Homework 2: Projectile Motion

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1. Introduction

If you drop a bowling ball and a feather from some height would they hit the ground at the same time? The average person (and even to intro physics students) will say something along the lines that the fall time is independent of mass and then say that bother the feather and bowling ball hit the ground at the same time. This is not quite the case because they are thinking of the ideal case in which we are neglecting air resistance and drag. What then happens if we don't neglect drag? Would changing parameters such as drop height or the objects speed in the x-direction also change the drop time?

In this paper we investigate the traditional projectile motion model but expanding upon it by including and varying drag in addition to varying launch speed, initial height, and object mass. However, an analytic solution for drag is hard to define, so we apply Euler's method to iteratively calculate a numeric solution. In Sec. 2 we describe and extend the simple projectile motion model using Euler's method. In Sec. 3 we discuss the influence that varying launch speed, initial height, object mass, and the drag coefficient has on the fall time and conclude in Sec. 4.

2. Model

Simple Projectile Motion. For projectile motion we want to know how the path an object takes usually in terms of the distance traveled horizontally (Δx) and vertically (Δy). We write these as a Taylor expansion of x and y respectively, dropping terms past the third,

$$x(t) = x_0 + \frac{dx}{dt}\Delta t + \frac{1}{2}\frac{d^2x}{dt^2}\Delta t^2 \qquad [1]$$

$$y(t) = y_0 + \frac{dy}{dt}\Delta t + \frac{1}{2}\frac{d^2y}{dt^2}\Delta t^2 \qquad [2]$$

which are the analytic solutions for an objects position (x, y) at some time t.

Projectile Motion with Drag. The simple projectile motion model works nicely when dealing with larger Δt or if high accuracy isn't an issue. To be more precise requires some additional work; we need to deal with the reality that there is air resistance (let's assume we are on Earth) and therefore drag.

The drag force can be approximated as,

$$F_{drag} \approx -B_1 v - B_2 v^2 \qquad [3]$$

where B_1 and B_2 are determined by the shape of the object. For very low velocities the first term dominates but for any non-low velocity the second term dominates. While B_2 does depend on the shape of the object, it can't be calculated exactly and must be approximated,

$$B_2 = \frac{1}{2}C\rho A \qquad [4]$$

Eq. 1 and eq. 2 then become,

$$x(t) = x_0 + v_x\Delta t + F_{drag,x}\Delta t \qquad [5]$$

$$y(t) = y_0 + v_y\Delta t + (-g + F_{drag,y})\Delta t. \qquad [6]$$

Numeric Solution. Because F_{drag} can't be calculated exactly, we employ Euler's method. Eqs. 5 & 6 need to be split up into several components to iteratively calculate, starting with position and drag,

$$x_{i+1} = x_i + v_{x,i}\Delta t \qquad [7]$$

$$y_{i+1} = y_i + v_{y,i}\Delta t \qquad [8]$$

$$F_{drag,x} = -B_2 v v_x \qquad [9]$$

$$F_{drag,y} = -B_2 v v_y. \qquad [10]$$

After these calculations we then need to use these values to calculate the next step in velocity,

$$v_{x,i+1} = v_{x,i} + (F_{drag,x}/m)\Delta t \qquad [11]$$

$$v_{y,i+1} = v_{y,i} + (-9.81 + F_{drag,y}/m)\Delta t. \qquad [12]$$

And after updating the velocity we loop back through using the new velocity until the object hits the ground. Now that we have all the equations we need to iterate, we can vary parameters to see how much, if any, effect they have on the time it takes for the object to hit the ground. A time step of $\Delta t = 0.1$ s was chosen as a smaller value didn't provide any further useful information.

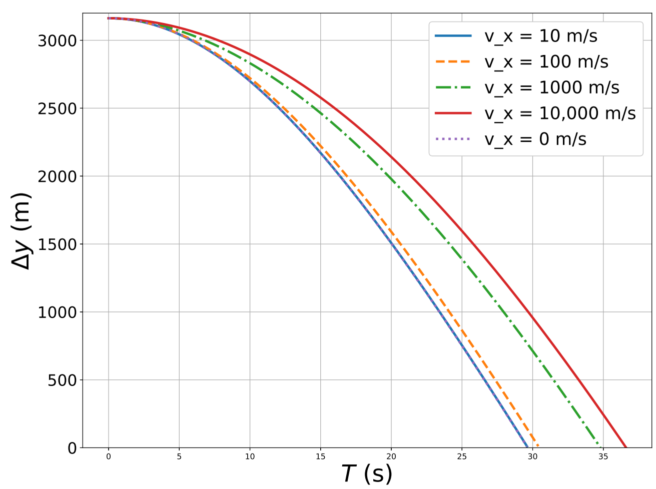

Fig. 1. v_x testing. As v_x gets larger, the curve seems to converge on a T value around 37.5. As v_x gets even larger it hits a limit and becomes not physical.

3. Results

Varying v_x. In the problem we are given that one of the objects is initially fired with a horizontal velocity of 10 m/s. In Fig. 1 several different values for v_x have been tested. v_x was varied over several orders of magnitude and several interesting things occurred.

First, v_x actually increased the drop time, which is counter to our naive intuition that v_x shouldn't influence the drop time at all. Given this new revelation, we would then expect that as v_x increases the time to hit the ground increases as well. This does occur but it seems to hit an upper limit. At $v_x = 10,000$ m/s, $T = 36.7$ s which is ~6 seconds longer than $v_x = 10$ m/s. However, increasing this further causes the drag term in Eq. 12 to be larger than gravitational effects. This causes the object to actually travel upwards ($T \to \infty$) and is not physical.

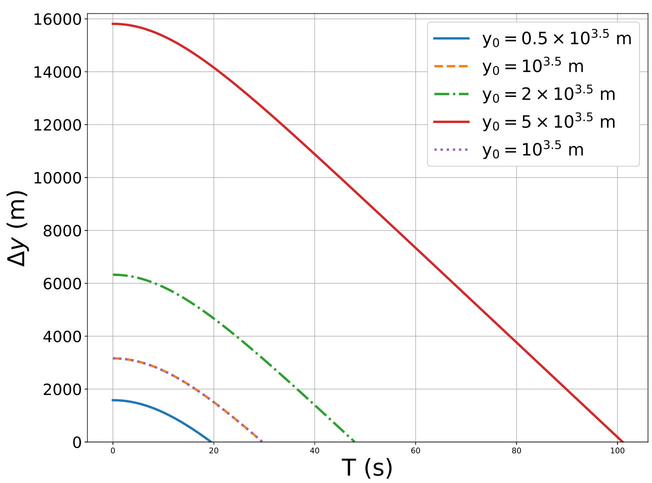

Fig. 2. y_0 testing. As you increase the initial height it takes longer to hit the ground. This doesn't deviate significantly from our idealized projectile motion expectations.

Varying y_0. The next case is how changing the initial height, y_0, that the objects are dropped from. As one would expect, as you increase y_0, T increases roughly linearly as shown in Fig. 2. While T does increase as y_0 increases, the line topology remains the same.

Varying m. Now that we know that mass actually does have an impact on how long an object takes to fall we need to look at what mass governs. Namely, mass scales the drag force. Given that the drag force is scaled by mass we can immediately intuit that as $m \to \infty$ the $F_{drag} \to 0$ which results in our idealized equation for $y(t)$ we are used to. We can see this effect visually in Fig. 3 where a mass range over several magnitudes is plotted. While the difference between $m = 100$ kg and $m = 10$ kg is roughly 2T, the difference between $m = 100$ kg and $m = 1000$ kg is only ~ 2.5 s. Comparing the difference between $m = 1000$ kg and $m = 10,000$ kg is even smaller at ~ 0.1 s, which is our time step and so we have encountered the limit of our numeric approximation and for any larger mass could just set $F_{drag} = 0$ with no loss of accuracy.

Varying F_{drag}. Just like varying y_0, we would expect that drag has a significant impact on T. Looking back at Eq. 11 & 12 we

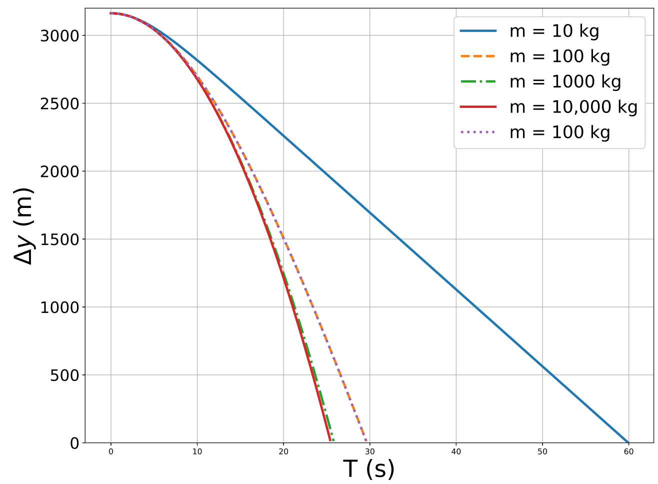

Fig. 3. Mass testing. As mass increases it also approaches a limit where it effectively makes the drag force term zero and becomes the idealized projectile motion.

see that drag accelerates the objects in the direction opposite to that of the direction of motion. Fig. 4 shows the effect increasing the drag coefficient several times over a magnitude has on T. Looking at Fig. 4 qualitatively we can see a linear relation to the drag coefficient and the difference between that T and the base T. $T_{5F} - T_F$ is ~2.5 times larger than $T_{5F} - T_F$ and $T_{10F} - T_{5F}$ is ~2 times larger again.

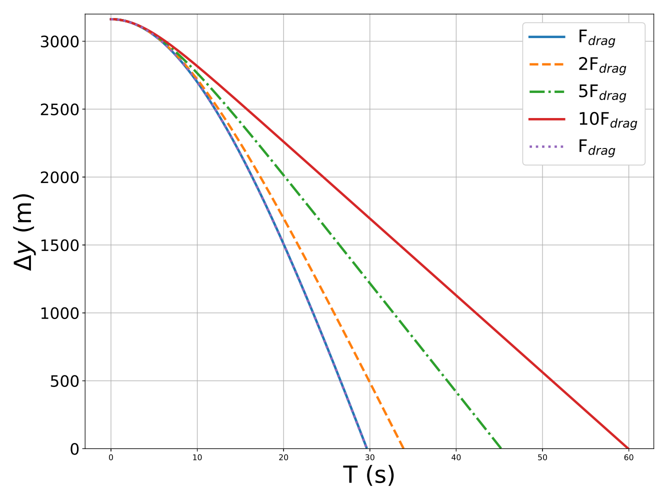

Fig. 4. Drag testing. As drag increases T increases linearly with the coefficient. There is also a limit to the drag coefficient as you can make it not physical if you aren't careful.

Numeric Accuracy. Since we are using an approximation instead of an analytic solution we want to ensure that our numeric approach has a high degree of accuracy. If it doesn't then we aren't approximating the physical system properly.

To check to make sure our numeric method works we can turn off our drag force to check against exact solutions from equations without drag. As seen in the table below, the average difference between the numeric and analytic solutions with $F_{drag} = 0$ is 0.0392% which is pretty accurate. In addition to checking our own calculations, we can also test against one value provided to us at $t = 3.0$ s. The given value for $t = 3.0$ s is $y = 3118.675$ while our numeric approximation for the same t is $y = 3119.604$ which is a 0.0297% difference.

4. Conclusion

If we don't consider drag we are only left with two parameters to vary to determine fall times: initial height and initial velocities. This, while providing a general approximation of projectile motion, can be improved upon by incorporating drag forces.

Because drag cannot be solved explicitly we need do numerically approximate the solution by employing Euler's method. In doing so we can see that the numeric method is on average within a 0.03% difference of the analytic solution so it approximates the analytic solution well.

In addition we can see that varying the four parameter mass, v_x, y_0, and the drag coefficient all have significant impacts into the fall time T. Beyond the expected influence changing y_0 has, the most influential is the drag coefficient itself. Because of this significance and the fact that the drag coefficient can't be explicitly calculated, one can see why it is usually neglected.

t (s)	y_exact (m)	y_numeric (m)	% Difference
2.0	3142.65766	3143.63866	0.031216
2.1	3140.64661	3141.67666	0.032797
2.2	3138.53746	3139.61656	0.034382
2.3	3136.33021	3137.45836	0.035970
2.4	3134.02486	3135.20206	0.037562
2.5	3131.62141	3132.84766	0.039157
2.6	3129.11986	3130.39516	0.040756
2.7	3126.52021	3127.84456	0.042359
2.8	3123.82246	3125.19586	0.043965
2.9	3121.02661	3122.44906	0.045576
3.0	3118.13266	3119.60416	0.047192

Table 1. Table of y values for both exact and numeric solutions for $2 \leq t \leq 3$ seconds.