Empirical Orthogonal Functions and their latest developments

Dealing with orthogonality and sparse data

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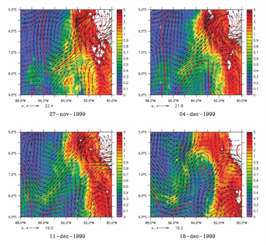
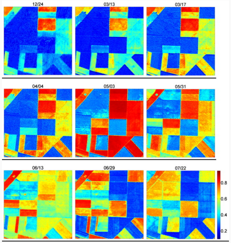
Context

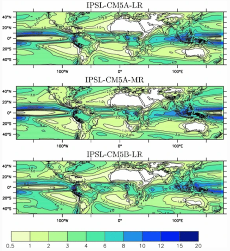
Oceanography



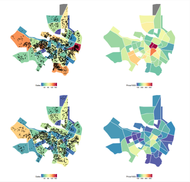
Agronomy



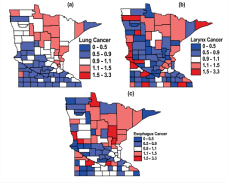
Climatology



Economics



Epidemiology



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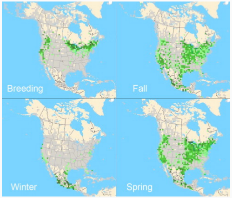

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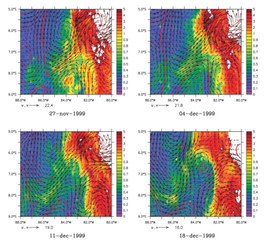

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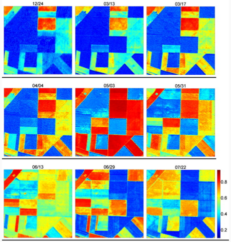

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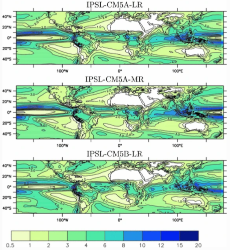

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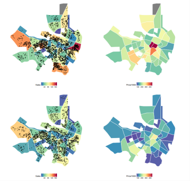

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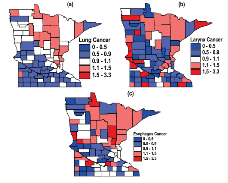

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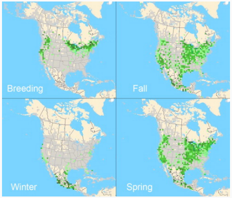

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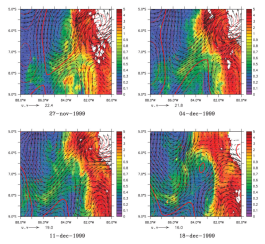

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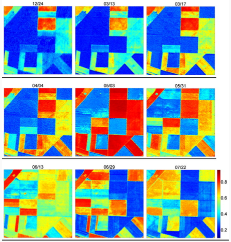

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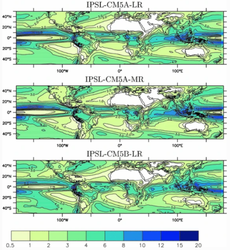

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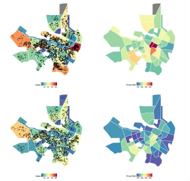

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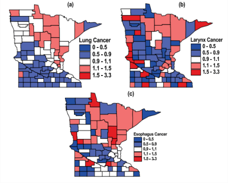

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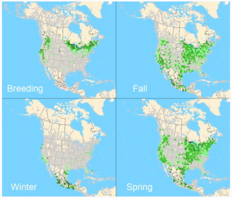

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Empirical Orthogonal Functions (EOFs) is the keystone method for dimension reduction of spatio-temporal data.

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Few extensions:

- Extended and multivariate EOFs [Fraedrich et al., 1997, Sparnocchia et al., 2003]
- Complex/hilbert EOFs [Bürger, 2021]
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New methods from ecology

Empirical Orthogonal Functions: the basics

Let's denote a spatio-temporal process $S = (S(x,t); x \in \mathbb{R}^2,$
$t \in \{t_1, \cdots, t_p\})$

The temporal average of S in denoted:

$$\overline{\mathbf{s}}^t(x) = \frac{1}{p}\sum_{k=1}^{p} S(x, t_k)$$

The time-centered space-time field:

$$\mathbf{S}' = \left(\mathbf{s}_{t_1} - \overline{\mathbf{s}}^t, ..., \mathbf{s}_{t_p} - \overline{\mathbf{s}}^t\right).$$

Then, $\mathbf{S}'$ has the form:

$$\mathbf{S}' = \begin{pmatrix} S'(x_1,t_1) & S'(x_1,t_2) & \cdots & S'(x_1,t_p) \\ S'(x_2,t_1) & S'(x_2,t_2) & \cdots & S'(x_2,t_p) \\ \vdots & \ddots & \ddots & \vdots \\ S'(x_n,t_1) & S'(x_n,t_2) & \cdots & S'(x_n,t_p) \end{pmatrix}$$

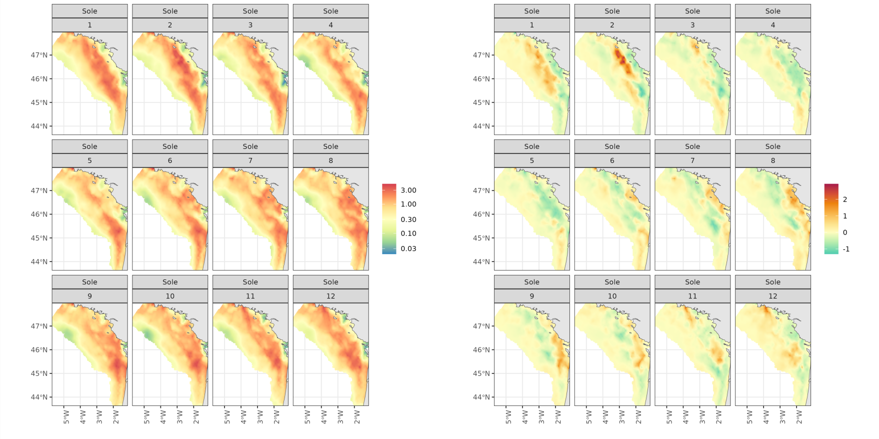

Figure 1: (left) Montly spatial log-predictions $\log S(x, t)$ of the hierarchical model. (right) Monthly anomalies of the spatial predictions $S^*(x, t)$. Each panel corresponds to the average distribution of prediction of anomalies for a month over the period 2008 - 2018.

The spatio-temporal field is decomposed so that:

$$S'(x,t) = \sum_{m=1}^{r} u_m(x) \cdot v_m(t)$$

where $\mathbf{u}_m$ capture the variance of $\mathbf{S}'$ and $\langle u_i(\cdot); u_j(\cdot)\rangle = 0$ and $\langle v_i(\cdot); v_j(\cdot)\rangle$ with $i \neq j$

This falls back to a diagonalisation through singular value decomposition:

$$\mathbf{S}' = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T \quad \text{(SVD)}$$

- $\mathbf{U}_{(n\times r)}$ contains the spatial factors,
- $\boldsymbol{\Sigma}_{(r\times r)}$ contains the singular values of $\mathbf{S}'$.
- $\mathbf{V}_{(p\times r)}$ contains the temporal loadings

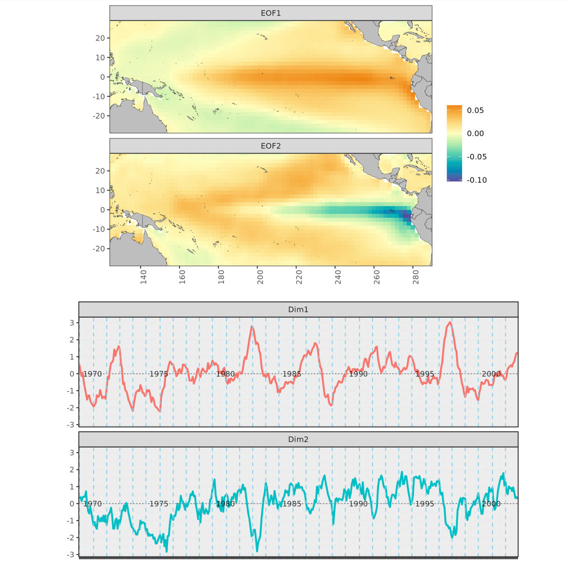

Figure 2: SST in the South Pacific. Results of the EOFs (Top) Spatial factors of two first dimensions. (Bottom) Related temporal loadings. Blue dashed vertical lines corresponds to the month of January for each year.

SST South Pacific

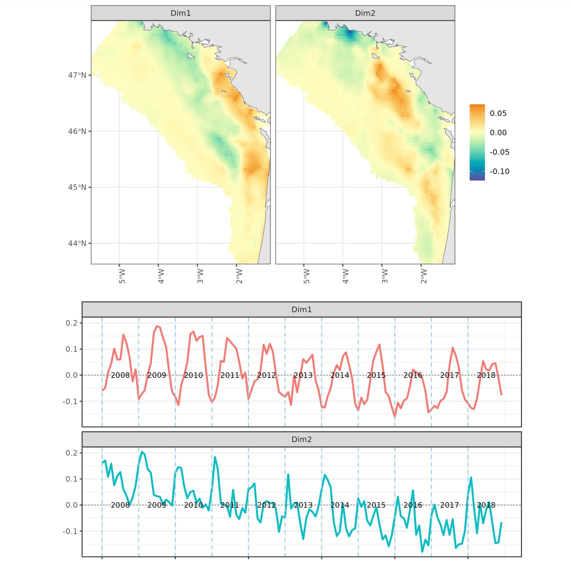

Figure 2: Sole in the Bay of Biscay. Results of the EOFs (Top) Spatial factors of two first dimensions. (Bottom) Related temporal loadings. Blue dashed vertical lines corresponds to the month of January for each year.

Sole
Bay of Biscay

Setting spatial orthogonal constraints on EOFs

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EOFs are statistically orthogonal *i.e.* for $i \neq j, \left\{\mathbf{u}_i, \mathbf{u}_j\right\} = 0$

Bu they are not spatially orthogonal *i.e.* the cross-covariance of the different EOFs maps is not necessarily 0.

$$\gamma_{ij}(r) = \frac{1}{2} E\left[\mathbf{u}_i(x) - \mathbf{u}_i(x+r)\right]\left[\mathbf{u}_j(x) - \mathbf{u}_j(x+r)\right]$$

EOM consist in a two step analysis:

- perform an EOFs on $\mathbf{S}'$ $\rightarrow$ obtain statistically decorrelated planes
- compute the covariogram and the cross-covariogram of $\mathbf{U}$ for a specific ditance r that we denote $\boldsymbol{\Gamma}_r$ of dimension $(p \times p)$.

$$\boldsymbol{\Gamma}_r = \boldsymbol{U}_{\mathrm{eom}} \boldsymbol{\Sigma}_{eom} \mathbf{V}_{\mathrm{eom}}^T$$

Planes are then ordered by increasing variance explained in $\mathbf{S}'$

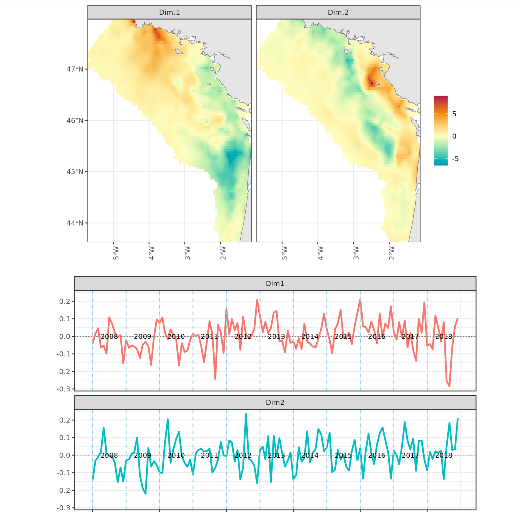
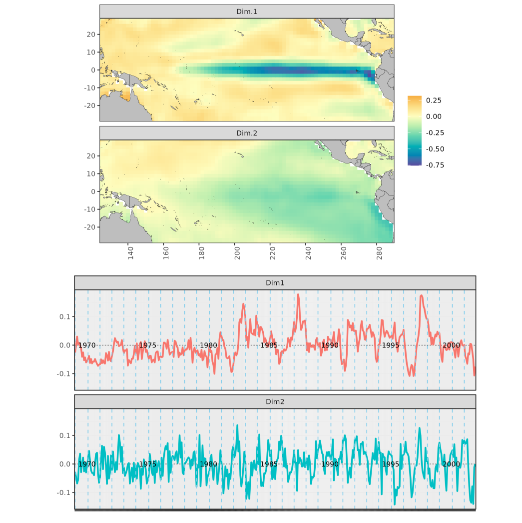


Figure 3: Sole in the Bay of Biscay. (Top) Spatial factors obtained by EOM for the two first dimensions. (Bottom) Temporal loadings for the first two dimensions.

Sole

Bay of Biscay



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SST

South Pacific

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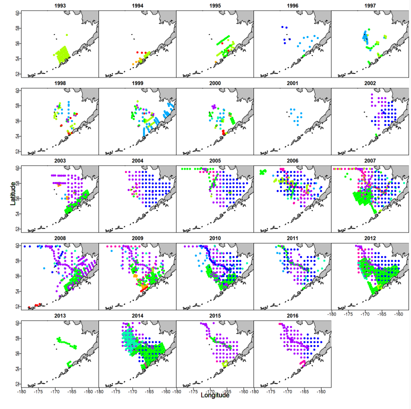

Figure 4: (Top) Location of survey tows for nine seasons (colors) in each year (panel) in the Bering Sea case study [Thorson et al., 2020].

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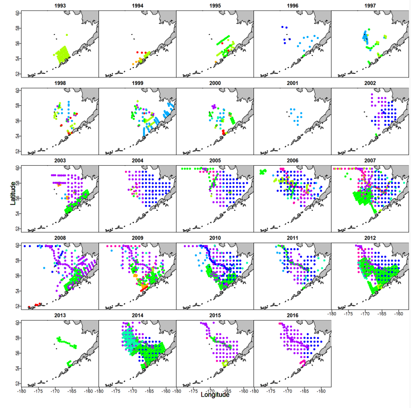

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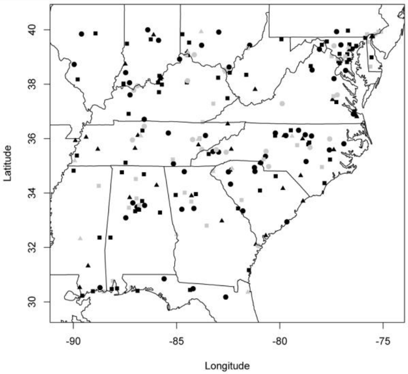


Figure 5: Sites reporting concentration of ozone (dots and squares) and PM2.5 (triangles and squares) used in our case study [Berrocal et al., 2010].

Let's define observations Y:

$$Y|S, \boldsymbol{\theta} \sim \mathcal{L}_Y(S, \boldsymbol{\theta}_{obs})$$

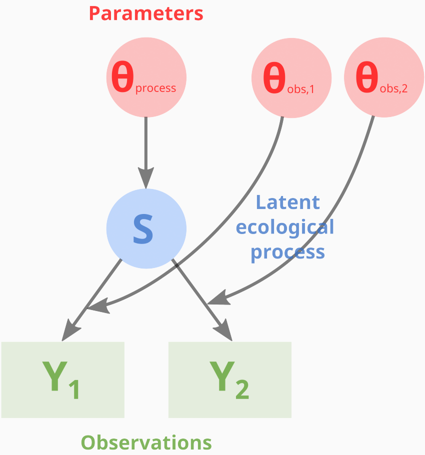
a latent field S:

$$S|\boldsymbol{\theta}_{process} \sim \mathcal{L}_S(\boldsymbol{\theta}_{process})$$

and parameters:

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_{obs}, \boldsymbol{\theta}_{process})$$

There can be several data sources Y_1 and Y_2, in which case each has its own probability distribution ($\mathcal{L}_{Y_1}$,$\mathcal{L}_{Y_2}$) and observation parameters ($\boldsymbol{\theta}_{obs,1}$, $\boldsymbol{\theta}_{obs,2}$).



We consider a **hierarchical model** where the **latent field** take the form:

$$log(S(x,t)) = \beta(t) + \omega(x) + \sum_{f=1}^{F} \lambda(t,f)\epsilon(x,f)$$

where f are the dimension of the EOFs with F dimensions, $\boldsymbol{\omega} \sim \mathcal{MG}(0, \boldsymbol{\Sigma}_\omega)$ and $\boldsymbol{\epsilon} \sim \mathcal{MG}(0, \boldsymbol{\Sigma}_\epsilon)$.

$\lambda(t,f)$ are the loadings and $\epsilon(x,g)$ are the EOF.

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and **the observations** are zero-inflated and takes the form:

$$\Pr(Y = y_i) = \begin{cases} 1 - p & \text{if } y_i = 0 \\ p \times \mathcal{L}(y_i; \log S(x_i, t_i); \sigma^2) & \text{if } y_i > 0 \end{cases}$$

Where Y is an observations as random variable and y_i is the realized observation. p is the probability to obtain a positive observation. $\mathcal{L}$ is the probability of the positive observations.

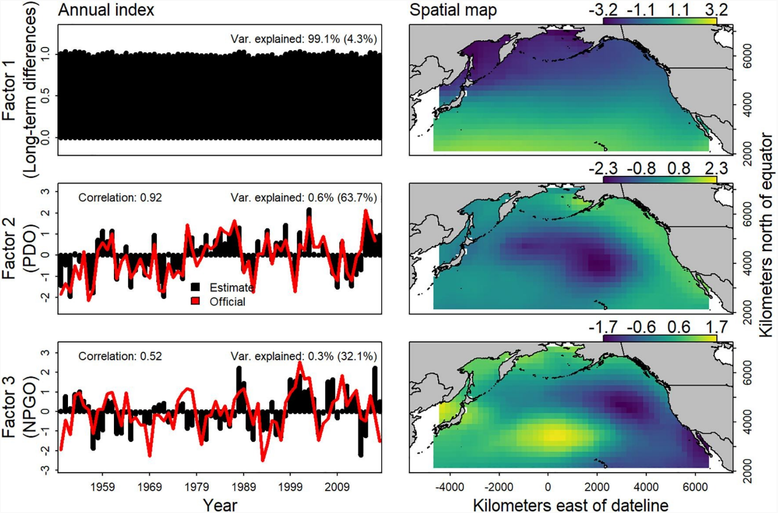


Figure 6: EOFs for sea surface temperature in the North Pacific (rows), where each axis includes the loadings (left columns) as well as a spatial factors (right column) [Thorson et al., 2021]. PDO: Pacific Decadal Oscillation. NPGO: North Pacific Gyre Oscillation.

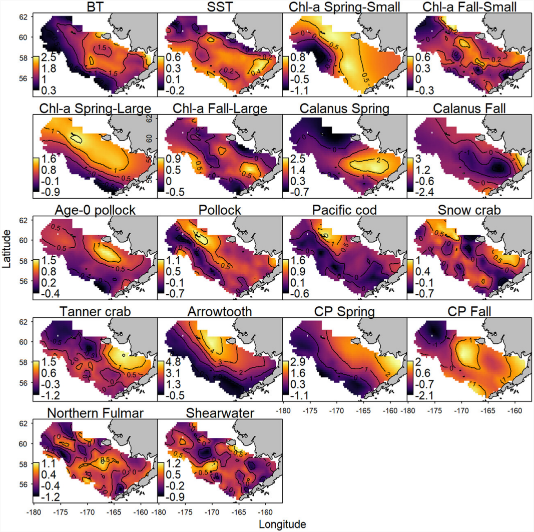


Figure 7: Spatial factors for each variable of the primary mode of ecosystem variability [Thorson et al., 2021].

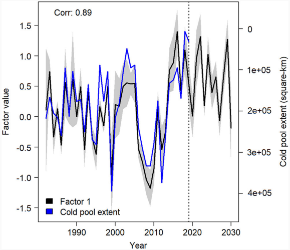
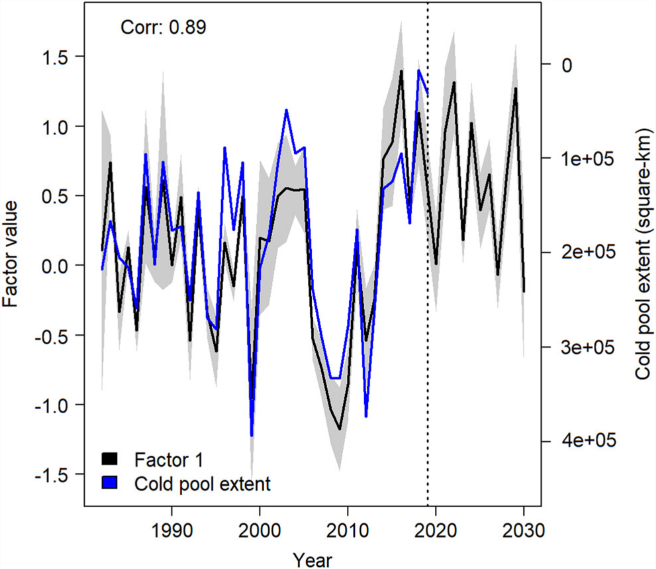


Figure 8: Dominant mode of ecosystem variability for spatio-temporal variation (black lines) and cold-pool extent (blue line) [Thorson et al., 2021].

Illustration: ecological data in the Bering sea.



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  ▲ The patterns are **not orthogonal**, this requires additional rotation.
- Further research avenues: perform EOFs that **accounts for both spatial and temporal correlation**.

Thank you for your attention!

Any questions?

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