Swarthmore Summer Puzzle Collective —June 27th, 2025

This week's theme: Miscellaneous!

2. Show that for positive integers a, b, c,

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

3. If a, b, c denote the lengths of the sides of a triangle, show that

$$3(ab + bc + ca) \leq (a + b + c)^2 \leq 4(ab + bc + ca).$$

4. If $0 \leq a, b, c \leq 1$, show that

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

5. Find $\min_{a,b\in\mathbb{R}} \max(a^2 + b,\, b^2 + a)$.

6. Find all the differentiable functions f defined for $x \geq 0$ which satisfy

$$f(xy) = f(x) + f(y), \quad x, y, \geq 0.$$

(Hint: Find out what $f(1)$ equals and what $f(x/y)$ equals. Then recall that $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and try to recover f.)

7. A real value function f, defined on the rational numbers, satisfies

$$f(x + y) = f(x) + f(y)$$

for all rational x and y. Prove that $f(x) = f(1) \cdot x$ for all rational x.

8. Given that a, b, c are odd integers, prove that the quadratic $r(x) = ax^2 + bx + c = 0$ cannot have a rational root.

9. Let p be a prime number, and n a positive integer. Prove that $n^p - n$ is divisible by p.

10. Let S be a set and $*$ be binary operation on S satisfying the two laws

$$x * x = x \quad \text{for all } x \text{ in } S.$$
$$(x * y) * z = (y * z) * x \quad \text{for all } x, y, z \text{ in } S.$$

Show that $x * y = y * x$ for all x, y in S.