Homework 1: Nuclear Decay

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1. Introduction

Radioactive decay is one of the staple population models used in physics. For single isotope models, they are easily and accurately expressed using exponentials but can also be expounded upon by adding another isotope. This added layer of complexity is great for introducing Euler's method of iteratively solving problems as the solutions to these problems are often very hard to solve analytically.

In this paper we investigate a model of radioactive decay for a parent isotope and its daughter by expanding upon the simple model of one isotope radioactive decaying. If the parent decays faster than the daughter, $\tau_A > \tau_B$, then the population ratio $N_B/N_A \to \infty$. Conversely, if $\tau_A < \tau_B$ then $N_B/N_A \to 0$.

In Sec. 2 we describe and extend the simple radioactive decay model using Euler's method. In Sec. 3 we discuss the influence that γ has on age estimates and conclude in Sec. 4.

2. Model

Single Isotope Radioactive Decay. Radioactive decay of an single species of atom is given by the ordinary differential equation (ODE),

$$\frac{dN(t)}{dt} = -\frac{N(t)}{\tau} \quad [1]$$

where $N(t)$ is the population at time t and τ is the decay constant. Integrating this equation then gives us,

$$N(t) = N_0 e^{-t/\tau} \quad [2]$$

where N_0 is the initial population.

Two Isotope Radioactive Decay. For a system of one parent species decaying and a population of a daughter species growing, the ODEs are,

$$\frac{dN_A}{dt} = -\frac{N_A}{\tau_A} \quad [3]$$

$$\frac{dN_B}{dt} = \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B} \quad [4]$$

with analytic solutions,

$$\frac{N_A(t)}{N_A(0)} = e^{-T} \quad [5]$$

$$\frac{N_B(t)}{N_A(0)} = \begin{cases} \frac{N_B(0)}{N_A(0)} e^{-\gamma T} + \frac{e^{-T} - e^{-\gamma T}}{\gamma - 1}, & \gamma \neq 1 \\ \frac{N_B(0)}{N_A(0)} e^{-T} + T e^{-T}, & \gamma = 1 \end{cases} \quad [6]$$

where these solutions are scaled by $N_A(0)$ with $T = t/\tau_A$ and $\gamma = \tau_A/\tau_B$.

Numerical Solution. For this problem we are unable to measure either population $N_A(t)$ or $N_B(t)$ alone but we can measure their ratio N_B/N_A. To accomplish this we need to take a numerical approach by employing Euler's method, iterating over small increases in time. Using Eq. 1 as a template for Euler's method we get,

$$N(t + \Delta t) \approx N(t) + \frac{dN(t)}{dt}\Delta t$$

Applying this to both Eq. 3 and Eq. 4 gives us,

$$N_{A,i+1} = N_{A,i} - \frac{N_{A,i}}{\tau_A}\Delta t \quad [7]$$

$$N_{B,i+1} = N_{B,i} + \frac{dN_{B,i}}{dt}\Delta t$$
$$= N_{B,i} - \frac{N_{B,i}}{\tau_B}\Delta t + \frac{N_{A,i}}{\tau_A}\Delta t \quad [8]$$

and scaling them by $N_A(0)$ gives us

$$\frac{N_{A,i+1}}{N_A(0)} = \frac{N_{A,i}}{N_A(0)} - T\frac{N_{A,i}}{N_A(0)} \quad [9]$$

$$\frac{N_{B,i+1}}{N_A(0)} = \frac{N_{B,i}}{N_A(0)}(1 - \gamma T) + T\frac{N_{A,i}}{N_A(0)} \quad [10]$$

where $T = \Delta t/\tau_A$, $\gamma = \tau_A/\tau_B$, and Δt is our time-step.

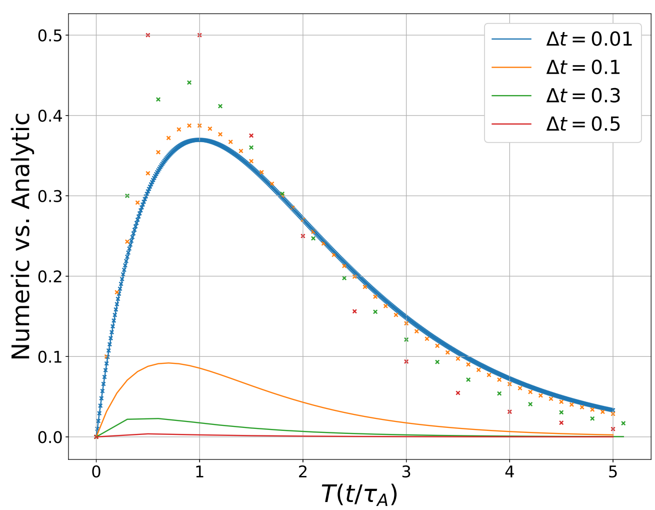

Fig. 1. Time step testing for $\gamma = 1$ (assuming an initial population of 1000 for testing purposes). The difference between the numeric and analytic solutions increases as Δt increases while the difference shrinks as the time-step shrinks. Here the time-step of $\Delta t = 0.01$ was chosen.

We now have two equations that iterate over time-steps to determine $N_A(t)$ and $N_B(t)$, scaled by $N_A(0)$ noting that we haven't found the analytic solution, just how to calculate an approximation step-by-step. The caveat here is that we would have to define $N_A(0)$ otherwise there is no iteration. To avoid this issue we can then get a ratio of B to A with $N_B(t)/N_A(t)$ which removes the need to know $N_A(0)$. Applying this we get,

$$\frac{N_{B,i+1}}{N_{A,i+1}} = \frac{N_{B,i}\,(1-\gamma T)}{N_{A,i}(1-T)} + \frac{T}{1-T} \qquad [11]$$

which uses the fraction that we can measure, N_B/N_A, to determine step-wise what the ratio will be at some future T.

Varying T. Now that we have equations to use we need to determine what the proper time-step is. In Fig. 1 several values for Δt were tested against their exact solutions (setting $\gamma = 1$ as it is the steady-state solution). As Δt gets smaller the difference between the numeric and analytic solutions also shrinks. $\Delta = 0.01$ was chosen as it seemed to fit the analytic solution reasonably well, only differing around the peak at $T = 1$, and decreasing it further didn't have any significant changes. In addition, having $\Delta t = 0.01$ means that over $5T$ there are 500 individual time-steps and the numeric solution matches the analytic solution's long-term behavior well.

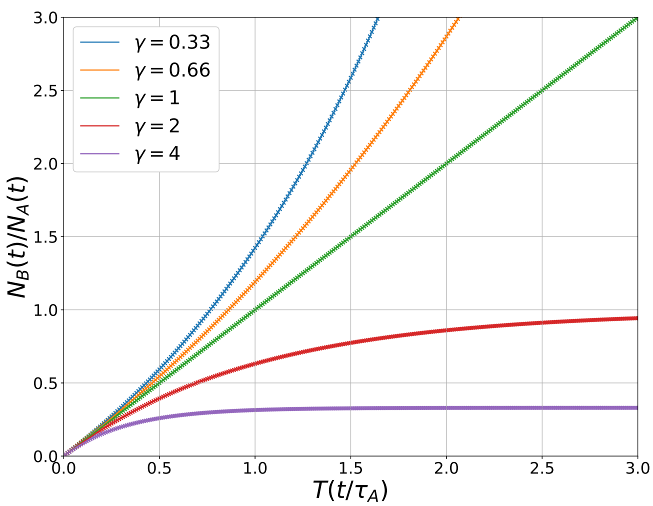

Fig. 2. Gamma testing for $N_B(t)/N_A(t)$. Several γ values were tested with the analytic solutions given as solid lines with the corresponding numeric solutions in the same color as x-ticks. γ didn't seem to affect the difference between numeric and analytic solutions appreciably.

3. Results

Varying γ. Now that we have nailed down what Δt is, we have reduced our equation to one variable, γ. In Fig. 2 five values for γ have been plotted: B decays faster than A $\tau_B > \tau_A$ ($\gamma < 1$), A decays faster than B $\tau_B < \tau_A$ ($\gamma > 1$), and the steady-state solution ($\gamma = 1$). As shown in the figure, γ seems to play little, if any, role in the numeric solution matching the analytic, which is surprising.

However, what γ does do is change the slope of the line, which directly determines how accurate an age estimate we can hope to get.

T Accuracy. To determine how γ changes the accuracy at which we can age of an object we first start by picking a point along a γ line. Then, adopting the 0.5% error specified, we find the y-values 0.05% away from our point and then trace those back to the x-axis. Doing so translates our 0.5% error from our measured ratio over to our age estimate T. As shown in Fig. 3 plotting windows for our y and x errors we get smaller windows as $\gamma \to 0$ and larger windows for $\gamma \to \infty$. Thus for $\gamma > 1$ our age estimate error grows while for $\gamma < 0$ it gets smaller due to the slope of the line (which γ determines).

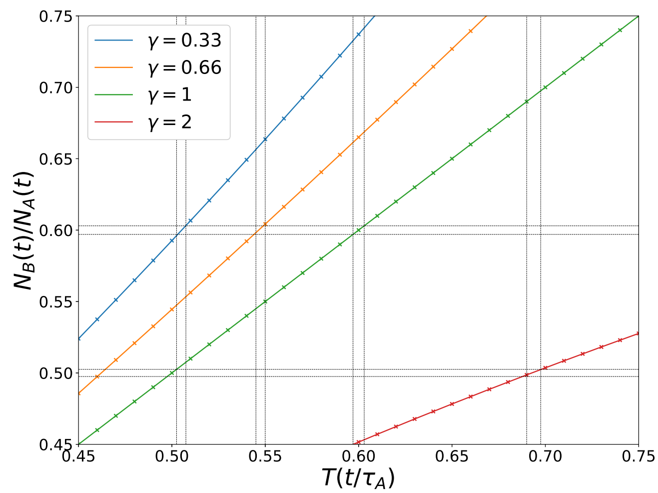

Fig. 3. Age estimation (T) error. Lines of 0.5% error were drawn horizontally to each γ line and then traced back to the x-axis to determine the error of T. The T error for increasing γ is (from left to right) ~ 0.005, ~ 0.05, ~ 0.06, and ~ 0.075 respectively.

4. Conclusion

We investigated a radioactive decay model with one parent isotope and one daughter isotope modeled by two ODEs. We solved these numerically by employing Euler's method and found that our age estimates depended not on our Δt but on γ. We could directly measure the ratio N_B/N_A directly and apply a static error threshold to determine how γ changes the age estimate error. As γ diverges from 1, the population ratios asymptotically approach their own steady-states. Thus, after a certain amount of half-lives using this method for an age estimate breaks down.

Physically, however, we would just pick an isotope with a very long half-life that also decays into a daughter isotope such that γ is as close to one (or slightly less than) as possible, thereby keeping the age estimate errors constrained.