LaTeX Rewrites #1

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Due Date: February 25, 2025

Artin Section 4: 4.1

♢ Let a and b be elements of a group G. Assume that a has order 7 and that $a^3b = ba^3$. Prove that $ab = ba$.

Proof. We shall prove directly that the group G with a multiplicative law of composition is abelian with the provided relation $a^3b = ba^3$ and that a has order 7.

To begin, consider the relation $b = b$. Re We substitute the right side identity with our order 7 element to show that:

$$\begin{aligned} b &= b \\ b1 &= 1b \\ ba^7 &= 1b \\ (ba^3)a^4 &= 1b \\ (a^3b)a^3a &= 1b \\ a^3(ba^3)a &= b \\ a^3a^3ba &= 1b \\ a^6ba &= 1b \\ a^6ba &= 1b \end{aligned}$$

Given that we can multiply both left sides of the equation by a, we yield

$$\begin{aligned} a^6ba &= 1b \\ a(a^6)ba &= ab \\ a^7ba &= ab \\ 1ba &= ab \\ ba &= ab \end{aligned}$$

Thus, we have successfully proven that the group G is indeed abelian. □

Artin Additional Problem 1:

$\Diamond$ Prove that a nonempty subset H of a group G is a subgroup if for all $x, y \in H$, the element xy^-1 is also in H.

Proof. We shall prove that subset H is indeed a subgroup of G—that is we will exemplify how the provided elements of H can be used to prove that the identity element 1, inverses, and needed closure are true in H under the induced multiplicative law of composition of group G.

It is assumed that elements x, y, xy^{-1} are elements that live in subset H, thus $H \neq \emptyset$.

Consider drawing upon an element a that lies in our group G. Let $a = x \in H$ and $a = y \in H$. It can be shown that

$$\begin{aligned} xy^-1 &\in H \\ (a)(a)^{-1} &\in H \\ aa^{-1} &\in H \\ 1 &\in H \end{aligned}$$

Thus, the identity element 1 lives in subset H.

Now, let us consider having $x = 1 \in H$. Then the following holds

$$\begin{aligned} xy^{-1} &\in H \\ 1y^{-1} &\in H \\ y^{-1} &\in H \end{aligned}$$

And given it is assumed that $y \in H$, $y, y^{-1} \in H$.

In a similar fashion, let $x = a^{-1}$ and $y = 1$, then

$$\begin{aligned} xy^{-1} &\in H \\ a^{-1}(1)^{-1} &\in H \\ a^{-1} &\in H \end{aligned}$$

Thus proving that for any element $x, y \in H$, there exist their inverses as well.

Finally, we know that $y^{-1} \in H$, so it is true that

$$\begin{aligned} x(y^{-1})^{-1} &\in H \\ xy &\in H \end{aligned}$$

Thus, H is closed and the subset H is indeed a subgroup of group G. $\square$