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Matrices Using Python

Problem Statement:

Consider a circle with it's centre lying on the focus of parabola $y^2 = 2px$.such that it touches the directrix of the parabola.The point of intersection of the circle and parabola is ?

SOLUTION:

Given:

Equation of Parabola is

$$y^2 = 2px \tag{1}$$

From (1) we can say that Parabola is concave towards positive x axis.
For the given equation of parabola,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -p \\ 0 \end{pmatrix} \ \& \ f = 0 \tag{2}$$

The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D} \qquad \mathbf{P} = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} \end{pmatrix} \tag{3}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{4}$$

On solving (3) with
$\mathbf{P_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\mathbf{P_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

where,

$$\lambda_1 = 0 \ and \ \lambda_2 = 1 \tag{6}$$

We have,

$$eccentricity, \quad e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{7}$$

from (6),

$$e = 1 \tag{8}$$

Normal vector of diretrix $\mathbf{n}$ is given by

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{P_1} \tag{9}$$

This gives,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{10}$$

For e=1, we have

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} \tag{11}$$

On solving we get,

$$c = -p/2 \tag{12}$$

Focii of a conic is given by the equation,

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{13}$$

$$\mathbf{F} = \begin{pmatrix} p/2 \\ 0 \end{pmatrix} \tag{14}$$

The directrix of the parabola is

$$\mathbf{n}^\top \mathbf{X} = \mathbf{c} \tag{15}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{X} = \mathbf{c} \tag{16}$$

To Find

To find the intersection points of the circle and parabola

STEP-1

The given circle and parabola can be expressed as conics with parameters,
For circle,

$$\mathbf{V}_1 = \mathbf{I} \tag{17}$$

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{18}$$

$$\mathbf{u_1} = \begin{pmatrix} -p/2 \\ 0 \end{pmatrix} \tag{19}$$

$$f_1 = -3p^2/4 \tag{20}$$

For Parabola,

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{21}$$

$$\mathbf{u_2} = \begin{pmatrix} -p \\ 0 \end{pmatrix} \tag{22}$$

$$f_2 = 0 \tag{23}$$

STEP-2

The intersection of the given conics is obtained as

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} \tag{24}$$
$$+ (f_1 + \mu f_2) = 0 \tag{25}$$

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \mu + 1 \end{pmatrix} \tag{26}$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = - \begin{pmatrix} p/2 + \mu p \\ 0 \end{pmatrix} \tag{27}$$

$$f_1 + \mu f_2 = -3p^2/4 \tag{28}$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{29}$$

And,

$$\tag{30}$$

Substituting equation (25),(26) and (27) in equation (28)

We get,

$$\implies \begin{vmatrix} 1 & 0 & -p/2 - \mu p \\ 0 & 1 + \mu & 0 \\ -p/2 - \mu p & 0 & -3p^2/4 \end{vmatrix} = 0 \tag{31}$$

Solving the above equation we get,

gives,

$$\mu = -1 \tag{32}$$

Thus, the parameters for a straight line can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{33}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} p/2 \\ 0 \end{pmatrix} \tag{34}$$

$$f = -3p^2/4, \tag{35}$$

$$\implies \mathbf{D} = \mathbf{V}, \mathbf{P} = \mathbf{I} \tag{36}$$

Thus, the desired pair of straight lines are

$$\left(\sqrt{|\lambda_1|} \quad \pm \sqrt{|\lambda_2|}\right) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{37}$$

$$\implies \begin{pmatrix} 0 & \pm 1 \end{pmatrix} \mathbf{x} - \mathbf{c} = 0 \tag{38}$$

$$\text{or, } \mathbf{x} = \mathbf{c} + \kappa \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \tag{39}$$

The points of intersection of the line is given by,

$$L: \quad \mathbf{x} = \mathbf{q} + \kappa \mathbf{m} \quad \kappa \in \mathbb{R} \tag{40}$$

with the conic section,

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{41}$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \kappa_i \mathbf{m} \tag{42}$$

where,

$$\kappa_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{43}$$

On substituting

$$\mathbf{q} = \begin{pmatrix} p/2 \\ -3p/2 \end{pmatrix} \tag{44}$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{45}$$

With the given circle

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{46}$$

$$\mathbf{u} = \begin{pmatrix} -p/2 \\ 0 \end{pmatrix} \tag{47}$$

$$f = -3p^2/4 \tag{48}$$

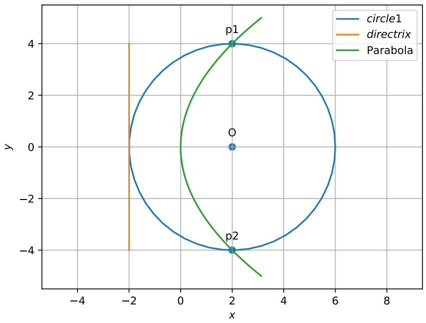
The value of κ_i

$$\kappa_i = 5p/2, p/2 \tag{49}$$

The points of intersection with Parabola along circle are

$$\mathbf{A} = \begin{pmatrix} p/2 \\ p \end{pmatrix} \tag{50}$$

$$\mathbf{B} = \begin{pmatrix} p/2 \\ -p \end{pmatrix} \tag{51}$$



Construction

Points	coordinates
A	$\begin{pmatrix} p/2 \\ p \end{pmatrix}$
B	$\begin{pmatrix} p/2 \\ -p \end{pmatrix}$

*Verify the above proofs in the following code.

`https://github.com/ballepu1994`