Part III Astrostatistics: Example Sheet 3
Example Class: Monday, 10 Mar 2017, 15:30pm, MR14

1 Doubly Lensed Quasar Time Delay Estimation

Quasar light curves (brightness time series) $f(t)$ (in magnitudes) are often modelled using the Orstein-Uhlenbeck (O-U) process, which has a stochastic differential equation of the form:

$$df(t) = \tau^{-1}[c - f(t)]\, dt + \sigma dW_t \tag{1}$$

where the second term is a Brownian motion (continuous-time limit of a random walk), with the variability scaled by σ, and the first term is a drag term that tends to return the brightness back to the mean level c. The O-U process is a Gaussian process

$$f(t) \sim \mathcal{GP}(m(t), k(t, t')\,) \tag{2}$$

with mean function $m(t) = \mu$ and covariance function or kernel:

$$\mathrm{Cov}[f(t), f(t')] = k(t, t') = A^2 \exp(-|t - t'|/\tau) \tag{3}$$

with characteristic amplitude $A^2 = \tau\sigma^2/2$. The characteristic time for the quasar brightness to revert to the mean c is τ. Hence, astronomers often called this a "damped random walk". In a doubly-lensed quasar system, two images of the same quasar are observed. However, their brightness time series will have a time delay and magnification relative to each other due to the gravitational lensing. Find in the accompanying dataset, time series measurements of the brightness time series of two images of a lensed quasar, $y_1(t)$ and $y_2(t)$. Assume the measurement errors are Gaussian with the given standard deviations. Where possible, write down and derive the relevant equations before you implement them in code.

1. Plot the data. For each image (y_1 or y_2) time series separately, fit an O-U process by optimising the marginal likelihood to estimate c, A, and τ for each image. Are these estimates consistent between the two time series? Estimate the overall relative magnification factor (difference in magnitudes) between the two images. (The multiplicative magnification μ due to the gravitational lens is related to the magnitude shift by $\mu = -2.5 \log \Delta m$).

2. Fixing the values of c, A, and τ you found for each image separately, overplot random light curves drawn from the GP prior on each separate time series dataset. Use the Gaussian Process machinery to estimate the underlying light curve of each image separately. Plot the expectation and standard deviation of the posterior prediction as a function of time.

3. Now write down a likelihood function for the two time series considered jointly, as two copies of the same realisation of the GP but shifted in time by the time delay Δt, and the magnification factor Δm (both relative to y_1), and measured with noise at the observed

times. Thus $y_1(t)$ is a noisy measurement of $f(t)$ and $y_2(t)$ is a noisy measurement of $f(t - \Delta t) + \Delta m$. Using suitable non-informative priors, write down a posterior density $P(\Delta t, \Delta m, c, A, \tau | \boldsymbol{y}_1, \boldsymbol{y}_2)$.

4. Sample from this posterior to estimate Δt and Δm using reasonable initial guesses as starting points. You may fix the O-U process parameters to reasonable values found previously.

5. Overplot the two time series datasets, with y_2 shifted to the y_1 frame by subtracting the estimated Δt and Δm. Now using the O-U parameters you found, plot the posterior estimate of the underlying light curve using the two combined datasets.

Numerical Clue: A proper covariance matrix admits a Cholesky decomposition: $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$, where $\boldsymbol{L}$ is the lower triangular Cholesky factor. The log of the determinant $|\boldsymbol{\Sigma}| = \det \boldsymbol{\Sigma}$ can stably be computed from Equation A.18 of Rasmussen & Williams. If you have computed the Cholesky factor $\boldsymbol{L}$, solutions $\boldsymbol{x}$ to linear equations of the form $\boldsymbol{\Sigma x} = \boldsymbol{b}$, i.e. $\boldsymbol{x} = \boldsymbol{\Sigma}^{-1}\boldsymbol{b}$, can be stably computed using forward/backward substitution, rather than by directly inverting $\boldsymbol{\Sigma}$, as described in Rasmussen & Williams, §A.4.