Self-Organized Criticality modelled with Cellular Automata

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1. Introduction

Cellular automata is a discrete modeling approach useful in computational approaches to problems like self-organized criticality which seems to be the underlying mechanism behind nonlinear diffusion dynamics (1). Self-organized criticality is defined by extended spatial and temporal degrees of freedom that evolve to a critical state without the need to tune any external parameters. A critical state is defined as an attractor in a dynamic system or as the state where each degree of freedom keeps each other in balance but are still susceptible to small perturbations or "noise".

This noise has several names: $1/f$ noise, flicker noise, and pink noise. Regardless of the name, the noise is defined with a power-law spectrum

$$S(f) \approx f^{-\beta}$$

for both spatial and temporal scaling at low frequencies. A sandpile is a good model to use as it evolves towards a critical state as you add more grains to it. Any added grain can cause nothing to occur, a single collapse, or an avalanche. Once the system has evolved to its critical state it is defined by the "lack of a characteristic length scale [which] leads directly to a lack of a characteristic time scale for the fluctuations" (1). This essentially means that at the critical state avalanches of all sizes and durations occur.

In this paper we explore self-organized criticality by investigating $1/f$ noise by adding grains of sand to a sand pile and looking at the statistics of the collapses. In Sec. 2 we discuss the sand pile model, its rule set, boundary conditions, and introducing damage to the lattice. Results are presented in Sec. 3 and conclusion in Sec. 4.

2. Sand Pile Model

Sand Piles. To explore nonlinear diffusion dynamics we simplify the problem and explore the behavior of a "sandpile", which as Bak, Tang, and Wiesenfeld (BTW) pointed out, is "much simpler to study than continuous partial differential equations" (1). We start out by defining our cellular automata on a square lattice of 100×100. On this lattice, every grid point is defined by a number, say z_{ij}. If $z_{ij} \geq z_c$, where z_c is the critical value or height, then that grid point collapses. This collapse causes the grid point in question to lose sand while its neighbors gain sand following the rules

$$\begin{aligned} z[i,j] &= z[i,j] - 4, \\ z[i \pm 1, j] &= z[i \pm 1, j] + 1, \\ z[i, j \pm 1] &= z[i, j \pm 1] + 1. \end{aligned} \quad [1]$$

While z_{ij} is the height of the sandpile the gradient is what actually determines the distribution of sand. However, in addition to capturing the qualitative properties, this model also captures some features of the self-criticality.

The model is driven by adding a single grain of sand to a random site on the sandpile. After a grain is added the system is allowed to relax if an avalanche occurred (see Fig. 1). Randomly adding grains and causing avalanches simulates "[t]he "$1/f$" noise [which] is the dynamical response of the sandpile to small random perturbations" (2). This process is repeated for a large number of grains, which for this paper was set to model Giordano and Nakanishi (GN) (3) at 10,000 grains.

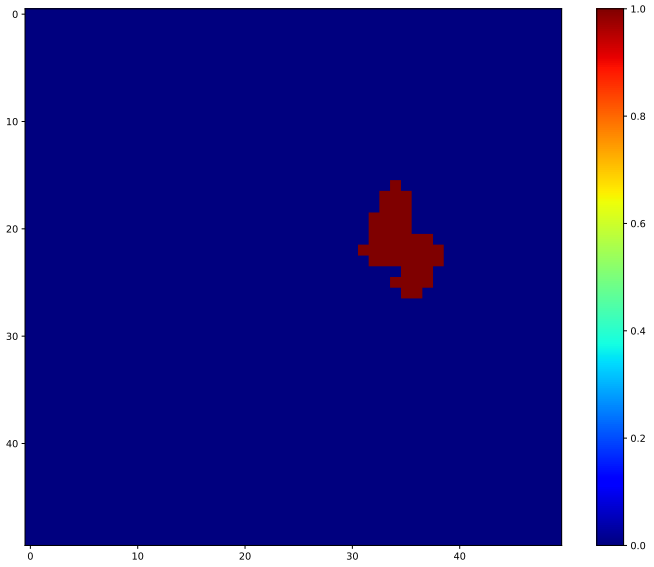

Fig. 1. Example of an avalanche (red) which occurred after 4913 grains were added.

Initial Conditions. As mentioned in Sec. 1, self-organized critical states do not require the tuning of initial parameters. Thus, after adding a large number of grains the sandpile will evolve towards a critical state. This means that we can initialize the model with any initial (external) conditions without affecting the critical state, as these states are virtually indistinguishable (3). For this paper each sandpile was initialized randomly with $2 \leq z_{ij} \leq 3$ as shown in the top left of Fig. 2.

Boundary Conditions. While the initial conditions do not play a large role in the sandpile's behavior, boundary conditions do. Two different boundary conditions could be implemented with two different outcomes: free boundaries and periodic boundaries. For periodic boundaries, any sand that fell off

the edges would appear on the opposite side of the pile. This means that the system would lose no grains (energy) and all grid points would eventually get to values of $z_{ij} > z_c$ causing avalanches everywhere constantly. This highly unstable system is not physical nor does it attain a critical state, so we allow the system to lose grains and employ free boundaries.

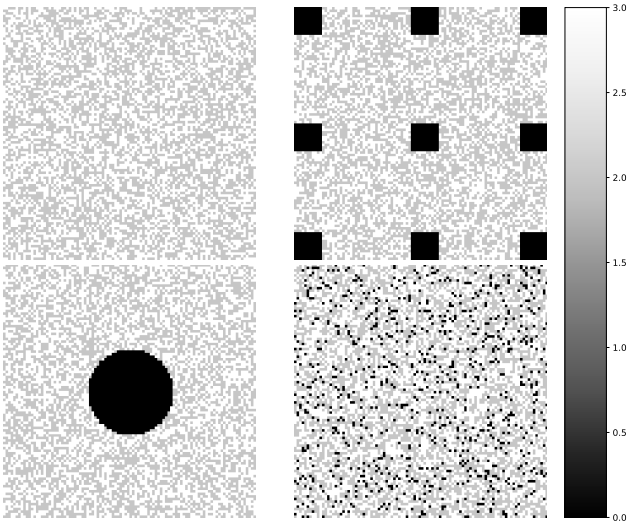

Fig. 2. Randomized 100×100 grids with $2 \leq z \leq 3$ with either no damage or 9% damage ($z = 0$, in black). *Top Left* features no damage, *top right* features nine "columns" of damage, *bottom left* features circular damage, and *bottom right* features random damage.

Damage. In addition to exploring a sandpile growing on a lattice, it is also interesting to explore how "damage", areas where grains are not allowed to be placed or collapse onto, affect the spatial and temporal degrees of freedom. To do this three types of damaged lattices were implemented: column, radial, and random. To construct damage for each type, a total damage area of 900 grid points or 9% of the lattice area was introduced by setting all z_{ij} in these areas to zero (see Fig. 2). The column type features nine 10×10 grids evenly spaced on the lattice, the radial type is a circle of radius 17 centered on the middle of the lattice and the random type picks 900 random points on the lattice.

The rule set for adding grains and collapsing is slightly different than that of the non-damage case. Eqn. 1 is modified to

$$z[i,j] = z[i,j] - n,$$
$$z[k,l] = z[k,l] + 1, \qquad [2]$$

where $z[k,l]$ corresponds to neighboring sites of $z[i,j]$ and n is the number of neighbors $z_{kl} > 0$. The number of non-zero neighbors is subtracted from the collapsing site and one grain is placed in each non-zero neighbor. These rules ensure that no grains are added onto the damage sites (one can think of them as unscalable mountains) while also allowing for grains to leave the system via the edges of the lattice.

3. Results

Several key statistics can be explored for each model such as the frequency of avalanches, the distribution of durations, and the frequency as a function of duration of avalanches. In addition to the single model statistics it is interesting to compare all the models to each other.

Avalanche Frequency vs. Size. The frequency of avalanches depends on their size by the power-law

$$N(s) \approx s^{-\tau} \qquad [3]$$

as shown by Bak, Tang, and Wiesenfeld (BTW) in their 1988 paper (1). To obtain the exponent of a power-law relation the log (base 10) of both number and size were taken and a linear regression applied to find a best-fit line. The slope of the best-fit line is the desired power-law exponent. This exponent was averaged over ten iterations for each model and the values are shown in the table below.

	BTW	No Damage	Column	Radial	Random
τ	-1.000	-1.079	-1.063	-1.216	-1.103

Table 1. Average values of τ over 10 iterations for all models.

The values for all the models are fairly close to that obtained by BTW (1) with the exception the radial model. This seems natural as restricting the area in which grains can be placed raises the average z in the non-damaged areas leading to many larger avalanches as shown in Fig. 3 below.

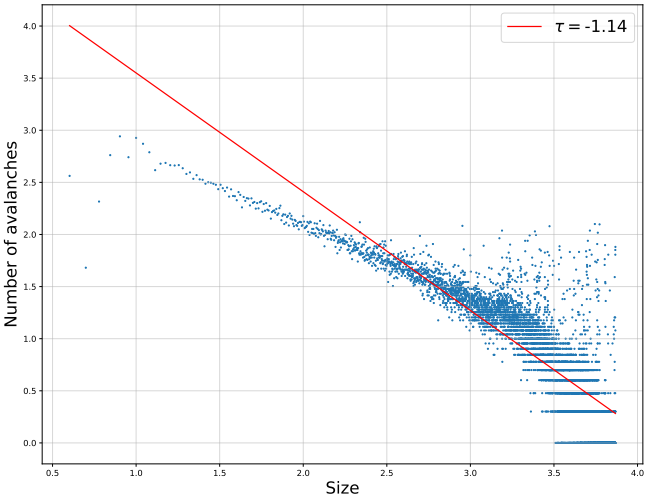

Fig. 3. Frequency vs. Size log−log plot for the random model after 10,000 grains added with τ being the slope of the line. Due to there being no characteristic length, many sizes are represented leading to a steeper slope.

Avalanche Duration vs. Size. While the frequency of avalanches is a power-law with a negative exponent, the duration has a positive slope

$$T(s) \approx s^{a} \qquad [4]$$

with $a \approx 0.65$ (3). As above, the log (base 10) was taken for both duration and size of avalanche and a linear regression applied. Each model was averaged over 10 iterations with the results in the table below.

	GN	No Damage	Column	Radial	Random
a	0.65	1.298	1.338	1.305	1.309

Table 2. Average values of a over 10 iterations for all models.

Of note here is that all the models returned a value of $a \sim 1.3$ while the value from GN was $a = 0.65$ (3). Fig. 4 sheds a little bit of light as to why this is the case.

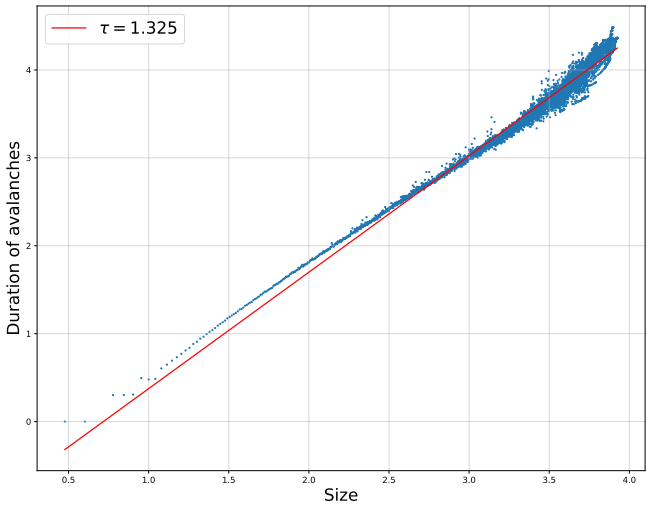

Fig. 4. Duration vs. Size log−log plot for the column model after 10,000 grains added with τ being the slope of the line. The critical state allows for all duration sizes (in a finite regime) rendering $\tau > 1$.

Between Fig. 4 and Fig. 12.46 (right) in GN (3) there is a difference in both the spatial and temporal dimensions. In Fig. 4 the size of an avalanche runs between (in log) $0.5 - 3.8$ whereas in GN it runs between $0.8 - 3.8$. The durations are also different as Fig. 4 runs from (in log) $0 - 4.7$ while in GN it runs from $1 - 2.7$. The plot is GN also only has nine points plotted using only one simulation and is vague on how exactly the plot was made. In contrast, Fig. 4 has more than 1000 points averaged over ten simulations.

Avalanche Frequency vs. Duration. Like the frequency above, the mean frequency per cell as a function of mean duration per cell is also given as a negative power-law

$$N(t) \approx t^{-b} \qquad [5]$$

with the average duration, t, replacing the size, s, from Eqn. 3. The log of both axes were taken and linear regression applied. The values for all models averaged over ten simulations are given in the table below.

	GN	No Damage	Column	Radial	Random
b	-1.1	-1.000	-1.000	-1.000	-1.000

Table 3. Average values of b over 10 iterations for all models.

GN calculated that $b \approx -1.1$ (3) whereas every model in this paper had an average of $b = -1.0$ (see Fig. 5). This is likely due to a variety of differences between this paper and GN such as the scale of the data and the number of simulations performed. However, these factors were mentioned in the previous section are encapsulated in the larger value for the exponent a.

Also of interest is that the average for each model was the same at $b = -1.0$. The no damage case, which is identical to that in GN, has a smaller b but GN approximates their b so the difference could be anywhere between $5 - 10\%$ which suggests the methods of this paper are robust. It is not clear why the damage cases also have $b = -1.0$ but it seems to imply that having damage on the lattice (at least to the level applied here) does not influence $N(t)$ at all.

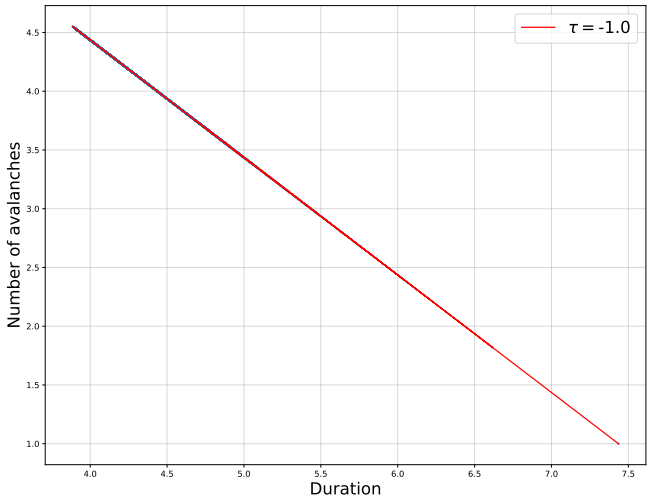

Fig. 5. Frequency vs. Duration log−log plot for the no-damage model after 10,000 grains added with τ being the slope of the line. This plot suggests that the number of avalanches is exactly inversely proportional to the duration of the avalanche.

4. Conclusion

Self-organized criticality is characterized by its spatial and temporal power-law scaling behavior called $1/f$ noise (alternatively flicker or pink noise). This noise in a dynamic system can be modeled using cellular automata on a 2D-lattice. We replicated the work done by Bak, Tang, and Wiesenfeld in their 1988 paper *Self-organized criticality* and Giordano and Nakanishi in the text *Computational Physics*, finding results similar in two of three cases. The significant difference occurred in the avalanche duration vs. size as the calculated exponent was, on average, twice as high as that of GN. To note, however, is that the data used more than one simulation, used more data points, and had a different scale. We also showed that initial conditions did not have a large impact on the models as they all evolved to a critical state, just with slightly different time scales. This confirmed BTW's result that critical states have no characteristic length or time scale for the avalanches over a finite regime.

In addition to replicating work done in BTW and GN we also added damage to the lattice to see what changed. We found that adding damage to the lattice did not significantly influence any of the statistics studied here. This could be due to lack of scaling for number of grains to the free area on the lattice in which the grains could be placed. Alternatively it could be due to removing area in which grains can be placed influences the time scale at which the models would reach the critical state.

1. Bak P, Tang C, Wiesenfeld K (1988) Self-organized criticality. *Phys. Rev. A* 38(1):364–374.
2. Bak P, Tang C, Wiesenfeld K (1987) Self-organized criticality: An explanation of the 1/f noise. *Phys. Rev. Lett.* 59(4):381–384.
3. Giordano NJ, Nakanishi H (2006) *Computational Physics.* (Pearson Education, Inc.).

Appendix

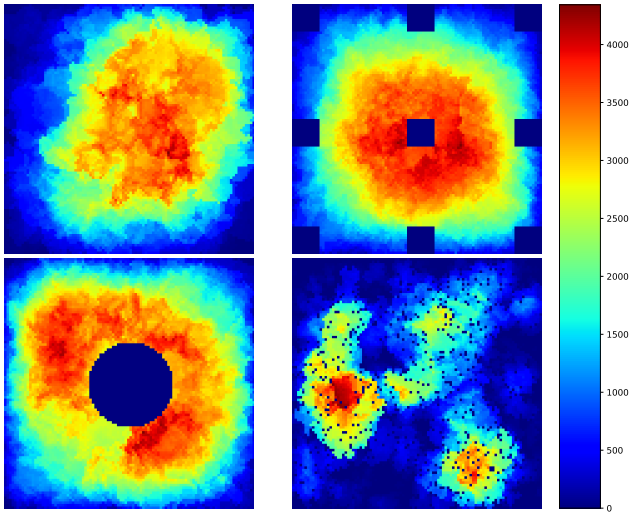

Fig. 6. Number of avalanches per cell after 10,000 grains added. *Top Left* No-damage model, *Top Right* column model, *Bottom Left* radial model, and *Bottom Right* random model.

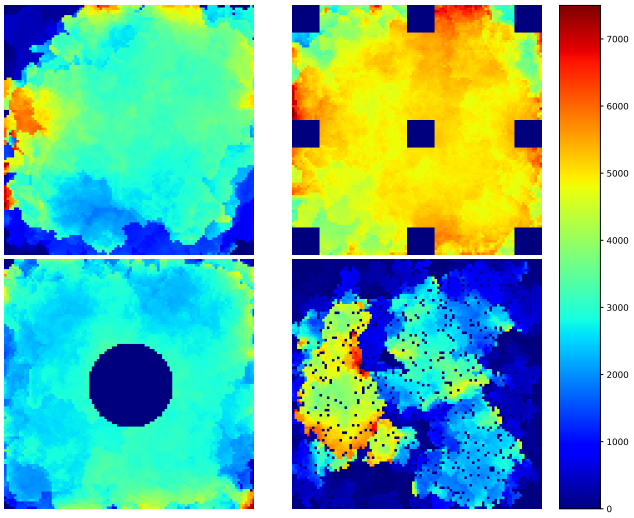

Fig. 7. Average size of avalanches per cell after 10,000 grains added. *Top Left* No-damage model, *Top Right* column model, *Bottom Left* radial model, and *Bottom Right* random model.

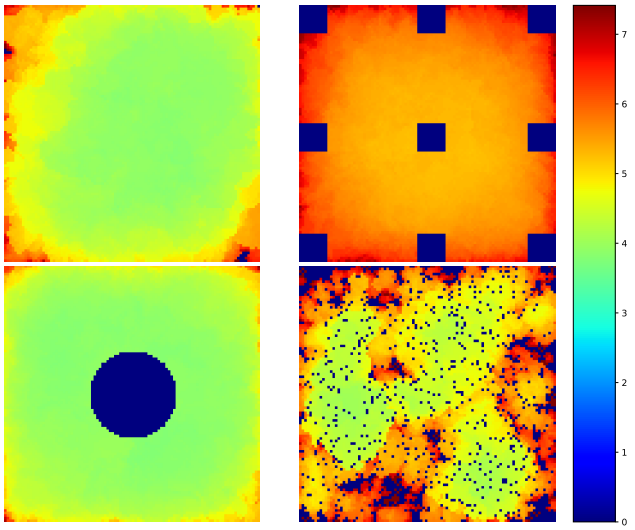

Fig. 8. Average duration of avalanches per cell after 10,000 grains added. *Top Left* No-damage model, *Top Right* column model, *Bottom Left* radial model, and *Bottom Right* random model.