = 2$, the boundary of the Milnor fiber has been described in [8]. We take $f(x, y) = x^a + y^b$ and $g(x, y) = xy$. We will assume a, b satisfying $a \geq b \geq 2$ and $a > 2$. We claim that the boundary of the Milnor fiber of this singularity is given by the plumbing graph
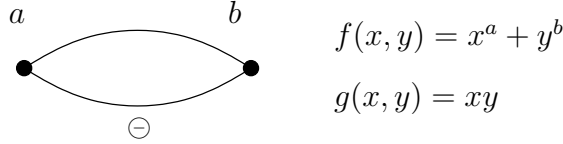

$$f(x, y) = x^a + y^b$$
$$g(x, y) = xy$$

Figure 3: A plumbing graph for the boundary of the Milnor fiber of the singularity $T_{a,b,*}$, $x^a + y^b + xyz$.

Let $\phi : V \to \mathbb{C}^2$ be the minimal resolution of the plane curve fg and let Γ be its resolution graph. Then Γ is a string with two arrowheads corresponding to g, one on each end of the string, as well as $d := \gcd(a, b)$ arrowheads corresponding to f. Name the nonarrowhead vertices of the graph $v_1, \ldots, v_s$ so that v_i, v_{i+1} are adjacent. Let $-b_i$ be the selfintersection number associated with the vertex v_i. There is a unique j so that $-b_j = -1$. The set $\mathcal{A}_f$ consists of d arrowheads, each connected to v_j, whereas $\mathcal{A}_g$ consists of two arrowheads, one connected to v_1 and the other to v_s. Write also m_i, l_i for the multiplicities of f and g on v_i.

Claim: We have $m_1 \geq l_1$ and $m_s > l_s$ and $m_i > l_i$ for $i = 2, \ldots, s - 1$.

In fact, using [3, Lemma 20.2], one finds $m_j = ab/d$ and $l_j = a/d + b/d$. It follows from our assumptions that $m_j - l_j > 0$. Now, define integers $r_i = m_i - l_i$ for $i = 1, \ldots, s$ and $r_0 = r_{s+1} = -1$. We then have

$$r_{i-1} - b_i r_i + r_{i+1} = 0, \quad i = 1, \ldots, \hat{j}, \ldots, s.$$

It follows easily that this sequence increases strictly from $r_0 = -1$ to $r_j = m_j - l_j$, and then decreases strictly from r_j to $r_{s+1} = -1$. Since these are integers, the claim follows.

We leave to the reader to show that the equality $m_i = l_i$ holds for $i = 1$ or $i = s$ if and only if $b = 2$, the case already covered by Némethi and Szilárd [8]. This can be achieved by calculating m_i and l_i explicitly using Lemma 20.2 of [3].

We start by showing how the above graph is obtained from the output of the algorithm in the case when $l_i > m_i$ for all i. Since $\mathcal{W} = \mathcal{W}_2$, the graph G_2 consists of two strings, one of them identical to Γ, the other one having Euler numbers with opposite signs and negative edges. In addition, we have $\mathcal{A}_{g,2} = \{a_x, a_y\}$, two arrowhead vertices corresponding to the strict transform of the factors x and y of g. As described in remark 6.4, the graph Γ can be taken as these two strings, connected on each end by vertices with Euler number m_x and m_y. These are the multiplicities of f along the components on the end of the string. It follows from [3] that these multiplicities are a and b. Furthermore, the two strings blow down (we can blow down the vertices one by one in the opposite order in which they appear during the process of resolving f). Each string is replaced by an edge, the first string by a positive edge, the second one by a negative edge. Below, we explicate the case when $a = 7$ and $b = 5$.
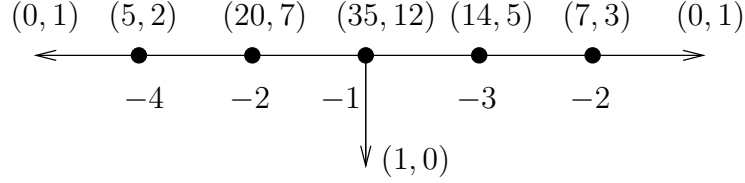

Figure 4: A resolution graph of the plane curves $f(x,y) = x^7 + y^5 = 0$ and $g(x,y) = xy$, along with their multiplicities.
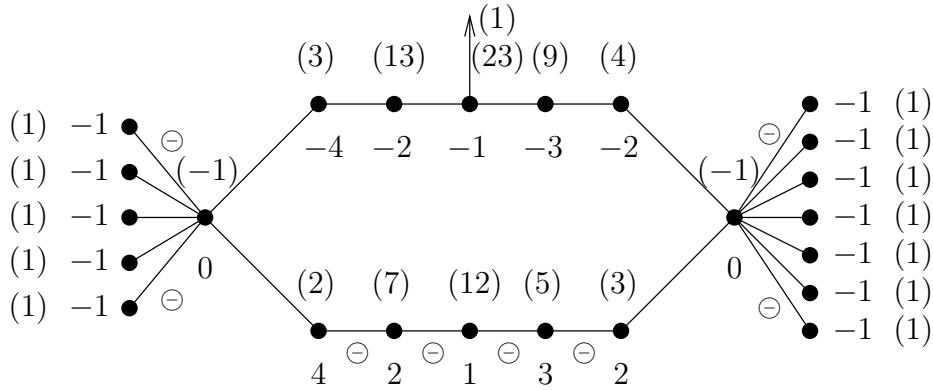

Figure 5: Output of the algorithm for $\Phi(x, y, z) = x^7 + y^5 + xyz$.

In the case when either $m_1 = l_1$ or $m_s = l_s$, the algorithm has, in fact,

the same output. We let it suffice to clarify this principle by considering an example. Take $a = 3$ and $b = 2$. A resolution graph Γ, decorated with the pairs of multiplicities (m_v, l_v) is shown in fig. 6.
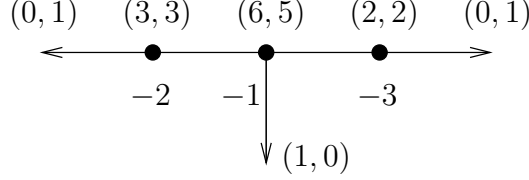

Figure 6: A resolution graph of the plane curves $f(x, y) = x^3 + y^2 = 0$ and $g(x, y) = xy$, along with their multiplicities.

We see that $\mathcal{W}_2$ now only contains the vertex v_2, whereas $v_1, v_3 \in \mathcal{W}_1$, each providing a connected component of Γ_1. We order the vertices in fig. 6 in such a way that $b_1 = -2$ and $b_3 = -3$. Applying definition 6.1(i) to the component Γ' of Γ_1, containing only the vertex v_1, as well as Γ'' containing only v_3, we get

$$d_{\Gamma'} = 3, \quad g_{\Gamma'} = 0, \quad -b_{\Gamma'} = -1, \quad d_{\Gamma''} = 2, \quad g_{\Gamma''} = 0, \quad -b_{\Gamma''} = -1.$$

We get five new vertices. The edges $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_2, v_3\}$ are of the form described in definition 6.1(iii). The five new vertices are connected to $v_{e_1,0}$ and $v_{e_2,0}$ to obtain the graph in fig. 7, which also shows the multiplicities of the function z. After blowing down, we obtain fig. 3.
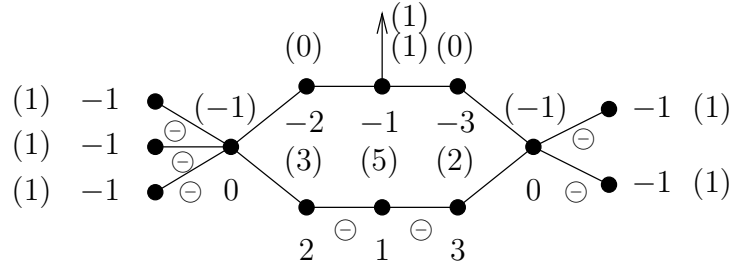

Figure 7: Output of the algorithm for $\Phi(x, y, z) = x^3 + y^2 + xyz$.

8.2 Example. Consider the plane curves

$$f(x, y) = (x^2 + \lambda_1 y^3)((x^2 + y^3)^2 + \mu_1 x^5)^2,$$
$$g(x, y) = (x^2 + \lambda_2 y^3)^3((x^2 + y^3)^2 + \mu_2 x^5),$$

where $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0, 1\}$ are distinct and $\mu_1, \mu_2 \in \mathbb{C} \setminus \{0\}$ are distinct. The resolution graph Γ, decorated with the multiplicities m and l is given in fig. 8. The set $\mathcal{W}_1$ consists of the first three vertices appearing during the resolution process, corresponding to the first Puiseux pair, where f and g have equal multiplicities. But, as f has more components with two Puiseux pairs, compared
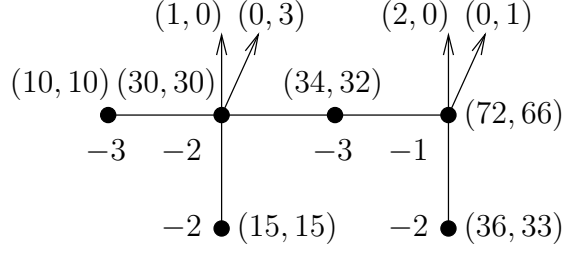
Figure 8: Plane curves whose branches have 1 and 2 Puiseux pairs.

with g, the multiplicities of f are higher along the second part of the resolution process, that is, the three vertices appearing last.

The graph Γ_1 is connected, and the invariants from definition 6.1(i) are easily computed using eq. 6.1 and eq. 6.2:

$$
\begin{aligned}
d_{\Gamma_1} &= \gcd\{10, 30, 15, 1, 34\}, & d_{\Gamma_1} &= 1, \\
1 \cdot (2 - 2g_{\Gamma_1}) &= 1 \cdot 15 + 1 \cdot 10 + (-2) \cdot 30 + 1 + 2, & g_{\Gamma_1} &= 17, \\
1 \cdot (-b_{\Gamma_1}) &= (30 \cdot 0 - 1 \cdot 30) + (30 \cdot 32 - 30 \cdot 34), & -b_{\Gamma_1} &= -90.
\end{aligned}
$$

Thus, we obtain a single vertex v_{Γ_1} with genus 17 and Euler number -90. Furthermore, $m_{\Gamma_1} = 1$.

The set $\mathcal{A}_{f,1}$ contains one element. Using the notation in definition 6.1(ii), we have $m_a = 1$, $m_w = 30$ and $l_w = 30$. We have $m_a/m_w = [1, 2, \ldots, 2]$ where the number of 2's is 29. Therefore, 61 new vertices are created. The vertex $v_{a,0}$ is connected to v_{Γ_1} by $m_e = 1$ edge with a negative sign. Furthermore,

$$
m_{a,0} = -30, \quad m_{a,1,+} = 0, \quad m_{a,1,-} = 30, \quad n_{a,1,+} = 1.
$$

Note that the vertices $v_{a,i,\pm}$ blow down, leaving only the vertex $v_{a,0}$.

There is one edge e connecting $\mathcal{W}_1$ and $\mathcal{W}_2$. In the notation of definition 6.1(iii), we have $m_e = 2$ and

$$
\begin{aligned}
m_{v_1} &= 30, & \tilde{m}_1 &= 15, & l_{v_1} &= 30, \\
m_{v_2} &= 34, & \tilde{m}_2 &= 17, & l_{v_2} &= 32,
\end{aligned}
$$

We have $17/15 = [2, 2, 2, 2, 2, 2, 2, 3]$, hence the creation of 17 new vertices. The vertices $v_{e,0}$ and v_{Γ_1} are connected by two edges with negative sign. Furthermore, we get

$$
m_{e,0} = -30, \quad m_{e,1,+} = 0, \quad m_{e,1,-} = 30.
$$

The set $\mathcal{W}_2$ contains three elements, inducing six new vertices with genus 0 and Euler number $\mp 3, \mp 1, \mp 2$. The ∓ 3 curves are connected with the vertices $v_{e,1,\pm}$ from the previous construction.

The set $\mathcal{A}_{g,2}$ contains one vertex, say $a \in \mathcal{A}_{g,2}$, as in definition 6.1(v). Set also $w = w_a$. Since $m_w = 72$ and $l_a = 1$, we get a vertex $v_{a,0}$ with genus 0 and Euler number 0, connected to the vertices $v_{a,1}, \ldots, v_{a,72}$ with genus 0 and Euler number -1, via a negative edge. We have $m_{a,0} = -1$ and $m_{a,i} = (1)$ for $1 \le i \le 72$.
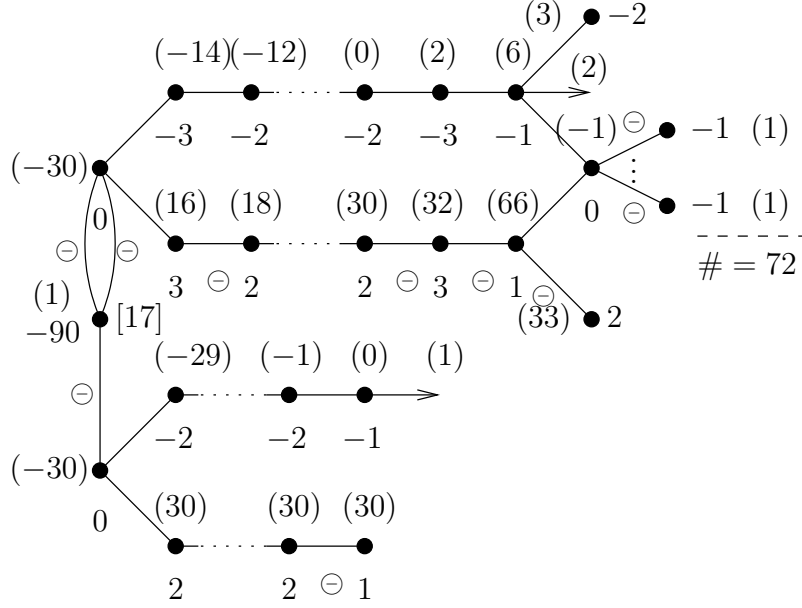
15

(3) $\bullet$ -2

$(-14)(-12)$ (0) (2) (6) (2)

-3 -2 -2 -3 -1 (-1) $\ominus$ $\bullet$ -1 (1)

(-30) (16) (18) (30) (32) (66) 0 $\ominus$ $\bullet$ -1 (1)

0 $\ominus$ $\ominus$ 3 $\ominus$ 2 2 $\ominus$ 3 $\ominus$ 1 $\ominus$ $\# = 72$

(1) $[17]$ (33) $\bullet$ 2

-90 (-29) (-1) (0) (1)

$\ominus$ -2 -2 -1

(-30) (30) (30) (30)

0 2 2 $\ominus$ 1

Figure 9: The dotted lines denote either a string of (-2)-pieces connected by positive edges or 2-curves connected by negative edges.

9 Proofs

To prove theorems 6.3 and 7.1, we start by defining pieces $M_v \subset M$ and projections $\pi_v : M_v \to \Sigma_v$ for all vertices $v \in \mathcal{V}$, using the description in theorem 2.5. From the construction, it will be clear that $M = \cup_{v \in \mathcal{V}} M_v$, and that individual pieces intersect according to the edges of G. Finally, we verify the formulas for genera and selfintersection numbers. In fact, it will be clear that the genus decoration is zero, except for in the case of $v_{\Gamma'}$, where an argument similar to A'Campo's formula [1] is used. Similarly as in [7, 8], nontrivial Euler numbers are determined using the multiplicities of z and lemma 4.3. We note that the proof of theorem 7.1, can be carried out as soon as the projections π_v are defined. In particular, this proof does not use the Euler numbers, which are computed using the multiplicities of z.

Proof of theorem 6.3. We start by providing sets M_v for each vertex of the graph G. We then prove that these pieces provide a plumbing structure on M with the plumbing graph G.

(i) Let X_1' be the closure of the set

$$\overline{T}_{f,g} \cap T_1 \setminus (T_{f,1} \cup T_2).$$

By construction, this is a closed tubular neighbourhood of

$$F_{f,1} := F_f \cap T_1 \setminus (T_{f,1} \cup T_2),$$

in particular, we have a disk bundle $X_1' \to F_{f,1}$. We can assume that the intersection of this disk bundle with the divisor associated with g is a set of

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disks. Let X_1 be the four manifold obtained from X_1' by twisting along these disks as in definition 2.4 and let $M_1 \subset \partial X_1$ be the associated S^1 bundle. It is then clear that we have $M_1 \subset M_{f,g}$, that the boundary of M_1 consists of tori and that M_1 is in a natural way an S^1 bundle over $F_{f,1}$.

Let $\Gamma' \subset \Gamma_1$ be a component as in definition 6.1(i). Setting

$$X_{\Gamma'}' = X_1' \cap \bigcup_{v \in \mathcal{V}(\Gamma')} T_v,$$

we obtain correspondingly $X_{\Gamma'} \subset X_1$ and $M_{\Gamma'} \subset M_1$. This way, $M_{\Gamma'}$ is an S^1 bundle over the surface $F_{f,\Gamma'} = F_f \cap X_{\Gamma'}'$.

Firstly, we note that the number of connected components of $F_{f,\Gamma'}$ is precisely $d_{\Gamma'}$ and that, furthermore, the monodromy permutes these components cyclically. This follows from Proposition 2.20 of [7], see also 2.21 of the same article.

Secondly, the genus of the components of $F_{f,\Gamma'}$ is $g_{\Gamma'}$, satisfying eq. 6.1. This follows from a small generalization of A'Campo's formula [1] which gives

$$\chi(F_{f,\Gamma'}) = \sum_{v \in \mathcal{V}(\Gamma')} m_v (2 - \hat{\delta}_v).$$

What is more, $F_{f,\Gamma'}$ has m_e boundary components close to the intersection of E_v and E_{w_e} for $e \in \hat{\mathcal{E}}(\Gamma')$. Thus, $F_{f,\Gamma'}$ has a total of $\sum_{e \in \hat{\mathcal{E}}(\Gamma')} m_e$ boundary components.

The formula eq. 6.2 is verified below.

(ii) Let $a \in A_{1,f}$ and set $w = w_a$. Define $M_a = \partial \overline{T}_{f,g} \cap \overline{T}_a$ We have coordinates u, v on T_a so that $T_a \cong \{(u,v) \mid |u| \leq 1, |v| \leq 1\}$ and so that $E_a \cap T_a$ and $E_w \cap T_a$ are the vanishing sets of u and v, respectively. We can then write $M_a = M_{a,+} \cup M_{a,-} \cup M_{a,0}$ where for certain $0 < \eta \ll \varepsilon \ll 1$, we have

$$M_{a,+} = \{(u,v) \mid |v| = 1, |u| \leq 1\}, \quad M_{a,-} = \{(u,v) \mid |v| = \eta, |u| \leq 1\}$$

and $M_{a,0}$ is defined by setting

$$M_{a,0}' = \{(u,v) \mid |u| = 1, \eta \leq |v| \leq 1\}, \quad M_{a,0} = M_{a,0}' \setminus N,$$

where N is an ε neighbourhood around $F_f \cap M_{a,0}'$. The projection of this picture via $(u,v) \mapsto (|u|,|v|)$ is shown in fig. 10. We can assume that in the coordinates u, v, we can write $f|_{T_a}(u,v) = u^{m_a} v^{m_w}$. Define $\tilde{m}_a = m_a/m_e$ and $\tilde{m}_w = m_w/m_e$. We find

$$F_f \cap M_{a,0}' = \coprod_{j=0}^{m_e-1} \left\{ (e^{(-t+j/m_a)\tilde{m}_w 2\pi i}, e^{t\tilde{m}_a 2\pi i}) \,\middle|\, t \in [0,1] \right\}$$

In fact, we have an S^1 bundle projection $\pi_{a,0}'$ mapping $M_{a,0}'$ to an annulus by the formula $\pi_{a,0}'(u,v) = u^{\tilde{m}_a} v^{\tilde{m}_v}$. This way, $F_f \cup M_a'^0$ consists of m_e fibers of $\pi_{0,a}'$. In particular, we can assume that $\pi_{a,0}'$ restricts to an S^1 bundle $\pi_{0,a} = \pi_{0,a}'|_{M_{0,a}}$. We orient the fibers so that one of them is parametrized as $t \mapsto (t^{-\tilde{m}_w}, t^{\tilde{m}_a})$, which induces an orientation on the target space of $\pi_{0,a}$.
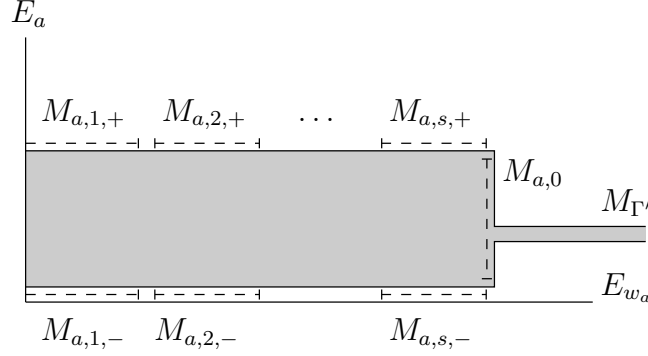
Figure 10: A diagram showing what happens near $E_a \cap E_{w_a}$.

By lemma 5.4, the manifold $M_{a,+}$ can be given as a plumbed manifold with plumbing graph as in fig. 2, where $[k_1, \ldots, k_s] = m_a/m_w$, so that the section corresponding to the arrowhead to the right can be chosen to coincide with a fiber of $M_{a,0}$, with the opposite orientation. Furthermore, we have an orientation reversing diffeomorphism $M_{a,+} \to M_{a,-}$ given by $(u, v) \mapsto (u, \eta v)$. This way, we see $M_{a,-}$ as a plumbed manifold with the same plumbing graph, modified by changing signs on all selfintersection numbers as well as edges.

At this point, we have shown that the manifold M_a is a plumbed manifold with plumbing graph G_a, with m_e dashed arrows added to $v_{a,0}$, except we did not specify a section corresponding to these arrowheads. Furthermore, (and this cannot be done without the sections) we have not determined the Euler number associated with $v_{a,0}$. Since $M_{0,a}$ is obtained by removing a tubular neighbourhood around a fiber of the projection $\pi'_{0,a}$, we can choose as a section a meridian around this fiber. Note that this section is exactly a fiber of the projection $\pi_{\Gamma',i}$ for a suitable $1 \leq i \leq d_{\Gamma'}$, where w is a vertex of the component Γ' of Γ_1. But we can be more specific. Let $\psi : S^1 \to M_{a,+}$ be a parametrization of a fiber in the boundary component of $M_{a,s,+}$. This induces a map $S_1 \times [\eta, 1] \to M'_{a,0}$ which is a global section to the fibration of $M'_{a,0}$, restricting to a global section to the fibration of $M_{a,0}$, which again restricts to a parametrization of the fibers of $M_{a,s,\pm}$, as well as a parametrization of a meridian around $F_f \cap M_{a,0}$. This shows that with this choice of sections on the boundary, the Euler number of the bundle $M_{a,0}$ is 0.

Finally, we note that $M_{a,0}$ intersects $M_{\Gamma'}$ in exactly m_e tori, and that the number of these tori in each component of $M_{\Gamma'}$ is the same. It follows from the construction that in each of these tori, a fiber of $M_{a,0}$ and a fiber from $M_{\Gamma'}$ form an integral basis on homology. Furthermore, one verifies that an oriented fiber of $M_{a,0}$ in such a torus is an oriented section of $M_{\Gamma'}$. This can be seen by noting that both wind around E_a with multiplicity $-\tilde{m}_a$. Therefore, these tori yield edges with a negative sign, by lemma 3.7. These are the edges defined in definition 6.2(i).

(iii) Let e and $v_i \in \mathcal{W}_i$ be as in 6.1(iii). Let D be a disc in E_{v_1} with center the intersection point of E_{v_1} and E_{v_2} corresponding to e which is a slightly bigger than the corresponding disk in $E_{v_1} \cap \overline{T}_{v_2}$. We can add the preimage of D in

$\overline{T}_{v_1} \setminus T'_{v_2}$ to $\overline{T}_{f,g}$ without changing its diffeomorphism type. From here, the proof follows similarly as in the previous case. A schematic picture is shown in fig. 11.
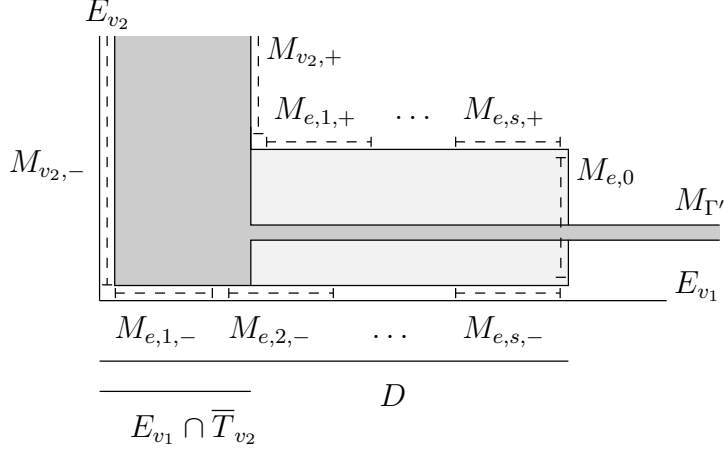

Figure 11: A diagram showing what happens near $E_{v_1} \cap E_{v_2}$.

(iv) Let $w \in \mathcal{W}_2$ and define M_w as the closeure of $\overline{T}_w \setminus \cup_v \overline{T}_v$, where the union $\cup_v$ ranges over $v \in \mathcal{W} \cup \mathcal{A}_g \setminus \{w\}$. It follows from construction that M_w consists of two copies of an S^1 bundle over the surface E_w with a disk removed for each neighbour in $\mathcal{W} \cup \mathcal{A}_g \setminus \{w\}$. Indeed, these are the corresponding subsets of the boundaries of T_w and T'_w (recall definition 2.2). Write $M_{w,+}$ and $M_{w,-}$ for the outer and inner components. These components will correspond to the vertices $v_{w,\pm}$.

These fibrations extends canonically over the disks removed from E_w by taking T_w and T'_w, and we can take a meridians around central fibers as the trivializing section on the boundary. It follows immediately that the two S^1 bundles have relative Euler numbers $-b_{w,\pm} = \mp b_w$. Furthermore, since E_w is a rational curve, the two vertices $v_{w,\pm}$ have associated genus 0.

As both components of M_w are boundaries of similar tubular neighbourhoods, they can be identified, but the inner one, i.e. the boundary of T'_w has its orientation reversed. We will consider this part as a fibration over the same base as the outer component. Therefore, a fiber in $M_{w,+}$ is a meridian around E_w, whereas a fiber in $M_{w,-}$ is a (relatively small) meridian around E_w with the orientation reversed.

If $w, w' \in \mathcal{W}_2$ are joined by an edge, it follows easily that the two components corresponding to w intersect with those of w' in the same way as prescribed by the resolution graph Γ.

(v) Finally, we describe what happens close to a component of the strict transform of g corresponding to $a \in \mathcal{A}_{g,2}$.

Let $M'_a = M' \cap \overline{T}_a$ and M_a the corresponding twisted subset of M. Let $M_{a,0} = M \cap \partial T_a$. It follows from construction that M_a fibers by a map $\pi_{a,0}$ over the

disk E_a with $m_a + 1$ smaller disks removed, one corresponding to T'_{w_a}, and m_a of them corresponding to T_ε. We orient the fiber to coincide with that of a meridian around E_a. This chooses an orientation of the base space of $\pi_{a,0}$, the opposite of the standard one on E_a. This bundle is trivialized in a similar way as in (ii), yielding Euler number $-b_{a,0} = 0$. It is also clear that $g_{a,0} = 0$.

The closure of $M_a \setminus M_{a,0}$ is an S^1 bundle over $F_f \cap T_a \cong \amalg_{m_w} \overline{D}$. This gives m_w pieces $M_{a,1}, \ldots, M_{a,m_w}$, ordered arbitrarily. The fibers are meridians around F_f.

We see that M_a is a plumbed manifold with plumbing graph G_a (with some dashed arrows added, corresponding to the boundary). Using lemma 3.7, we see that the edges between $v_{a,0}$ and $v_{a,i}$ have a negative sign, whereas the edges connecting $v_{a,0}$ and $v_{w,\pm}$ are positive.

In (i-v) above we have assigned subsets $M_v \subset M$ to each vertex v of the graph G constructed in definitions 6.1 and 6.2. It is clear that each piece is connected and that each boundary components of any of the pieces are tori. Furthermore, the components of intersection of two pieces correspond to the edges connecting the corresponding vertices. The base space of each fibration is a surface of the genus specified, or zero otherwise.

The only part which remains to prove is eq. 6.2. But this follows immediately from lemma 4.3 and theorem 7.1. ∎

Proof of theorem 7.1. (i) Let Γ' be as in definition 6.1(i). It follows from the proof in [10] that $|z|$ is constant on $M_{\Gamma',i}$ (and nonzero). It follows that the dual multiplicities $n_{\Gamma',i}$ vanish. A fiber in $M_{\Gamma',i}$ is an oriented meridian around F_f. The restriction of $z = (f - \varepsilon)/g$ to such a fiber is a map of degree 1, thus $m_{\Gamma',i} = 1$.

(ii) Let $a \in \mathcal{A}_{1,f}$ as in definition 6.1(ii). We start by observing that the vanishing set of the function $z = (f - \varepsilon)/g$ is contained in the piece $M_{a,1,+}$. The vanishing set of z is the Milnor fiber F_f of f. The intersection $F_f \cap \partial T_a$ consists of two parts, contained in neighbourhoods around $E_w \cap \partial T_a$ and $E_a \cap \partial T_w$. By construction, the former is not included in M_a. The latter is homologous to a meridian around E_w with multiplicity m_w. We can take this meridian as $E_a \cap \partial T_a$. Therefore, the dual multiplicities vanish on all vertices of G_a except for $v_{a,1,+}$ and we have $n_{a,1,+} = m_a$.

It follows from the explicit calculations in (ii) in the proof of theorem 6.3 that the restriction of $f - \varepsilon$ to a fiber of $M_{a,0}$ has degree zero. Indeed, in the coordinates u, v introduced there for the polydisk T_a, we have $f|_{T_a} = u^{m_a} v^{m_w}$ and a fiber in $M_{a,0}$ is parametrized in these coordinates by $[0,1] \to T_a$, $t \mapsto (e^{-t\tilde{m}_w 2\pi i}, e^{t\tilde{m}_a 2\pi i})$. Since g vanishes with order l_w along E_w, and does not vanish along E_a, it follows that the multiplicity $m_{a,0}$ equals $-l_w \tilde{m}_a$.

Now, the sequence $m_{a,i,+}$, $i = 1, \ldots, s$ satisfies

$$
\begin{array}{rcccccl}
& - & b_{a,1,+} m_{a,1,+} & + & m_{a,2,+} & = & n_{a,+,1}, \\
m_{a,i-1,+} & - & b_{a,i,+} m_{a,i,+} & + & m_{a,i+1,+} & = & 0, \qquad i = 2, \ldots, s-1, \\
m_{a,s-1,+} & - & b_{a,s,+} m_{a,s,+} & & & = & m_{a,0}.
\end{array}
$$
(9.1)

The same equations are satisfied by the sequence $(l_w - \tilde{m}_w)\mu_i - m_a \tilde{\mu}_i$, as is easily checked. It follows that the two sequences coincide, since the matrix associated

with this system of linear equations is negative definite. A similar argument proves the statement for the multiplicities $m_{a,i,-}$.

(iii) Let e be an edge in G connecting v_1 and v_2 as in definition 6.1(iii). We start by observing that z does not vanish on M_e, and so all dual multiplicities are vanish for the vertices of G_e.

Similarly as above, we find that the map $f - \varepsilon$ restricted to a fiber $C_{e,0} \subset M_{e,0}$ has degree zero, and g has degree $\tilde{m}_2 l_{v_1} - \tilde{m}_1 l_{v_2}$. It follows that $m_{e,0} = \tilde{m}_1 l_{v_2} - \tilde{m}_2 l_{v_1}$. Now, similar reasoning as above determines the multiplicities $m_{e,i,\pm}$. Namely, we have linear equations

$$
\begin{array}{rcll}
- b_{e,1,\pm} m_{e,1,\pm} + m_{e,2,\pm} & = & m_{e,0}, & \\
m_{e,i-1,\pm} - b_{e,i,\pm} m_{e,i,\pm} + m_{e,i+1,\pm} & = & 0, & i = 2,\ldots,s-1, \\
m_{e,s-1,\pm} - b_{e,s,\pm} m_{e,s,\pm} & = & m_{w_2,\pm}. &
\end{array}
\tag{9.2}
$$

The result follows as soon as we determine the multiplicities $m_{w_2,\pm}$:

(iv) Let $w \in \mathcal{W}_2$. Above, we have determined that $C_{w,-}$ is a meridian around E_w, small with respect to ε and having the opposite orientation than the standard meridian. It follows that $z = (f - \varepsilon)/g$ restricted to $C_{w,-}$ has degree l_w, i.e. $m_{w,-} = l_w$. Furthermore, z does not vanish on $M_{w,-}$, thus $n_{w,0} = 0$.

On the other hand, $C_{w,+}$ is an oriented meridian around E_w, with respect to which ε is chosen small. It follws that $m_{w,+} = m_w - l_w$. Furthermore, the vanishing set of z in $M_{w,+}$ is homologous to the strict transform of f, with multiplicities. Therefore, each $a \in \mathcal{A}_{f,2}$ contributes m_a to $n_{w,+}$, resulting in the sum given.

(v) Let $a \in \mathcal{A}_{g,2}$ and set $w = w_a$. Similarly as in (i), we find that z does not vanish on M_a. Therefore, $n_{a,i} = 0$ for $i = 0,\ldots,m_w$. In (v) of the proof of theorem 6.3 we found that $C_{a,0}$ is a meridian around E_a. We can assume that the restriction of $f - \varepsilon$ to such a meridian has degree 0. It is also clear that g restricts to a degree $-l_a$ map on such a fiber. It follows that $m_{a,0} = -l_a$.

For $i = 1,\ldots,m_w$, the fiber $C_{a,i}$ is a small meridian around F_f. It follows that $m_{a,i} = 1$. ∎

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