Assessment No. 3

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Question 1:

(a) We note that the set contains the additive identity element $(0, 0)$, as $0 \equiv 0 \mod 3$.

Let $a \equiv b \mod 3$ and $c \equiv d \mod 3$. Then we see that $(a, b) + (c, d) = (a + c, b + d)$ and claim that $a + c \equiv b + d \mod 3$. Because $a \equiv b \mod 3$, $a = b + 3n$ for some integer n. Similarly, $c \equiv d \mod 3$ implies that $c = d + 3m$ for some integer m. Adding these two equations gives us $a + c = b + d + 3(n + m)$. Thus, $a + c \equiv b + d \mod 3$. Therefore, the set is closed under addition.

Let $a \equiv b \mod 3$ so that (a, b) is an element in S. Since modularity is preserved under multiplication, $-a \equiv -b \mod 3$ and $(-a, -b)$ is also in S. Adding these two elements yields $(a, b) + (-a, -b) = (0, 0)$, and thus the set contains additvive inverses.

Thus, the set S is a subgroup of $\mathbb{Z} \times \mathbb{Z}$ under addition. And since the elements of S are integers, it is an abelian group.

Since $1 \equiv 1 \mod 3$, the multiplicative identity $(1, 1)$ is in S.

Let $a \equiv b \mod 3$ and $c \equiv d \mod 3$ be two elements in the set. Then we see that $(a, b)(c, d) = (ac, bd)$ and we claim that $ac \equiv bd \mod 3$. Multiplying $a \equiv b \mod 3$ by c yields $ac \equiv bc \mod 3$. Multiplying $c \equiv d \mod 3$ by b yields $bc \equiv bd \mod 3$. Thus, by transitivity, $ac \equiv bd \mod 3$. Therefore, the set is closed under multiplication.

Since the components of the elements in S are integers, the set is inherently abelian under multiplication.

Therefore, the set S is a subring of $\mathbb{Z} \times \mathbb{Z}$.

However, the set is not an ideal. For example, the element $(1, 1)$ in S multiplied by $(1, 0)$ in the ring yields $(1, 0)$, which is not in S. Thus, the set cannot be an ideal.

Question 1:

(b) Since $0 \equiv 0 \equiv 0 \mod 3$, we see that $(0,0)$ is in S, and thus the set contains the additive identity.

Let $a \equiv b \equiv 0 \mod 3$ and $c \equiv d \equiv 0 \mod 3$ be two elements in the set. Then we see that $(a,b) + (c,d) = (a+c, b+d)$ and we claim that $a + c \equiv b + d \equiv 0 \mod 3$. Notice that $a \equiv b \equiv 0 \mod 3$, $a = b + 3n$ for some integer n Similarly, $c \equiv d \equiv 0 \mod 3$ implies that $c = d + 3m$ for some integer m. Adding these two equations gives us $a + c = b + d + 3(n + m)$, which is another element in S. Thus, $a + c \equiv b + d \equiv 0 \mod 3$ is in S and the set is closed under addition.

Let $a \equiv b \equiv 0 \mod 3$ so that (a,b) is an element in S. Since modularity is preserved under multiplication, $-a \equiv -b \equiv 0 \mod 3$ and $(-a, -b)$ is also in S. Adding these two elements yields $(a,b) + (-a,-b) = (0,0)$, which implies $0 \equiv 0 \equiv 0 \mod 3$. Thus, the set contains additvive inverses.

Therefore, the set S must be an abelian subgroup of $\mathbb{Z} \times \mathbb{Z}$ under addition.

Let $a \equiv b \equiv 0 \mod 3$ and $c \equiv d \equiv 0 \mod 3$ be two elements in the set. Then we see that $(a,b)(c,d) = (ac, bd)$ and we want to show that $ac \equiv bd \equiv 0 \mod 3$ is in the set. Multiplying $a \equiv b \equiv 0 \mod 3$ by c yields $ac \equiv bc \equiv 0 \mod 3$. Multiplying $c \equiv d \equiv 0 \mod 3$ by b yields $bc \equiv bd \equiv 0 \mod 3$. Thus, by transitivity, $ac \equiv bd \equiv 0 \mod 3$. Therefore, the set is closed under multiplication.

However, $(1,1)$ cannot be in the set, as $1 \equiv 1 \equiv 0 \mod 3$ is not true. Thus, the set does not contain the multiplicative identity.

Therefore, the set S is NOT a subring of $\mathbb{Z} \times \mathbb{Z}$.

Question 2:

(a) This statement is false, as per Theorem 11.4.3 Correspondence Theorem, the ideal must be contained within the kernel of the map.

Counter Example:

We shall disprove this claim via a counter-example. Let φ be the map defined by $\varphi : \mathbb{Z} \to \mathbb{Q}$ where $\varphi(1) = 1$ and let I be an ideal in the domain $I = 2\mathbb{Z}$. Consider the element $2 \in I$ and $\frac{1}{3}$ in $\mathbb{Q}$. We see that $\frac{1}{3}2 = \frac{2}{3} \notin \varphi(I)$. Thus, the claim is false.

Question 2: (b) This statement is true, and we shall prove it directly.

Since φ is a ring homomorphism, $\varphi(0) = 0$ and consequently $0 \in \varphi^{-1}(J)$. Thus, $\varphi^{-1}(J)$ is non-empty and contains the additive identity.

Let $x, y \in \varphi^{-1}(J)$, then $\varphi(x) \in J$ and $\varphi(y) \in J$. Since φ is a ring homomorphism, we see that $\varphi(x) + \varphi(y) = \varphi(x + y) \in J$. Thus, $x + y \in \varphi^{-1}(J)$ and the set is closed under addition.

Let $r \in R$ and $a \in \varphi^{-1}(J)$, then $\varphi(a) \in J$. Since φ is a ring homomorphism, we see that $\varphi(r)\varphi(a) = \varphi(ra) \in J$. Thus, $ra \in \varphi^{-1}(J)$ and the set is closed under multiplication with elements of the ring.

Therefore, the set $\varphi^{-1}(J)$ is an ideal of R.

Question 3:

(a) Is $\mathbb{Z} \cong \frac{\mathbb{Z}[x]}{(x^2-3x+2)}$?

We note that $(x^2 - 3x + 2) = (x - 1)(x - 2)$, and thus the ideal is generated by the two elements $I = x - 1$ and $J = x - 2$. Per Artin's Chapter 11.6.8 (c), since $IJ = 0$, we see that $\frac{\mathbb{Z}[x]}{(x^2-3x+2)} \cong \frac{\mathbb{Z}[x]}{(x-1)} \times \frac{\mathbb{Z}[x]}{(x-2)}$.

Given that the image of $\frac{\mathbb{Z}[x]}{(x-1)}$ is in $\mathbb{Z}$ and the image of $\frac{\mathbb{Z}[x]}{(x-2)}$ is in $\mathbb{Z}$, by the first isomorphism theorem, we see that $\frac{\mathbb{Z}[x]}{(x^2-3x+2)} \cong \mathbb{Z} \times \mathbb{Z}$.

Thus, we want to show whether or not $\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$.

This is true for a mariad of reasons, one such reason is that both objects differ in the number of idempotent elements. The only idempotent elements in $\mathbb{Z}$ are 0 and 1, while the idempotent elements in $\mathbb{Z} \times \mathbb{Z}$ are $(0, 0)$, $(1, 1)$, $(0, 1)$ and $(1, 0)$.

Thus, the two rings CANNOT be isomorphic.

Question 3:

(b) Is $\mathbb{Z} \cong \frac{\mathbb{Z}[x]}{(2x-1)}$?

Let us assume, for the sack of a contradiction, that they are isomorphic. Then we have a ring homomorphism $\varphi : \frac{\mathbb{Z}[x]}{(2x-1)} \to \mathbb{Z}$ such that $\varphi(1) = 1$.

Note that the kernel of our of our domain is then $\overline{2x-1} = \bar{0}$, which implies that $\overline{2x} = \bar{1}$. Under the map φ, we see that $\varphi(\overline{2x}) = \varphi(\bar{2})\varphi(\bar{x}) = 2\varphi(\bar{x}) = 1$, since φ s bijective and $\varphi(1) = 1$. However, we arrive at a contradiction, as there exist no integer solutions to the equation $2n = 1$ for $n \in \mathbb{Z}$. Thus, we see that φ cannot be a ring homomorphism.

Therefore, we conclude that the two rings are NOT isomorphic.

Question 3:

(c) Is $\frac{F_3[x]}{(x^2+x+1)} \cong \frac{F_3[x]}{(x^2+2x+2)}$?

Consider the first ring $\frac{F_3[x]}{(x^2+x+1)}$. Notice that $x^2 + x + 1$ is reducible over F_3, with the factorization $x^2+x+1 = (x-1)(x+2)$. In the quotient ring, this implies $\overline{(x-1)(x+2)} = \overline{x^2+x+1} = \overline{0}$. Since $\overline{x-1}$ and $\overline{x+2}$ are non-zero elements in $\frac{F_3[x]}{(x^2+x+1)}$, this ring has non-zero zero divisors. Therefore, $\frac{F_3[x]}{(x^2+x+1)}$ is not a field.

Now consider the second ring $\frac{F_3[x]}{(x^2+2x+2)}$. Notice that $x^2 + 2x + 2$ is irreducible over F_3. In a field, if the product of two elements is zero, then at least one of the elements must be zero. Thus, the ideal generated by $x^2 + 2x + 2$ does not result in a non-zero zero divisor in the quotient ring.

Since one of the rings, $\frac{F_3[x]}{(x^2+x+1)}$, is not a field (because it has non-zero zero divisors), and the other ring, $\frac{F_3[x]}{(x^2+2x+2)}$ is a field, they cannot be isomorphic.

Therefore, the two quotient rings are NOT isomorphic.

Question 3:

(d) Is $\frac{\mathbb{R}[x]}{(x^2+1)} \cong \frac{\mathbb{R}[x]}{(3x^2+5)}$?

We note that the solutions to the ideal of the first ring generated by $(x^2 + 1)$ are $\pm i$. Thus, we can recognize the first ring as $\mathbb{R}[i]$.

Similarly, the solutions to the ideal of the second ring generated by $(3x^2 + 5)$ are $\pm i\sqrt{\frac{5}{3}}$. And letting $\alpha = \sqrt{\frac{5}{3}}$, we can recognize the second ring as $\mathbb{R}[\alpha i]$.

Thus, we want to show whether $\mathbb{R}[i] \cong \mathbb{R}[\alpha i]$.

Per Artin's Chapter 11 Section 3, evaluating real polynomials at a complex number yields ring homomoprhims to the complex numbers. Thus, $R[i] \cong \mathbb{C}$ and $R[\alpha i] \cong \mathbb{C}$.

Indeed, since both rings can be recognized as $\mathbb{C}$, we see that they ARE isomorphic.

Question 4:

(a)

Suppose that $p(t) \equiv q(t) \mod f(t)$, then $p(t) - q(t) = h(t)f(t)$ for some polynomial $h(t)$ in $\mathbb{C}[t]$.

Under ϕ_f, we see that $\phi_f(p(t) - q(t)) = \phi_f(p(t)) - \phi_f(q(t)) = \phi_f(h(t)f(t))$. Since $f(t)$ is in the kernel of ϕ_f, we see that $\phi_f(h(t)f(t)) = h(t)f(t) \in (f(t))$. Thus, we see that $\phi_f(p(t)) - \phi_f(q(t)) \in (f(t))$, or equivalently, $\phi_f(p(t)) \equiv \phi_f(q(t)) \mod (f(t))$. Thus, the map is well-defined.

Question 4:

(b) No. I

Let $f(t) = t$. Since t can generate all powers of t, we notice that if we quotient $C[t]$ by t, we are left with the ring of constant polynomials, which is isomorphic to $\mathbb{C}$.

Similarly, quotienting $C[[t]]$ by t yields the ring of a constant series, which is also isomorphic to $\mathbb{C}$.

Thus, by the first isomorphism theorem, we see that ϕ_f can be recognized as the map $\phi_f : \mathbb{C} \to \mathbb{C}$, which is the identity map of $\mathbb{C}$.

Because $1 \mapsto 1$ and $0 \mapsto 0$, the map is indeed injective.

Furthermore, since the image is the entirety of $\mathbb{C}$, we see that the map is surjective.

Question 4:

(b) No. II

Let $f(t) = t + 1$. Since $t + 1$ can generate all powers of t, we notice that if we quotient $C[t]$ by $t + 1$, we are left with the ring of constant polynomials evaluated at -1, which is isomorphic to $\mathbb{C}$.

Furthermore, under $C[[t]]$, $(t + 1)$ is a unit as the inverse $(t + 1)^{-1}$ is the formal power series $\sum_{n=0}^{\infty}(-t)^n$. Thus, the resulting quotient of $C[[t]]/(t + 1)$ is $\{0\}$, as that is the only coset remaining.

Therefore, we can recognize that the map ϕ_f as a map from $\mathbb{C}$ to $\{0\}$, which is the zero map.

Thus, the map is NOT injective, as the kernel is the entirety of $\mathbb{C}$. And since the image is $\{0\}$, we see that the map IS surjective.