PSET5

Student: Ryan Baldwin, `rbaldwi2@swarthmore.edu`
Professor: Dr. Hsu
Collaborators:
Peer Reviewers:
Due Date: March 7, 2025

Artin Chapter 2: 10.2

◊ Let H and K be subgroups of a group G.

- **(a)** Prove that the intersection $xH \cap yK$ of two cosets of H and K is either empty or else is a coset of the subgroup $H \cap K$.
- **(b)** Prove that if H and K have finite index in G then $H \cap K$ also has finite index in G.

Proof. (a)

Case 1: Consider the case where the cosets have no overlapping elements. That is,

$$xH \cap yK = \emptyset.$$

Case 2: Consider the case where the intersection between cosets is nonempty. That is, let $a \in xH \cap yH$ such that $a = xh$ for some element $h \in H$ and $a = yk$ for some $k \in K$. What must follow is that:

$$aH = (xh)H = xH.$$

Similarly,

$$aK = (yk)K = yK.$$

The key question now becomes whether or not

$$aH \cap aK \quad \text{is a coset of} \quad H \cap K. \tag{1}$$

We will show that

$$a(H \cap K) \subseteq aH \cap aK.$$

Let the element of the intersection of $aH \cap aK$ be $w = ah = ak$. Then w is always in the form of some left aH and aK, hence the subset holds.

We now want to show that

$$aH \cap aK \subseteq a(H \cap K).$$

Let $z \in aH \cap aK$. Per the definition of an intersection, $z = ah = ak$ for some $h \in H$ and for some $k \in K$. In other words,

$$z = ah = ak$$

Left multiplying by a^{-1} gives,

$$h = k.$$

Since $h = k$, it follows that $h \in H \cap K$. Thus, $z = ah$ when $h \in H \cap K$ (i.e. $z \in a(H \cap K)$).

Hence, we have shown that

$$aH \cap aK = a(H \cap K). \tag{2}$$

Applying this idea to equation (1) gives

$$aH \cap aK = a(H \cap K), \quad \text{a coset of} \quad H \cap K.$$

Ergo, $xH \cap yH$ is either empty or a coset of the subgroup $H \cap K$. □

Proof. (b)
Let the index of H be $[G : H] = n$ and let the index of K be $[G : K] = m$.

Recall the resulting equation from part (a), where we previously showed that the coset of the intersection of two subsets, when operated on by $a \in G$ is equal to the intersection of the two subsets then operated on by a. This equation defines a relation between the cosets of of $H \cap K$ and H and K.

Considering the coset $H \cap K$ and equation (2), it must be true then that the number of cosets of $H \cap K$ in G is bounded by the product of the number of cosets of H and K, meaning

$$[G : H \cap K] \leq [G : H][G : K] = n \cdot m.$$

Thus, $H \cap K$ has finite index in G. □

Artin Chapter 2: 11.6

$\Diamond$ Let G be a group that contains normal subgroups of order 3 and 5, respectively. Prove that G contains an element of order 15.

Proof. Let H be of order 3 and K be of order 5 respectively.

Given that any group with prime order is cyclic, H and K are both cyclic subgroups.

Since H is normal in G, the product set HK is a subgroup of the group G.

We will show that $HK \cong H \times K$, meaning that G contains a subgroup $C_3 \times C_5 \cong C_{15}$.

(i) Observe that $H \cap K = \{1\}$ as both subgroups are cyclic of coprime order.

(ii) $HK = G$ as sets when we limit the size of the codomain for some subgroup in G.

(iii) We are provided that $H \trianglelefteq G$ and $K \trianglelefteq G$.

From **Artin Chapter 2: 11.1**, given two groups of order r and s, the resulting direct product of the two groups will have order $lcm(r, s)$.

Indeed, we have shown that the product group $H \times K \cong G$, we can be certain G contains a subgroup with an element of order 15 given $|H \times K| = lcm(3, 5) = 15$.

Thus, given that $H \times K$ is isomorphic to G, the group G must contain a subgroup that contains an element of order 15. $\square$

Additional Problem 5

$\Diamond$ Find all subgroups of $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ that are isomorphic to the Klein 4-group.

Proof. First, we notice that this direct product group has 16 elements total: 1 element of order 1, 8 elements of order 4 and 7 elements of order 2.

Since V_4 is generated by 2 elements of order 2, we consider the set of all elements of order 2 in this direct product group:

$$\{(1,0,0),(0,1,0),(0,0,2),(1,1,0),(0,1,2),(1,0,2),(1,1,2)\}.$$

Given that each subgroup of this direct product group is abelian, we know that all of these elements could be valid generators. Hence, the number of ways to choose 2 elements from 7 are:

$$\binom{7}{2} = \frac{7!}{2!(7-2)!} = \frac{7\times 6}{2} = 21 \quad \text{ways to pick generators for} \quad V_4.$$

However, if we look closely, we will notice that there are in fact 3 different ways to define the map from the direct product group to V_4. In other words, there are 7 possible images of $\varphi(V_4)$ with 3 different ways to represent this injection.

So in fact, there are

$$\binom{7}{2}\frac{1}{3} = 7 \quad \text{ways to pick generators for} \quad V_4.$$

Thus, the subgroups that are isomorphic to the Klein 4-group are:

$$\{(1,0,0),(0,1,0),(1,1,0),(0,0,0)\},$$

$$\{(1,0,0),(0,0,2),(1,0,2),(0,0,0)\},$$

$$\{(1,0,0),(0,1,2),(1,1,2),(0,0,0)\},$$

$$\{(0,1,0),(0,0,2),(0,1,2),(0,0,0)\},$$

$$\{(0,1,0),(1,0,2),(1,1,2),(0,0,0)\},$$

$$\{(0,0,2),(1,1,0),(1,1,2),(0,0,0)\},$$

$$\{(1,1,0),(0,1,2),(1,0,2),(0,0,0)\}.$$

$\square$