PSET8

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Artin Chapter 11, 3.7: Determine the automorphisms of the polynomial ring $\mathbb{Z}[x]$

Let $\varphi : \mathbb{Z}[x] \to \mathbb{Z}[x]$ be a ring automorphism. This is true if φ is determined by the image of x, say $\varphi(x) = f(x)$. Since φ must be invertible and $\mathbb{Z}[x]$ is ring generated by x, $f(x)$ must also be a generator of the polynomial ring, that is the images $\mathbb{Z}[x]$ and $\mathbb{Z}[f(x)]$ one in the same.

Since $\mathbb{Z}[x]$ generates all polynomials of any degree, $f(x)$ must be a polynomial of degree 1 in order to generate all degree polynomials. That is,

$$\varphi(x) = ax + b, \quad \text{where } a, b \in \mathbb{Z}.$$

In order to confirm whether this map is bijective, we must find ψ such that $\psi(\varphi(x)) = x$ and that $\varphi(\psi(x)) = x$.

Let $\psi(x) = a^{-1}(x - b)$. Since the only unit in $\mathbb{Z}[x]$ is 1, a must be ± 1. And thus inverse map must have the form $\pm x + c$ for some $c \in \mathbb{Z}$.

Therefore, the automorphisms of $\mathbb{Z}[x]$ are given by:

$$x \mapsto \pm 1x + b, \quad b \in \mathbb{Z}.$$

Artin Chapter 11, 3.9:(a) Show that if x is nilpotent, then $1 + x$ is a unit

Assume $x^r = 0$ for some positive integer r. Consider the element:

$$1 - x^r = (1 + x)(1 - x + x^2 - \cdots + (-1)^{r-1}x^{r-1}).$$

Since $x^r = 0$, we have:

$$(1 + x)(1 - x + x^2 - \cdots + (-1)^{r-1}x^{r-1}) = 1.$$

Thus, $1 + x$ is a unit with inverse given by $(1 - x + x^2 - \cdots + (-1)^{r-1}x^{r-1})$.

Artin Chapter 11, 3.9:(b) If R has prime characteristic p and a is nilpotent, then $1+a$ is unipotent.

Assume $a^r = 0$. Since R has characteristic p, for any k:

$$(1+a)^p = \sum_{i=0}^{p} \binom{p}{i} a^i.$$

But all binomial coefficients $\binom{p}{i}$ for $0 < i < p$ are divisible by p, so they vanish in characteristic p. Thus:

$$(1+a)^p = 1 + a^p.$$

By induction:

$$(1+a)^{p^n} = 1 + a^{p^n}.$$

Since a is nilpotent, $a^{p^n} = 0$ for some large n, so

$$(1+a)^{p^n} = 1^p + a^{p^n} = 1 + 0 = 1.$$

Therefore, $1+a$ is unipotent.

Artin Chapter 11, 4.3(a): Identify the ring $\mathbb{Z}[x]/(x^2-3, 2x+4)$.

We are given the ideal $(x^2-3, 2x+4)$ in $\mathbb{Z}[x]$.

First, from $2x+4=0$, we have

$$2(2x+4) - x(2x+4) = 8 - 2x^2 = 8 - 6 = 2 = 0$$

So $2 = 0$, which implies that we can reduce modulo 2. Hence, the original ring is isomorphic to the following:

$$\mathbb{Z}[x]/(x^2-3, 2x+4) \cong \mathbb{F}_2[x]/(x^2-3).$$

And since $-3 \equiv 1$ in $\mathbb{F}_2$, it is true that $x^2 - 3 \equiv x^2 + 1$. Therefore,

$$\mathbb{Z}[x]/(x^2-3, 2x+4) \cong \mathbb{F}_2[x]/(x^2+1).$$

Artin Chapter 11, 4.3(c): Identify the ring $\mathbb{Z}[x]/(6, 2x-1)$

We are quotienting $\mathbb{Z}[x]$ by the ideal $(6, 2x-1)$. First, we can immediately reduce modulo 6 to see that

$$\mathbb{Z}[x]/(6, 2x-1) \cong \mathbb{Z}/6\mathbb{Z}[x]/(2x-1).$$

And since $3(2x = 1)$ is in the ideal, $0 = 3$ is in the ideal. Thus, we can further reduce down to modulo 3.

$$\mathbb{Z}[x]/(6, 2x-1) \cong \mathbb{Z}/3\mathbb{Z}[x]/(2x-1).$$

In $\mathbb{Z}/3\mathbb{Z}$, 2 is invertible (since $2 \cdot 2 = 1$), so:

$$\mathbb{Z}[x]/(6, 2x-1) \cong \mathbb{Z}/3\mathbb{Z}[x]/(2x-1).$$

Indeed, we also know that $x = 2$ in $\mathbb{F}_3$, so this is can be recognized as the field

$$\mathbb{Z}[x]/(6, 2x-1) \cong \mathbb{F}_3.$$