Research Statement

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1 Background and starting points

1.1 Topological characterization of analytic invariants

Since the sixties, many results characterizing analytic invariants of normal surface singularities via their topology have been discovered. In [46], Mumford proved that the germ $(X, 0)$ of a normal surface is smooth if and only if the link M is simply connected. Artin proved that rational singularities can be characterized by their links [4, 5]. Laufer extended this program to minimally elliptic singularities [31]. Némethi continued this program for elliptic singularities [47]. Usually, in order to obtain topological characterizations of analytic properties of singularities one needs to impose certain (weak) topological and analytic conditions; e.g. X should be Gorenstein and/or the link M is a rational homology sphere. One may also work with specific classes of singularities like Newton nondegenerate [57] or superisolated hypersurfaces [39], or specific constructions such as splice quotient singularities [56].

1.2 Multi index filtrations and associated powerseries

The exceptional divisor of a resolution $\tilde{X}$ of a singularity X consists of finitely many irreducible components. This yields a set of valuations on the local ring of X, and therefore a multi index filtration. The associated multivariable Hilbert series H is a strong analytic invariant. The multivariable Poincaré seris P is the 'multivariable graded version' of the Hilbert series, and a a nontrivial theorem states that they are, in fact, equivalent [49]. The Poincaré series has the advantage that its support is contained in the anti-nef (Lipman) cone, hence it has similar properties as the Taylor expansion of a rational function (hence one might hope to find its 'topological candidate'). It also has subtle connections with the associated analytic semigroup.

A topological analogue of P is the 'zeta function' Z. It is a rational function given by a combinatorial formula from the plumbing graph. In [49, 51], Némethi proves that the Poincaré series coincides with the zeta function in the cases of rational, minimally elliptic singularities, and more generally for splice quotients (providing in this way topological description of P and H).

The Hilbert series is obtained from the Poincaré series using a function that adds up some of its coefficients. Analogously, one may construct from Z a series Q, whose coefficients are given by a similar counting function. This is therefore a topological candidate for the Hilbert series. This series Q was studied by László and Némethi [30] using Ehrhart theory of polytopes, showing that certain coefficients can be identified with the Seiberg-Witten invariants of the link.

Our goal is to generalize these identifications, or to describe the differences between P and Z, or H and Q.

In a preprint [65], submitted for publication, I compare three multi-indexed filtrations for Newton nondegenerate hypersurface singularities in $(\mathbb{C}^3, 0)$ having a rational homology sphere, and show that they coincide in a cone of full dimension inside the Lipman cone. Nonetheless, these filtration do not coincide in general. In this article, I also show that these filtration coincide if the singularity is a suspension, and that in this case, the zeta function and the Poincaré series coincide. One of the filtration is the one discussed above, and the two other are related with the embedding of the hypersurface, and the Newton polyhedron. These filtrations and their associated Poincaré series have been studied in [37, 19, 20]

1.3 Motivic zeta functions and Milnor fibers

Using arc spaces and motivic integration one defines different zeta functions associated with local singularities (at different level of complexity, 'motivic', 'topological', etc.), see the long list of articles of Denef and Loeser and their mathematical school [15, 16, 17, 18]. Several results and conjectures indicate a strong relationship with the topology of the singularity and other analytic invariants: e.g., the motivic Milnor fiber lifts all the classical invariants of the Milnor fibration package, or the (positive) solution of the Nash problem relates the irreducible components of the arc spaces with the irreducible exceptional curves [21, 25], similar relation is realized for holomorphic small arcs in terms of the link [29], or the *monodromy conjecture* relates the poles of the topological zeta function with monodromy eigenvalues.

1.4 The Seiberg-Witten invariant conjecture

The Casson invariant conjecture of Neumann and Wahl predicts that the signature of the Milnor fiber of a complete intersection singularity with integral homology sphere link is eight times the Casson invariant of the link [55]. This identity is generalized to singularities with rational homology sphere links and without the complete intersection assumption as follows. On one hand, the Seiberg-Witten invariants are numbers associated with three manifolds and their $spin^c$ structures via gauge theory. It reduces to the Casson invariant in the integral homology sphere case. On the other hand, in the Gorenstein case, the signature of a Milnor fiber and the geometric genus can be related [68].

Némethi and Nicolaescu used these identifications to generalize the Casson invariant conjecture to a larger class of singularities [52]. This is *the Seiberg-Witten invariant conjecture*, which relates the Seiberg-Witten invariant of the link equipped with the canonical $spin^c$ structure and the geometric genus. More generally, one can compare the Seiberg-Witten invariant of any $spin^c$ structure with the rank of the first cohomology of a suitable line bundle, see [51].

Restricted to the anti-nef cone, shifted by the anticanonical cycle, the coefficients of the Hilbert function are given by a polynomial function. The constant term of this function calculates the geometric genus, see e.g. [50]. Similarly, the counting function Q provides the Seiberg-Witten invariants. Therefore, showing that the Poincaré series coincides with the zeta function proves the Seiberg-Witten invariant conjecture.

1.5 Lattice cohomology and path lattice cohomology

Another topological invariant is the lattice cohomology [48]. These groups conjecturally equal the Heegaard-Floer homology groups of the link [58]. The Seiberg-Witten invariants can be read off the lattice cohomology as a normalized Euler characteristic. A different, but related construction is the minimal path lattice cohomology, also described in [48]. This is related to computation sequences, as used by Laufer [31], with certain minimality conditions. In this way the lattice cohomology theory creates a bridge between topological and analytical machineries. Using such computation sequences, one obtains a topological upper bound on the geometric genus, or, more generally, ranks of first cohomologies of certain line bundles. In fact, the smallest of these bounds is the rank of the minimal path lattice cohomology. One therefore has two connections between the geometric genus and lattice cohomology constructions. The Seiberg-Witten invariant conjecture relates it with its Euler characteristic, while certain analytic structures make the choice for their geometric genus the rank of the minimal path lattice cohomology. These invariants do not always coincide (though under certain conditions they do).

1.6 The boundary of Milnor fibers

In the case of nonisolated singularities, the link is no longer a smooth manifold. For hypersurface singularities, however, one may consider instead the boundary of the Milnor fiber, which in the case of isolated singularities coincides with the link. That this boundary is a graph manifold, i.e. having only Seifert pieces in its JSJ decomposition (equivalently, being given by a plumbing graph) was stated in [41], whose proof contains a gap, but was proved in [54, 42] by separate methods. Further computations are provided in [44, 43]. A more general result is obtained in [24], where a sufficient and necessary condition is given for a function $f\overline{g} : (X, 0) \to (\mathbb{C}, 0)$ to have a Milnor fibration, where $(X, 0)$ is the germ of a three dimensional analytic space having (at most) an isolated singularity, and with $f, g \in \mathcal{O}_{X,0}$. Furthermore, it is proved that in this case, the boundary of the Milnor fiber is a graph manifold. Most recently, Curmi [14] showed that the boundary of the Milnor fiber of any smoothing is a graph manifold.

An interesting aspect of this theory is that the class of three-manifolds considered is widened considerably, as the plumbing graphs of these boundaries are typically not negative definite, as is the case for links of isolated singularities. Explicit computations for Hirzebruch singularities are found in [43] and for suspension singularities in [44]. The book [54] also contains many examples. In [63] we provide an algorithm which produces a plumbing graph representing the boundary of the Milnor fiber of the singularity $f(x, y) + zg(x, y) = 0$, in terms of a resolution of the plane curve singularity $fg = 0$. Using this method, we construct a boundary of a Milnor fiber whose normal plumbing graph consists of several copies of $S^1 \times S^2$, and two larger graphs.

1.7 Automorphisms of surfaces

The topology of plane curves and isolated surface singularities can be studied from the point of view of automorphisms of (real) surfaces. This theory is not only used to obtain rich invariants such as the monodromy, but also to prove theorems in their own right.

The Milnor fibration associated with a germ of an isolated plane curve singularity, defined by $f = 0$, induces the geometric monodromy, an automorphism on the Milnor fiber which fixes the boundary. By the monodromy theorem, its action on homology—the algebraic monodromy—has as eigenvalues roots of unity, and Jordan blocks of size not exceeding 2. Lê [34] showed that for irreducible f, the algebraic monodromy is in fact semisimple, and so it is determined by its eigenvalues. A'Campo [1], however, proved that in this case, the geometric monodromy is not of finite order unless the singularity has only one Puiseux pair.

In general, one may associate a Milnor fibration with any holomorphic function f on an isolated surface singularity $(X, 0)$, see e.g. [59, 12]. If f defines an isolated singularity, then this fibration provides an open book decomposition of the link of $(X, 0)$. Furthermore, the conntact structure determined by the analytic structure on X is supported by this open book decomposition. These facts were used by Caubel, Popescu-Pampu and Némethi [12] to show that this contact structure is in fact well defined by the link alone. Pichon [59] showed that monodromies constructed from holomorphic germs of functions on isolated surface singularities are distinguished by a condition on a topological invariant introduced by Nielsen, the *Nielsen graph*, see e.g. [59] and references therein. For this, one must note that these monodromies are indeed pseudo-periodic.

A'Campo introduced tête-à-tête graphs to describe the monodromy automorphism in an unpublished manuscript. Their theory has been developed further by Graf [28], de Bobadilla, Pe and Portilla [26] and Portilla and Sigurðsson [13].

1.8 Lagrangian spines and symplectic monodromy

As a continuation of the development of tête-à-tête graphs, A'Campo imagined a higher dimensional variant of the concept in order to describe the monodromy associated with hypersurface singularities in higher dimension [2]. This means finding in the Milnor fiber a stratified compact subspace which is a homotopy retract and whose strata are isotropic submanifolds, as well as developing a theory of constructing the monodromy from this information, and the symplectic form on the Milnor fiber.

It is already known that the Milnor fiber contains a polyhedral space onto which it retracts [36]. In the case of Brieskorn–Pham singularities, the above ideas fit with the classical construction of the join inside the Milnor fiber, described in [45, §9].

1.9 Deformations of surface singularities

Deformation theory provides a rich setting in algebraic and analytic geometry. In the case of surface singularities, various notions of equisingularity play a role, and are related to appropriate notions of simultaneous resolution (see e.g. [66]). Among several results lending weight to the study of numerical invariants are the following two. Wahl [69] shows that a deformation $\tilde{X}_t$ of a resolution $\tilde{X} = \tilde{X}_0$ of a singularity X blows down if and only if the geometric genus $h^1(\mathcal{O}_{\tilde{X}_t})$ is constant. On the other hand, Laufer [32] proves that, up to base change, deformations of Gorenstein singularities admit very weak simultaneous resolution if and only if the numerical invariant K^2 is constant, where K denotes the canonical cycle on the minimal resolution, supported on the exceptional divisor.

2 Current and future work

2.1 Newton nondegenerate hypersurfaces and the SWIC

During my PhD program in CEU Budapest, under the supervision of András Némethi, I have proved that for an isolated hypersurface singularity in $(\mathbb{C}^3, 0)$ given by a function which is nondegenerate with respect to its Newton boundary, the geometric genus coincides with the rank of the minimal path cohomology, provided that the link is a rational homology sphere. A similar formula is proved for the Seiberg–Witten invariant, proving the Seiberg-Witten invariant conjecture in this case. This shows that for these singularities, the geometric genus has two topological characterizations, the minimal path lattice cohomology rank and the Seiberg-Witten invariant. These results are described in my thesis [64].

I am currently finishing an article, jointly with Némethi, where we generalize the condition of Newton nondegeneracy to Weil divisors in toric varieties. In this manuscript, we prove that for this class, the geometric genus is computed similarly from a computation sequence, in particular, it depends only on the topology of the singularity. In order to prove this, we calculate the geometric genus and give an identification of the Gorenstein property in terms of the Newon polyhedron, which may is a result which may be of use on its own.

In order to answer the Seiberg–Witten invariant conjecture for this class, I believe that it is possible to compute the Seiberg–Witten invariant for Weil divisors of this type which are Gorenstein from the Newton diagram. Although this result has not been proved yet, I have already found some results in the direction of a proof using lattice cohomology and induction on the number of facets of the Newton diagram.

2.2 Mixed tête-à-tête graphs and surface automorphisms

In [26], the authors introduce *mixed* tête-à-tête graphs and use them to codify the monodromy of any irreducible plane curve. In a joint manuscript with Portilla [13], we prove that mixed tête-à-tête graphs codify precisely the Milnor fibrations associated with isolated holomorphic function germs on isolated surface singularities. In particular, this class includes all isolated plane curves. This class can further be extended to germs having at least one component of multiplicity one, by allowing *relative* mixed tête-à-tête graphs.

In further collaboration with Pablo Portilla, we hope to describe a similar picture in higher dimension. It is already known in any dimension that the Milnor fiber has a *Lê polyhedron*, that is, a finite CW coplex, of real dimension matching that of the complex dimension of the fiber [35, 36]. A'Campo has already suggested a very explicit realization of such a complex using symplectic methods [2]. In this language, we speak of a *Lagrangian spine*, seeing that the cells are smoothly embedded isotropic submanifolds, in particular, the top dimensional strata are Lagrangian. Using methods from the tête-à-tête construction, the more combinatorial data of this complex should realize not just the homotopy type of the Milnor fiber, but a symplectic neighbourhood, and therefore the entire Milnor fiber as a symplectic manifold, together with the monodromy. This construction should therefore be a powerful tool in the study of the Milnor fibration.

2.3 The mapping class group and singularities

Pichon gave a topological classification of automorphisms of oriented real surfaces with boundary, appearing as the monodromy associated with the Milnor fibration of a reduced function germ $f : (X, 0) \to (\mathbb{C}, 0)$ on an isolated surface singularity [59]. These turn out to be those $\phi : \Sigma \to \Sigma$ for which there exists an $n \in \mathbb{Z}_{\geq 0}$ so that ϕ^n is the product of right handed Dehn twists along disjoint embedded curves, including each boundary component. In particular, such a monodromy ϕ has a power which factorizes as the product of right handed Dehn twist.

It is still an open question whether such a ϕ itself already factors as the product of right handed Dehn twists along not necessarily disjoint embedded curves. This question has been worked on, and was suggested to me as a possible joint effort, by Pablo Portilla.

If ϕ appears as the monodromy of a reduced function germ on a *hypersurface*, then this question has a surprisingly simple answer. Indeed, by moving the Milnor fibration from the link of the singularity to the boundary of the Milnor fiber, Picard-Lefschetz theory identifies the monodromy as the product of monodromies associated with singularities of type A_1, and these are right handed Dehn twists. In fact, this method works for any smoothing of $(X, 0)$. Now, there exist topological obstructions to smoothing certain $(X, 0)$, see e.g. [38]. On the other hand, there exist smoothings of non-normal isolated surface singularities, whose normalization admits no smoothing, see e.g. [60]. This means that one of two roads should be open: either one can force the Picard-Lefschetz theory to work on smoothings of nonnormal singularities, providing a positive answer to the question of factorization, or there is an analytic obstruction related to the nonsmoothability of $(X, 0)$ which can be translated to a construction of an automorphism ϕ of the above type, which does not admit a factorization into right handed Dehn twists. In either case, this would provide an answer to an entirely topological question, whose proof would depend completely on analytic function theory.

2.4 Deformations of surface singularities

A collaborative effort with Bobadilla, Lázsló and Némethi has yielded partial results towards describing those normal surface singularities having no topologically interesting deformations not altering the numerical invariant K^2. For Gorenstein singularities, the condition for K^2 being constant is equivalent to the deformation to having a weak simultanious resolution [32]. A satisfactory answer to this question is expected to provide new steps towards proving relations between topological invariants of singularities, by means of reduction to this special case.

3 Future research topics

3.1 Superisolated singularities

Recent results on superisolated singularities [39] provide interesting counterexamples to the Seiberg-Witten invariant conjecture. Nevertheless, we proved in [53] that the geometric genus equals the 'other candidate', the rank of the minimal path lattice cohomology (as the Newton nondegenerate germs). This identification provides strong combinatorial restrictions for the form of the equation of the singularity. The point is that these equations are closely related to the equations of projective plane curves, hence to the hard open problem of existence and classification of such curves with prescribed topological data. More precisely, the question is whether there exists a curve embedded in $\mathbb{CP}^2$ with a fixed degree and singularities of fixed local topological types. This problem and the above strategy is addressed in [23, 22] (among other questions and conjectures). A partial answer to one of these conjectures is given in [10] using deep properties of Heegaard-Floer homology. The hope is that the strong rigidity properties provided by the lattice cohomology approach will provide classification results for certain classes. This direction has been taken in [8, 7, 11, 9].

3.2 The Oka-Teissier-Varchenko algorithm

The results on Newton nondegenerate singularities depend heavily on the algorithm of Oka-Teissier-Varchenko on toric resolutions [57, 67]. Suitable generalisations of this construction might provide insight into more general singularities (even for general hypersurface equations). The methods used for superisolated singularities should apply for certain degeneracies along the Newton boundary faces, thus providing a larger class of hypersurface singularities for which one can develop a combinatorial description of the link (as a combination of toroidal and plane curve resolution pieces). For them, one may ask to which extent the construction and results of [53] (as well as other general results on Newton nondegenerate singularities) can be generalised.

3.3 The analytic semigroup

The semigroup consists of the values of the valuations associated with the components of the exceptional divisor of a resolution. The support of the Poincaré series is contained in this semigroup, but the converse does not hold in general. Recently, however, Gorsky and Némethi [27] proved such a result with the valuations replaced by those of branches of a plane curve and the Poincaré series replaced by a generalisation, namely, the motivic Poincaré series. Adopting these ideas to the case of surface singularities should provide a similar strong link between a suitable generalisation of the Poincaré series and the analytic semigroup.

A crucial question targets the relationship of the analytic semigroup with the fixed topological type, and the characterization of the analytic type (up to equisingularity) by the semigroup.

3.4 Classification of hypersurface singularities

A very difficult question posed by Laufer [33] asks how the topological types of hypersurface singularities can be identified. The expectation is that results on the analytic semigroup, or the Poincaré series, compared with the topological semigroup and the zeta function will produce key steps in this direction. There are already some (unpublished) partial supporting results valid for certain families of singularities.

3.5 The spectrum

The computation sequence for Newton nondegenerate singularities in [53] provides more data than just the geometric genus. Using a modified version of this computation sequence, one can in fact determine a part of the spectrum. More precisely, let $\{\alpha_i\}_i$ denote the spectrum, normalised so that $0 < \alpha_i < 3$. Then the geometric genus is the number of α_i with $\alpha_i \leq 1$. These numbers, or exponents, are provided in [62] in terms of the Newton diagram, but it is not known in general how to obtain them directly from the link. This is, however, possible using a modification of the construction in [53], described in [64], which requires only little information directly from the Newton diagram. This coincidence should be studied further in order to obtain a topological characterisation of the spectrum or related (Hodge or motivic type) analytic invariants.

3.6 Mixed tête-à-tête graphs and known invariants

As the theory of mixed tête-à-tête graphs is relatively new, it provides an opportunity to review older theory to see if it can be improved. On one hand, the mixed tête-à-tête graph is a strong deformation retract of the surface. Therefore, homotopy invariants can be studied directly in terms of this graph. On the other hand, a calculus should be developed in order to determine when two mixed tête-à-tête graphs induce the same surface automorphism. In light of the Giroux correspondence, stabilization should be included in this calculus.

3.7 Motivic zeta functions and Milnor fibers

We believe that a conceptual connection should exists between the series P and Z described and analyzed above and the motivic zeta functions and motivic Milnor fibers. For special families of singularities, the above discussion (sometimes via nontrivial theorems) supports the fact that the Poincaré series P and the combinatorial Z provide (part of the) the Hodge spectrum, the Milnor number or even the monodromy eigenvalues. Since the motivic invariants are high generalizations of these, a deep relationship should exist.

3.8 Local tropicalization and numerical invariants of singularites

The language of tropicalization of varieties has been adopted to the local study of singularities by Popescu-Pampu and Stepanov [61]. This theory should be a powerful tool to keep track of the combinatorial data associated with resolutions of singularities, especially in the Newton nondegenerate case. In particular, we expect to see many topological invariants of hypersurface singularities expressed in this language.

Stable intersection of tropical varieties forms an important part of their study, and allows for the expression of numerical invariants by simple formulas [40, 3]. The counterpart of this theory in the local case does not exist, and it is not clear what kind of complications may arise in its development. Nonetheless, this is a road worth pursuing, in order to find new and simpler formulas for invariants of singularities. As an example, if $(X, 0) \subset (\mathbb{C}^{n+1})$ is a hypersurface singularity, then the Milnor number can be seen as the dimension of the Milnor algebra, the quotient of the local ring at $0 \in \mathbb{C}^{n+1}$ by the partial derivatives of f. This dimension is therefore an intersection number obtained by intersecting the varieties defined by the partial derivatives of f. Another example is the multiplicity of any n-dimensional singularity $(X, 0)$, seen as the dimension of the local ring of $(X, 0)$, divided by n generic linear functions. In the tropical world, this should be the n-fold intersection of the tropicalization of a generic linear function.

3.9 Lipschitz classification of surface singularities

Singularities have been studied from the point of view of Lipschitz equivalence. The embedding of any germ $(X, 0)$ to a smooth space $(\mathbb{C}^N, 0)$ induces two metrics on the germ, the *outer metric*, induced by restriction, and the *inner metric*, obtained by taking infimum of lengths of paths contained in $(X, 0)$ between two points. The case of normal complex surface singularities, in particular, has been studied in [6] (see also references therein) where the authors obtain a complete classification of Lipschitz equivalence classes arising from the inner metric of such singularities. This classification is closely related to the JSJ-decomposition of the link of $(X, 0)$, and in a certain sense, how fast the JSJ-pieces shrink, when approaching the origin. An interesting continuation of this subject would be to study nonisolated hypersurfaces, or, more generally, smoothings of singularities, and see what similar behaviour arises for the JSJ-pieces of the boundary of the Milnor fiber. The boundary of the Milnor fiber is a graph manifold, whose pieces belong either to the *trunk*, or the *vanishing zone*. We expect that the pieces in the trunk, which can be seen as pieces of the link of the normalization of $(X, 0)$ behave similarly as in the normal case. The vanishing zone, however, is the part of the boundary close to the singular locus of $(X, 0)$, and behaves differently than a link of a singularity. Although the definition of the Lipschitz behaviour of those pieces has to be defined concretely before making statements, we do expect to see different behaviour for those pieces than in the normal case.

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