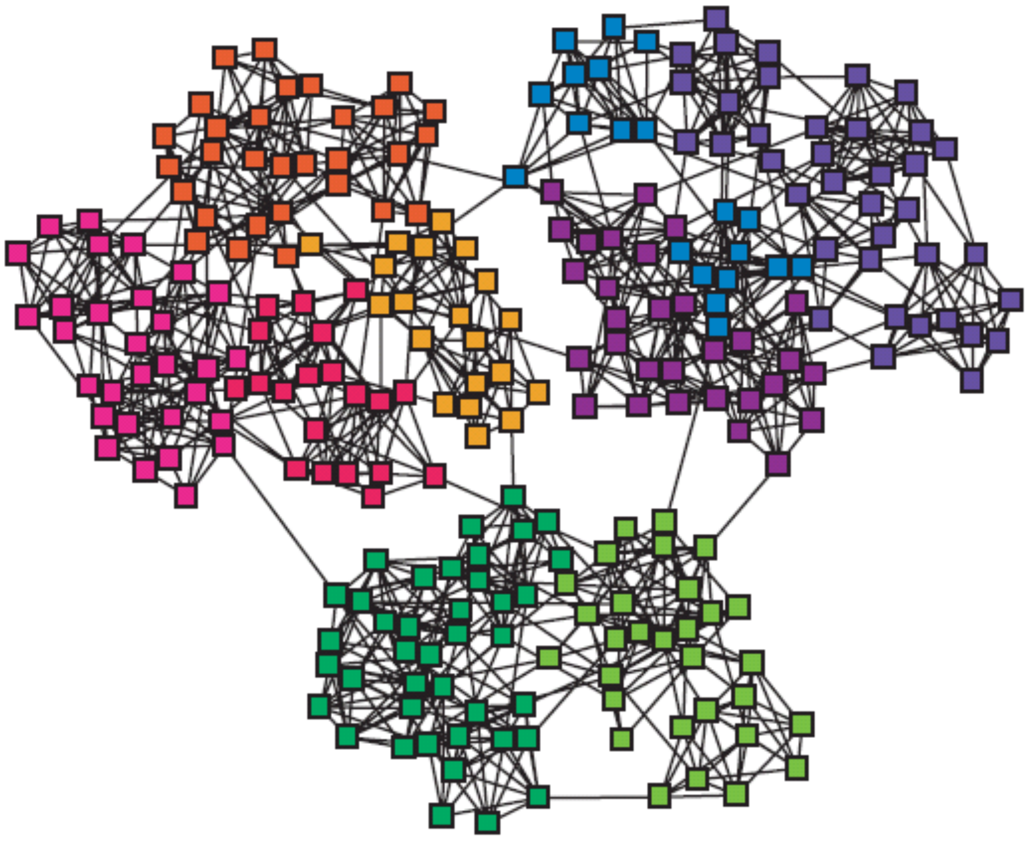
Mining Social-Network Graphs

Analysis of Large Graphs: Community Detection

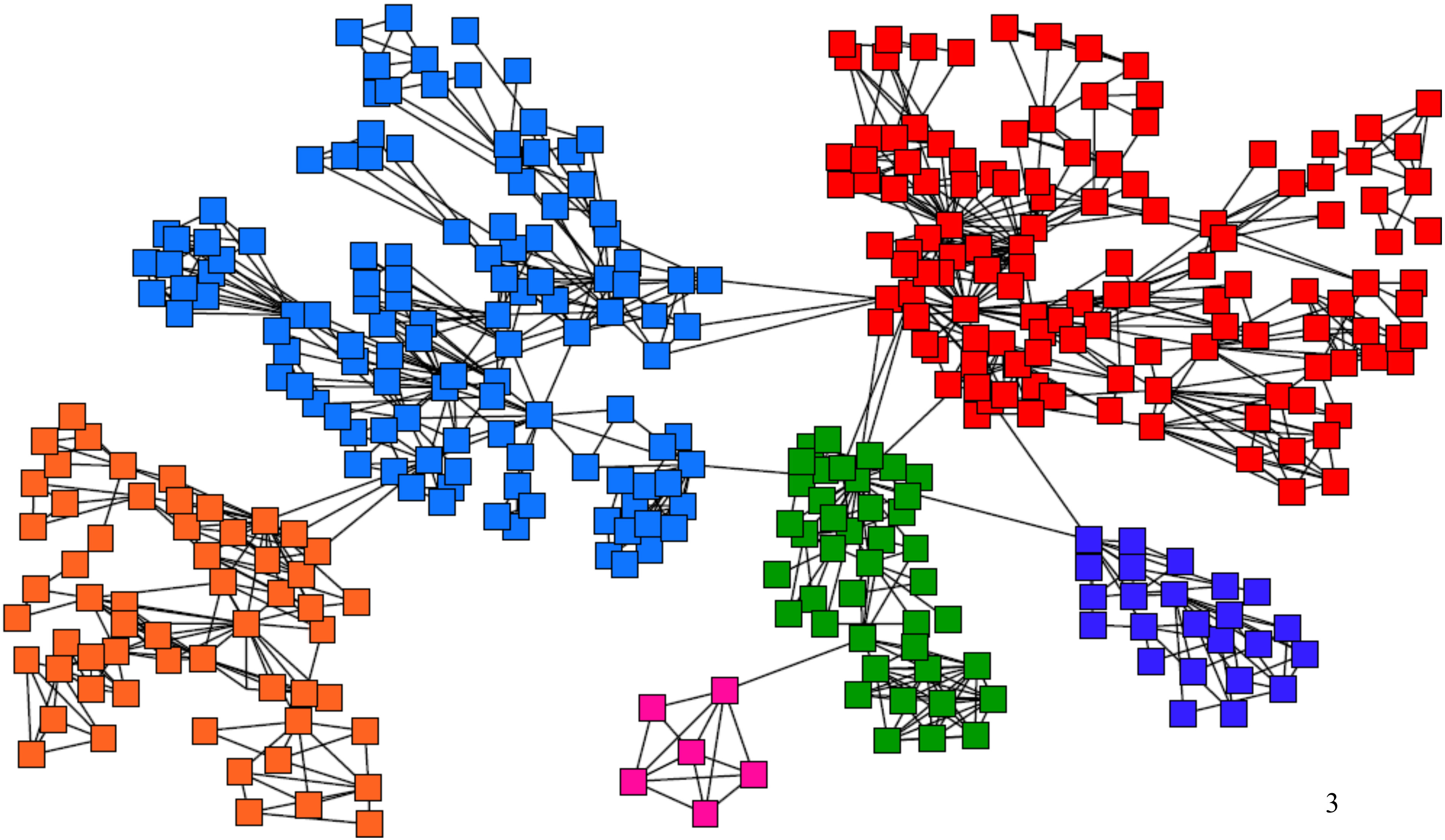
With slide contributions from J. Leskovec, Anand Rajaraman, Jeffrey D. Ullman

Networks & Communities

- We often think of networks being organized into **modules, cluster, communities:**

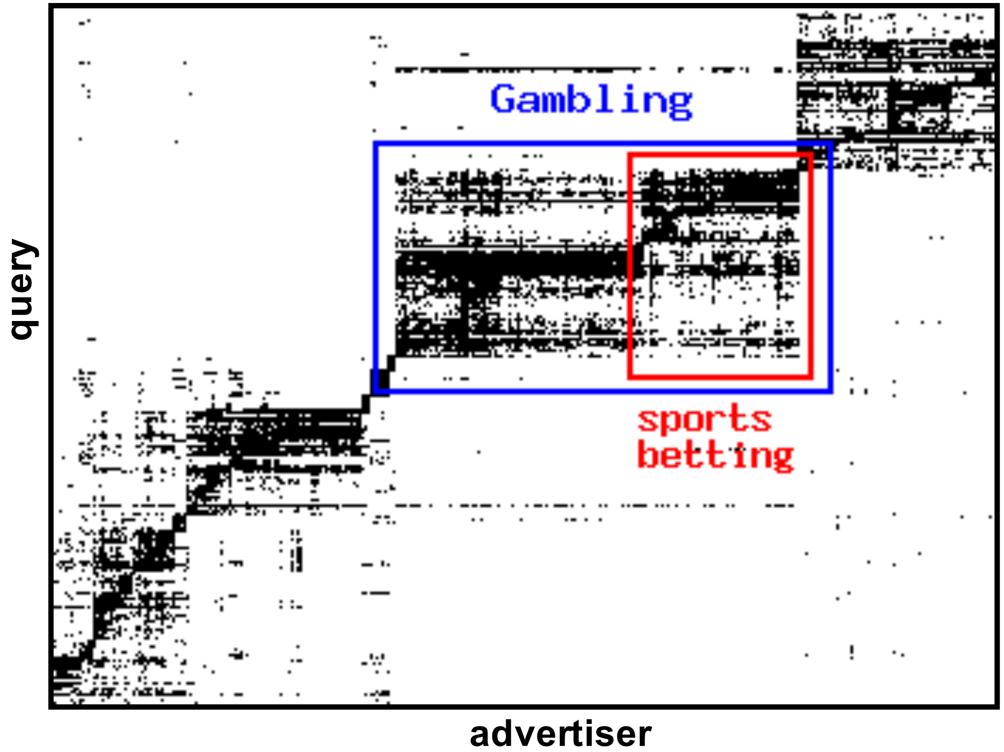


Goal: Find Densely Linked Clusters



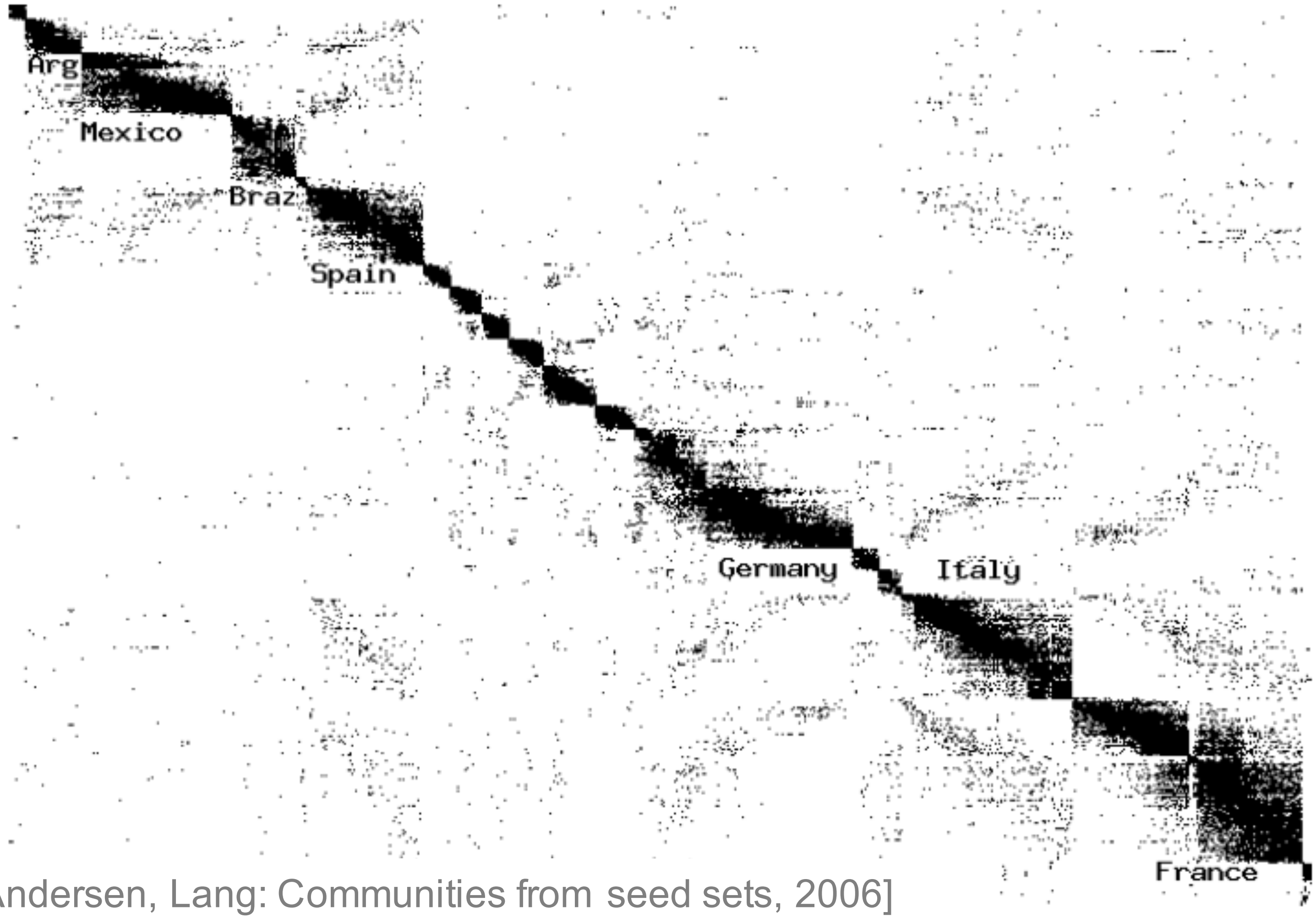
Micro-Markets in Sponsored Search

- Find micro-markets by partitioning the query-to-advertiser graph:



Movies and Actors

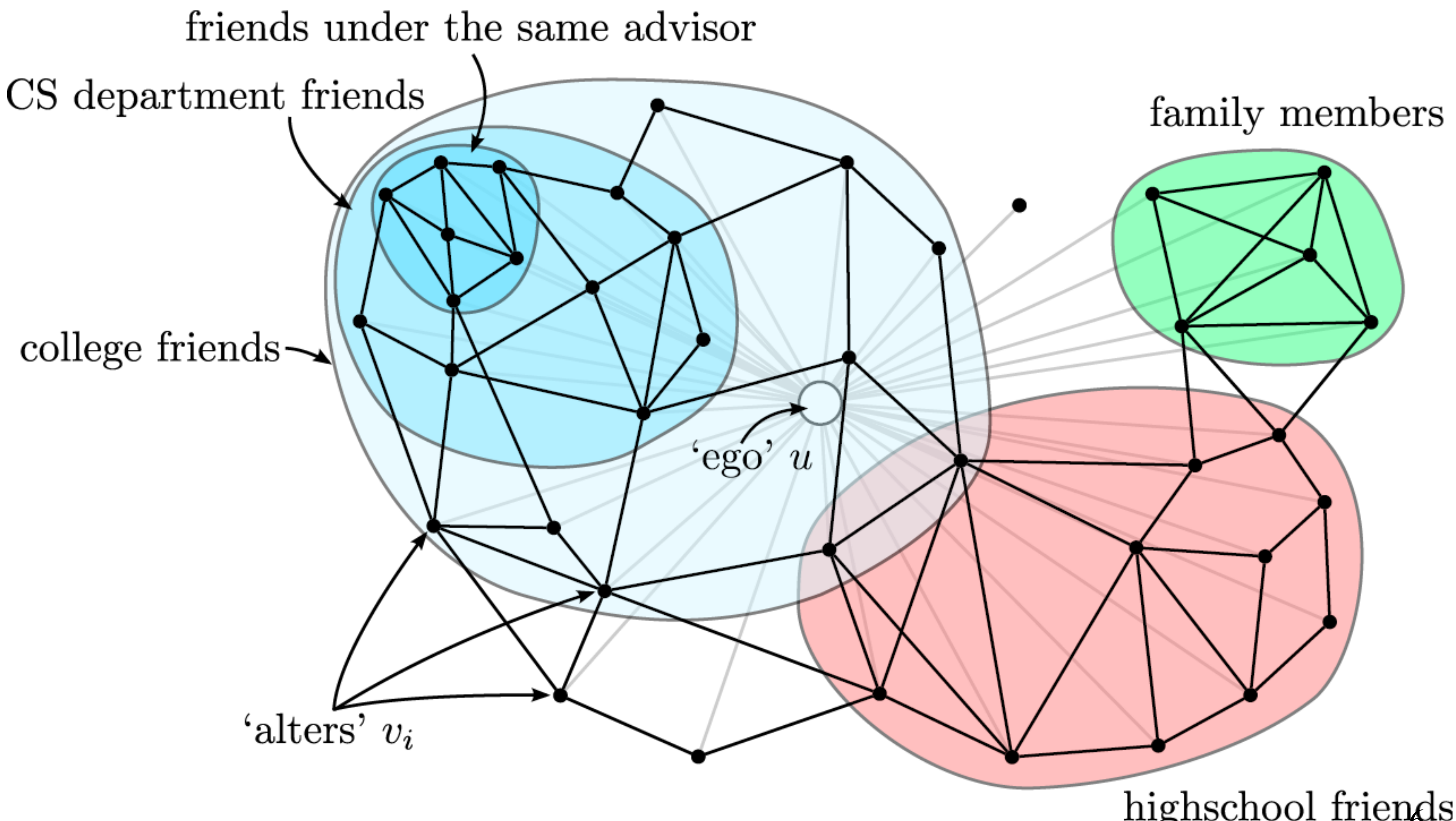
- Clusters in Movies-to-Actors graph:



[Andersen, Lang: Communities from seed sets, 2006]

Twitter & Facebook

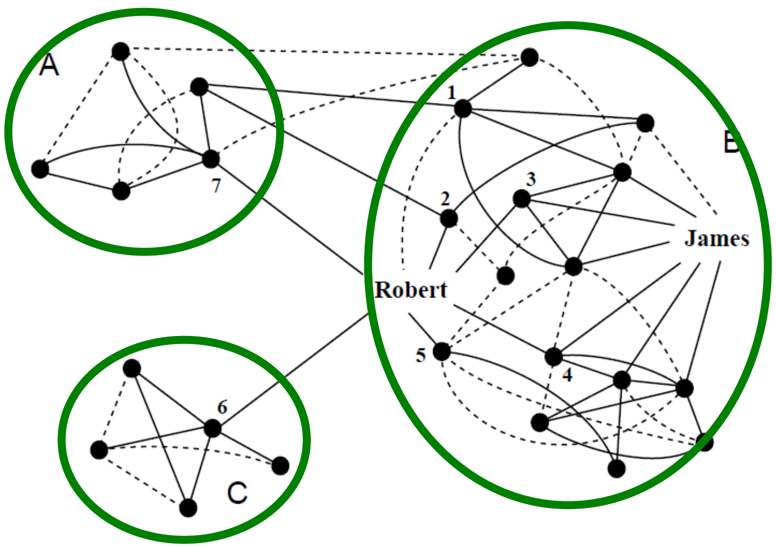
- Discovering social circles, circles of trust:

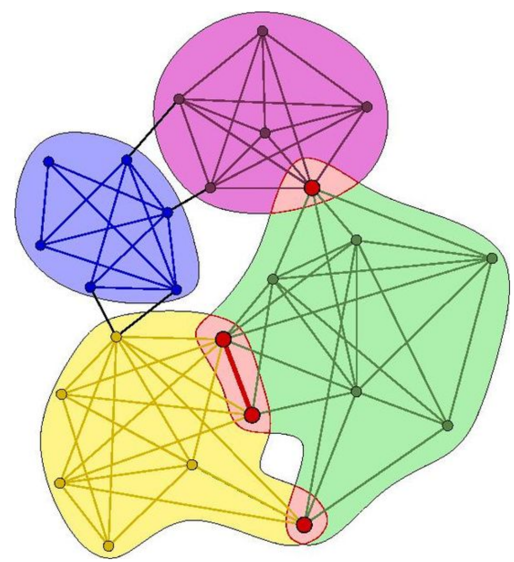


[McAuley, Leskovec: Discovering social circles in ego networks, 2012]

COMMUNITY DETECTION (GRAPH BASICS)

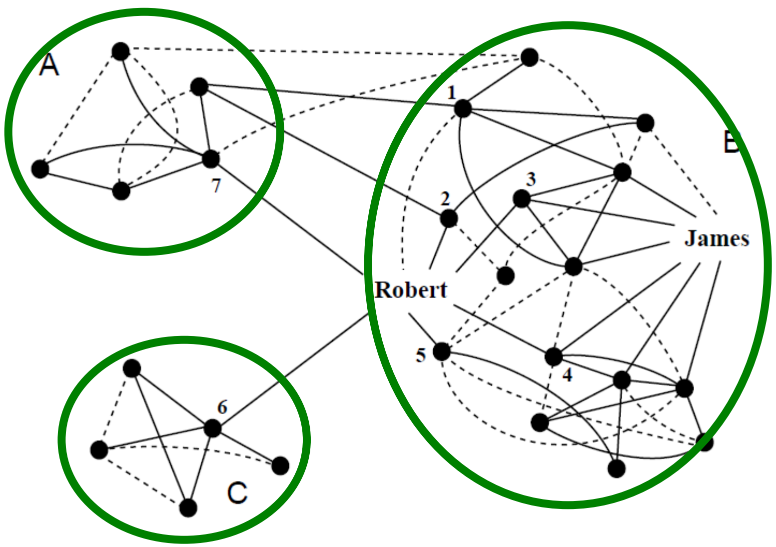
How to find communities?

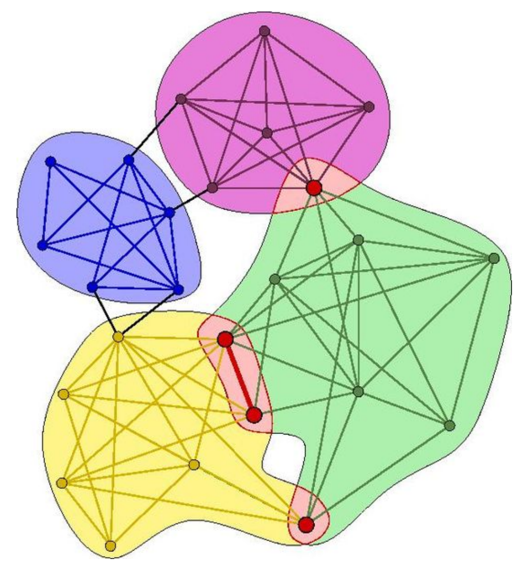




COMMUNITY DETECTION (ALGORITHMS AND METHODS)

How to find communities?

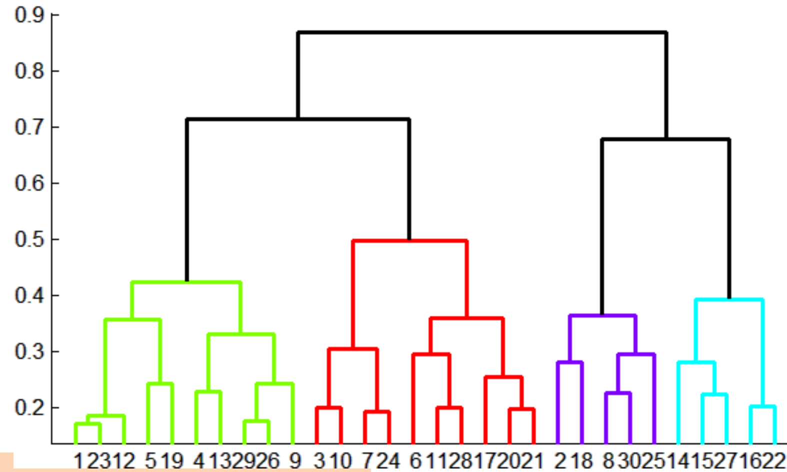




We will work with **undirected** (unweighted) networks

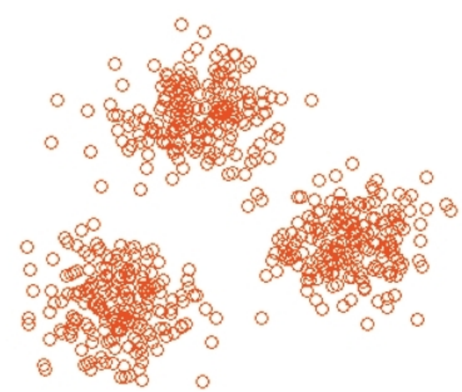
Recall: Methods of Clustering

- **Hierarchical:**
  - **Agglomerative** (bottom up):
    - Initially, each point is a cluster
    - Repeatedly combine the two "nearest" clusters into one
    - **Used a distance metric**



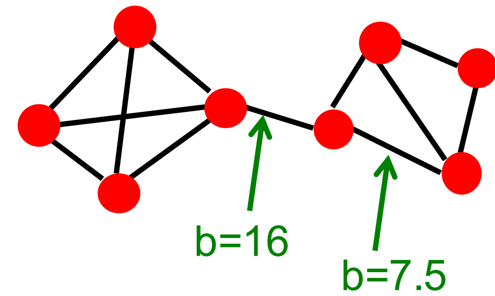
Today:
  - **Divisive** (top down):
    - Start with one cluster and recursively split it

- **Point assignment:**
  - Maintain a set of clusters
  - Points belong to "nearest" cluster
  - **Used a distance metric**

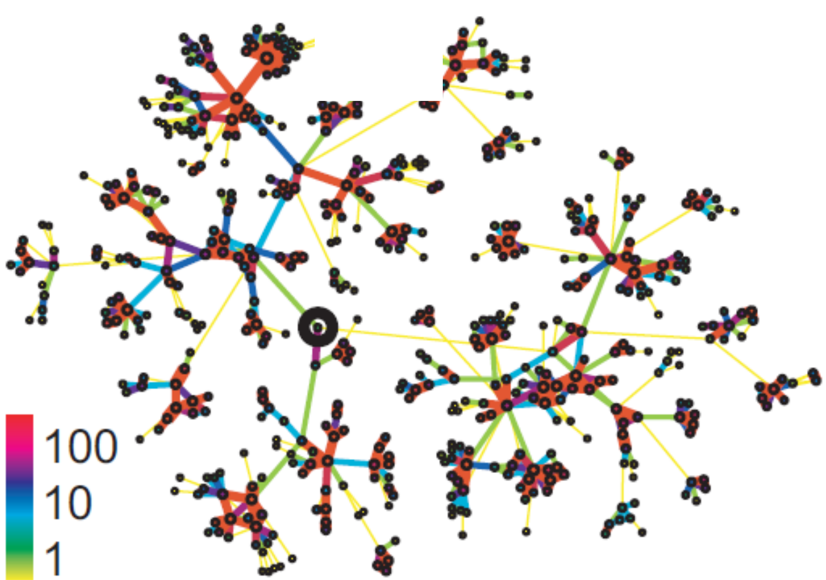


Betweenness Concept

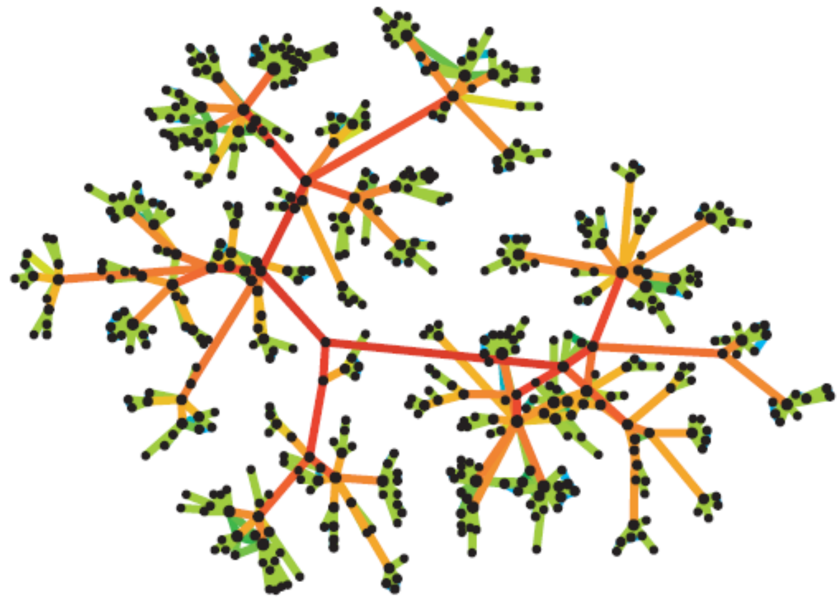
- **Edge betweenness:** Number of shortest paths passing over the edge



- **Intuition:**



Edge strengths (call volume) in a real network

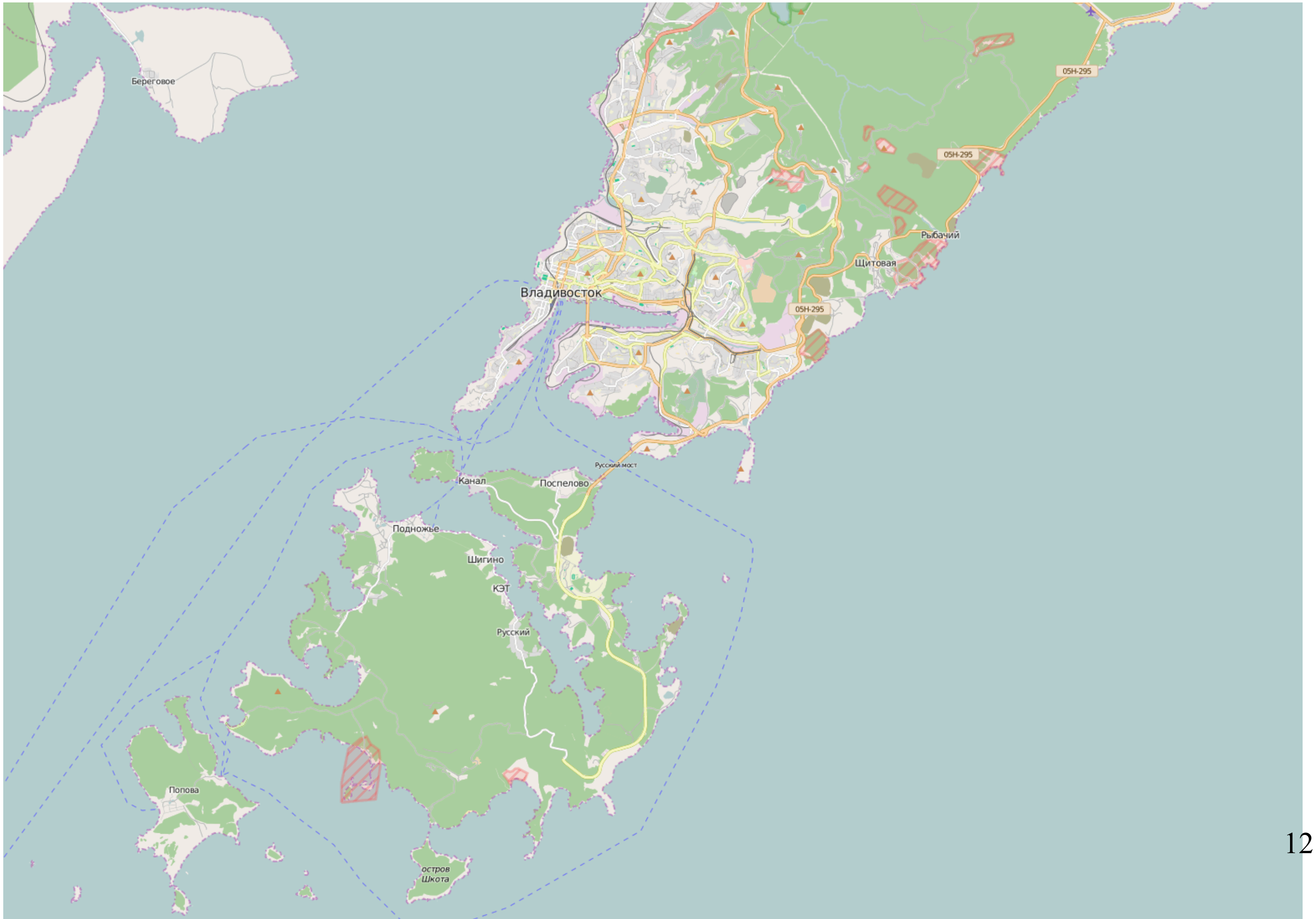


Edge betweenness in a real network

Betweenness Concept (Cont'd)

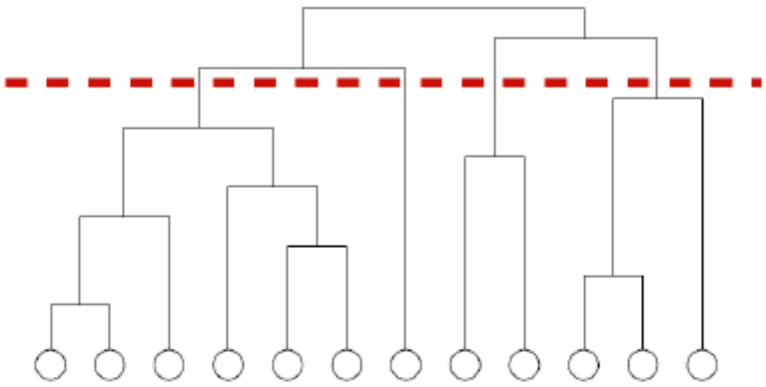
- Find edges in a social network graph that are least likely to be inside a community
- Betweenness of edge (a, b):
  - number of pairs of nodes x and y -> x, y ∈ C
  - **edge (a,b) lies on the shortest path between x and y**
- If there are several shortest paths between x and y, edge (a,b) is credited with the fraction of those shortest paths that include edge (a,b)
- **A high score is bad:** suggests that edge (a,b) runs between two different communities
  - **a and b are in different communities.**

The Russian Bridge



WE NEED TO RESOLVE 2 QUESTIONS

1. How to compute betweenness?
2. How to select the number of clusters?



The Girvan-Newman Algorithm

- Want to **discover communities using divisive hierarchical clustering**
  - Start with one cluster (the social network) and recursively split it
- **Will do this** based on the notion of edge **betweenness**:

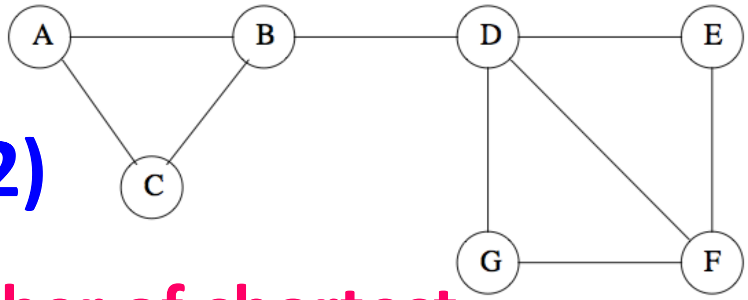
  Number of shortest paths passing through the edge
- **Girvan-Newman Algorithm:**
  - Visits each node X once
  - Computes the number of shortest paths from X to each of the other nodes that go through each of the edges
- **Repeat:**
  - Calculate betweenness of edges
    1. Thresholding to remove high betweeness edges, or
    2. Remove edges with highest betweenness: **between** communities
- **Connected components are communities**
- Gives a hierarchical decomposition of the network.

Girvan-Newman Algorithm (1)

- Visit each node X once and **compute the number of shortest paths from X to each of the other nodes that go through each of the edges**

- **1) Perform a breadth-first search (BFS) of the graph, starting at node X**
  - The level of each node in BFS is length of the shortest path from X to that node
  - So edges that go between nodes on the same level can never be part of a shortest path from X
  - **Edges between levels are called DAG edges** (DAG = Directed Acyclic Graph)
  - **Each DAG edge is part of at least one shortest path from root X.**

Girvan-Newman Algorithm (2)



- 2) Label each node by the number of shortest paths that reach it from the root node
  - Example: BFS starting from node E, labels assigned

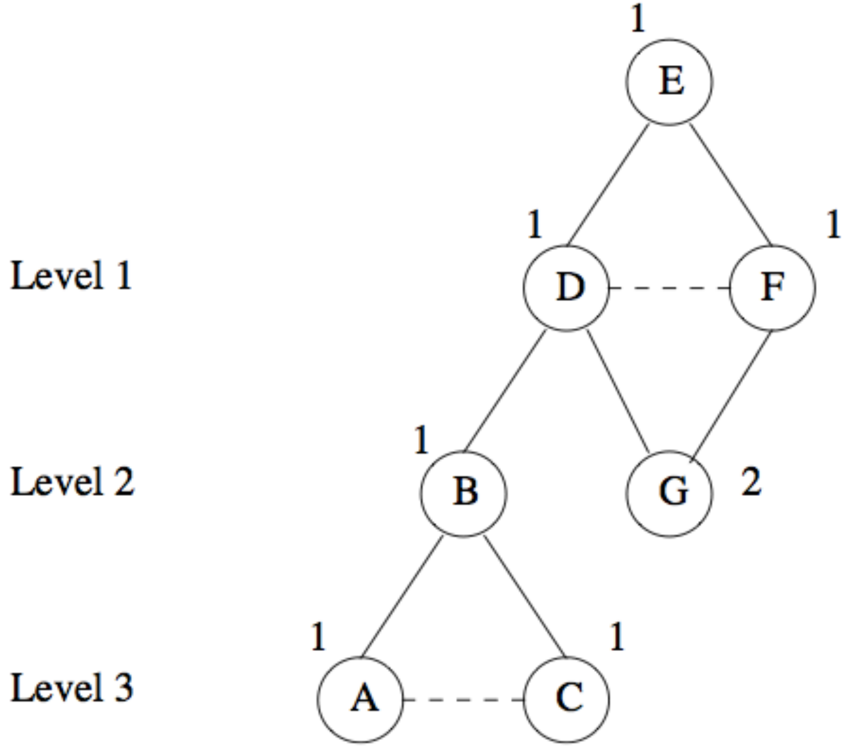


Figure 10.4: Step 1 of the Girvan-Newman Algorithm

Girvan-Newman Algorithm (3)

- **3) Calculate for each edge *e*, the sum over all nodes Y (of the fraction) of the shortest paths from the root X to Y that go through edge *e***
  - **Compute this sum for nodes and edges, starting from the bottom of the graph**
  - **Each node** other than the root node **is given a credit of 1**
  - **Each leaf node in the DAG gets a credit of 1**
  - Each **node** that is not a leaf gets credit = **1 + sum of credits of the DAG edges from that node to level below**
  - A DAG **edge** e entering node Z (from the level above) is **given a share of the credit of Z proportional to the fraction of shortest paths from the root to Z that go through e**

Girvan-Newman Algorithm (4)

- Assign node and edge values starting from bottom

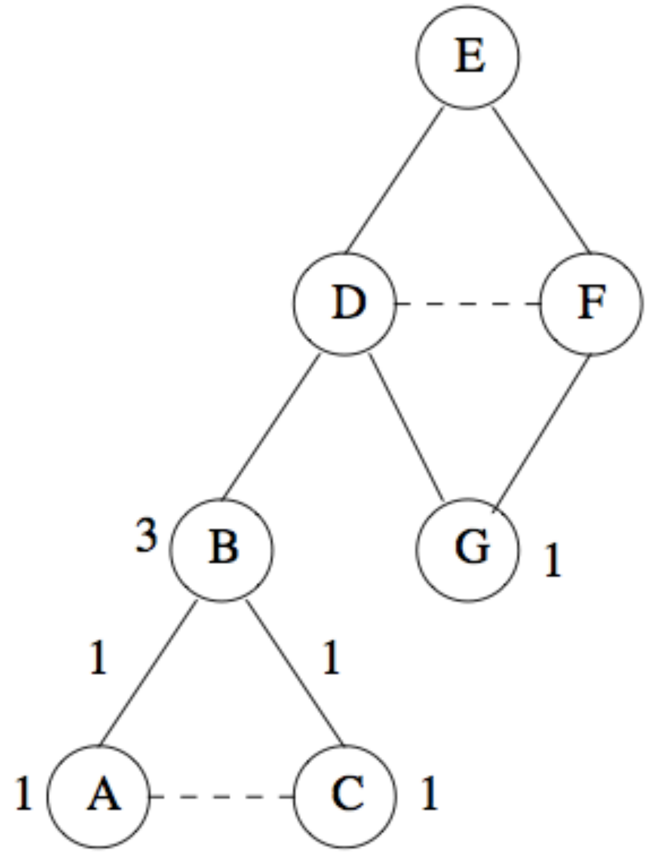
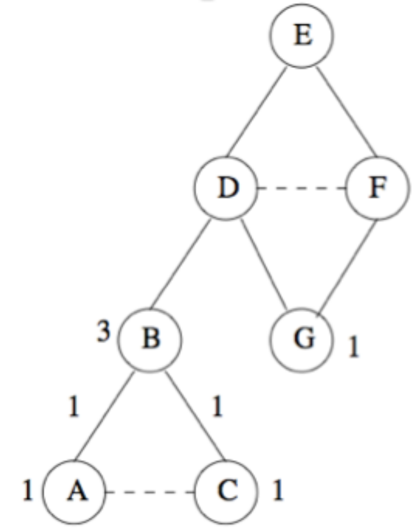


Figure 10.5: Final step of the Girvan-Newman Algorithm – levels 3 and 2

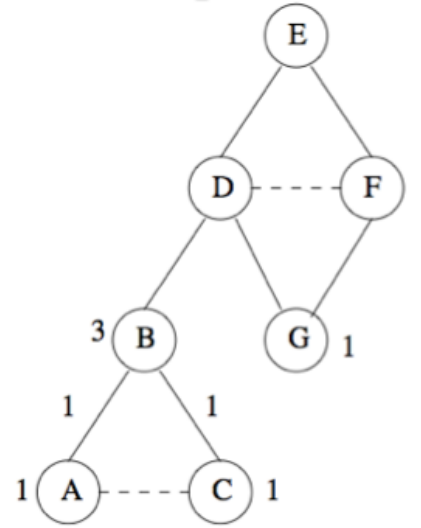
Girvan-Newman Algorithm (5)



Assigning credits:

- A and C are **leaves**: get credit = 1
- Each of these nodes **has only one parent**, so their credit=1 is **given to edges** (B,A) and (B,C)
- At level 2, G is a **leaf**: gets credit = 1
- B gets credit **1 + credit of DAG edges entering from below** = 1 + 1 +1 = 3
- B has only **one parent**, so edge (D,B) **gets entire credit of node** B = 3
- **Node G has 2 parents (D and F): how do we divide credit of G between the edges?**

Girvan-Newman Algorithm (6)



- In this case, **both D and F have just one path from E to each of those nodes**
  - So, **give half credit of node** G **to each of those edges**
  - Credit = 1/(1 + 1) = 0.5
- **In general, how we distribute credit of a node to its edges depends on number of shortest paths**
  - **Say there were 5 shortest paths to D and only 3 to F**
  - **Then credit of edge (D,G) = 5/8 and credit of edge (F,G) = 3/8**
- Node D gets credit = **1 + credits of edges below it** =
  1 + 3 + 0.5 = 4.5
- Node F gets credit = 1 + 0.5 = 1.5
- D has **only one parent**, so Edge (E,D) gets credit = 4.5 from D
- Likewise for F: Edge (E,F) gets credit = 1.5 from F.

Girvan-Newman Algorithm (7): Completion of Credit Calculation starting at node E

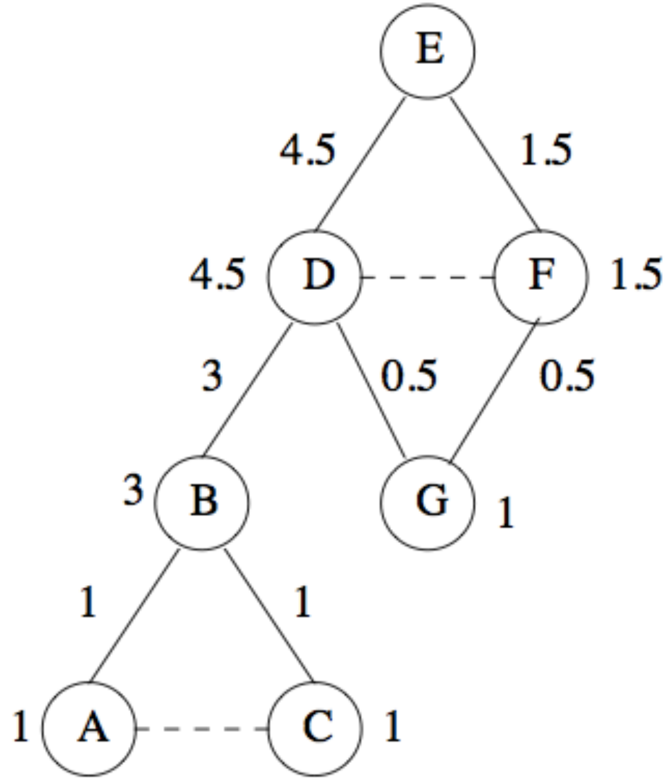


Figure 10.6: Final step of the Girvan-Newman Algorithm – completing the credit calculation

Girvan-Newman Algorithm (8): Overall Betweenness Calculation

- To complete betweenness calculation, must:
  - **Repeat this for every node as root**
  - **Sum the contributions** on each edge
  - **Divide by 2** to get true betweenness
    - since every shortest path will be counted twice, once for each of its endpoints

Using Betweenness to Find Communities: Clustering

- **Betweenness scores for edges of a graph behave something like a distance metric**
  - Not a true distance metric
- **Could cluster by taking edges in increasing order of betweenness and adding to graph one at a time**
  - At each step, connected components of graph form clusters

- **Girvan-Newman: Start with the graph and all its edges and remove edges with highest betweenness**
  - **Continue until graph has broken into suitable number of connected components**
  - Divisive hierarchical clustering (top down)
    - Start with one cluster (the social network) and recursively split it.

Using Betweenness to Find Communities (2)

- (B,D) has highest betweenness (12)
- Removing edge would give natural communities we identified earlier: {A,B,C} and {D,E,F,G}

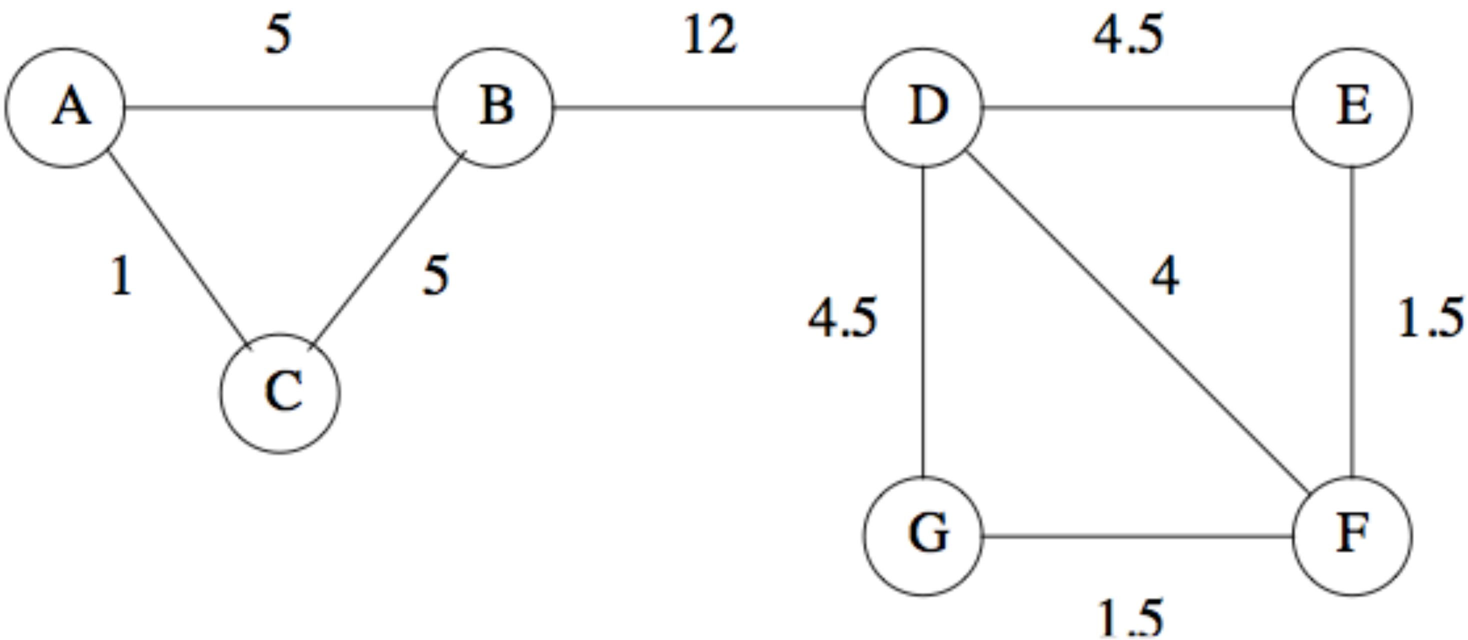


Figure 10.7: Betweenness scores for the graph of Fig. 10.1

Using Betweenness to Find Communities (3): Thresholding

- Could continue to remove edges with highest betweenness

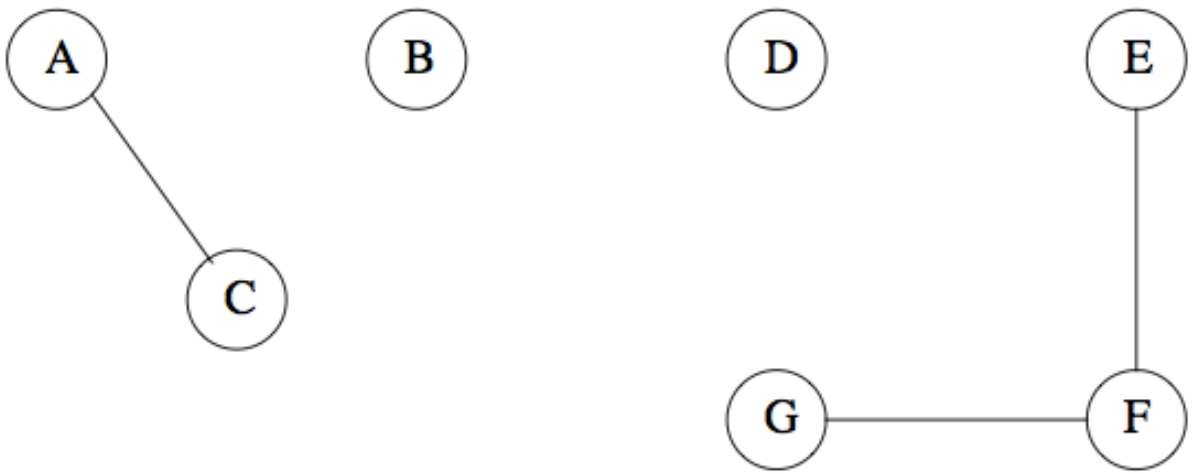


Figure 10.8: All the edges with betweenness 4 or more have been removed

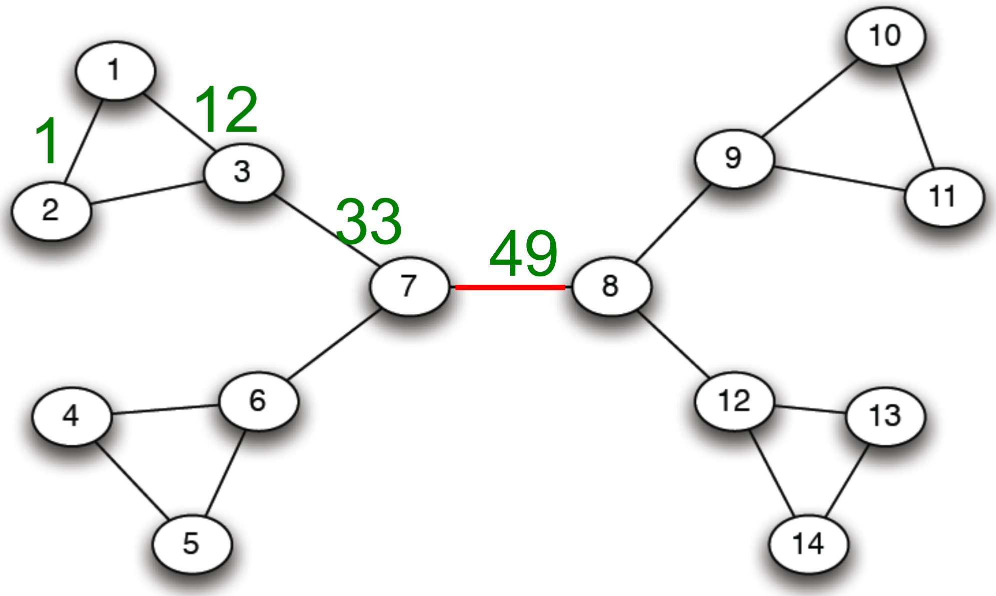
Run Girvan-Newman Iteratively for Community Detection

- Recall: Divisive hierarchical clustering based on the notion of edge **betweenness**:

  Number of shortest paths passing through the edge

- **Girvan-Newman Algorithm:**
  - » **Undirected unweighted networks**
  - **Repeat until no edges are left:**
    - Calculate betweenness of edges
    - This time: remove edges with highest betweenness
  - Connected components are communities
  - Gives a hierarchical decomposition of the network.

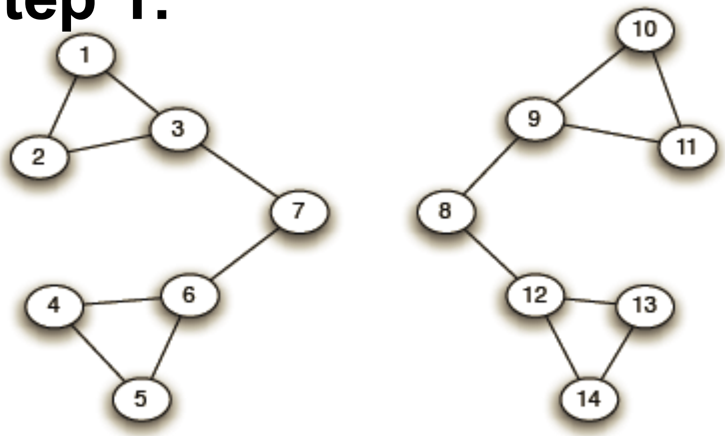
Girvan-Newman: Example



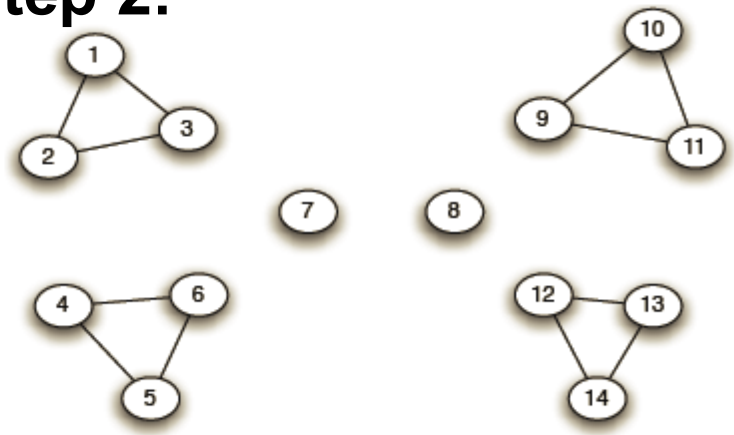
Need to re-compute betweenness at every step

Girvan-Newman: Example

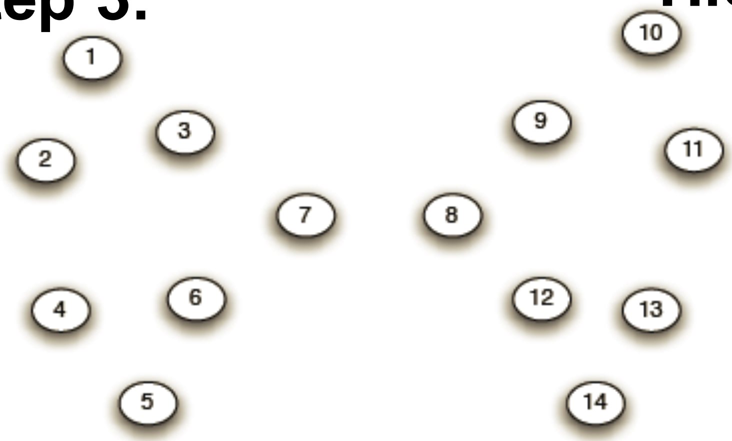
Step 1:



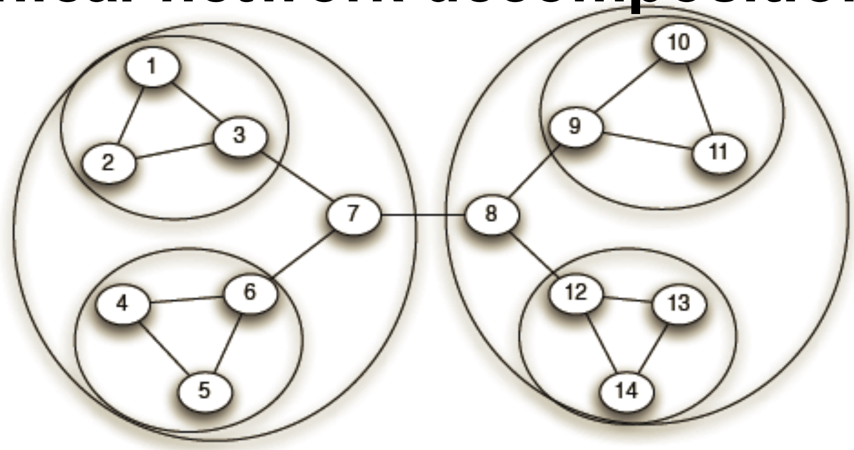
Step 2:



Step 3:

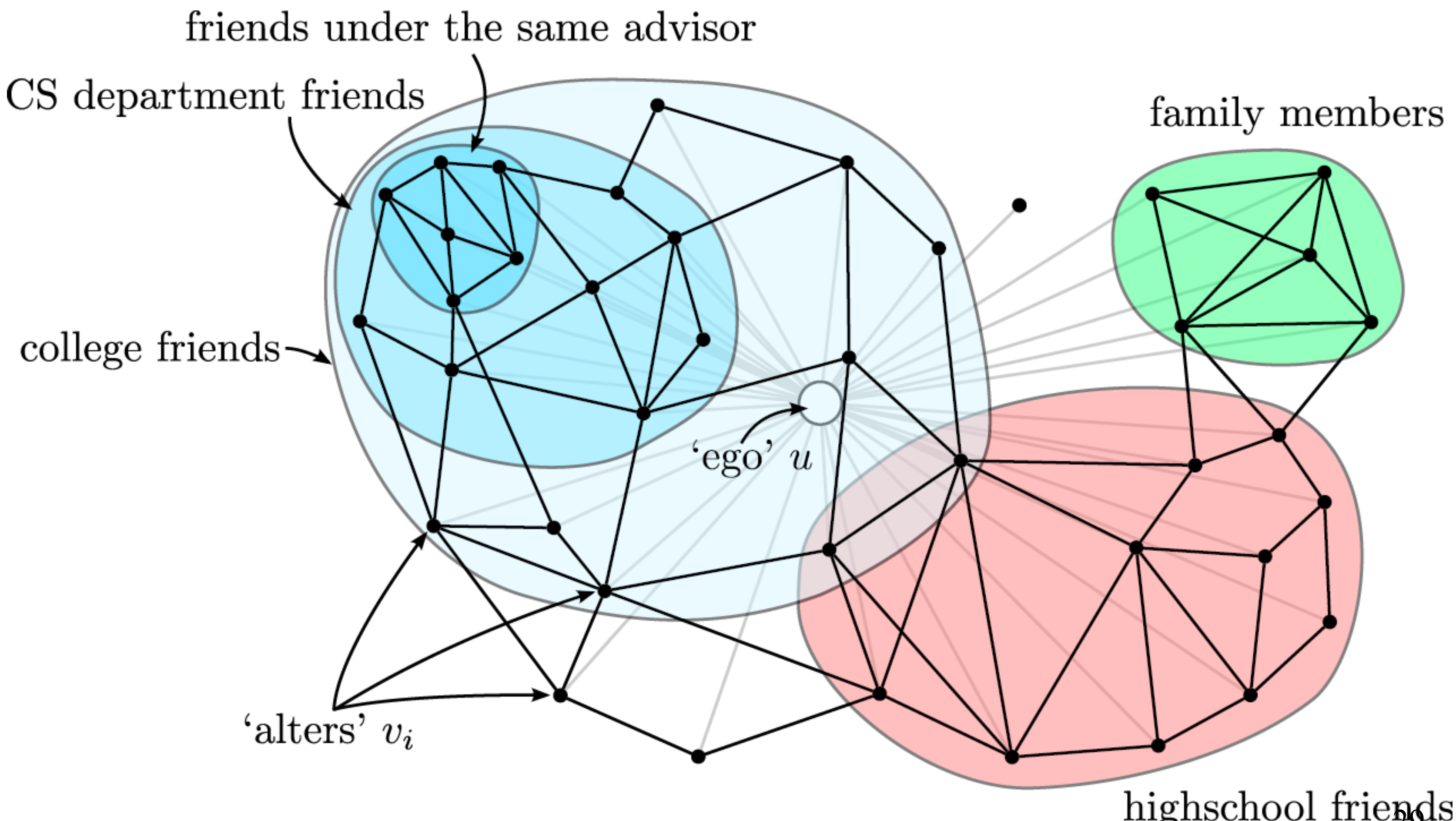


Hierarchical network decomposition:



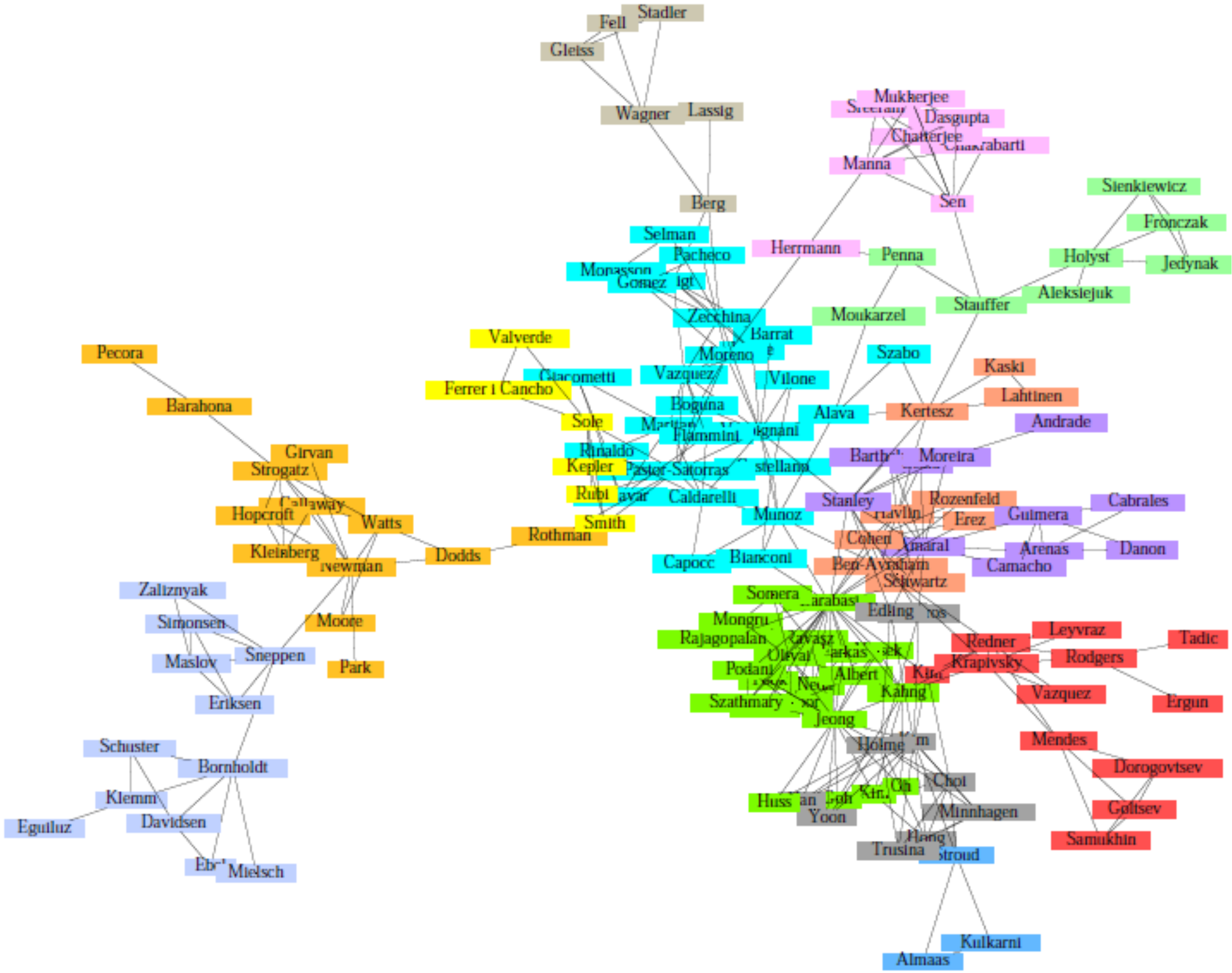
Recall: Twitter & Facebook

- Discovering social circles, circles of trust:



[McAuley, Leskovec: Discovering social circles in ego networks, 2012]

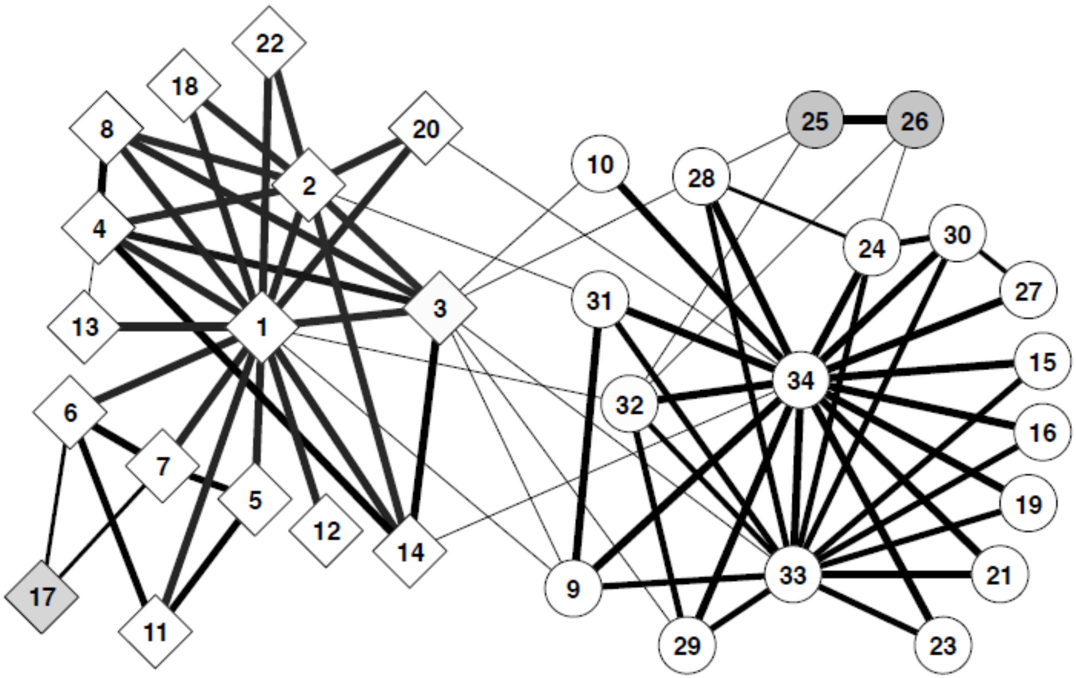
Girvan-Newman: Results

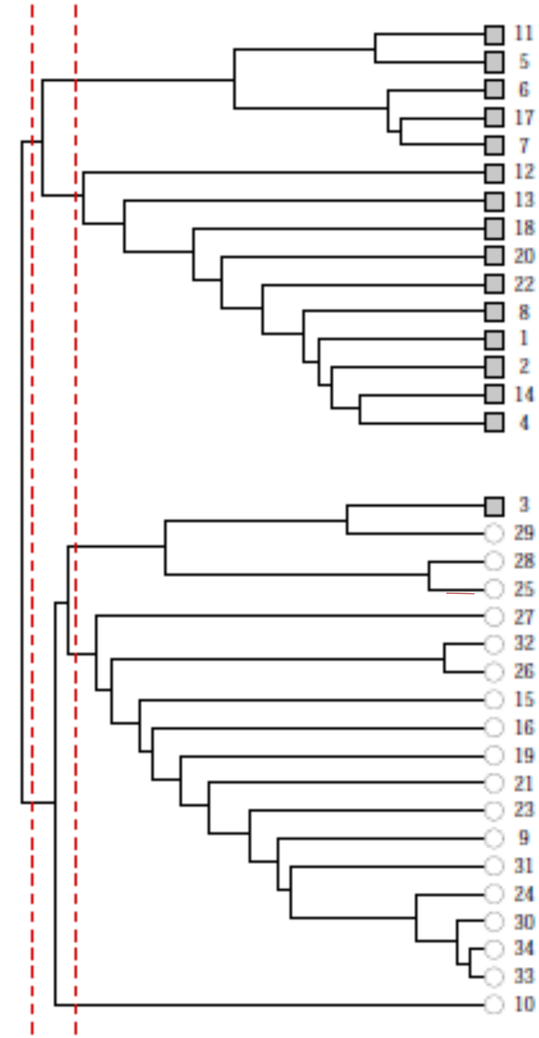


Communities in physics collaborations

Girvan-Newman: Results

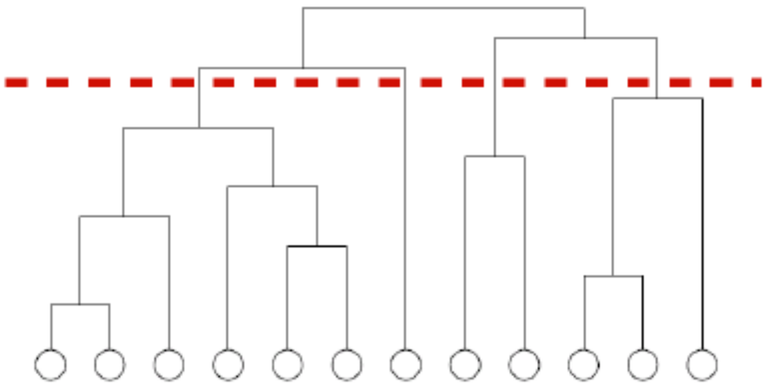
- **Zachary's Karate club:** Hierarchical decomposition





WE NEED TO RESOLVE 2 QUESTIONS

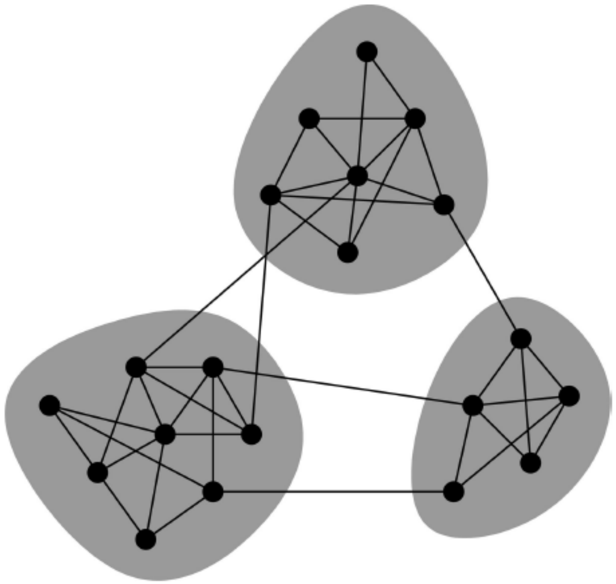
1. **How to compute betweenness?**
2. **How to select the number of clusters?**



Network Communities

- **Communities:** sets of **tightly connected nodes**
- Define: **Modularity Q**
  - A measure of how well a network is partitioned into communities
  - Given a partitioning of the network into groups $s \in S$:

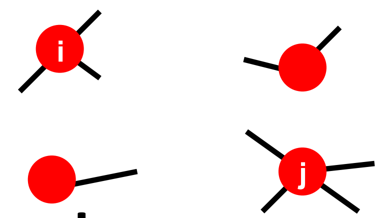
$$Q = \sum_{s \in S} [\ (\#\ \text{edges within group } s) - (\text{expected}\ \#\ \text{edges within group } s)\ ]$$



Need a null model!

Null Model: Configuration Model

- **Given real G on n nodes and m edges, construct rewired network G'**
  - Same degree distribution but random connections
  - Consider G' as a multigraph
  - **The expected number of edges between nodes i and j** of degrees k_i and k_j equals to: $k_i \cdot \frac{k_j}{2m} = \frac{k_i k_j}{2m}$
    - The expected number of edges in (multigraph) **G'**:
      - $= \frac{1}{2} \sum_{i \in N} \sum_{j \in N} \frac{k_i k_j}{2m} = \frac{1}{2} \cdot \frac{1}{2m} \sum_{i \in N} k_i \left(\sum_{j \in N} k_j \right) =$
      - $= \frac{1}{4m} 2m \cdot 2m = m$



Note: $\sum_{u \in N} k_u = 2m$

Modularity

- **Modularity of partitioning S of graph G:**
  - **Q = $\sum_{s\in S}$ [(# edges within group *s*) – (expected # edges within group *s*)]**
  - $Q(G,S) = \underbrace{\frac{1}{2m}}_{\text{Normalizing cost.: -1<Q<1}}\sum_{s\in S}\sum_{i\in s}\sum_{j\in s}\left(A_{ij} - \frac{k_i k_j}{2m}\right)$

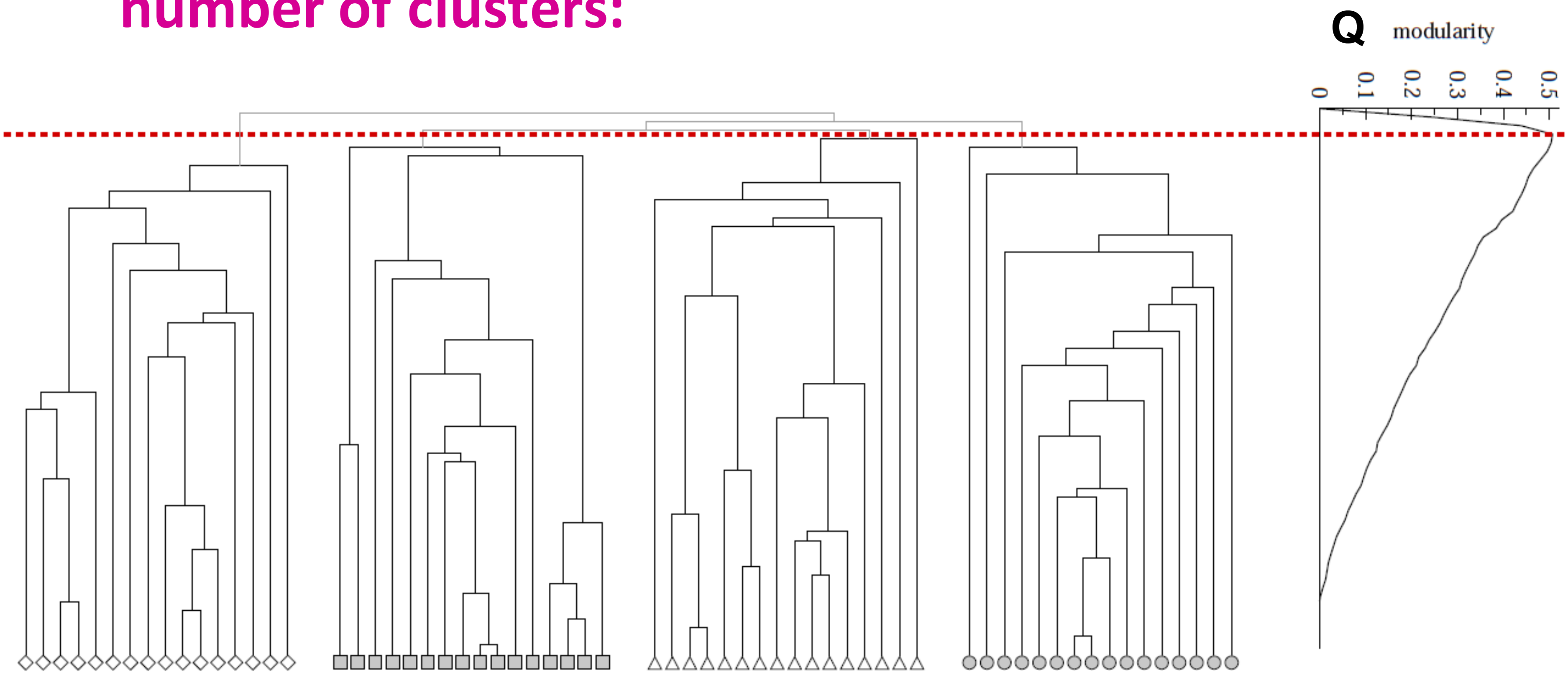
Normalizing cost.: -1<Q<1

A_{ij} = 1 if i connects j, 0 else

- **Modularity values take range [−1,1]**
  - It is positive if the number of edges within groups exceeds the expected number
  - **0.3-0.7<Q** means significant community structure.

Modularity: Number of clusters

- Modularity is useful for selecting the number of clusters:



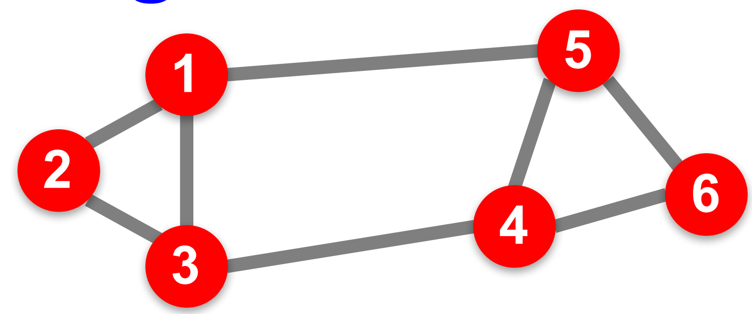
Another approach to organizing social-network graphs

SPECTRAL CLUSTERING

Partitioning Graphs

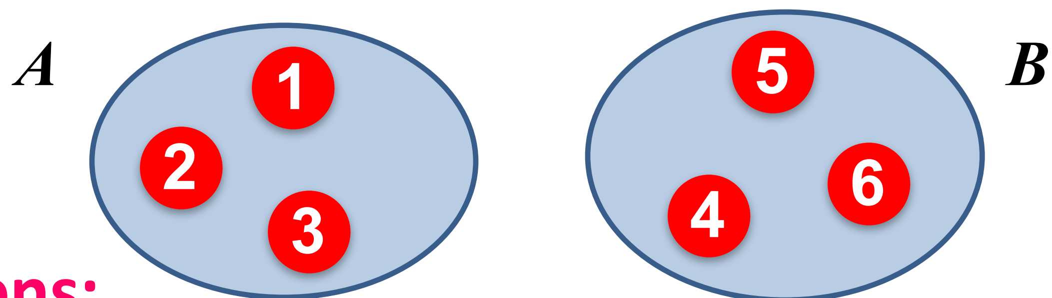
- Another approach to organizing social networking graphs
- Problem: partitioning a graph to minimize the number of edges that connect different components (communities)
- Goal of minimizing the cut size
- If you just joined Facebook with only one friend
  - Don't want to partition the graph with you disconnected from rest of the world
  - Want components to be not too unequal in size.

Graph Partitioning



- **Undirected graph**

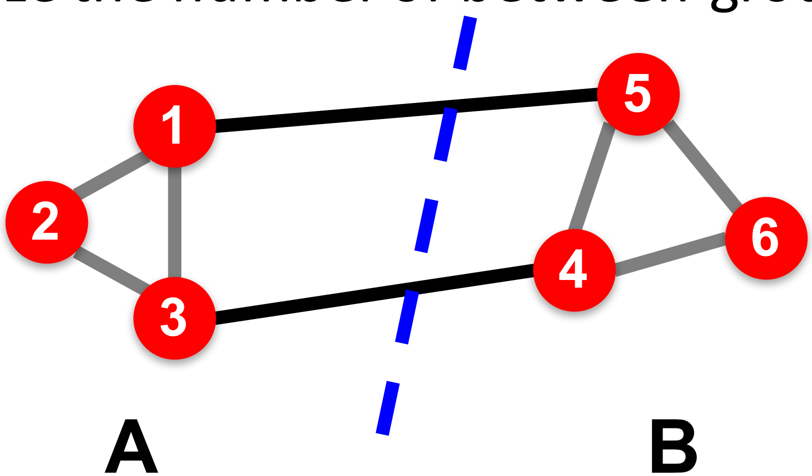
- **Bi-partitioning task:**
  - Divide vertices into two disjoint groups



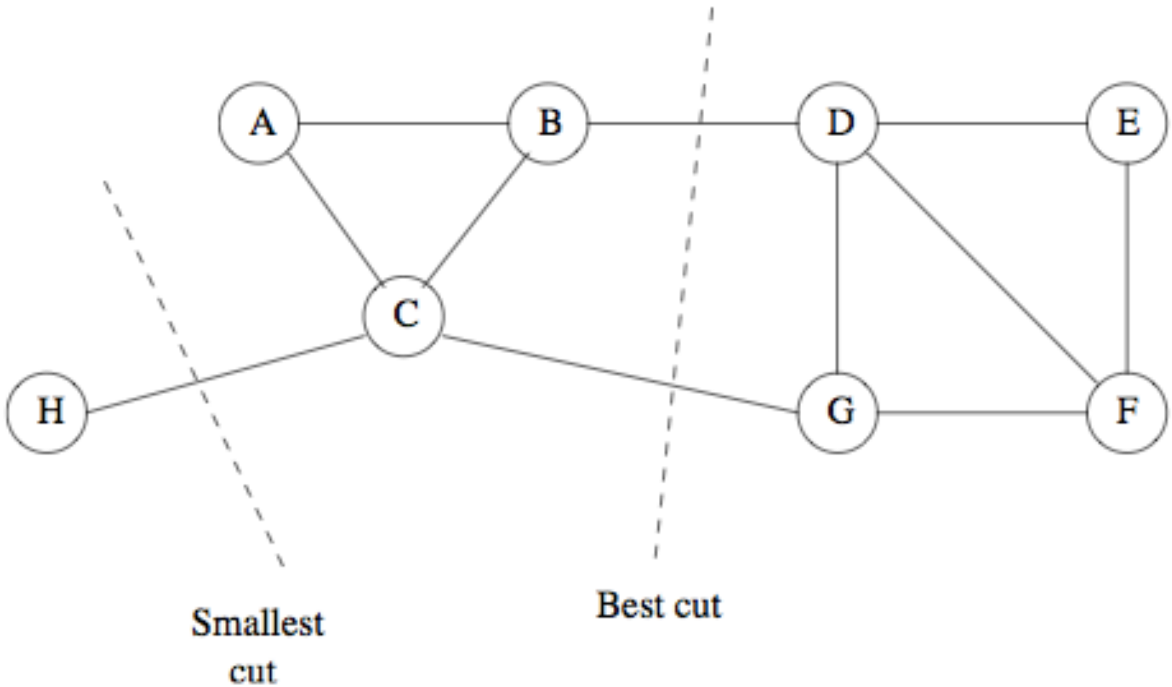
- **Questions:**
  - How can we define a "good" partition of ?
  - How can we efficiently identify such a partition?

Graph Partitioning

- **What makes a good partition?**
  - Divide nodes into two sets so that the **cut (set of edges that connect nodes in different sets) is minimized**
  - Want the two sets to be approximately equal in size
  - Maximize the number of within-group connections
  - Minimize the number of between-group connections



Example 10.14

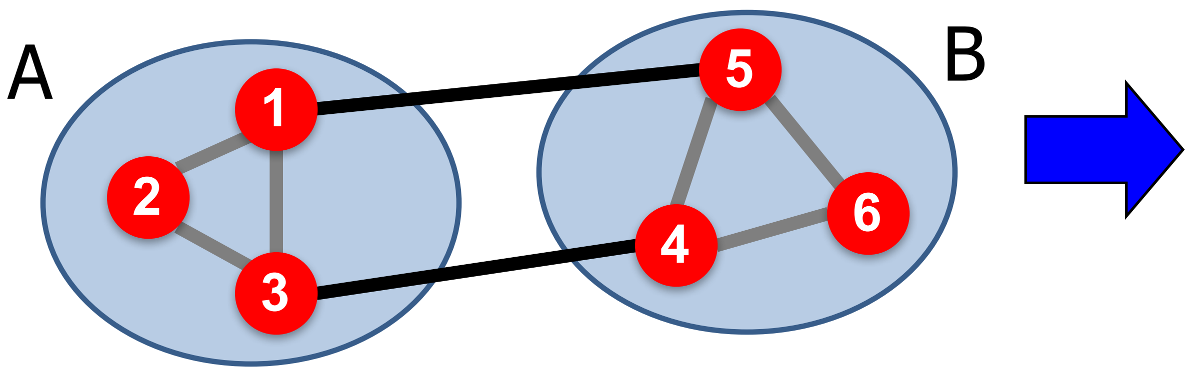


- If we minimize cut: best choice is to put H in one set, other nodes in other set
- But: we reject partitions where one set is too small
- Better is to use cut with (B,D) and (C,G)
- **Smallest cut is not necessarily the best cut**

Graph Cuts

- **Express partitioning objectives as a function of the "edge cut" of the partition**
- **Cut:** Set of edges with only one vertex in a group:

$$cut(A,B) = \sum_{i \in A, j \in B} w_{ij}$$



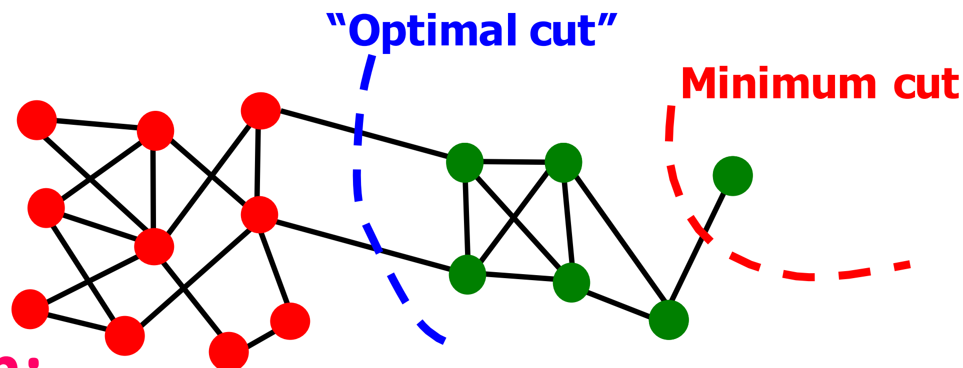
$cut(A,B) = 2$

Graph Cut Criterion

- **Criterion: Minimum-cut**
  - Minimize weight of connections between groups

$$\arg\min_{A,B} \, cut(A,B)$$

- **Degenerate case:**



- **Problem:**
  - Only considers external cluster connections
  - Does not consider internal cluster connectivity

Graph Cut Criteria

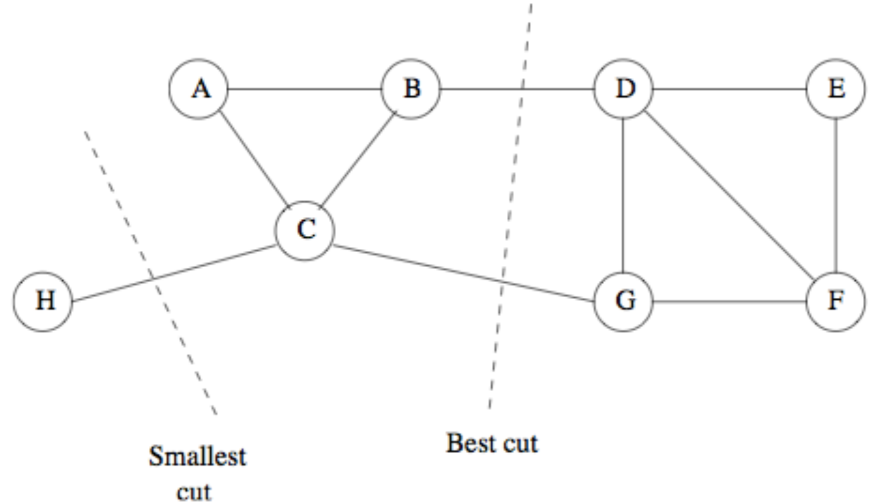
- **Criterion:** **Normalized-cut** [Shi-Malik, ’97]
  - Connectivity between groups relative to the density of each group

$$ncut(A,B) = \frac{cut(A,B)}{vol(A)} + \frac{cut(A,B)}{vol(B)}$$

$\boldsymbol{vol(A)}$**:** total weight of the edges with at least one endpoint in $\boldsymbol{A}$: $\boldsymbol{vol(A) = \sum_{i \in A} k_i}$

- **Why use this criterion?**
  - Produces more balanced partitions
- **How do we efficiently find a good partition?**
  - **Problem:** Computing optimal cut is NP-hard

Example 10.15



- Partition nodes of graph into two disjoint sets S and T
- Normalized Cut for S and T is:

$$\frac{\text{Cut}(S,T)}{\text{Vol}(S)} + \frac{\text{Cut}(S,T)}{\text{Vol}(T)}$$

- If we choose S={H} and T={A,B,C,D,E,F,G} then Cut(S,T) = 1
  - Vol(S) = 1 (number of edges with at least one end in S)
  - Vol(T) = 11: all edges have at least one node in T
  - Normalized cut is 1/1 + 1/11 = 1.09
- For cut (B,D) and (C,G): S = {A,B,C,H}, T = {D,E,F,G}, Cut(S,T) = 2
- Vol(S) = 6, Vol(T) = 7, normalized cut: 2/6 + 2/7 = 0.62

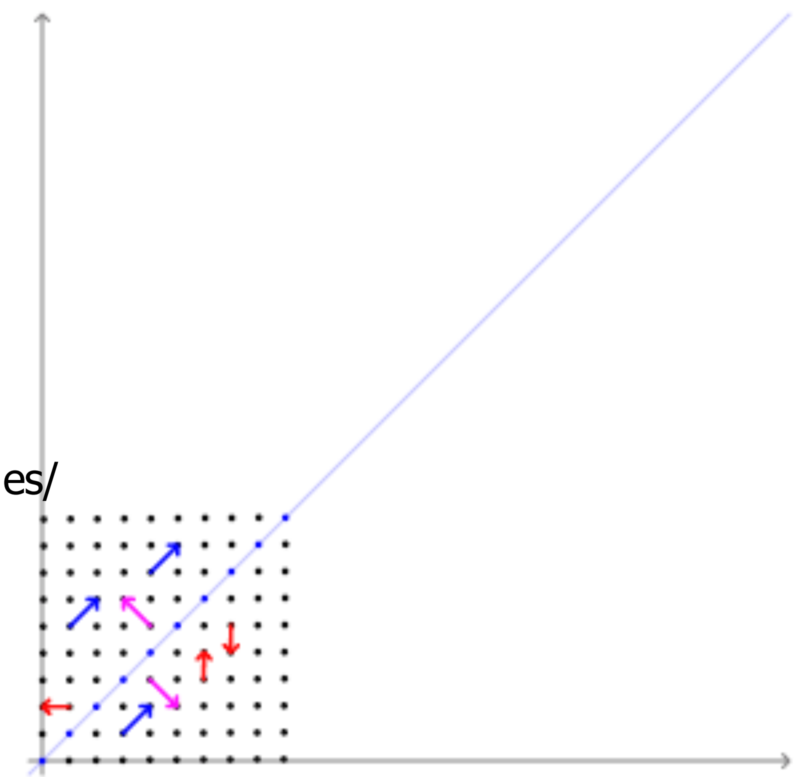
Using Matrix Algebra to Find Good Graph Partitions

- Three matrices that describe aspects of a graph:
  - Adjacency Matrix
  - Degree Matrix
  - Laplacian Matrix: difference between degree and adjacency matrix
- Then get a good idea of how to partition graph from eigenvalues and eigenvectors of its Laplacian matrix.

Recall: Eigenvalues and Eigenvectors

- The transformation matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ preserves the direction of vectors parallel to $\mathbf{v} = (1,-1)^T$ (in purple) and $\mathbf{w} = (1,1)^T$ (in blue). The vectors in red are not parallel to either eigenvector, so, their directions are changed by the transformation.

$$A\mathbf{v} = \lambda\mathbf{v}$$



http://setosa.io/ev/eigenvectors-and-eigenvalues/

https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

Spectral Graph Partitioning

- A: adjacency matrix of undirected **G**
  - A_{ij} **=1** if $(\boldsymbol{i},\boldsymbol{j})$ is an edge, else **0**
- x is a vector in $\Re^n$ with components $(\boldsymbol{x_1},\ldots,\boldsymbol{x_n})$
  - Think of it as a label/value of each node of $\boldsymbol{G}$
- **What is the meaning of $A\cdot x$?**

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$y_i = \sum_{j=1}^{n} A_{ij} xj = \sum_{(i,j)\in E} x_j$$

- **Entry y_i is a sum of labels x_j of neighbors of i**

What is the meaning of *Ax*?

- ***j**^{th}* **coordinate of $A \cdot x$:**
  - Sum of the x-values of neighbors of j
  - Make this a new value at node j

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{A \cdot x = \lambda \cdot x}$$

- **Spectral Graph Theory:**
  - Analyze the "spectrum" of matrix representing $\boldsymbol{G}$
  - **Spectrum:** Eigenvectors $\boldsymbol{x_i}$ of a graph, ordered by the magnitude (strength) of their corresponding eigenvalues $\boldsymbol{\lambda_i}$:

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

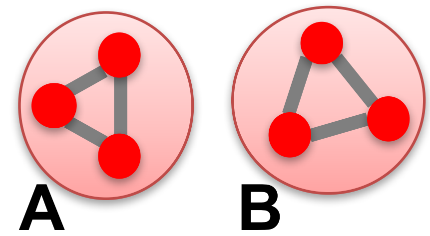
Example: d-regular graph

- Suppose all nodes in $\boldsymbol{G}$ have degree $\boldsymbol{d}$ and $\boldsymbol{G}$ is connected
- **What are some eigenvalues/vectors of $\boldsymbol{G}$?**
- $\boldsymbol{A}.\boldsymbol{x} = \boldsymbol{\lambda} \cdot \boldsymbol{x}$ What is $\boldsymbol{\lambda}$? What $\boldsymbol{x}$?
  - **Let's try:** $\boldsymbol{x} = (\mathbf{1}, \mathbf{1}, \dots, \mathbf{1})$
  - **Then:** $\boldsymbol{A} \cdot \boldsymbol{x} = (\boldsymbol{d}, \boldsymbol{d}, \dots, \boldsymbol{d}) = \boldsymbol{\lambda} \cdot \boldsymbol{x}$. **So:** $\boldsymbol{\lambda} = \boldsymbol{d}$
  - **We found eigenpair of $\boldsymbol{G}$:** $\boldsymbol{x} = (\mathbf{1}, \mathbf{1}, \dots, \mathbf{1})$, $\boldsymbol{\lambda} = \boldsymbol{d}$

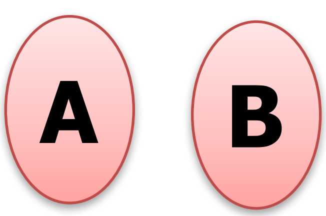
Remember the meaning of $\boldsymbol{y} = \boldsymbol{A}.\boldsymbol{x}$:

$$y_j = \sum_{i=1}^{n} A_{ij} x_i = \sum_{(j,i)\in E} x_i$$

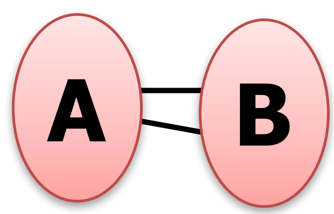
Example: Graph on 2 components



- **What if G is not connected?**
  - G has **2** components, each d-regular
- **What are some eigenvectors?**
  - $x =$ Put all 1s on A and 0s on B or vice versa
    - $x' = (1, \dots, 1, 0, \dots, 0)$ **then** $\mathbf{A} \cdot x' = (d, \dots, d, 0, \dots, 0)$
    - $x'' = (\underbrace{0, \dots, 0}_{|A|}, \underbrace{1, \dots, 1}_{|B|})$ **then** $A \cdot x'' = (0, \dots, 0, d, \dots, d)$
    - And so in both cases the corresponding $\lambda = d$
- **A bit of intuition:**



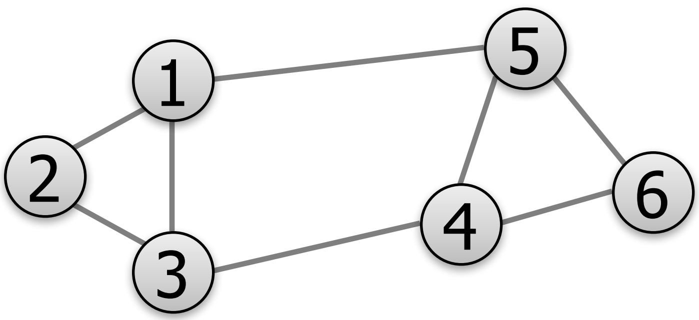
$\lambda_1 = \lambda_2$



$\lambda_1 \approx \lambda_2$

Matrix Representations

- **Adjacency matrix (A):**
  - $n \times n$ matrix
  - $A=[a_{ij}]$, $a_{ij}=1$ if edge between node i and j

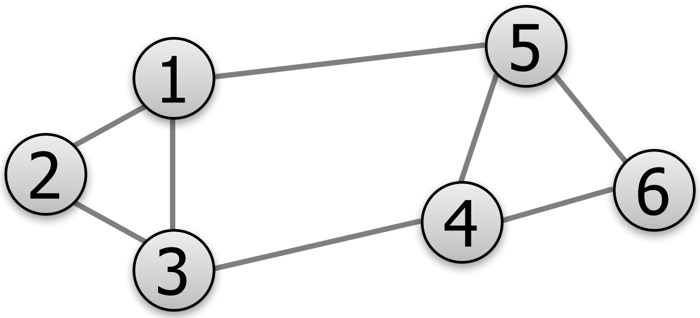


	1	2	3	4	5	6
1	0	1	1	0	1	0
2	1	0	1	0	0	0
3	1	1	0	1	0	0
4	0	0	1	0	1	1
5	1	0	0	1	0	1
6	0	0	0	1	1	0

- **Important properties:**
  - Symmetric matrix
    - because it's an undirected graph
  - Eigenvectors are real and orthogonal
    - orthogonal means dot_product(Eigenvectors_i, Eigenvectors_j) = 0

Matrix Representations

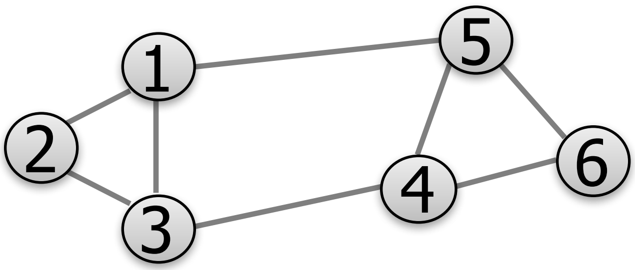
- **Degree matrix (D):**
  - $n \times n$ diagonal matrix
  - $D=[d_{ii}]$, d_{ii} = degree of node i



	1	2	3	4	5	6
1	3	0	0	0	0	0
2	0	2	0	0	0	0
3	0	0	3	0	0	0
4	0	0	0	3	0	0
5	0	0	0	0	3	0
6	0	0	0	0	0	2

Matrix Representations

- **Laplacian matrix (L):**
  - $n \; x \; n$ symmetric matrix



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

$$L = D - A$$

- **What is trivial eigenpair?**
  - $x = (1, \ldots, 1)$ then $L \cdot x = 0$ and so $\lambda = \lambda_1 = 0$ **(smallest eigenvalue)**

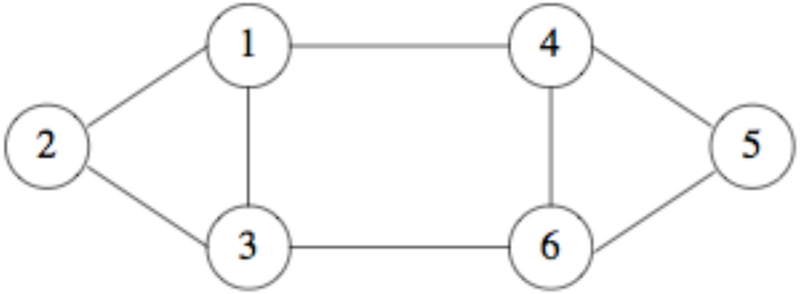
- **Important properties of symmetric matrices:**
  - **Eigenvalues** are non-negative real numbers
  - **Eigenvectors** are real and orthogonal

$$\mathbf{x}^{\mathrm{T}}\mathbf{1} = \sum_{i=1}^{n} x_i = 0$$

Partitioning Graphs Using Eigenvalues and Eigenvectors of Laplacian Matrix

- **Smallest eigenvalue** for every Laplacian matrix is **0**
- Its corresponding eigenvector is [1,1,1,…1]
- To find **second-smallest eigenvalue** for symmetric matrix (such as Lapalcian): (READ THE PAPER! [Shi-Malik, '97]
  - Second smallest eigenvalue is the minimum of $x^T L x$ where $x = [x_1, x_2, \ldots, x_n]$ is a column vector (Rayleigh quotient)
  - Sum of $x_i^2 = 1$
  - x is orthogonal to the eigenvector associated with smallest eigenvalue
- Value of x that achieves this minimum is the **second eigenvector**
- This second smallest eigenvector x will have some positive and some negative components (why?)
- **Partition the graph by taking one set to be the nodes I whose corresponding vector component xi is positive.**

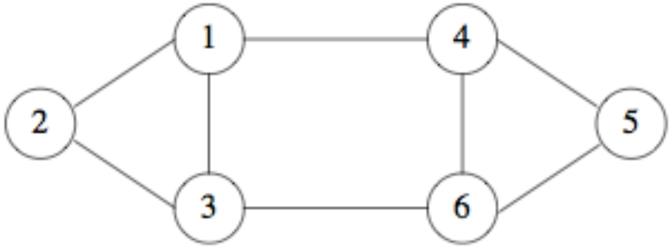
Example 10.19



$$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

- Graph and its Laplacian matrix

Example (cont.)



Eigenvalue	0	1	3	3	4	5
Eigenvector	1	1	-5	-1	-1	-1
	1	2	4	-2	1	0
	1	1	1	3	-1	1
	1	-1	-5	-1	1	1
	1	-2	4	-2	-1	0
	1	-1	1	3	1	-1

- Use standard math package to find all eigenvalues and eigenvectors
  - (Have not scaled eigenvectors to length 1, but could)
- Second eigenvector has three positive and three negative components
- Suggest obvious partitioning of {1,2,3} and {4,5,6}

λ_2 as optimization problem

- **Fact:** **For symmetric matrix M:**

$$\lambda_2 = \min_x \frac{x^T M x}{x^T x}$$

- **What is the meaning of min $x^T L x$ on G?**
  - $x^T L x = \sum_{i,j=1}^{n} L_{ij} x_i x_j = \sum_{i,j=1}^{n} (D_{ij} - A_{ij}) x_i x_j$
  - $= \sum_i D_{ii} x_i^2 - \sum_{(i,j)\in E} 2 x_i x_j$
  - $= \sum_{(i,j)\in E} (\overbrace{x_i^2 + x_j^2}\underbrace{}{} - 2 x_i x_j) = \boldsymbol{\sum_{(i,j)\in E} (x_i - x_j)^2}$

Node $\boldsymbol{i}$ has degree $\boldsymbol{d_i}$. So, value $\boldsymbol{x_i^2}$ needs to be summed up $\boldsymbol{d_i}$ times.
But each edge $(\boldsymbol{i}, \boldsymbol{j})$ has two endpoints so we need $\boldsymbol{x_i^2} + \boldsymbol{x_j^2}$

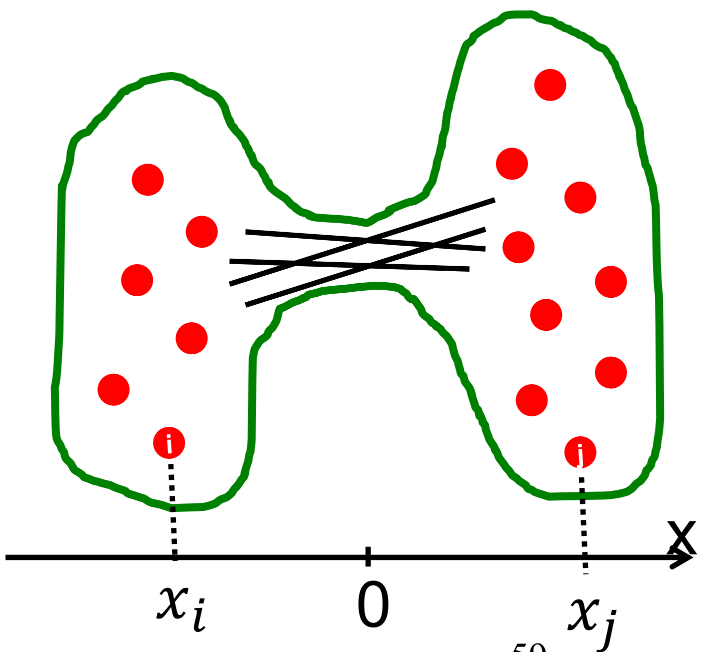
λ_2 as optimization problem (cont'd)

- **What else do we know about x?**
  - $\boldsymbol{x}$ is unit vector: $\sum_i x_i^2 = 1$
  - $\boldsymbol{x}$ is orthogonal to $\mathbf{1}^{\text{st}}$ eigenvector $(\mathbf{1}, \dots, \mathbf{1})$ thus: $\sum_i x_i \cdot 1 = \sum_i x_i = 0$

- **Remember:**

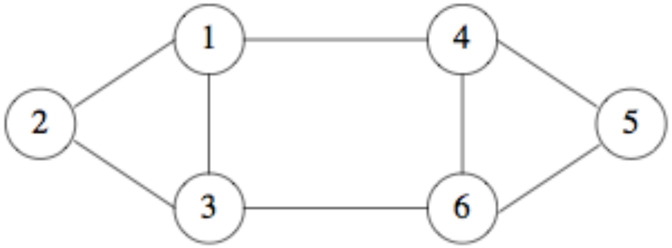
$$\lambda_2 = \min_{\substack{\text{All labelings} \\ \text{of nodes } i \text{ so} \\ \text{that } \sum x_i = 0}} \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_i x_i^2 \; {=1}}$$

We want to assign values x_i to nodes i such that few edges cross 0.
(we want x_i and x_j to subtract each other)



Balance to minimize

Recall: Example



Eigenvalue	0	1	3	3	4	5
Eigenvector	1	1	-5	-1	-1	-1
	1	2	4	-2	1	0
	1	1	1	3	-1	1
	1	-1	-5	-1	1	1
	1	-2	4	-2	-1	0
	1	-1	1	3	1	-1

- Use standard math package to find all eigenvalues and eigenvectors
  - (Have not scaled eigenvectors to length 1, but could)
- Second eigenvector has three positive and three negative components
- Suggest obvious partitioning of {1,2,3} and {4,5,6}

So far…

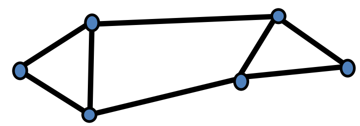
- **How to define a "good" partition of a graph?**
  - Minimize a given graph cut criterion
- **How to efficiently identify such a partition?**
  - Approximate using information provided by the eigenvalues and eigenvectors of a graph
- **Spectral Clustering**
  - Naïve approache:
    - Split at **0**

Spectral Clustering Algorithms

- **Three basic stages:**
  - **1) Pre-processing**
    - Construct a matrix representation of the graph
  - **2) Decomposition**
    - Compute eigenvalues and eigenvectors of the matrix
    - Map each point to a lower-dimensional representation based on one or more eigenvectors
  - **3) Grouping**
    - Assign points to two or more clusters, based on the new representation

Spectral Partitioning Algorithm

- **1) Pre-processing:**
  - Build Laplacian matrix L of the graph



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

- **2) Decomposition:**
  - Find eigenvalues λ and eigenvectors x of the matrix L

$\lambda =$

0.0
1.0
3.0
3.0
4.0
5.0

$x =$

0.4	0.3	-0.5	-0.2	-0.4	-0.5
0.4	0.6	0.4	-0.4	0.4	0.0
0.4	0.3	0.1	0.6	-0.4	0.5
0.4	-0.3	0.1	0.6	0.4	-0.5
0.4	-0.3	-0.5	-0.2	0.4	0.5
0.4	-0.6	0.4	-0.4	-0.4	0.0

  - Map vertices to corresponding components of λ_2

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

How do we now find the clusters?

Spectral Partitioning

- **3) Grouping:**
  - Sort components of reduced 1-dimensional vector
  - Identify clusters by splitting the sorted vector in two
- **How to choose a splitting point?**
  - Naïve approaches:
    - Split at **0** or median value
  - More expensive approaches:
    - Attempt to minimize normalized cut in 1-dimension (sweep over ordering of nodes induced by the eigenvector)

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

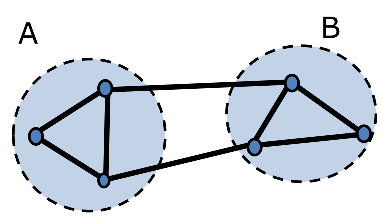
Split at 0:

Cluster A: Positive points

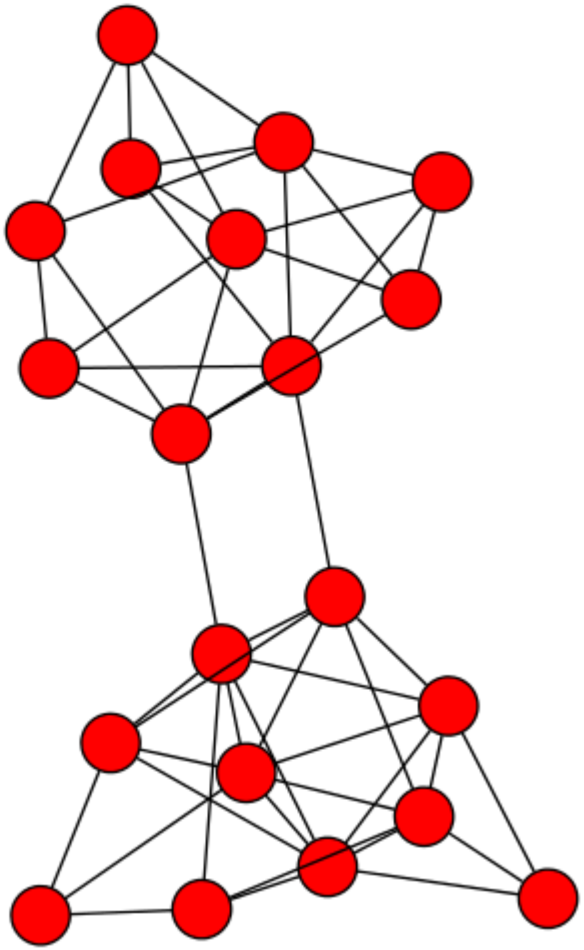
Cluster B: Negative points

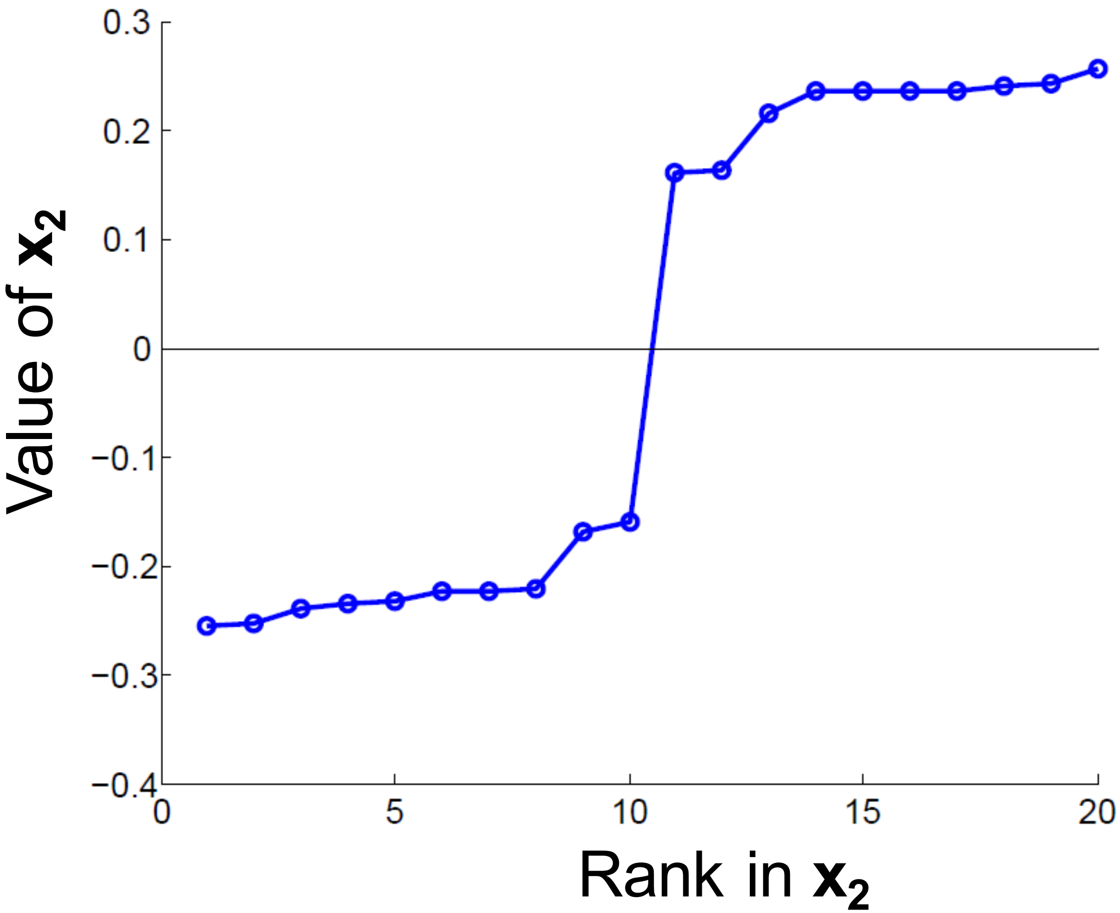
1	0.3
2	0.6
3	0.3

4	-0.3
5	-0.3
6	-0.6

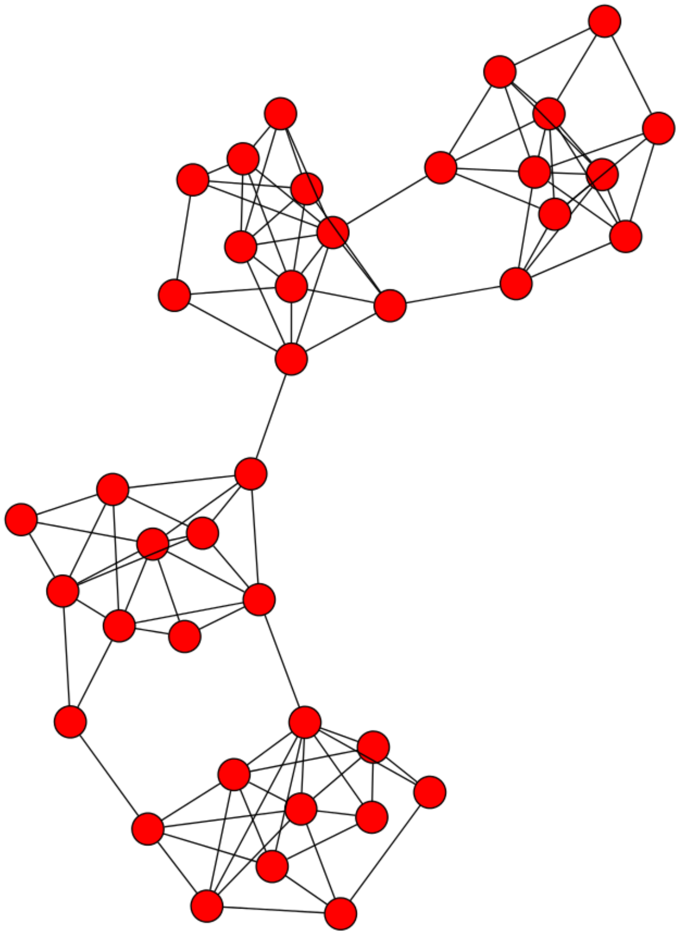


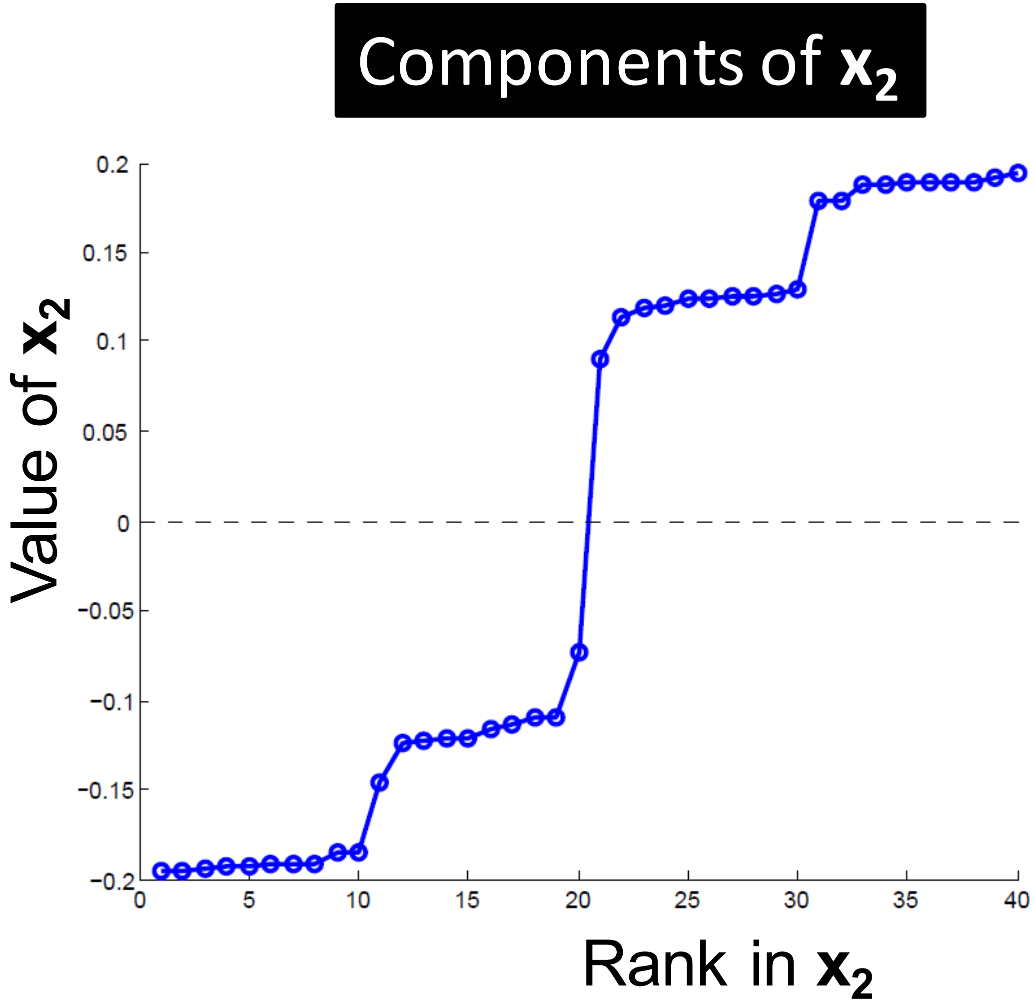
Example: Spectral Partitioning



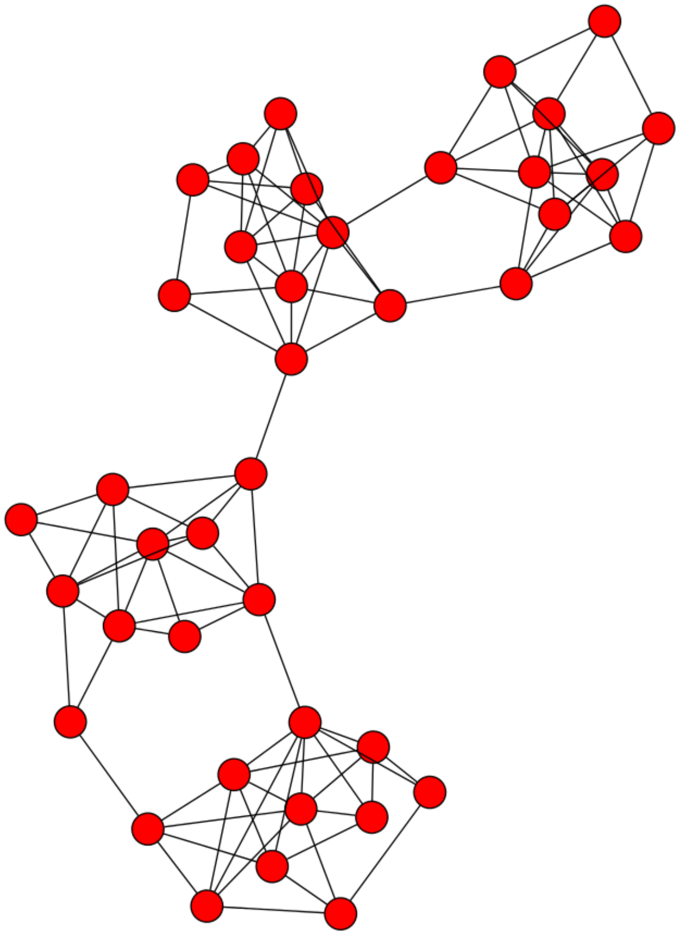


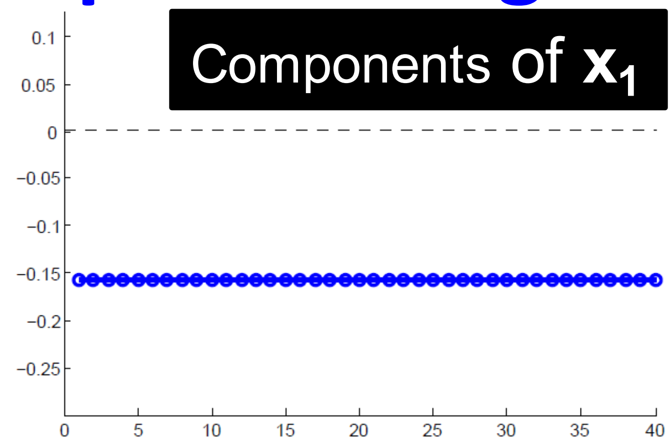
Example: Spectral Partitioning

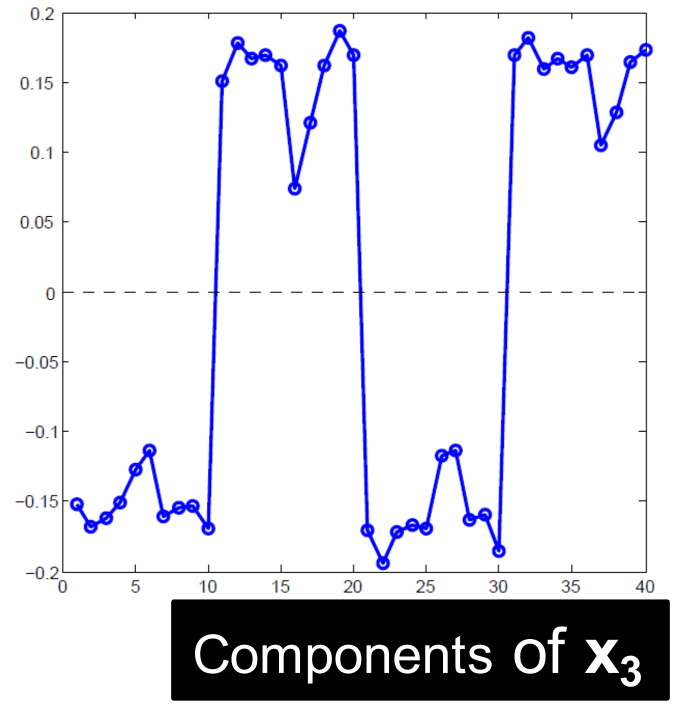




Example: Spectral partitioning







k-Way Spectral Clustering

- **How do we partition a graph into k clusters?**

- **Two basic approaches:**
  - **Recursive bi-partitioning** [Hagen et al., '92]
    - Recursively apply bi-partitioning algorithm in a hierarchical divisive manner
    - Disadvantages: Inefficient, unstable
  - **Cluster multiple eigenvectors** [Shi-Malik, '00]
    - Build a reduced space from multiple eigenvectors
    - Commonly used in recent papers
    - Multiple eigenvectors prevent instability due to information loss
    - A preferable approach…

DIRECT DISCOVERY OF COMMUNITIES: TRAWLING

With slide contributions from P. Desikan; http://www-users.cs.umn.edu/~desikan/

Web community

- Groups of individuals who share common interests, together with the web pages most popular among them

- Web page collections with a shared topic.

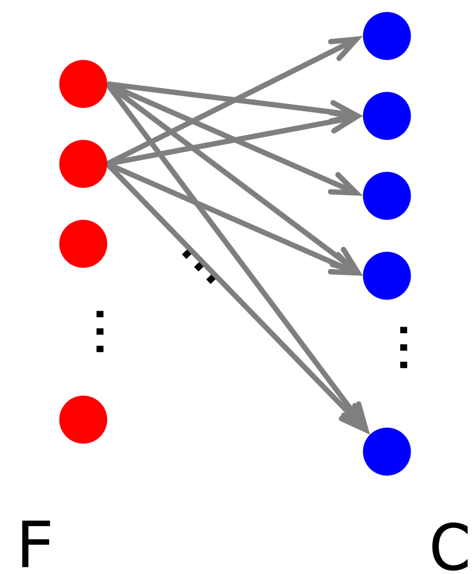
Types of Communities

- Explicitly- defined
  - Communities that manifest themselves as newsgroups or as resource collections on directories such as Yahoo!

- Implicitly- defined
  - Communities that result from nature of content-creation of the web.

Terms and Definitions (1)

- *Directed Bipartite Graph:* A graph whose nodes set can be partitioned into two sets *F* and *C*, and every directed edge in the graph is from a node *u* in *F* to a node *v* in *C*



Terms and Definitions (2)

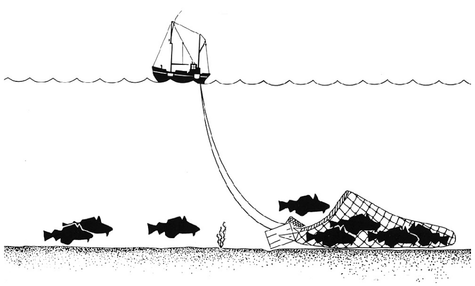
- *Completed Bipartite Graph:* A bipartite graph that contains all possible edges between a vertex of *F* and a vertex of *C*
- *Core:* A complete bipartite sub-graph with at least i nodes from *F* and at least *j* nodes from *C*
  - In the web world, the *i* pages the contains the links are referred to as '*fans*' and the *j* pages that are referenced as '*centers*'

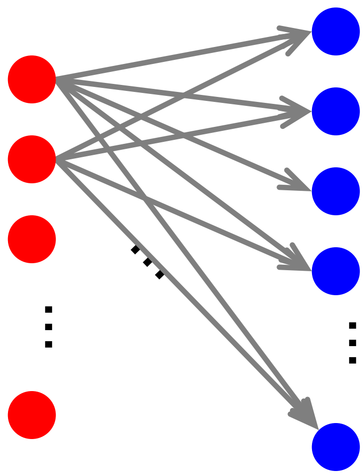
Trawling the Web for Emerging Web Communities

- *Trawling*: Systematic Enumeration of emerging communities from web crawl
- Scan through a web crawl and identify all instances of graph structures that are indicative signatures of communities.

Trawling

- Searching for small communities in the Web graph
- What is the signature of a community / discussion in a Web graph?





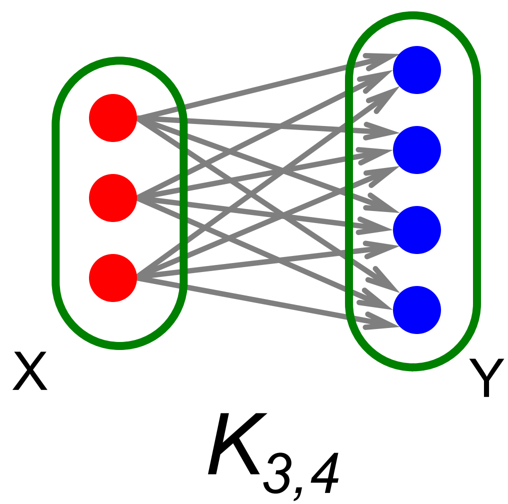
Dense 2-layer

Use this to define "topics":
What the same people on the left talk about on the right
Remember HITS!

Intuition: Many people all talking about the same things

Searching for Small Communities

- **A more well-defined problem:**
  Enumerate complete bipartite subgraphs $\boldsymbol{K_{s,t}}$
  - Where $\boldsymbol{K_{s,t}}$: s nodes on the "left" where each links to the same t other nodes on the "right"



|X| = s = 3
|Y| = t = 4

Fully connected

Frequent Itemset Enumeration

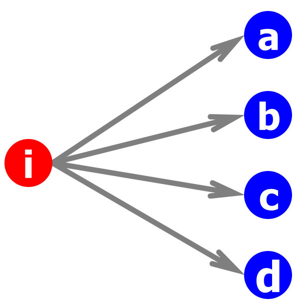
- **Market basket analysis.** Setting:
  - **Market:** Universe U of n items
  - **Baskets:** m subsets of U: $S_1, S_2, \ldots, S_m \subseteq U$
    (S_i is a set of items one person bought)
  - **Support:** Frequency threshold f
- **Goal:**
  - **Find all subsets T s.t. $T \in S_i$ of at least f sets S_i**
    (items in T were bought together at least f times)
- **What's the connection between the itemsets and complete bipartite graphs?**

From Itemsets to Bipartite $K_{s,t}$

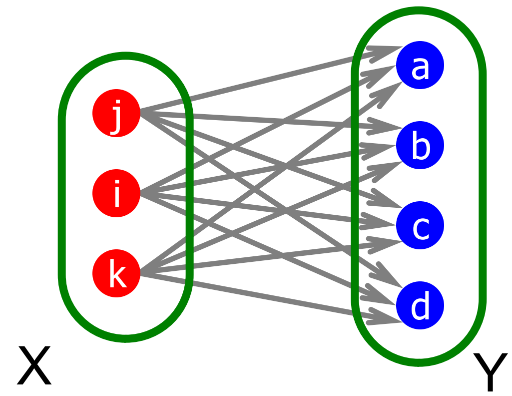
Frequent itemsets = complete bipartite graphs!

How?

- View each node i as a set S_i of nodes i points to
- $K_{s,t}$ = a set Y of size t (all items) that occurs in s (a basket) sets S_i
- Looking for $K_{s,t}$ → set of frequency threshold to s and look at layer t – all frequent sets of size t



S_i={a,b,c,d}

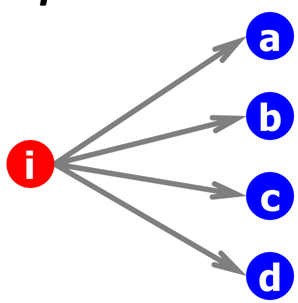


s … minimum support (|X|=s)

t … itemset size (|Y|=t)

From Itemsets to Bipartite $K_{s,t}$

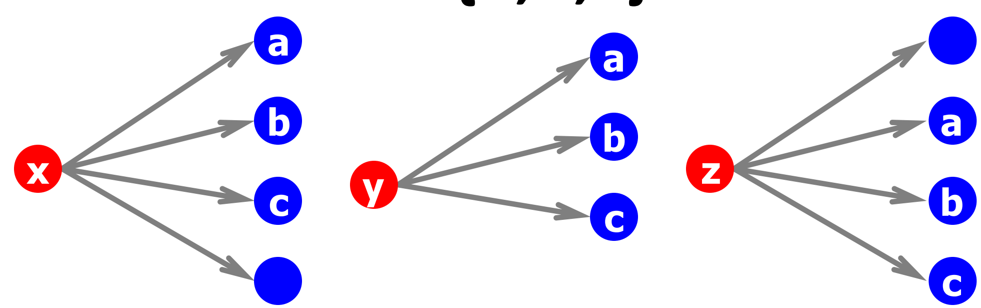
View each node ***i*** as a set $\boldsymbol{S_i}$ of nodes ***i*** points to



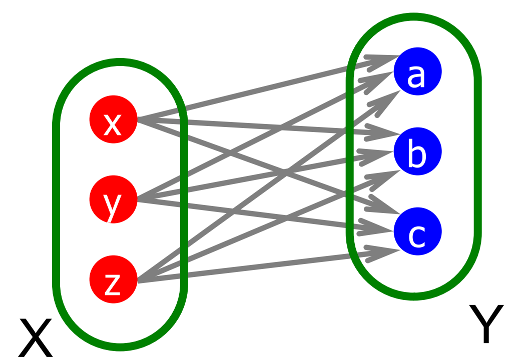
S_i={a,b,c,d}

Find frequent itemsets:
s … minimum support
t … itemset size

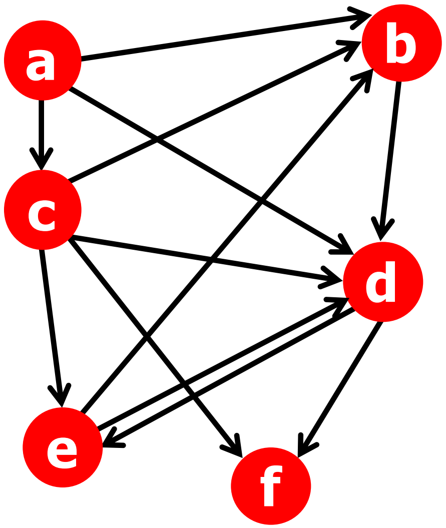
Say we find a **frequent itemset *Y={a,b,c}*** of supp ***s***
So, there are ***s*** nodes that link to all of **{a,b,c}**:



We found *$K_{s,t}$*!
$K_{s,t}$ = a set ***Y*** of size ***t*** that occurs in *s* sets $\boldsymbol{S_i}$



Example



Itemsets:

a = {b,c,d}

b = {d}

c = {b,d,e,f}

d = {e,f}

e = {b,d}

f = {}

- **Support threshold s=2**
  - **{b,d}**: support 3
  - **{e,f}**: support 2
- **And we just found 2 bipartite subgraphs:**