Homework 5: Stochastic Processes

Richard Ballantyne[a,1]

[a]Dept. of Physics & Astronomy, Western Washington University, 516 High Street, Bellingham WA 98225; [1]E-mail: ballanr@wwu.edu

1. Introduction

Stochastic systems are those in which the properties of the system are determined probabilistically. In physics the typical stochastic (or random) process is a diffusion problem, such as heat diffusion in a gas. The models typically involve too many particles to make calculations for all of them feasible so we look at bulk properties. An interesting example of stochastic system is the growth of clusters that exhibit fractal like qualities (i.e., snowflakes, coastlines, etc.).

In this paper we investigate the growth of a DLA (diffusion limited aggregation) cluster in a 100×100 grid space, comparing clusters with walkers being generated solely along the x-axis to those with walkers being generated radially. In Sec. 2 we describe two growth models and the iterative approach. In Sec. 3 we discuss the visual differences between clusters and their dimensionality, and conclude in Sec. 4.

2. Model

DLA Clusters. DLA (diffusion limited aggregation) Clusters are a random process where particles undergo random walks (simulating Brownian motion) and generate clusters. To set up a cluster we initialize a 2D-array with a nucleus at the origin (center of grid). Then, walkers are generated depending on whether the cluster is going to grow linearly or radially. In either case, the walkers are spawned in an empty grid space within a radius of the origin,

$$r_{\text{init}} = r_{\text{cluster}} + 5 \tag{1}$$

which was calculated every time a walker was spawned. The walker is then allowed to explore by taking steps up, down, left, or right. To keep things simple the probability for choosing any direction was kept even at 1/4 for each direction.

As the walker moved around the grid it would either end up running into the cluster or walk too far away from the origin. Running into the cluster would cause the walker to stop exploring and to stick to the cluster, increasing the cluster size. To speed up computational time the walkers were restricted by both distance from origin and the size of the grid space. The walker was deleted if it moved off of the grid it or if it moved beyond a maximum radius

$$r_{\max} = 1.5 \times r_{\text{init}} \tag{2}$$

After all the walkers had been allowed to explore, and the cluster grown, the dimensionality of the cluster can be calculated. For traditional shapes like lines, discs, and spheres the dimensionality is given by an integer value (1, 2, and 3 respectively) but fractals exhibit fractional values. Dimensionality of a cluster is approximately

$$d_f \approx \log(m - r) \tag{3}$$

where m is the mass of the cluster and r is the radius. In this simulation the mass of each walker was defined as one which means that the mass of the cluster m is the number of walkers in the cluster.

Linear Growth. For a linear growth DLA cluster we restrict the walkers to being generated on the x-axis (center). A random integer of 1 or 2 is then selected (simulating a coin flip) to determine which side of the cluster the new walker spawns on. Then another random integer is determined within r_{init} and the walker is spawned there, repeating this process until the walker finds an empty space. To keep the walkers from sticking to the edge of the grid space the number of walkers used is kept at 10% of the area of the grid space (1000 walkers).

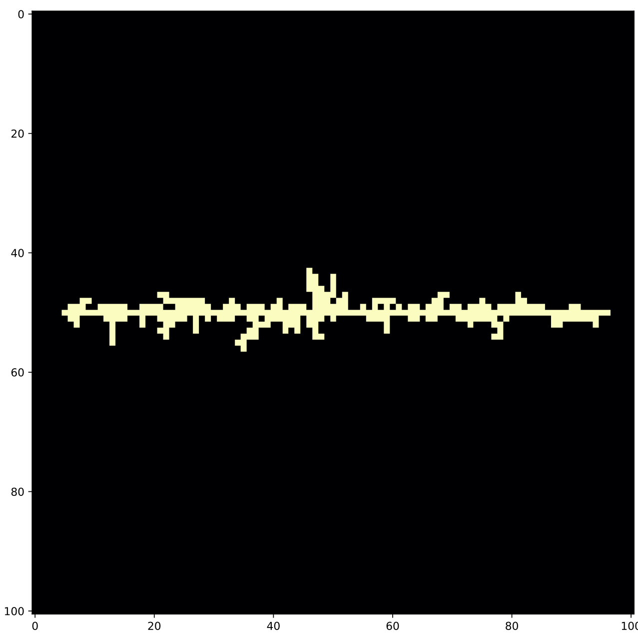

Fig. 1. Linear cluster (yellow) grown on a 100×100 grid space with 1000 walkers.

Radial Growth. For a radially grown DLA cluster we allow the walkers to be generated on a circle centered about the origin (middle of the grid). A random angle θ ($0 < \theta \leq 2\pi$) is generated then r_{init} is multiplied by $\cos(\theta)$ and $\sin(\theta)$ to get the x and y positions to spawn a walker. A radial cluster has a larger surface area than a linear cluster, which means that the walkers also have a larger area they can explore before reaching the edges of the grid space. Thus, the number of walkers was set at 30% of the area of the grid space.

3. Results

Shape and Structure. In Fig. 1 is shown an example of a linear cluster. The cluster shape is roughly that of a line, having only several minor branches off the x-axis. It seems that as walkers are generated along a single line, they are much more likely to aggregate along the x-axis than to make branches in the y direction. There are also very few holes inside the cluster structure itself.

Fig. 2 shows an example of a radially grown cluster. It has an interesting structure, resembling a disc, with four main trunks with many branches. This overall disc shape is to be expected as the walkers are being spawned randomly along a circle centered on the origin. Unlike an Eden growth model which grows roughly as a disc with few holes the radial DLA, as shown here, has many large holes in its interior. This difference is likely due to the DLA walkers being allowed to randomly take a step while the Eden clusters aren't. Another interesting quirk of Fig. 2 is that the main trunks aren't along rotations of 90° but seem to be closer to 120° rotations.

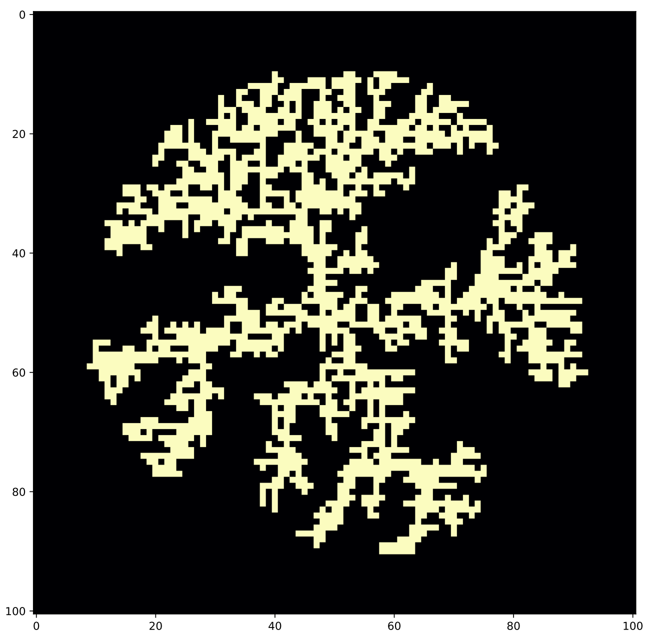

Fig. 2. Radial cluster (yellow) grown on a 100×100 grid space with 3000 walkers.

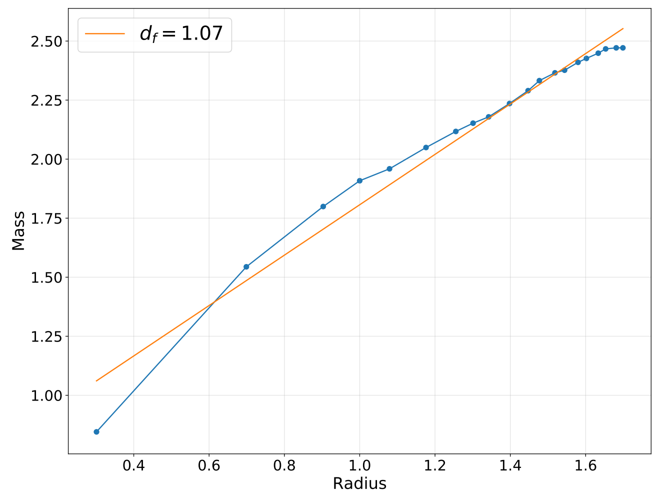

Fig. 3. Log − Log plot of mass vs. radius for the linear DLA cluster (blue) with best-fit dimensionality line (orange). An ideal line has a $d_f = 1$ while the linear cluster has a $d_f \approx 1.07$.

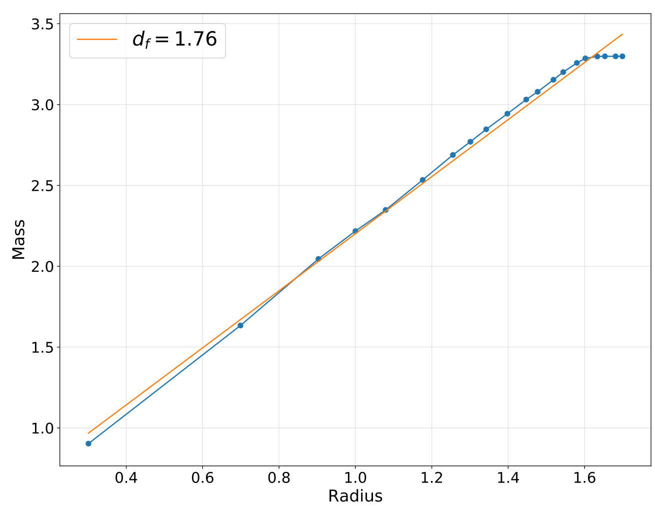

Fig. 4. Log − Log plot of mass vs. radius for the radial DLA cluster (blue) with best-fit dimensionality line (orange). An ideal disc has a $d_f = 2$ while the linear cluster has a $d_f \approx 1.76$.

Dimensionality. Once the clusters were grown it was possible to determine their dimensionality. Because the process of growing DLA clusters is essentially random and the clusters aren't traditional shapes, Eq. 3 is an approximation similar to integrating the gravitational potential of a star over different radii. So, the mass of the cluster (number of particles) was calculated for larger and larger radii. To determine the dimensionality, $\log m$ vs. $\log r$ plots were made and a best fit line was calculated. As best fit line gives the dimensionality of the cluster because if Eq. 3 is unpacked slightly,

$$d_f \approx \log(m - r) = \frac{\log m}{\log r} \quad [4]$$

which is the slope of the best fit line.

Fig. 3 shows the log − log plot for the linearly grown cluster. While the best-fit line doesn't seem to follow the data strikingly well it does give us a dimensionality of 1.07. This is close to what a normal line's dimensionality of 1, but it higher because of the growths above and below the x-axis.

Fig. 4 shows the log − log plot for the radially grown cluster. The mass vs. radius seems to scale linearly which is intuitive as we are summing over discs and the shape of the cluster is roughly as disc. The best-fit line follows the data very well and gives a dimensionality of 1.76. Unlike the linear cluster that has a dimensionality larger than the shape it is similar to, a line, the radial cluster has a dimensionality smaller than a disc's 2. This is natural as a line is a 1D object while the linear cluster is a 2D version of a 1D object, whereas the radial cluster is a 2D version of a 2D object. If the walkers were allowed some height into or out of the figure, then the dimensionality of the radial cluster would go above 2.

4. Conclusion

Fractals are interesting objects because while physics usually uses simplified objects (approximating objects as the closest easy to use shape), objects in nature objects are not perfect, they are fractals. The DLA cluster is an example of a fractal modeling a stochastic process (diffusion limited aggregation).

In the two models shown, the walkers are initiated plays a key part in the overall final structure of the cluster. Restricting the walkers to being spawned along a line resulted in a shape similar to a line while radially spawning the walkers resulted in disc. These cluster shapes had non-integer dimensionalities either larger or smaller than the shapes they are similar to. By allowing walkers to randomly explore a grid space we replicate growths in nature from coastlines to bacterial colonies to the root systems of plants, showing that objects are rarely perfect shapes.