A Discussion on Symmetry and Awesome Sets

An Introduction to the Isometries of $\mathbb{R}^3$

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Outline

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What are *aba*-sets?

Also known as "awesome"-sets, A-sets

Definition

Let G be a group.
A set $S \subseteq G$ is an *aba-set* if for all $a, b \in S$, $aba \in S$.

- Previous results:
  - If $x \in S$ is an element of odd order, then $\langle x \rangle \in S$.

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How much can these sets generate?

Questions to consider:

- Can an *aba*-set with only generators and the identity recover the whole group?
- Is there a more geometric quality that *aba*-sets induce?

Let's looks at an example to further explore this!

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The Dihedral Group $\forall a, b \in S, \; aba \in S$

Example

Let $S = \{1, \sigma, \zeta\} \subset D_n$ be a set with property *aba*-closure. What must follow is that S must contain ('generate') all powers of the flips and rotations.

Proof.

- Flips have order 2, meaning $\sigma = \sigma^{-1} \in S$ a priori.
- We can build-up all powers of *zeta* through different recursive combinations with the identity.
  For example, $\zeta 1 \zeta = \zeta^2$ and $\zeta\zeta\zeta = \zeta^3$ and so forth.
- Thus, we can generate all powers of the flip-σ and rotation-ζ elements.

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Subgroups of $\mathbb{R}^3$ $\quad \forall a, b \in S,\ aba \in S$

To find groups similar to the dihedral group, let's impose firm positions on the vertices of D_n on a plane.

In other words, let us look at what space D_n lives in!

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- The dihedral groups are all *subgroups* of the group of rotations in $\mathbb{R}^3$ (SO3).
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Another Interesting Example

$$\forall a, b \in S, \ aba \in S$$

Remark

Recall that if $|x|$ is odd and in S, $\langle x \rangle \in S$. We can extend this to *any* cyclic subgroup, so long as we start with the identity! That is to say:

Let $x \in G$ have finite order. If $\{1, x\} \in S$, then $\langle x \rangle \in S$.

Furthermore, if G is a cyclic group, then the entire group is an *aba-group!*

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Groups to Inspect

- Focus on investigating the *aba*-sets of subgroups and semi-direct products of primarily $\mathbb{R}^3$ (a topic for next week).
- Observe if there is any correlation between said *aba* sets (specifically with sets only containing the generators and identity of these groups).

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Programming

- Working on code that will speed up the process of finding *aba*-sets.
- *Potentially attempt to create a 3D program that may enumerate any *geometric* qualities of *aba*-sets.

Summary

- I am now not only investigating *aba*-sets from an algebraic viewpoint, but seeing if the space where a group lives in induces a constraint in algebraic results.
- Observing how *aba*-sets act on symmetries may lead into questions like how much of an original group is guaranteed to be encoded in an *aba*-set.
- While larger conjectures in $\mathbb{R}^n$ would be awesome, I will likely continue to work with grups in $\mathbb{R}^2$ and $\mathbb{R}^3$ due to some computational concerns (things get *big* and confusing).

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Thank you!