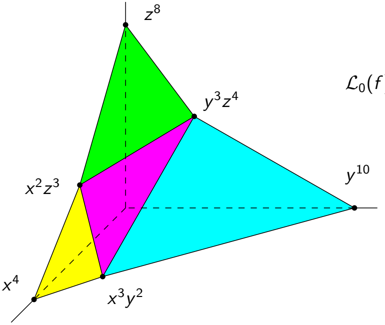
The Brzostowski-Krasiński-Oleksik conjecture on the Łojasiewicz exponent for Newton nondegenerate hypersurface singularities

Baldur Sigurðsson

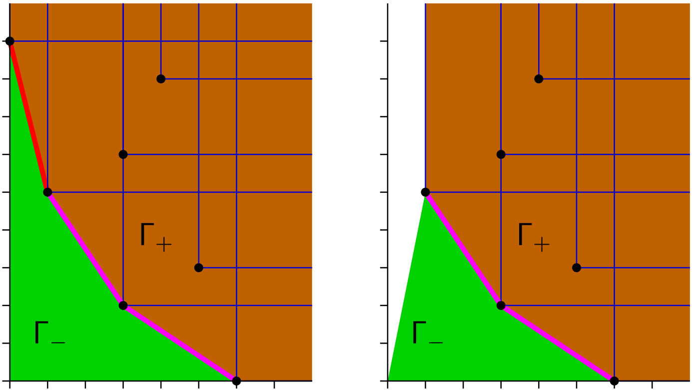
IX RGAS Yearly Meeting, 13-15 January 2025

Newton diagrams

- Given a series $f(z) = \sum_i a_i z^i \in \mathbb{C}\{z_1, \ldots, z_n\}$
- Define the *support* $\mathrm{supp}(f) = \{i \in \mathbb{Z}_{\geq 0} \mid a_i \neq 0\}$
- $\Gamma_+(f)$ is the convex closure of $\mathrm{supp}(f) + \mathbb{R}^n_{\geq 0}$.
- $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$.



$\mathcal{L}_0(f) = 9$



$$\mathcal{L}_0(f) = 5\frac{1}{2}$$

A general question of Arnold

- Let $f \in \mathbb{C}\{z_1, \dots, z_n\}$ be a holomorphic function
- Denote its Newton diagram by $\Gamma(f) \subset \mathbb{R}^n_{\geq 0}$.
- Denote by $A(f)$ be some numerical analytic invariant of f,
- Denote by $a(\Gamma)$ some numerical invariant of a Newton diagram $\Gamma \subset \mathbb{R}^n_{\geq 0}$.

Question

Given A is there an a such that

$$a(\Gamma(f)) = A(f) \quad \textit{if } f \textit{ is nondegenerate}$$

Similarly, given a, is there an A such that the above holds?

Examples

$A(f)$	$a(\Gamma)$	
Multiplicity	$\min_{i\in\Gamma(f)} \lvert i\rvert_1$	
Milnor nº μ	Newton nº $\nu(\Gamma_-(f))$	Kouchnirenko 1976
Monodromy zeta	[formula]	Varchenko 1976
Geometric genus p_g	$\lvert\Gamma_-(f)\cap\mathbb{Z}^n_{>0}\rvert$	Merle and Teissier 1977
Spectrum $\cap(-1,0]$	Newton filtration	Saito 1988
Oscillatory index	remoteness*	Varchenko 1976

*some conditions

The Łojasiewicz exponent

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ, defining and *isolated* singularity $(X, 0) \subset (\mathbb{C}^n, 0)$.

Theorem (Łojasiewicz)

There exists $\theta, c > 0$ such that for z small:

$$|\nabla f(z)| \geq c|z|^\theta.$$

Definition

Denote by $\mathcal{L}_0(f)$ the infimum of such θ: $\boxed{\text{the } \textit{Łojasiewicz exponent}}$.

Theorem (Teissier, Teissier / Lejeune-Jalambert)

This infimum is a minimum, realized by the maximal polar quotient. In particular, the $\mathcal{L}_0(f)$ is rational.

Polar invariants

- Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a *generic* linear function, defining a hyperplane $H = \ker g$.
- Define the *polar curve* as the germ of the analytic set

$$P = \{z \in \mathbb{C}^n \mid H \subset \ker df_z\} = P_1 \cup P_2 \cup \ldots \cup P_l$$

Theorem (Teissier)

Decompose the polar curve into irreducible components $P = P_1, \ldots, P_l$. Then

$$\mathcal{L}_0(f) = \max_q \frac{(X, P_q)_0}{(H, P_q)_0} - 1.$$

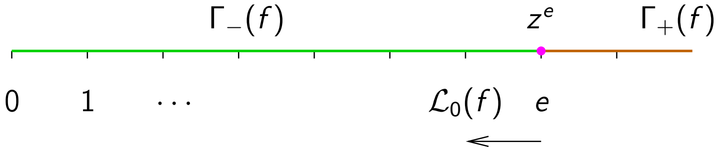
The case $n = 1$

If $n = 1$, then we can assume that $f(z) = z^e$ where e is the multiplicity. Thus,

$$|\nabla f| = |ez^{e-1}| \sim |z|^{e-1}, \qquad \mathcal{L}_0(f) = e - 1,$$

and so

$$\mathcal{L}_0(f) = \min_{i \in \Gamma(f)} |i|_1 - 1,$$



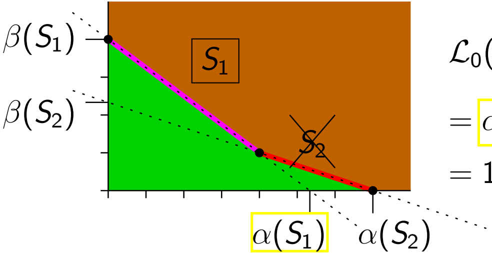
The case $n = 2$ (Lenarcik)

Theorem (Lenarcik 1998)

Let $f \in \mathbb{C}\{x, y\}$ be holomorphic and Newton nondegenerate. If $\Gamma(f)$ has $\boxed{\textit{nonexceptional}}$ *segments, then*

$$\mathcal{L}_0(f) = \max_S \max\{\alpha(S), \beta(S)\}$$

were S runs through $\boxed{\textit{nonexceptional segments}}$.



$$\mathcal{L}_0(y^4 + x^4 y + x^7)$$

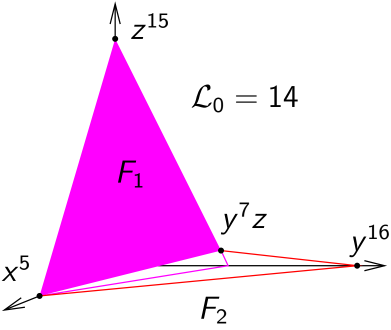
$$= \boxed{\alpha(S_1)} - 1$$

$$= 13/3$$

One Brzostowski-Krasiński-Oleksik definition ($n > 0$)

Definition

If $F \subset \Gamma(f)$ is a *facet* of the Newton diagram of f, denote by $x_i(F)$ the coordinate of the intersection point of the hyperplane containing F and the z_i-coordinate axis, and let $m(F)$ be the maximum.



$\mathcal{L}_0 = 14$

$$f(x, y, z) = x^5 + y^7z + z^{15}$$

$$x_1(F_1) = 5 \qquad x_1(F_2) = 5$$

$$x_2(F_1) = 15/2 \qquad x_2(F_2) = 16$$

$$x_3(F_1) = 15 \qquad x_3(F_2) = 16/9$$

$$\boxed{m(F_1) = \max\{5, 15/2, 15\} = 15}$$

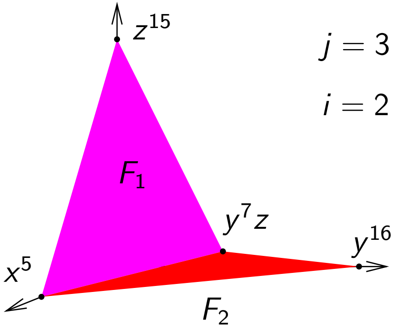
$$\cancel{m(F_2) = \max\{5, 16/9, 16\} = 16}$$

Another Brzostowski-Krasiński-Oleksik definition ($n > 0$)

Definition

A facet $F \subset \Gamma(f)$ is *exceptional* if there exist $i \neq j$ such that

- all vertices of F lie on the j-th coordinate hyperplane, except for one
- and that one has distance 1 to the z_i-axis.



$j = 3 \qquad (5,0,0),(0,16,0) \in \{z = 0\}$

$i = 2 \qquad |(0,7,1) - (0,7,0)| = 1$

$\boxed{F_2 \text{ is exceptional}}$

The Brzostowski-Krasiński-Oleksik conjecture

Conjecture (Brzostowski-Krasiński-Oleksik 2012)

Assume that $f \in \mathbb{C}\{z_1, \dots, z_n\}$ defines a Newton nondegenerate isolated singularity, and that $\Gamma(f)$ contains a nonexceptional facet. Then

$$\mathcal{L}_0(f) = \max_F m(F) - 1$$

where F runs through nonexceptional facets.

Theorem (Brzostowski-Krasiński-Oleksik 2020)

The conjecture is true for $n = 3$.

Theorem (S)

The conjecture is false for $n = 4$.

A counterexample

The function

$$f(x, y, z, w) = x^2 + y^2 + xz + yz + xw + 2yw + z^3 + w^3$$

is Newton nondegnerate, and *nondegenerate*, i.e. the Hessian matrix is invertible. Therefore,

$$\mathcal{L}_0(f) = 1$$

The Newton diagram has two facets, both nonexceptional

$$\begin{aligned} F_1 &: x^2, y^2, xz, yz, xw, yw, & m(F_1) &= 2 \\ F_2 &: xz, yz, xw, yw, z^3, w^3, & m(F_2) &= 3 \end{aligned}$$

A new conjecture

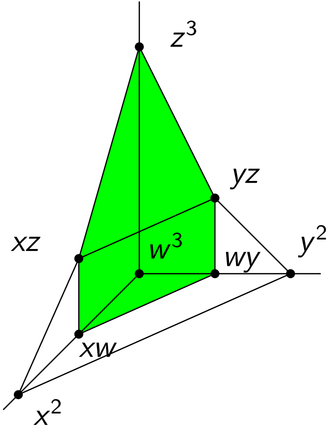
Definition

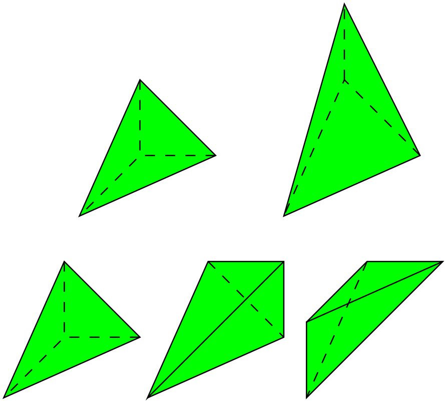
A facet F of a Newton diagram is *(exceptionally) decomposable* if it can be triangulated by exceptional simplices.

Conjecture (S)

The conjecture by Brzostowski-Krasiński-Oleksik holds if we replace exceptional with exceptionally decomposable.

F_2 is weakly decomposable





¡Muchas gracias!